Numerical Linear Algebra
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Lecture - 33
Convergent Matrices- I

Hello friends. Welcome to the lecture, in this lecture, we will discuss some concept related to convergent matrices. This concept is very very useful in the analysis of some algorithm in numerical linear algebra. So, if you recall, in previous class, we have discussed the concept of norms and vector norms a vector norms and matrix norm. Now with the help of these norms, we try to define, what do you mean by convergent matrix, but before that let us discuss, what do we mean by convergent of a sequence of matrices.

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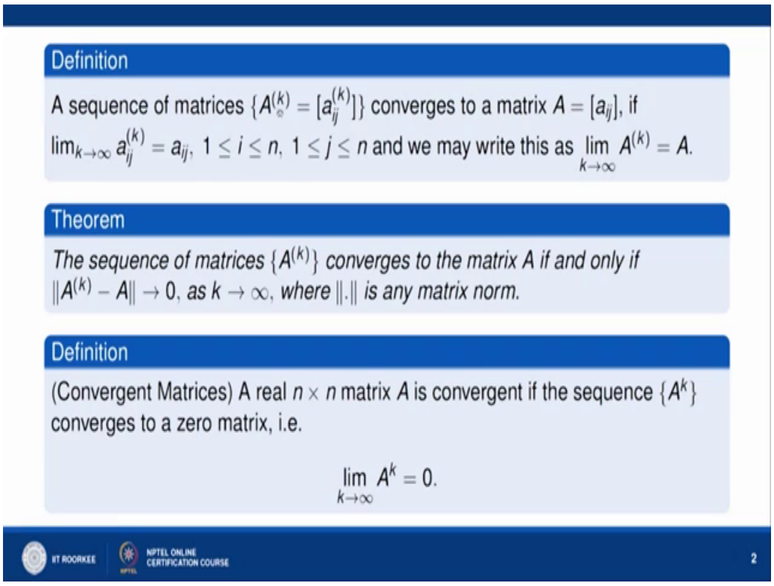

So, a sequence of matrices A k whose elements are looking like a i J k converges to a matrix A whose element is represented as a i j, if limit k tending to infinity, a i J k is equal to a i j for all i and j between 1 to n and we write this convergence that this A k converges to a as limit k tending to infinity A k as A. So, basically this sequence converges this sequence A k converges to A, if this convergence is happening element wise. So, it means that every i jth element of A k converges to i jth element of A here.

So, if you want to write in terms of norm, then we have this following theorem, we say that the sequence of matrices A k converges to the matrix A if and only if this norm of A k minus a is tending to 0, ask any k tending to infinity where this norm is any matrix norm. We have discussed a certain matrix norm one to provenance and infinity norm. So, this sequence converges to A k converges to matrix A if this convergence is happening in any of matrix norm. So, this is not very difficult if you look at what do we mean by that limit k tending to infinity A k is equal to a this means that this convergence of i jth element to the i jth element of the limit matrix. So, we can say that here we can apply the Frobenius norm and we can say that this is nothing, but that limit k tending to infinity Frobenius norm of A k minus A is tending to 0.

Now, and we have also discussed that the infinite dimension case all the matrix norms are same. So, if it is happening in terms of Frobenius norm then we can use any equivalent matrix norm and we know that in finite dimension all matrix norms are equivalent. So, it means that if it this convergence follows in Frobenius norm then this convergence will follow in any matrix norm..

So, we can justify this theorem, I am not providing any proof of this theorem now with the help of this general concept of convergence of matrices let us define the concept known as convergent matrices. So, here a real n cross n matrix A is said to be convergent if the sequence of powers of A means A to the power k converges to a 0 matrix. So, it means that if you look at what you have A, consider the sequence of powers of this. So, here A; A k is basically powers of A kth powers of A and we say that it can if it converges to 0 matrix, then we say that this matrix A is a convergent matrix and in the notation of this we can say that limit k tending to infinity A k is equal to 0.

So, this is how we define converges matrix.

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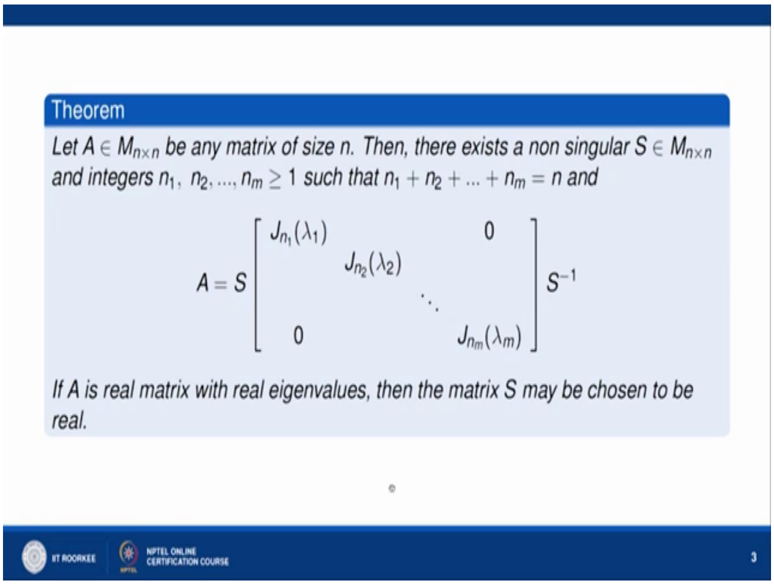

So, to find out say condition that under what condition this matrix A is convergent we have to recall, we will recall the concept known as Jordan canonical form. So, let us recall this theorem that let A is be any matrix of size n square matrix of size n, then they exist a non singular matrix S of size n and integers n 1 to n m greater than or equal to 1 says that some of these n i's is equal to n and A can be written as S J in j S inverse here this j is this matrix which is known as the Jordan canonical form corresponding to this matrix A here if is a real matrix with real Eigen values, then this matrix S which we are considering here may be chosen to be real. So, it means that if is real matrix with real Eigenvalues, then we can say that A can be written as S J S inverse where j is given by this block matrix.

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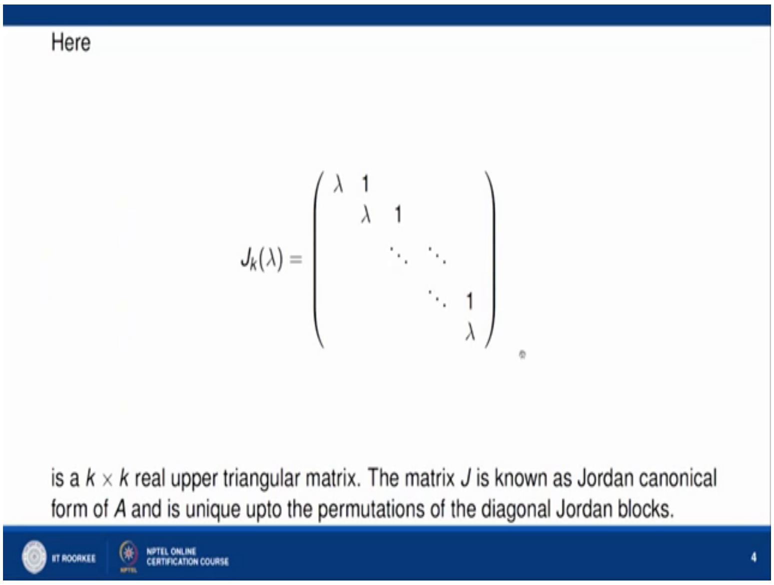

Now, here this block this J n 1 lamb J n 1 lambda 1 J n 2 lambda 2, these are block matrices corresponding to the Eigen values lambda 1, lambda 2, lambda m. So, here how this is J n i lambda i is it looking like. So, here this a Jordan block corresponding to Eigenvalue lambda is given by this, this is a upper triangular matrix. So, it is all zeros here and here we have zeros in diagonally we have lambda and in upper diagonal we have one rest all rest all the entries are 0 and this is A k cross k real upper triangular matrix the matrix j is known as Jordan canonical form of a and it is unique up to the permutation of the diagonal Jordan blocks. So, it means that it may happen that here this may commute do you here in place of the J n 1 lambda 1 we may have J n 2 lambda 2 and so on. So, this is unique up to the permutation of this Jordan block.

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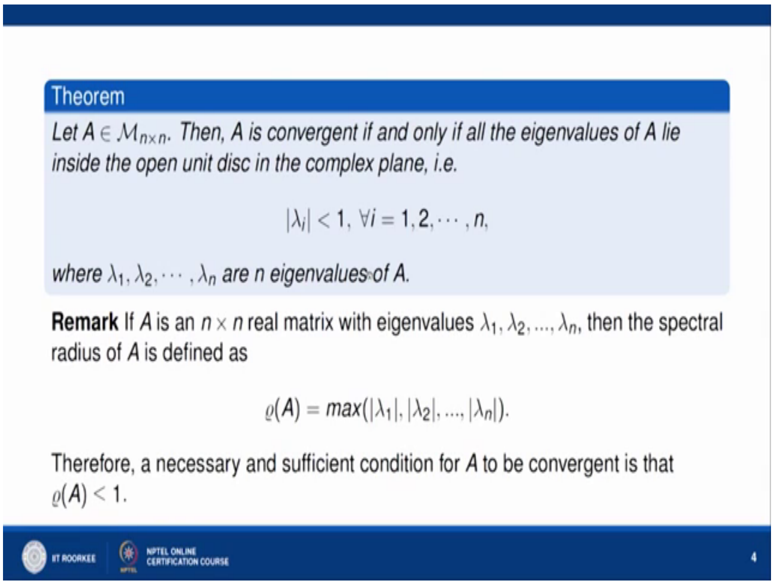

So, now once we know the Jordan canonical form representation of a metric, then with the help of Jordan canonical form, we can prove the following theorem. This theorem says that let A be the square matrix of size n then is convergent if and only if all the Eigenvalues of a lie inside the open unit disk in the complex plane that is that modulus of lambda i is less than one for every i from 1 to n. So, it means that every Eigenvalue of A has a modulus value less than 1. So, here lambda i's are and Eigenvalues of m and we can associate a definition with these disk condition that if A is and any n cross n real matrix with Eigenvalues lambda 1 to lambda n.

Then we may define a spectral radius of a as maximum of modulus of lambda i's. So, here if you look at this condition this is true for every i. So, we can say that if it is true for every i, we can say that it is true for maximum of modulus of lambda. So, it means that this condition can be restated as that a necessary and sufficient condition for a to be convergent is that spectral radius of a is less than or equal to 1. Now there is a small problem here, it is a spectral radius of a is less than 1.

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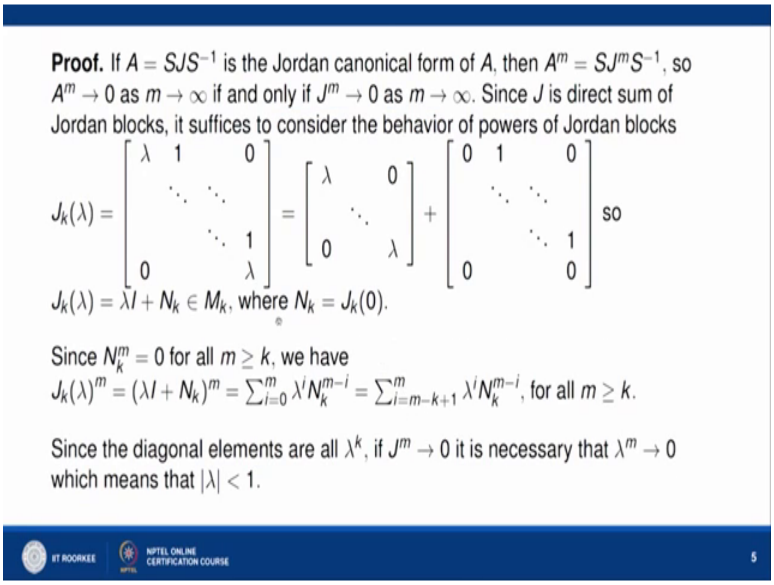

So, let us have a small proof of this theorem it says that if A is S J S inverse where j is the canonical form of A, then we can write a to power m as S J m S J to power m S inverse this, we can look at; as you take a square then a square can be written as S J S inverse into S J S inverse you can say that as it is happening with the dime diagnosable form.

So, the same thing we can consider for Jordan canonical form. So, powers of A can be written as powers of Jordan canonical form. So, A to power m is written as S J to power m S inverse. So, it means that if we one that a to power m is tending to 0 as n tending to infinity this will happen if and only if this j to the power m is tending to 0 as n tending to infinity. So, it means that to look at the say that if is a convergent matrix or not we have to look at the corresponding Jordan canonical form.

So, now if you look at Jordan canonical form j then it is direct sum of Jordan blocks. So, if you recall here, this is our J. So, J is considered as direct sum of these Jordan blocks J n 1 lambda 1, J n 2 lambda 2, J n m lambda m. So, we have to look at the powers of like J. So, j to the power k is the powers or corresponding powers of these blocks. So, we need to consider or what do we mean by J n 1 and to power k lambda 1 and J n 2 to power k lambda 2 and so on. So, we have to focus on powers of these Jordan blocks. So, to see this let us consider a particular block say J k lambda and J k lambda is given as in diagonal we have lambda upper diagonal one and rest all 0 and this can be written as this diagonal matrix having diagonal entries as lambda plus this matrix whose diagonal entries is 0 only upper diagonal one is we have 1, 1, 1 and rest all other entries are 0.

So, this; we can write as they are like J k lambda as lambda into y, this is nothing, but a scalar matrix we can write it lambda i plus n k. So, this is some matrix which we denote as n and size is k. So, we are writing J k lambda as lambda i plus n k now this belongs to the set of all matrices of size k cross k now here we can consider that n k is nothing, but Jordan block corresponding to lambda equal to 0. So, n k is nothing, but J k 0. So, we know that if you look at this then this is an important matrix. So, of size k. So, it means that this n to power m k is equal to 0 for all m greater than or equal to k..

So, we can easily verify that this n to power k is going to be 0. So, this is a nilpotent matrix of our order k. So, using this that n m k is equal to 0 for all m greater than equal to k look at the power say m of this J k lambda. So, this can be considered as lambda i plus n k to power m. So, using binomial theorem this can be written as summation i equal to 0 to power m 0 to m lambda to power i and k to power m minus i here.

So here, since we know that this n m k is equal to 0 for all m greater than or equal to k. So, it means that we can consider only powers m which is less than or equal to k minus 1. So, m is strictly less than or equal to k minus 1. So, it means that here I can write it that i is not only is running from m minus k plus 1, rather than, it is running over 0 because between i equal to 0 to m minus k it is basically 0. So, we here if you look at let me consider this. So, here we need to look at the powers of this Jordan block.

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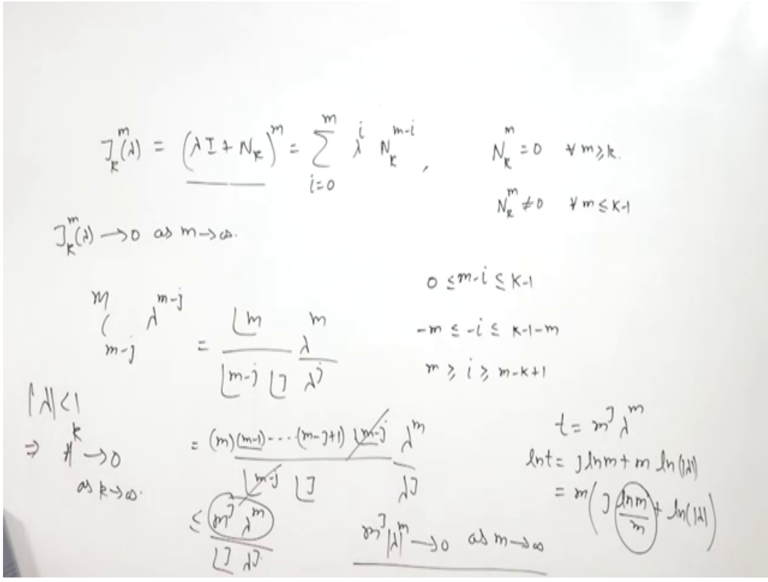

So, J k lambda to power m can be written as lambda i plus n k to power m. So, this with the help of binomial theorem we can say that it is i equal to 0 to m lambda to power i and n k to power m minus i here. Now we already know that this n k is a nilpotent matrix of order k. So, it means that n k to power m is equal to 0 for every m greater than equal to k. So, it means that this is nonzero only when m is less than or equal to k minus 1.

So, it means that here the power should vary from 0 to k minus one. So, m minus i should vary from 0 to k minus 1. So, it means that if you simplify this, then we can write that this minus i should run from minus m to k minus 1 minus m or we can say that I should run from m minus k plus 1 to m. So, it means that here we can say that this J k lambda to power m is equal to i from m minus k plus one to m lambda to power i and k m minus i for all m greater than or equal to k. Now if you look at the entries of n k to power m minus i here, then we can say that all the diagonal elements are all lambda to power k..

So, if you look at here this is your n k here then if you take the power of n k m minus i then diagonal entries is nothing, but 0 and if you look at this then diagonal entries are lambda to power k. So, it means that the J k lambda to power m a power are what lambda to power m if J m is tending to 0 it is necessary that lambda to power m is tending to 0 and this will happen only when modulus of lambda is less than 1.

So, it means that J k lambda to power m is tending to 0 provided that modulus of lambda is less than one. So, that proves the necessary condition and that A is a convergent matrix if modulus of lambda is less than 1.

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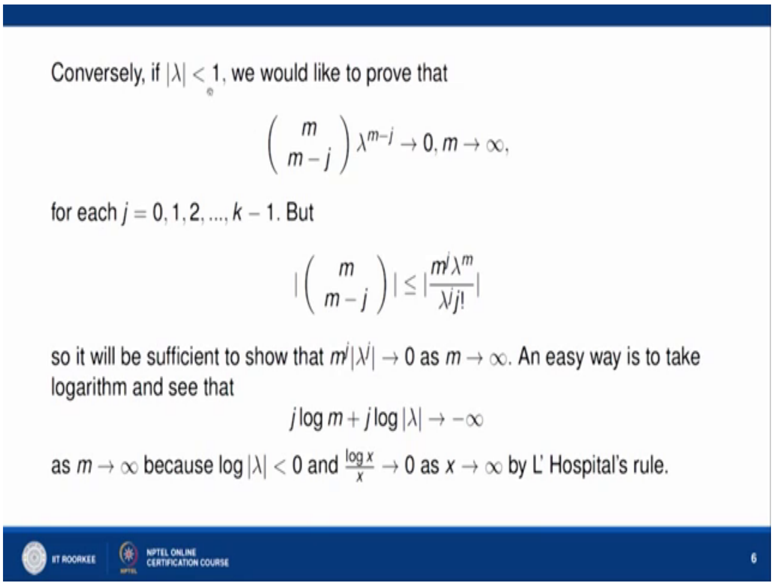

Now, now let us assume that modulus of lambda is less than one and now we want to prove that our A matrix is a convergent matrix. So, again we have to look at the Jordan block and we want to show that this J k lambda to power m is standing to 0. So, here we have to look at that this is tending to 0 here. So, here I can say that here in place of lambda to power i, I can start with lambda to power m minus n i, we can interchange the rule of this. So, here we can say that if modulus of lambda is tending to a less than one then we have to show that m c m minus j lambda to power m minus j is tending to 0 as m tending to infinity and this is true for every j here j from 0 to k minus one, but what is this quantity.

If you look at this quantity m choose m minus a, this can be written as factory m to power j lambda to power m divided by lambda to power j a factorial j here. So, this can be bounded by this if you look at I am just writing this is what m c m minus j lambda to power m minus j here. So, this is what this is factorial m divided by factorial n minus j factorial j lambda to power m, I am writing here as lambda j. Now this can be written as m m minus 1 m minus 2 and here, we can write m minus j plus 1 and factorial m minus j divided by factorial m minus j and factorial j lambda to power m upon lambda to power j.

So, this-this will cancel out and here we can say that m minus 1 is less than m; m minus j plus 1 is less than and so, we can say that this is m to power j lambda to power m divided by factorial j and lambda to power j. So, here if you look at the term which is depending on m is this. So, here if we can show that this m to power j lambda to power m is standing to 0 as m tending to infinity then we are done then we can say that this J k lambda power m is standing to 0 as m tending to infinity. So, we have to show that this term is tending to 0 as n tending to infinity. So, let us say that you take this as t. So, t is basically m to power j lambda to power m. So, here this since m is also coming in power. So, let us take log here. So, l n t we you can say that it is j l n m plus m l n lambda here. So, here let me say that it is modulus of lambda. So, here we had to look at modulus of lambda.

So, here we are considering modulus of lambda here because otherwise this is not going to define. So, here I can write this as m and we can write it j l n m divided by m plus l n modulus of lambda here and we already know that as m tending to infinity this l m; m divided by m is tending to 0 that you can prove by L Hospital's rule. So, it means that this is going to be 0 as m tending to infinity and if you look at l n of models modulus of lambda is basically minus y because modulus of lambda is given as less than 1. So, l n of modulus of lambda is minus this is tending to 0 and m is tending to infinity.

So, it means that this term is tending to minus infinity. So, if this term is tending to minus infinity, then our t is tending to 0 e to power minus infinity is tending to 0. So, your t is tending to 0 as m tending to infinity. So, it means that our J k lambda power m is tending to 0 as m tending to infinity. So, this implies that if modulus of lambda is less than 1, then this implies that each Jordan block is tending to 0 it means that you j standing to 0 and hence we can say that a is tending to 0 A k is tending to 0 as k tending to infinity, right. So, that proves the necessary and sufficient condition given in previous theorem.

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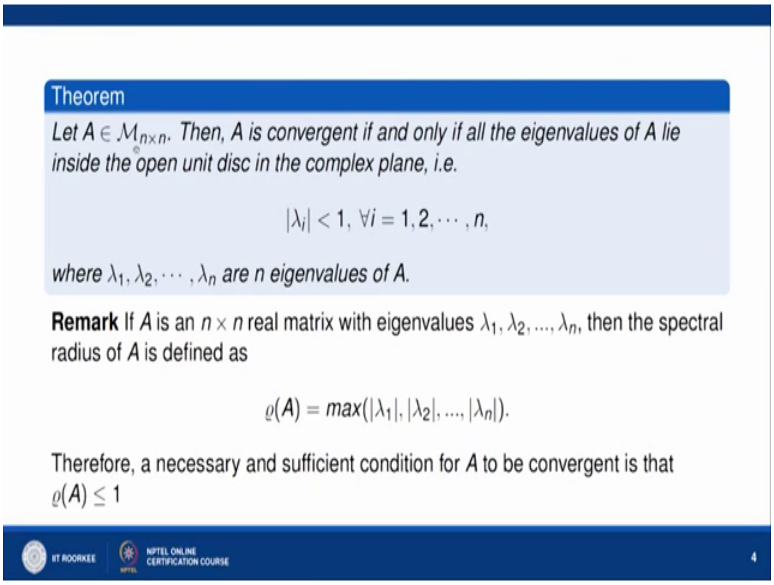

So, let me recall again here that if we have a square matrix of size n then is convergent if and only if all the Eigenvalues of a lie inside the open unit disk or we can say that a spectral radius of a is less than 1.

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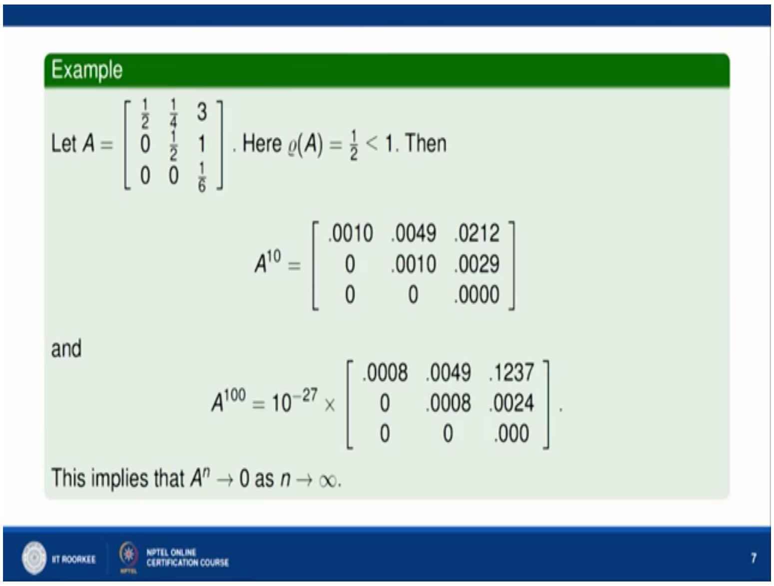

So, that is the proof given here. Now let us take a simple example based on this theorem. So, let us say that a is given as upper triangular matrix whose diagonal trees are 1 by 2; 1 by 2 and 1 by 6 and upper diagonal entries are 1 by 4 and 1 and this is given as 3.

So, here we can easily see that all the Eigenvalues are given as 1 by 2, 1 by 2, 1 by 6. So, maximum Eigenvalues is basically 1 by 2. So, you can say that the spectral radius is 1 by 2 which is less than 1, then we can easily see that a to power 10 is given by and this metric 0.001000. This we can easily check with the help of Matlab. So, by using Matlab, we can easily verify that a to power 10 is of this and if we look at more higher power of A we can say that a to power 100 is 10 to power minus 27 in to this.

So, it means that you look at that this is very small here spectral radius is less than one then this A to power n is tending to 0 as n tending to infinity means this metric A is our convergent matrix. So, here we have seen the condition of converging matrix in terms of spectral radius now let us redefine the definition of convergence matrix in terms of any matrix norm.

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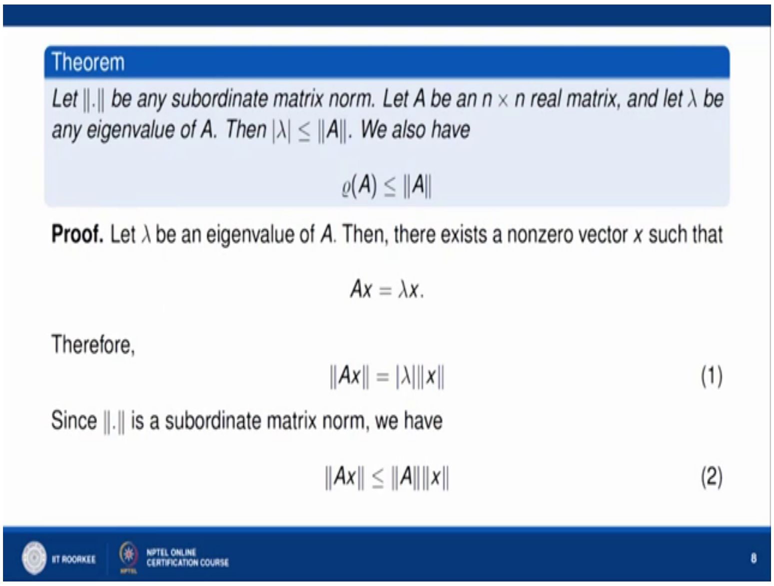

So, here we have following theorem which relate the spectral radius and any subordinate matrix norm. So, it says that let this norm be any subordinate matrix norm and let A be any n cross n real matrix and let lambda be any Eigen value of a then modulus of lambda is less than or equal to norm of A.

So, it means that if it is true for any Eigenvalue of a matrix A, then with the help of this we can say that maximum of modulus of lambda is less than or equal to norm of A which is nothing, but saying that spectral radius of a is less than or equal to norm of a. So, this simply says at how this spectral radius is related to your matrix norm. In fact, spectral radius A of A given matrix A is less than or equal to any matrix norm of A. So, let us just look at the proof of this..

So, let lambda be an Eigenvalue of a then they exist a non 0 vector x such that A x equal to lambda x. So, here we are assuming that lambda x is an Eigen pair of a. So, it means it satisfies the condition A x equal to lambda x. So, taking the norm on both the sides we have norm of A x equal to norm of modulus of lambda norm of x now we can see that norm of A x. Now this can be written as that norm is less than or equal to norm of A into norm of x. So, here norm of A is here we are using any matrix norm of A. So, here using this relation 2, we can say that modulus of lambda norm of x is less than or equal to norm of A into norm of x which is listed here.

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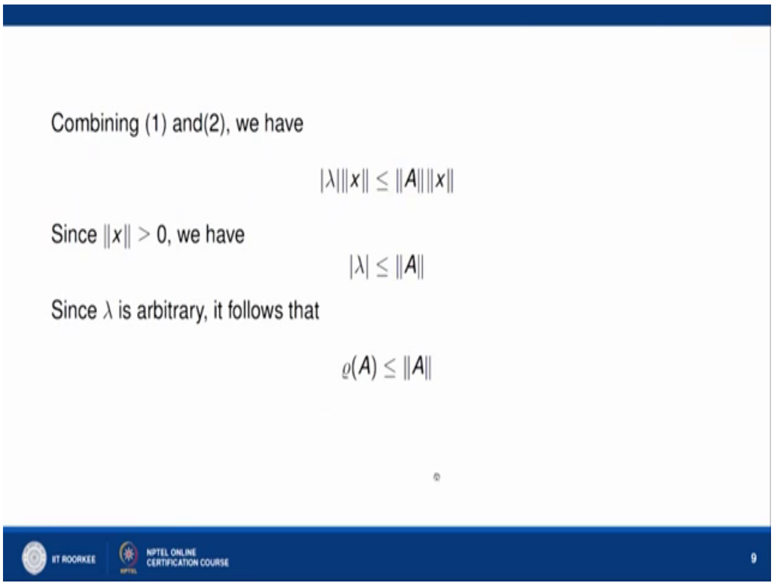

Now, here since x is an Eigenvector, modulus of x should be strictly greater than 0. So, we have modulus of lambda less than or equal to norm of A here.

So, now this is true for any Eigenvalue of matrix A. So, we can take the maximum over lambda and we can say that spectral radius of A is less than or equal to norm of A. So, here, we relate expect a spectral radius of a matrix with the any matrix norm of a given matrix.

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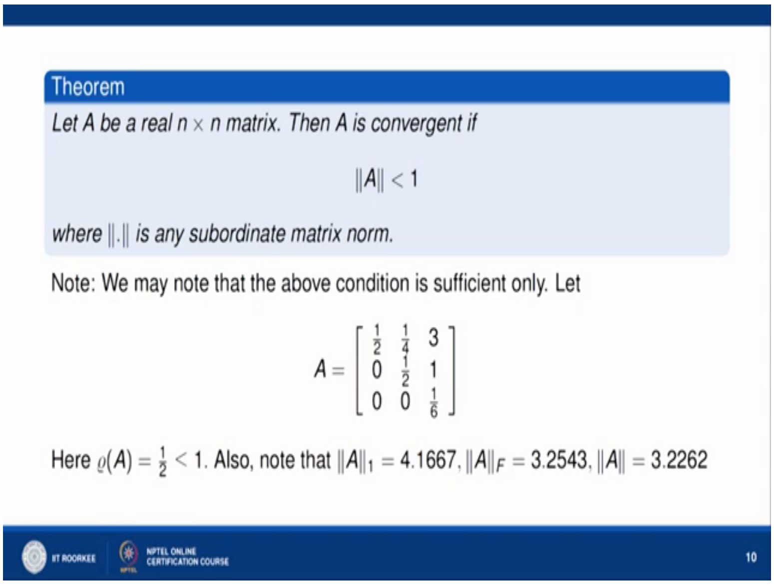

So, with the help of this result, we can define new result in terms of converging matrix. So, we say that let A be a real n cross n matrix, then is convergent if norm of A is less than one here this norm is any subordinate matrix norm. So, if you look at this is a sufficient condition because if norm of A is less than one then spectral radius of A is less than 1 and we already know that is a spectral radius is less than one implies that A is a convergent matrix..

So, here this result is only a sufficient result, but this previous result that is convergent if and only if spectral radius is less than 1 is so necessary and sufficient condition. So, we have a sufficient condition in terms of matrix norm that A is convergent if norm of A is less than 1. So, that is only a sufficient condition. So, for that let us consider this example which we have just consider A as 1 by 2, 1 by 4, 3 0 1 by 2 0 1, 0 1 by 6. So, if you look at a spectral radius is 1 by 2 which is less than 1 and if you look at some metric subordinate norms say 1 norm of a 1 norm of A is this 3 plus 1 plus 1 by 6. So, it is 4.1667 and if you look at the Frobenius norm; it is again 3.2543 and 2 norm of A if you do not write anything then default norm is 2 norm.

So, 2 norm of A is given as 2.6. So, here for this matrix A, we do not have that matrix norm of a is less than one, but still we have seen that this matrix is a convergent matrix. So, it means that if matrix norm is less than 1 implies that it is convergent matrix, but if matrix norm of A is not less than 1, we cannot say anything, we have to check the spectral radius of this matrix A. So, here we have discussed the concept of converging matrix and we have seen the necessary and sufficient condition of a converging matrix in terms of a spectral radius that if a spectral radius is strictly less than 1, then metric is convergent and we also see the converging condition in terms of any matrix norm and that condition is sufficient condition and we have seen some example of a converging matrix.

So, here we stopped and in next lecture, we will discuss the perturbation analysis of a given matrix. So, it means that if we have a non singular matrix and if we perturbed our non singular matrix then how far; what should be the perturbation says that it remains non singular matrix.

So, that we are going to discuss in next lecture here we stop our lecture thank you for listening this.

Thank you.