Trägheitsmomente

Trägheitstensor

im körperfesten System $\mathbf{I}' = -\int_K \tilde{\mathbf{r}}' \cdot \tilde{\mathbf{r}}' \, \mathrm{dm} = \begin{bmatrix} I'_{xx} & I'_{xy} & I'_{xz} \\ I'_{xy} & I'_{yy} & I'_{yz} \\ I'_{xz} & I'_{yz} & I'_{zz} \end{bmatrix} = \mathbf{I}'^T$

im Inertialsystem (Koordinatentransformation) $\mathbf{I} = \mathbf{S} \cdot \mathbf{I}' \cdot \mathbf{S}^T = \mathbf{I}^T$

Direkte Berechnung

Massenträgheitsmomente

$$I'_{xx} = \int_K (y^2 + z^2) \, \mathrm{dm}$$
$$I'_{yy} = \int_K (z^2 + x^2) \, \mathrm{dm}$$
$$I'_{zz} = \int_K (x^2 + y^2) \, \mathrm{dm}$$

Massendeviationsmomente

$$I'_{xy} = -\int_K xy \, \mathrm{dm}$$
$$I'_{yz} = -\int_K yz \, \mathrm{dm}$$
$$I'_{xz} = -\int_K xz \, \mathrm{dm}$$

Dreiecksungleichungen

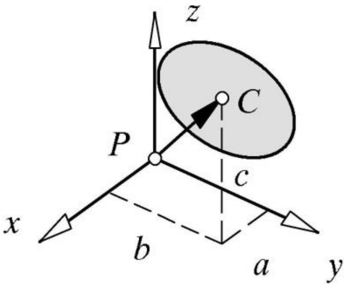
$$I'_{xx} + I'_{yy} \geq I'_{zz}$$
$$I'_{yy} + I'_{zz} \geq I'_{xx}$$
$$I'_{zz} + I'_{xx} \geq I'_{yy}$$

speziell: Im Hauptachsensystem verschwinden die Deviationsmomente (Achsen senkrecht zu Symmetrieebenen sind Hauptachsen)

$$I^H = \begin{bmatrix} I^H_{xx} & 0 & 0 \\ 0 & I^H_{yy} & 0 \\ 0 & 0 & I^H_{zz} \end{bmatrix}$$

Vereinfachte Berechnung für zusammengesetzte Körper

$$K = \cup_i K_i \quad : \quad \mathbf{I}' = \sum_i \mathbf{I}'_i$$

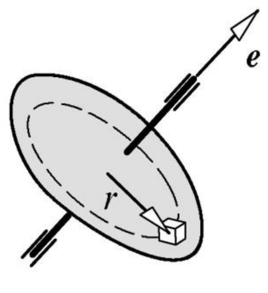


Huygens−Steiner Gleichungen

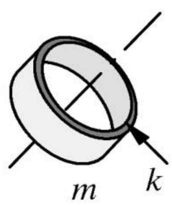
(Wechsel des Bezugspunktes, Schwerpunkt C) $\mathbf{I}_P = \mathbf{I}_C + m \begin{bmatrix} b^2 + c^2 & -ab & -ac \\ -ab & c^2 + a^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{bmatrix}$

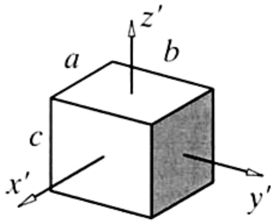
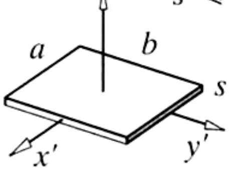
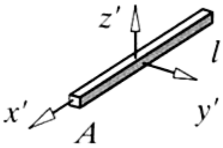
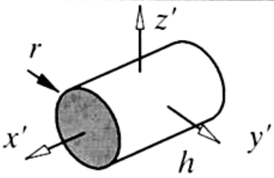
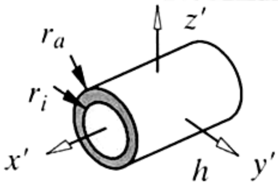
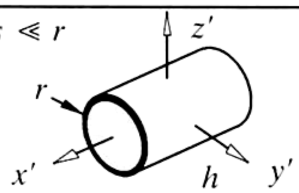
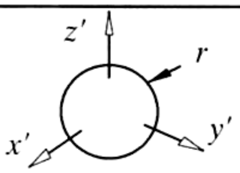
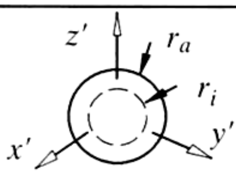
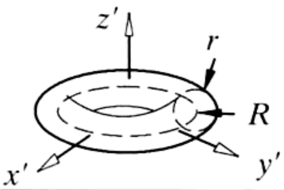
Trägheitsmoment bezüglich einer Achse $\mathbf{e}$ ($\|\mathbf{e}\| = 1$)

$$I_e = \int_K r^2 \, \mathrm{dm} = \mathbf{e} \cdot \mathbf{I} \cdot \mathbf{e}$$



Trägheitsradius k: $I_e \overset{!}{=} k^2 m \Rightarrow k = \sqrt{\frac{I_e}{m}}$



Körper	Geometrie	Masse	Trägheitsmomente
Quader		$m = \rho abc$	$I^{\prime}_{xx} = \frac{m}{12}(b^2 + c^2)$ $I^{\prime}_{yy} = \frac{m}{12}(c^2 + a^2)$ $I^{\prime}_{zz} = \frac{m}{12}(a^2 + b^2)$
Platte	$s \ll a, b$ 	$m = \rho abs$	$I^{\prime}_{xx} = \frac{m}{12} b^2$ $I^{\prime}_{yy} = \frac{m}{12} a^2$ $I^{\prime}_{zz} = \frac{m}{12}(a^2 + b^2)$
dünner Stab		$m = \rho Al$	$I^{\prime}_{xx} = 0$ $I^{\prime}_{yy} = I^{\prime}_{zz} = \frac{ml^2}{12}$
Kreis-zylinder		$m = \rho\pi r^2 h$	$I^{\prime}_{xx} = \frac{1}{2} mr^2$ $I^{\prime}_{yy} = I^{\prime}_{zz} = \frac{m}{12}(3r^2 + h^2)$
Hohl-zylinder		$m = \rho\pi(r_a^2 - r_i^2)h$	$I^{\prime}_{xx} = \frac{m}{2}(r_a^2 + r_i^2)$ $I^{\prime}_{zz} = I^{\prime}_{yy} = \frac{m}{4}(r_a^2 + r_i^2 + h^2/3)$
Zylinder-schale	$s \ll r$ 	$m = \rho 2\pi rsh$	$I^{\prime}_{xx} = mr^2$ $I^{\prime}_{yy} = I^{\prime}_{zz} = \frac{m}{12}(6r^2 + h^2)$
Kugel		$m = \rho\frac{4}{3}\pi r^3$	$I^{\prime}_{xx} = I^{\prime}_{yy} = I^{\prime}_{zz} = \frac{2}{5}mr^2$
Hohl-kugel		$m = \rho\frac{4}{3}\pi(r_a^3 - r_i^3)$	$I^{\prime}_{xx} = I^{\prime}_{yy} = I^{\prime}_{zz} = \frac{2}{5}m\frac{r_a^5 - r_i^5}{r_a^3 - r_i^3}$
Kreis-torus		$m = \rho 2\pi^2 r^2 R$	$I^{\prime}_{xx} = I^{\prime}_{yy} = \frac{m}{8}(4R^2 + 5r^2)$ $I^{\prime}_{zz} = \frac{m}{4}(4R^2 + 3r^2)$