Tone Apperception and Weber-Fechner's Law

Thomas Noll
Technical University of Berlin

September 12, 2002

Das Vorstellen der Tonbewegungen ist ein wirkliches Mitmachen derselben mit dem Willen; die Seele, der lebendige Menschengeist, führt selbst diese Bewegungen aus und erfreut sich in ihnen seines Daseins, seiner Wirkungskräfte. (Hugo Riemann [23], p.15).

Abstract

In extrapolation of Wilhelm Wundt's suggestion to apply Weber-Fechner's law to apperception and in slight modification of Davis Lewin's definition of generalized interval systems we propose a four-dimensional space $\mathbb{A}$ as a model for an active tone system. This space $\mathbb{A}$ is also a group acting on itself. Its elements are interpreted in two ways, (1) as tone loci, i.e. as states of apperceptive processes and (2) as apperceptive acts which can be aquired by such processes as transitions between tone loci. The four dimensions are constituted by semitone steps, semitone alterations, fifth steps and fifth alterations. A fundamental ingredience to the model is the distinction between actual and virtual apperception. The former is based on the group operation of $\mathbb{A}$ while the latter is based on the addition of tone vectors, i.e. elements of the tangent space $\mathbb{V}$ of $\mathbb{A}$ at its identity element. Tone vectors, active tones and tone loci are connected by the apperception formula $App(X, v) = X \cdot exp(v)$ which adapts Wundt's formula to the four-dimensional situation. The interpretation of this model includes general aspects of apperception, namely a study of linear and circular apperception as well as an economy principle of reason and decision. The specific music-theoretical interpretations focus on enharmonicity. As an alternative to group factorisation as an algebraic means to describe enharmonic identification we show that the discrete quintic group $\mathbb{G} \subset \mathbb{A}$ already yields enharmonic relations as a consequence of the apperception model. It is argued that the relations of hallucination and Escher-staircase provide suitable explanations for synquintic and enharmonic identifications. As a more sophisticated case of enharmonicity we motivate the concept of synchromatic identification. Finally, attention is paid to the geometrical study of the active tone system. A pseudo-Riemannian[1] metrics supports the economy principle and offers relationsships to relativity.

[1]We mention that mathematicians use the attribute *Riemannian* in honor of Bernhard Riemann (1826 - 1866), while music theorists use it in honor of Hugo Riemann (1849 - 1919). The present article intends to contribute with a pseudo-Riemannian apperception model to the ongoing Neo-Riemannian discourse.

1 Introduction

The crux of investigations into tones and tone relations is that their experience seems to be closely related to the perception of physical sound on the one hand and to active experience on the other. The present article proposes a speculative theoretical approach to the latter aspect. This first section aims at generally clarifying what the article is about and what it is *not* about.

1.1 Disciplinary and Ontological Preliminaries

In this paper we argue in favour of a concept of *tone apperception* as an indispensable ingredient of the investigations into musical tones and their relations. The phenomenon of music is accessible through several levels of description: physical, psychological, cognitive, cultural, and not to forget, music theoretical. There are two ways of distinguishing different ontological levels, namely (a) to attribute a specialized ontology to each of the disciplines in accordance with their denotation systems or (b) to consider *general ontology* on its own as another level of description among the others. Both choices are problematic and refer to deeper epistemological problems that cannot be dealt with in this paper. The subject domain must be entered with caution but without prejudice to the disciplinary competences. Physics, physiology, psychology, semiotics and philosophy are relevant foundational sciences or disciplines for music studies, but one cannot be sure that their contemporary foundations are already capable of jointly supporting a satisfying conceptual framework for the study of *musical tones*.

From the disciplinary point of view the present approach to *tone apperception* is basically intended to be a contribution to a music-theoretical level of description. This means the main line of argument in this paper intends to contribute to the solution of inner music-theoretical problems. However, from the ontological point of view the main focus of this study is on active musical experience and therefore involves issues of musical mental activity. We therefore have to admit that the present approach is not embedded in current psychological research nor is it adapted to the accepted methodology of this discipline. Nevertheless we hope that our reformulation and interpretation of Wilhelm Wundt's proposal to derive a law of apperception from Weber-Fechner's law will eventually stimulate further cognitive studies. It seems that Wundt's speculative idea has more or less abandoned by contemporary psychology rather than falsified.

There are two other rather formal disciplinary aspects to be mentioned. The formulation of the present approach includes the step by step construction and investigation of a mathematical model. To a certain degree, mathematical competence is required in order to judge the correctness of mathematical propositions. But there is no mathematical novelty involved. The model is an application of standard knowledge onto linear groups and their geometrical aspects. The other point is to notify that some ideas behind the present approach are loosely inspired by ideas from theoretical physics. But a transfer of such ideas has neither immediate disciplinary nor ontological implications nor is it neccessary to be familiar with these ideas in order to follow the lines of argument. The paper is a selfcontained proposal for a mathematical theory of specific musical mental acts, namely tone apperceptions.

1.2 Intonation-Apperception Analogy

Music is mediated primarily through physical sound. For occidental music it is furthermore characteristic that the frequency parameter occupies a prominent role. This is paralleled in music theory through a prominent interest in musical intervals, scales, harmony, etc. Hence, at first sight, musical tones appear to be mental counterparts of physical sounds and manifold attempts to descriptively or normatively systematize frequency ratios have influenced both domains, musical accoustics and music theory. But closer investigations lead to various objections against a simple analogy between the two domains. Music-theoretical investigations of tone systems must primarily be evaluated with regard to their explanatory power within music theory. A crux for the whole discussion - and a likely source of confusion - is therefore the existence of at least two music-theoretically interesting models, which have parallel acoustic interpretations:

- *Riemann-Tone-Net* and *Just Intonation.* A two-dimensional scheme is commonly used to denote relative frequencies in so-called 'just intonation'.[2] The primarily accoustic systematisation became merged with music-theoretic issues through the denotative usage of note names. Hugo Riemann later turned the scheme into a proper music-theoretical one by considering it as an apperceptive configuration space. This idea still plays a role in Neo-Riemannian approaches and leaves open interesting problems. We mention that Riemann himself did not insist on a parallelism of apperception and intonation. To him equal temperament was a practical solution on the physical level and not a tone system. To avoid confusion we use the label *Euler-Net* for the physical Log-Frequency interpretation and *Riemann-Tone-Net* for the apperceptive interpretation.

- *Abstract 12-Tone System and Equal Temperament.* The speculative consideration of an abstract 12-tone system on a theoretical level became manifest when atonal musical styles emerged. But there are historical and numerous contemporary attempts to study music of various styles with respect to an abstract 12-tone-system. As in the case of Riemann's ontological shift towards apperception one must carefully distinguish between the description of equal temperament on the one hand and investigation of a 12-tone-system on the other. This becomes evident when sophisticated structural properties of $\mathbb{Z}_{12}$ are interpreted music-theoretically. We recall Guerino Mazzola's mathematical models for modulation and counterpoint, Eytan Agmon's study of diatonism with respect to the direct

[2] This scheme goes back to Leonhard Euler and has been favorized again in the second half of the 19th century. The frequencies differ by rational factors $2^a 3^b 5^c$ with $a, b, c \in \mathbb{Z}$. Their logarithms $log(2^a 3^b 5^c) = a \cdot log(2) + b \cdot log(2) + c \cdot log(5)$ form a three-dimensional lattice. A two-dimensional scheme is obtained according to octave identification.

product $\mathbb{Z}_{12} \times \mathbb{Z}_7$ or Richard Cohn's discovery of the voiceleading parsimony among major and minor triads and among pentatonic/diatonic scales with respect to the study of particular transformations.

Philologically, one might consider analogies between intonation and apperception as possible sources of music-theoretical inspiration. But the question, as to whether a given model fits with a given phenomenon is not a philological, but an empirical one. Both tone systems, the Riemann-Tone-Net as well as the abstract 12-Tone system must be evaluated in music-theoretical investigations. It is neither acceptable to legitimate these structures as tone-systems simply through an intonation-apperception analogy nor it is acceptable to reject them across the board simply through the attribution of such an analogy.

1.3 Associative and Pure Apperception

The study of mental activity is sometimes divided into associative and apperceptive aspects. *Association* stands for the connectivity of mental contents. Conventions and issues of learning are typical associative phenomena. *Apperception* stands for active experience of the mind as a consequence of its selfconscious awareness and is characterised by the occurrence of decision making under the influence of free will. A comprehensive study of mental activity behind musical experience must take both aspects into account: the association of conventional contents as well as the apperceptive acts that give access to these contents. *associative apperception* labels such a comprehensive approach.

We recall two music-theoretical examples for the associative evocation of context-dependent parameters:

- In Hugo Riemann's *functional harmony* single tones are associated with positions within or relative to prime chords. These prime chords constitute an associated tonal context. In the context of a C-major-tonality, a single tone C may be associated with the positions *prime of tonic* or *fifth of subdominant*, a single tone D with *fifth of dominant* or *added sixth to subdominant*, etc. Associative apperception of a tone succession $D \to C$ includes a decision between a succession such as *fifth of dominant* $\to$ *prime of tonic* or *added sixth to subdominant* $\to$ *fifth of subdominant* and others. The situation becomes more complicated when modulatory processes are taken into consideration.

- In *diatonism* single tones are associated with scale degrees within contextual diatonic scales. Associative apperception of a tone succession $D \to C$ includes a decision between a succession such as $\hat{2}_C \to \hat{1}_C$ or $\hat{6}_F \to \hat{5}_F$ or even $\hat{6}_F \to \hat{1}_C$

As a central subject of investigation for the present article we introduce the idealized concept of *pure apperception*. With this concept we intend to minimise the consideration of associated contexts. But at the same time we emphasise the aspect of active decision making. The task is to find an appropriate amount of contextual assumptions to explain given apperceptive phenomena such as enharmonic identification. In other words, we do by no means question the relevance of conventional contexts such as triadic vocabularies or diatonic scales, nor the neccessitiy of a concept of associative apperception. We try rather to simplify the situation within our model and hope to gain insights from an idealisation.

2 Prolegomena to a Theory of Apperception

The section presents basic assumptions and arguments behind the mathematical model of the *Active Tone System* (see section 3). The ingredients are formulated in such a way that the specificity of *tone apperception* does not come to the foreground any further than necesssary.

2.1 Actual and Virtual Apperception

When Gustav Theodor Fechner (cf. [2]) proposed his mathematical formulation of Ernst Heinrich Weber's law concerning the correspondence of stimulus intensity and sensation noticability he initiated an heated debate about the ontological interpretation of these findings. Among others Wilhelm Wundt (cf. [30], S. 629 ff.) argued in favour of a purely psychological interpretation of Weber-Fechner's law which he puts as:

$$M = C \cdot ln(E)$$

In this formula E denotes the *intensity of a sensation*, M its *degree of noticability* and C a constant. The formula presupposes a distinction between *sensations as such* on the one hand and their *noticability* on the other. Furthermore, Wundt considers this law as a special case of a hypothetical general *law of apperception*.[3] Wundt's idea is of course highly debatable. As already mentioned in the introduction, it has been abandoned rather then falsified. Therefore it is interesting to further elaborate this idea in a concrete situation.

A first consequence of Wundt's idea is the distinction of two apperceptive modes. We call them the *actual* and the *virtual*. The actual mode is exemplified through sensations as such, i.e. through their intrinsic processuality, while the

[3] "Die psychologische Deutung sucht das Webersche Gesetz weder aus den physiologischen Eigenschaften der Nervensubstanz noch aus einer eigentümlichen Wechselwirkung des Physischen und Psychischen, sondern zunächst aus den psychischen Vorgängen abzuleiten, die bei der Vergleichung der Empfindungen wirksam sind. Sie bezieht also dasselbe nicht auf die Empfindungen an und für sich, sondern auf die Apperzeption derselben, ohne die ihre quantitative Schätzung niemals stattfinden kann. Psychologisch läßt sich nämlich offenbar das Webersche Gesetz auf die allgemeinere Erfahrung zurückführen, daß wir in unserem Bewußtsein nur ein relatives Maß besitzen für die Intensität der in ihm vorhandenen Zustände, daß wir also je einen Zustand an einem anderen messen, mit dem wir ihn zunächst zu vergleichen veranlaßt sind. Wir können auf diese Weise das Webersche Gesetz als einen Spezialfall eines allgemeineren Gesetzes der Beziehung oder der Relativität der Bewußtseinszustände auffassen. Danach ist das Webersche Gesetz nicht sowohl ein Empfindungsgesetz als ein Apperzeptionsgesetz." (cf. [30], p. 632)

virtual mode is examplified through degrees of noticability. At an abstract level we may postulate a domain $\mathbb{A}$ of *apperceptive acts* and a domain $\mathbb{V}$ of *apperception vectors*.

- We speak of *acts* in order to emphasise the processual ontology of this domain. Mathematically we will assume that $\mathbb{A}$ carries the structure of a group, i.e. we assume that apperceptive acts can be concatenated in a specific way. We call this operation *actual apperception*. Mathematically one considers the group action of $\mathbb{A}$ on itself:

  $$act : \mathbb{A} \times \mathbb{A} \to \mathbb{A} \quad \text{with} \quad act_A(X) = X \cdot A$$

  This formula expresses the idea that any apperceptive process situated at a locus X can acquire a chosen act A in order to reach the locus $X \cdot A$. This ontology is somewhat strange because the group elements themselves are interpreted in two ways: as single acts and as accumulated processes.

- We speak of *vectors* in order to generalise from a one-dimensional axis of possible degrees of noticability to more dimensions where the aspect of difference or length is complemented by the aspect of direction. Mathematically we will assume that $\mathbb{V}$ carries the structure of a vector space, i.e. we assume that apperception vectors can be added to one another or stretched. The operation of vector addition is called *virtual apperception*. Mathematically one considers the action of the group $\mathbb{V}$ on itself:

  $$virt : \mathbb{V} \times \mathbb{V} \to \mathbb{V} \quad \text{with} \quad virt_v(x) = x + v$$

  This formula expresses the idea that any chosen noticable vector can be added to any given noticable vector in order to reach the virtually noticable vector $x + v$. This apperceptive mode gives virtual access to apperception vectors without assuming acts of making them noticable, i.e. actual apperceptions corresponding to these vectors.

Following Wundt's fundamental idea we try to link apperceptive acts to apperception vectors through a logarithmic function, or conversely, to link vectors to acts through an exponential function. We may formalise the one-dimensional situation, which Wundt perhaps had in mind, by setting

$$\mathbb{A} = \mathbb{R}^*_+ \quad \text{and} \quad \mathbb{V} = \mathbb{R}.$$

The exponential and the logarithmic functions are mutually inverse group isomorphisms between the additive group $\mathbb{R}$ of real numbers and the multiplicative group $\mathbb{R}^*_+$ of positive real numbers:

$$\mathbb{R} \quad \underset{exp}{\overset{log}{\leftrightarrows}} \quad \mathbb{R}^*_+$$

The theory of Lie-Groups and Lie-Algebras provides a more or less self evident generalisation to this situation. We do not assume the reader's familiarity with this theory and postpone sophisticated considerations to the last section **??** of this article. But in order to propose a general formulation of Wundt's apperception formula we mention that the exponential map $exp : \mathbb{V} \to \mathbb{A}$ translates vectors into group elements. The vectors are elements of the associated Lie-Algebra $\mathbb{V} = T_{Id}\mathbb{A}$, which is the tangent space to the group $\mathbb{A}$ at its neutral element Id. Suppose we want to mathematically model a given type of apperception in terms of a suitable Lie-Group $\mathbb{A}$ we may put Wundt's apperception formula as follows:

$$App(X, v) = X \cdot exp(v) = act_{exp(v)}(X)$$

This formula expresses that any appercpetion vector $v \in \mathbb{V}$ can be *apperceived* through a corresponding apperceptive act $act_{exp(v)}$. If an apperceptive process is at a locus $X \in \mathbb{A}$ one can acquire the new act $exp(v) \in \mathbb{A}$ in order to reach the locus $App(X, v)$, which is called the *apperception of* v *at* X.

A suitable music-theoretical point of departure to the investigation of tone apperception is David Lewin's model of a *Generalised Interval System* (see [15]). The distinction between virtual and actual apperception can be immediately tied up to this model which comprises a musical space S, a group $IVLS$ of intervals, and a function $int : S \times S \to IVLS$, satisfying two axioms

- **A:** For all selections of three points $r, s, t \in S$ one has $int(r, s) \circ int(s, t) = int(r, t)$.
- **B:** For each point $s \in S$ and each interval $i \in IVLS$ there is a unique solution $x = T_i(s)$ to the equation $int(s, x) = i$.

The two conditions **A** and **B** together with the group properties of $IVLS$ give rise to the definition of a group $STRANS := \{T_i : S \to S \,|\, i \in IVLS\}$ of transformations being anti-isomorphic to $IVLS$ due to $T_i \circ T_j = T_{j \circ i}$.

It is warrantable to characterise various applications of this model as investigations into musical apperception. Particularly in the Neo-Riemannian discourse authors pay attention to processuality of musical mental activity. Although the types of musical objects vary (i.e. the points of the specific musical spaces are tones, chords, tonalities or other objects), the study of pathways through these musical spaces are not seldomly interpreted in terms of mental processes.[4]

We now set to work on combining Wundt's apperception formula with Lewin's GIS-model. We first recall the one-dimensional situation which yields a GIS-model ($S = \mathbb{A} = \mathbb{R}^*_+, IVLS = \mathbb{V} = \mathbb{R}, int$) where

$$int : \mathbb{R}^*_+ \times \mathbb{R}^*_+ \to \mathbb{R} \quad \text{with} \quad int(X, Y) = log(Y) - log(X)$$

Condition **A** is obviously fulfilled and condition **B** is fulfilled by virtue of the apperception formula because $int(X, App(X, v)) = log(X \cdot exp(v)) - log(X) = v$. Especially the space S coincides with the group of transpositions $STRANS$.

[4] The term 'intuition' is more frequently used than 'apperception'.

The exponential function is a group isomorphism between $IVLS$ and $S = STRANS$ and hence guarantees a full parallelism of actual and virtual apperception.

One way of generalisation would be to interpret any GIS-model as a possible apperception model. Condition B would embody a particular apperception formula and condition A would guarantee a full parallelism of virtual and actual apperception. But this idea is not compatible to our original idea, namely to generalise Wundt's particular apperception formula to higher dimensional groups. There are two reasons:

1. The exponential map is generally neither injective nor surjective, and hence does not have a well defined inverse map log. So one cannot define the function int properly.

2. The additive group $\mathbb{V}$ of a vector space is always commutative, while a Lie-Group $\mathbb{A}$ is generally not, i.e. the familiar functional equation of the exponential function $exp(v + w) = exp(v) \cdot exp(w)$ does not hold in such a case.

We mention that Lewin also considers non-commutative groups, but the anti-isomorphy of $IVLS$ and $STRANS$ still guarantees the full parallelism of actual and virtual apperception. In the present approach we dismiss this strong assumption. While actual apperception is not assumed to be commutative, we instead assume virtual apperception to be always commutative. Even Lewin's own characterisation of intervals as 'extensional relations' an the one hand and transformations as 'internal gestures' on the other hand does not immediately suggest a full compatibility between the two concepts. The following table summarises the correspondences between the GIS-model and our present apperception model.[5]

Space S	Lie-Group $\mathbb{A}$
Interval Group IVLS	Lie-Algebra $\mathbb{V} = T_{Id}(\mathbb{A})$
Transformation group $STRANS$	Lie-Group $\mathbb{A}$
Group action $STRANS \times S \to S$	Group action $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$
$int : S \times S \to IVLS$ $T_{-} : IVLS \to STRANS$	$exp : \mathbb{V} \to \mathbb{A}$
condition **B**	$App(X, v) = X \cdot exp(v)$
condition **A**	Campbell-Baker-Hausdorff Formula

In our concrete application of the apperception model we are concerned with a discrete subgroup $\mathbb{D} \subset \mathbb{A}$ of a particular Lie-Group, rather than with the full continuous group $\mathbb{A}$. For the central part of the paper (sections 3 and 4) we therefore will not make use of arguments and techniques from differential geometry. But with regard to the formulation and conveyance of certain ideas life is much simpler within an continuous ambient space. We therefore switch

[5] The Campbell-Baker-Hausdorff-Formula yields the appropriate generalisation to the functional equation $exp(v + w) = exp(v) \cdot exp(w)$ and therefore corresponds to condition **A**.

from a 'naive' discussion within these prolegomena into more sophisticated one in section **??**.

Geometrically, the main idea is simply this. The Lie-Group $\mathbb{A}$ 'as a space' is understood as a configuration space. Within this kinematic viewpoint *apperception processes* are modelled as sequences $(X_1, X_2, ..., X_n)$ of loci in this space (i.e. as discrete 'curves'). The successions from one locus X_i to the next X_{i+1} are understood as resulting from apperceptive acts $X_{i+1} = App(X_i, v_i) = act_{exp(v_i)}(X_i)$. Recall that these apperceptions are controlled by tangent vectors v_i at the neutral element $Id \in \mathbb{A}$. Geometrically speaking, if a process were at locus Id instead of X_i these vectors would approximate the direction and distance of the locus $exp(v_i)$. But $exp(v_i)$ is not interesting *as a locus* but *as an acqusition of an act* to a process that is actually at locus X_i. Hence the vector pointing to X_{i+1} is not v_i but $X_i \cdot v_i \in X_i \cdot \mathbb{V} = T_{X_i}\mathbb{A}$.[6]

The figure below displays an apperception process (Id, X_1, X_2, X_3, Id) of four successions that virtually corresponds to the vector-sequence $(v, 2v, 3v, 4v)$.

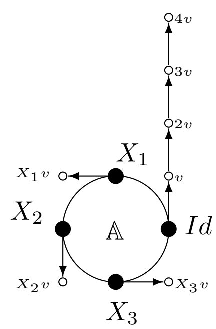

Figure 1: Actual and virtual apperception

The vectors v_i and $X_i \cdot v_i$ nominally represent the 'same' direction at different loci, namely at Id and X_i, respectively. But with respect to a suitable ambient space for the entire *tangent bundle* $T\mathbb{A}$, comprizing all tangent spaces for all loci $X \in \mathbb{A}$ these two vectors may literally differ from one another. In the figure there are four one-dimensional tangent spaces to the (one-dimensional) circle $\mathbb{A}$ within a two-dimensional ambient space. The vectors v, X_1v, X_2v, X_3v nominally represent the same direction (call it 'go to the left') at different loci. The virtual apperception sequence $(v, v+v, v+v+v, v+v+v+v)$ at the locus Id is not sensitive to the literal change of direction. In just means 'keep on going to the left'. For example the apperception vector $4v \in \mathbb{V}$ is different from the zero vector, while actually $X_4 = Id \in \mathbb{A}$. This illustrative example is also a central one to the present approach. The following subsection is therefore dedicated to its further explication.

[6]The action of $\mathbb{A}$ on itself induces an action on the tangential bundle $T\mathbb{A} \times \mathbb{A} \to T\mathbb{A}$.

2.2 Linear and Circular Apperception

The idea of a circular configuration space for apperception refers to the *circle* as an archetypical mental structure. We focus our attention on two aspects, namely (a) elements having finite order and (b) the characteristic concept of distance on a circle. These two aspects can of course be studied independently. Even discrete subgroups on a circle need not to be finite and finite cyclic groups need not to be studied with respect to their embedding into a circle. However, both aspects are prominent in studies on tone relations and it may therefore be interesting to study them in combination.

The idea of a circular organisation of musical tones has a ramified tradition with two main branches. Firstly, the spread of well-tempered tuning in the 18^{th} century was accompanied by theoretical treatises reflecting upon kinship relations along the circle of fifths (c.f [5] or [12]). This tradition is being continued in manifold music-theoretic interpretations of the cyclic group $\mathbb{Z}_{12}$. Secondly, models of musical pitch with a second cyclic component besides the linear[7] *pitch-height* have been proposed since the early days of music psychology (Opelt 1934, Drobisch 1855, Revesz 1913)). The second component is often called *chroma*. There are also attempts to bring these two branches together. We mention Roger Shepard ([27]), who deduced four- and five-dimensional models from empirical distance judgements by a multidimensional scaling method and interpreted these results in terms of abstract pitch spaces. His models involve two discrete cycles, namely a chromatic cycle as well as a cycle of fifths which are arranged on circles. We also mention Martin Ebeling ([1]) who tackles the problem of hearing adjustment. Both authors make considerable efforts to take ontological and disciplinary difficulties into account. A critical discussion of these arguments would require its own study and cannot be dealt with in the present article. But as a contribution to such a discussion it seems advisable to take the possibility of *circular apperception* seriously, i.e. as an archetype of mental activity - perhaps beyond sensual modes.

Ebeling's approach includes a particular idea, which can be tied up with the general apperception formula. He proposes to investigate the interaction of pitch-height and chroma within the sensation of pitch in terms of the *complex logarithm*. What Ebeling has in mind is a two-dimensional version of Weber-Fechner's law. On the one hand he focusses on psycho-accoustic aspects of tone perception, but on the other hand he tries to explain musical phenomena such as modulation in his generalised psycho-accoustic terms (c.f [1] p. 88). With Wilhelm Wundt's reading of Weber-Fechner's law in mind, we therefore suggest to adapt Ebeling's generalisation to the domain of apperception.[8]

Recall the exponential function for complex numbers, mapping the additive group $\mathbb{C}$ of complex numbers onto the multiplicative group $\mathbb{C}^*$ of all non-zero

[7] Our usage of 'linear' as opposed to 'circular' is meant geometrically in the sense of 'straight' (rather than algebraically).

[8] We do not intend to reject a psycho-accoustic interpretation, but we suggest elaborating a purely apperceptive approach.

complex numbers

$$exp : \mathbb{C} \to \mathbb{C}^* \quad \text{with} \quad exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The following figure shows that the corresponding apperception model with $\mathbb{A} = \mathbb{C}^*$ and $\mathbb{V} = \mathbb{C}$ involves both archetypical kinds of actual apperception: *rectilinear* and *circular*.

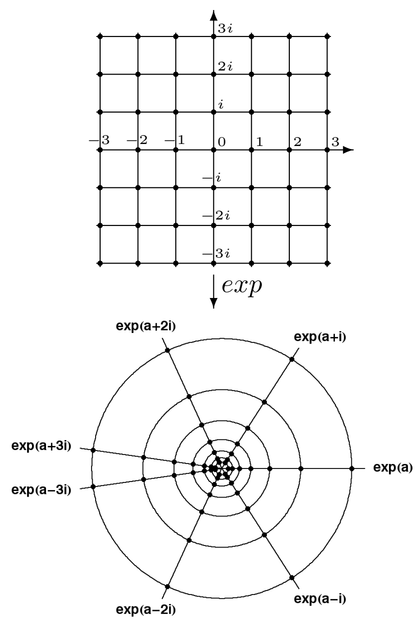

Figure 2: Exponential Function for Complex Numbers. The upper figure displays $7^2 = 49$ points and $6^2 = 36$ meshes of the integral lattice in the additive complex plane. The lower figure shows their images with respect to the exponential function.

The embedding $\mathbb{R} \subset \mathbb{C}$ of the real line yields a rectilinear type of apperception as discussed in the beginning of subsection 2.1. All horizontal lines $\mathbb{R} + bi$ in the tangent space (i.e. parallels to the real axis) are mapped onto rays $\mathbb{R}^*_+ exp(bi)$ around 0 (the missing point in $\mathbb{C}^*$), while all vertical lines in the tangent space, i.e. the imaginary axis $\mathbb{R}i$ and all its parallels $a + \mathbb{R}i$ are mapped onto concentric circles with radius exp(a) around 0. The discrete lattice in the tangential space displays integral points $a + bi$ with $a, b \in \mathbb{Z}$. It is important to observe that integral steps in the imaginary direction are not(!) related to finite apperceptive cycles. Their exponentials are all different and of infinite order, except for $exp(0) = 1$. In other words, the restriction of the above apperception model to solely integral apperception vectors still yields a proper GIS-model. The exponential function satisfies the functional equation $exp(x+y) = exp(x) \cdot exp(y)$ for all $x, y \in \mathbb{C}$ and it is injective for integral vectors.

Apperception cycles of finite order n such as a regular cycle of fifths correspond to the imaginary apperception vectors $v = \frac{2\pi i}{n}$. In order to combine discrete linear apperception with an arbitrarily given regular n-cycle for *direct* circular apperception, one might consider the discrete apperception vectors $\mathbb{V} = \{a + b \cdot \frac{2\pi i}{n} \,|\, a, b \in \mathbb{Z}\}$. However, can one assume a free availability of arbitrary transcendental scaling factors $\frac{2\pi i}{n}$? In a pointed formulation one may ask whether such an assumption would implicitly impose a kind of 'squaring the circle' to virtual apperception. We prefer to dispense with such an assumption in the present approach. Instead we concentrate on the investigation of *indirect* creation of specific elements of finite order, especially of order 6 and 4.

2.3 Economy of Reason and Decision

In the introduction we already mentioned the problem the important role of active experience in the study of tone relations. In order to measure up to the close connection between sound perception and musical experience, we have to define a suitable interface between sound perception and tone apperception. Perceived sound stimuli somehow *give reasons* for apperceptions. We avoid the term 'cause' to emphasise the active nature of apperception. The interface consists of the ontological distinction between sound stimuli that *give reason* and apperceptive acts that *are reasons*. As indicated in subsection 1.2 we are not concerned with the former, i.e. we are not concerned with sound stimuli and their perceptions, nor we are concerned with mental representation of sound. Apperceptive reasons are elementary acts within an autonomous domain. We speculate upon their role within apperceptive processes. By chosing the term 'reason' instead of 'representation' we have in mind that these are embedded in a larger domain of apperceptive acts, where reasons occur with *decisions* that are motivated by these reasons. We formulate the following two general principles:

1. **Dialogue Principle:** Apperceptive acts are composed of reasons and decisions.

2. **Economy Principle:** The amount of decision in an apperceptive act does not exceed the amount of reason.

'Amount' stands for a quantitative measure that will be specified in subsection 5.1 in terms of a suitable pseudo-Euclidean metrics. Here we suggest a heuristics. Suppose we are given a one-dimensional discrete space and a certain distance $\overline{XY}$ in this space as an apperception reason. A suitable way of experiencing such a distance would be a movement from X to Y. But the metaphor of a *continuous* movement is not appropriate in that case. We are looking for a *discrete* apperceptive act. Therefore suppose that two agents, call them metaphorically 'past' and 'future', are located at X and Y respectively. In order to jointly experience the distance $\overline{XY}$ they perform an instantaneous dialogue consisting of a single act of decision about the locus of the 'present' between them. If there is exactly one locus Z in between X to Y they may locate the 'present' there in a fair-minded decision. But if the distance $\overline{XY}$ consists of exactly one unit in the discrete space a non-trivial decision is forced. The 'present' has to be located either at the location X of the 'past' or at the loaction Y of the 'future'. A proper 'in between' does not exist. Generally speaking, the decision can be measured as the deviation from the unmarked midpoint between X and Y. In *monologic* apperceptive acts the amount of reason and the amount of decision coincide. Proper *dialogic* acts are characterised through the choice of a 'present' Z different from X and Y. The economy principle forbids *synlogic* acts, where Z is located outside of $\overline{XY}$.

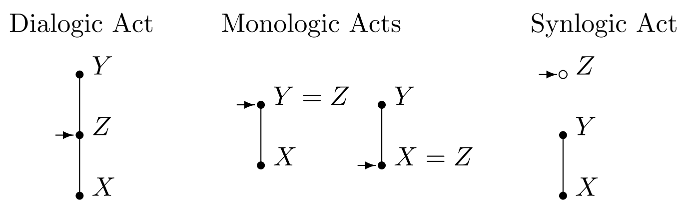

Figure 3: Typology of virtual apperceptions

The temporal metaphor has been chosen purely for heuristic reasons without further ontological intent. Nevertheless we benefit from a fruitful idea in kinematics. Much as *time* is treated as an external parameter of *spacetime* we consider *decision* as an external parameter of *apperception*. According to the negative and the positive direction of the decision-axis we distinguish between two types of monologic acts which we call *alterations* and *steps*. Within our temporal metaphor alterations are illustrated by coincidences of past and present while steps are illustrated by coincidences of future and present. But again we have to emphasise that our principles do not include any temporal interpretation. We now set to work on combining Wundt's apperception formula with the idea of an enlarged configuration space, including reasons and decisions. In accordance with the previous considerations we have to distinguish between actual and virtual apperception (subsection 2.1), and have to define a suitable exponential function $exp : \mathbb{V} \rightarrow \mathbb{A}$ for generalised apperception vectors, including virtual reasons and decisions.

The following figure shows a two-dimensional discrete space $\mathbb{V}$ of apperception vectors, which are composed of two parameters, namely (virtual) reason and (virtual) decision. The filled points represent apperceivable vectors. These are called *dialogue-like* and *monologue-like* respectively. The unfilled points denote vectors violating the economy principle and are called *synlogue-like*. For the rest of this subsection we restrict our considerations to a one-dimensional axis of *real reason vectors*, i.e. we define a two-dimensional apperception model for decisive apperception with respect to a rectilinear space of reasons. In the next section 3 we upgrade the model to a four-dimensional space where alterations and steps are introduced for both components of complex reason vectors: real and imaginary.

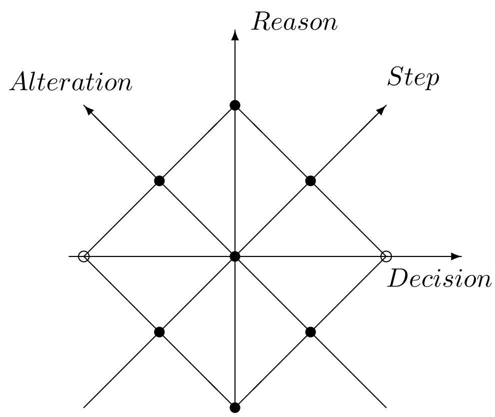

Figure 4: Two-dimensional space spanned by reason- and decision axes

Let r and d denote the elementary reason- and decision vectors and let s and a denote the elementary step- and alteration vectors respectively. Furthermore, call $\{a, s\}$ the *monologic basis* for $\mathbb{V}$ and $\{r, d\}$ the *diasynlogic basis* and consider the corresponding pair of mutually inverse coordinate transformations

$$\Phi : \mathbb{V}_{diasyn} \to \mathbb{V}_{mono} \qquad \Psi : \mathbb{V}_{mono} \to \mathbb{V}_{diasyn}$$

$$\Phi(u, v) := (u + v, u - v) \qquad \Psi(x, y) := (\tfrac{1}{2}(x + y), \tfrac{1}{2}(x - y)).$$

We define the exponential function $exp_{mono} : \mathbb{V}_{mono} \to \mathbb{A}_{mono}$ with respect to the mono-logic basis simply by applying the one-dimensional exponential to both coordinates.

$$exp_{mono}(x, y) := (exp(x), exp(y)).$$

From exp_{mono} we derive corresponding the formula $exp_{diasyn} : \mathbb{V}_{diasyn} \to \mathbb{A}_{diasyn}$:

$$\begin{aligned} exp_{diasyn}(u, v) &= \Psi(exp_{mono}(\Phi(u, v))) \\ &= \Psi(exp(u + v), exp(u - v)) \\ &= (\tfrac{1}{2}(exp(u + v) + exp(u - v)), \tfrac{1}{2}(exp(u + v) - exp(u - v))). \end{aligned}$$

For trivial the decision $v = 0$ we obtain $exp_{diasyn}(u, 0) = (exp(u), 0)$, i.e the exponential extends the ordinary exponential function for reasons. The figure below displays the corresponding exponential function $exp_{diasyn} : \mathbb{V}_{diasyn} \to \mathbb{A}_{diasyn}$ with respect to the diasynlogic basis.

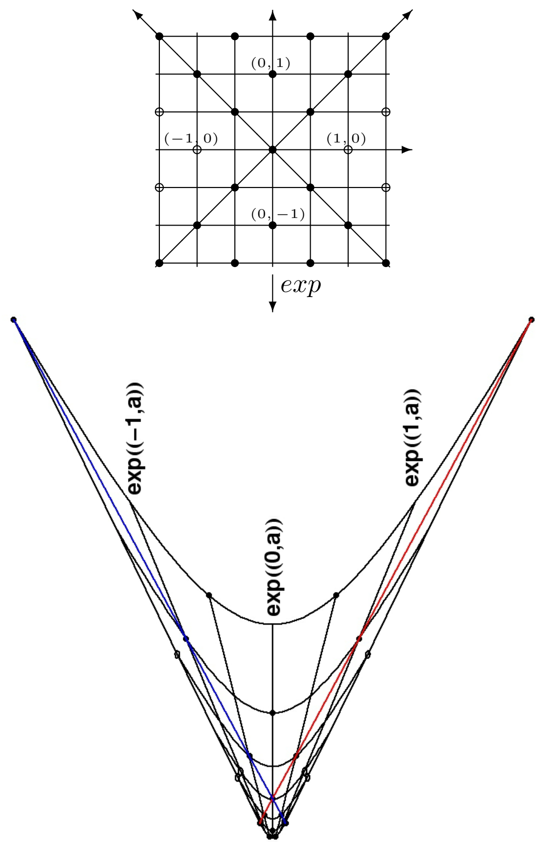

Figure 5: Exponential Function for Decision and Reason. The upper figure displays $7^2 = 49$ points and $6^2 = 36$ meshes of the virtual reason-decision lattice. The lower figure shows their images with respect to the exponential function.

3 Pure Tone Apperception

Our general prolegomena to apperception are now to be specified in terms of a concrete model for tone apperception. The first two subsections 3.1 and 3.2 are dedicated to the music-theoretical discussion of virtual tone apperception. In subsection 3.3 we define a discrete lattice $\mathbb{D}$ within a four-dimensional space $\mathbb{V}$ of tone vectors and present a four-dimensional group $\mathbb{A}$, which is called the *Active Tone System*.

3.1 Real and Imaginary Diatonic Height

We use the abstract notion of *height* in order to denote the reasons for (or within) tone apperceptions. Recall that we are not concerned with empirical questions regarding the interplay of sound perception and apperception. In other words, *height* is first of all a technical term to be interpreted music-theoretically and must not be confused with the psycho-accoustic concept of 'pitch height'. As a point of departure we study the discrete lattice of integral complex numbers (cf. subsection 2.2)

$$\mathbb{D}_{dia} = \{a + bi \,|\, a, b \in \mathbb{Z}\}.$$

A basic idea is to interpret *whole tone* intervals as whole integer vectors and *semi tone* intervals as half integer vectors. In subsection 2.3 we already saw, that the aspect of integrity goes beyond a mere bookkeeping of 'granularity'. An apperception vector with an integral reason coordinate may have a vanishing decision coordinate, while an apperception vector with a half-integer reason coordinate always implies a half-integer decision coordinate.

We further motivate the particular choice of the whole tone intervals as whole integer vectors by considering the phenomenon of octave identification in a specific way. The diatonic whole-tone is commonly qualified in two ways.

1. as a constitutive step of a certain size in a diatonic scale (besides the smaller semitone interval).

2. as a difference interval between two-fifths and one octave (which refers to an idealised frequency ratio of $\frac{3}{2}\frac{3}{2}\frac{1}{2} = \frac{9}{8}$). This qualifies the whole tone as a secondary interval and presupposes the fifth to be more elementary.

From the apperceptive viewpoint we should avoid circular definitions, i.e. we cannot interpret both qualifications just as paraphrases of one another. One can either define a *fifth* melodically as a concatenation of three whole steps and one semitone step or one can define a whole tone as an interval that is composed of two fifths (minus one octave).

We avoid the circularity by introducing two diatonic whole tone intervals, a

1. *real whole tone vector* $\delta_0 = 1 + 0i \in \mathbb{D}_{dia}$. The half $\frac{1}{2}\delta_0$ of this vector is interpreted as a *neutral semitone*.

2. *imaginary whole tone vector* $\delta_1 = 0+1i \in \mathbb{D}_{dia}$. The half $\frac{1}{2}\delta_1$ of this vector is interpreted as a *neutral fifth*.

The integral lattice $\mathbb{D}_{dia}$ spanned by δ_0 and δ_1 is named the *diatonic lattice*. The continuous two-dimensional ambient space $\mathbb{V}_{dia}$ is named the *diatonic plane*.[9]

The specific role of octave identification in our model is to interpret neutral fifths as 'imaginary semitones'. Much as a real whole tone can be divided into two 'real' semitones, an imaginary whole tone can be divided into two fifths. According to our definition not only do the resulting subdivisions differ, but the reasons to do so are already different vectors, namely δ_0 and δ_1. The same pair of physical tones with a frequency ratio in the neighbourhood of $\frac{9}{8}$ may give reason for a δ_0-apperception or a δ_1-apperception or both. It seems reasonable to assume real height apperception as a primary mode and imaginary height apperception as a secondary mode. Octave identifications add additional imaginary reasons to the primary real reasons. As a typical example we recall the apperception of harmonised passing tones:

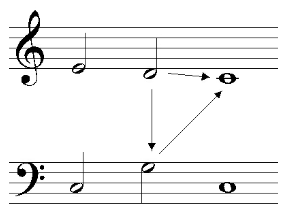

Figure 6: Apperception of harmonised passing tones

The step from the passing tone d to the final c can be described as an apperception of a real whole tone vector $-\delta_0$, but the apperception of an intermediate G between the d to the final c presupposes the (virtual) apperception of an imaginary whole tone vector $-\delta_1$. With regard to our considerations in subsection 2.3 the definition of the diatonic plane implies rectilinear apperception of the real height vectors and circular apperception of the imaginary height vectors.

Our definition has to be further discussed in the light of similar distinctions in the literature. We mention Rudolf Louis und Ludwig Thouille and their distinction between *tone neighborhood* and *tone kinship* as complementary modes of relating tones to one another ([14], p. 278). Even closer to our approach is Jacques Handschin's distinction of *tone height* and *tone character*. [10]

3.2 Steps and Alterations

In the previous subsection 3.1 we introduced *neutral semitone* and *neutral fifth* as half integral diatonic vectors $\frac{1}{2}\delta_0$ and $\frac{1}{2}\delta_1$. With regard to our considerations of subsection 2.3 these are not(!) considered as proper apperception vectors,

[9] The complex plane $\mathbb{C}$ is understood as a two-dimensional Lie-Algebra over $\mathbb{R}$.

[10] The following remark suggests the consideration of a free two-dimensional configuration space: "Die Durchkreuzung von Tonhöhe und Toncharakter, [...], beruht auf der begrifflichen Scheidung der beiden Dinge; und dasselbe tut die Affinität der beiden." (cf. [6], pp. 31)

because they imply a non-vanishing decision-coordinate. Before setting work on introducing *diasyntonic* and *monotonic* coordinate systems we provide music-theoretical interpretations of the *semitone lattice* and the *quintic lattice*. The following figures specify the reason-decision-lattice of subsection 2.3 for semitone vectors (left figure) and for fifth vectors or *quintic* vectors as we prefer to call them (right figure). The vertical half integer axes $\mathbb{Z}\frac{\delta_0}{2}$ and $\mathbb{Z}\frac{\delta_1}{2}$ are called the *diachromatic height axis* and the *diaquintic height axis* respectively. To both height axes we have corresponding horizontal decision axes. The integral decision vectors σ_0 and σ_1 are called the *synchromatic unit vector* and *synquintic unit vector* respectively.[11] The half integer axes $\mathbb{Z}\frac{\sigma_0}{2}$ and $\mathbb{Z}\frac{\sigma_1}{2}$ are called the *synchromatic width axis* and *synquintic width axis* respectively.

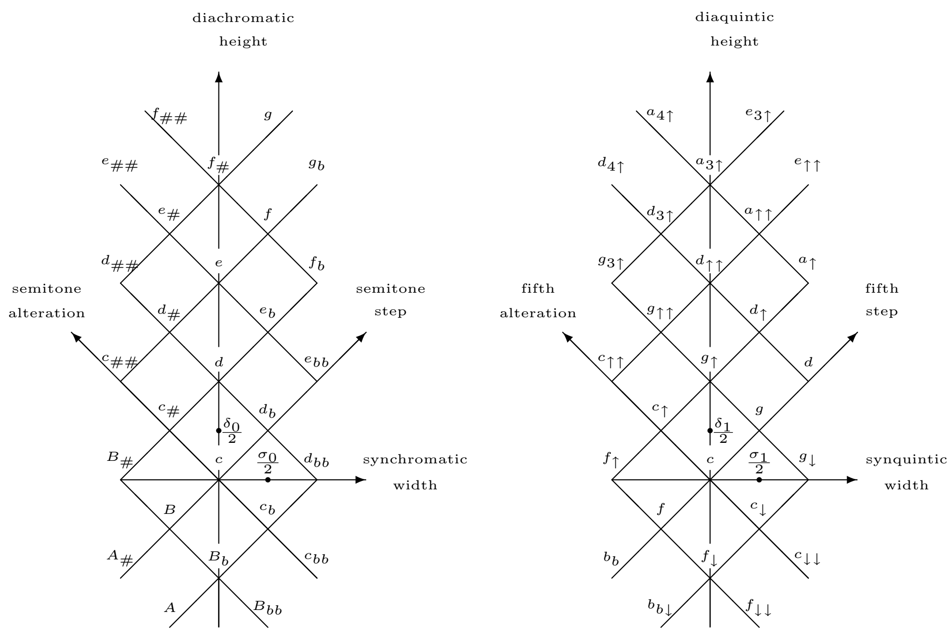

Figure 7: Semitone lattice $\mathbb{D}_s$ and quintic lattice $\mathbb{D}_q$

At first sight the semitone lattice $\mathbb{D}_s$ corresponds to musical notation. This implies the possibility of a 'prima vista'-interpretation of any single note-relation in the semitone-lattice. This is a practical starting point for apperceptive analysis of a musical piece, but we do not assume that 'prima vista' interpretations automatically do well. One must especially expect tone conflicts according to multiple tone relations. The quintic lattice $\mathbb{D}_q$ has been drawn in analogy to

[11]The synchromatic and synquintic unit vectors are the apperceptive correlates to what is traditionally described in terms of 'commata'. A curiosity of the present approach is to view them as integral quantities.

the semitone lattice. However, the distinction between fifth steps and fifth alterations does not correspond to musical notation and thus needs further explanation and motivation.

The labels of vectors in the quintic lattice are invented in formal analogy to the labels of vectors in the semitone lattice. Steps result in the change of the name, and alterations lead to change in the alteration sign. Fifth alterations are designated by vertical arrows in the subscript. For simplicity's sake we used the same basic note-names in both lattices, but they designate different vectors.

The music-theoretical question as to whether it is useful to distinguish two elementary types of fifths has to be answered on the basis of musical analyses. For the present paper we will especially emphasise the explanatory power of this assumption in the study of enharmonicity (section4). As a speculative example we interpret the succession of the bass tones in a typical $I \quad IV \quad I \quad V \quad I$ cadence as follows:

$$c \quad \rightarrow \quad c_{\downarrow} \quad \rightarrow \quad g_{\downarrow} \quad \rightarrow \quad g \quad \rightarrow \quad c$$

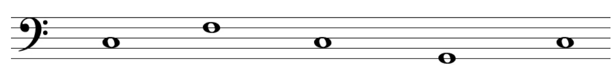

Figure 8: Apperception of Fifths

In order to better understand this interpretation, recall that the apperceptive approach presupposes a processual ontology. This implies that the identity of objects is not given apriori. Rather, identities are results or properties of apperceptive processes. In our interpretation of this short sequence we assume virtual identity of the first and the last tone. Furthermore, we assume an imaginary whole tone vector between the 4^{th} and 5^{th} degree and have therefore two choices for the corresponding vectors namely either f and $c_{\uparrow}$ or $c_{\downarrow}$ and g. The strong quality of the $V \rightarrow I$ succession motivates the interpretation of the last succession as a proper fifth step. Consequently the opening $I \rightarrow IV$ succession is interpreted as a quintic alteration. Finally, we have to interpret the middle occurrence of the 1^{st} degree. There are two choices, namely to assume an identity between the first and third tone or to prefer the other possible subdivision of the imaginary whole tone. The latter interpretation leads to a virtual cycle. We come back to this example in subsection **??**. In Riemannian terms, we would interpret the *dominant* $\rightarrow$ *tonic* succession as a fifth step and the *Tonic* $\rightarrow$ *Subdominant* succession as a fifth alteration. The non-identical middle tone would represent a dominant of the subdominant as well as a subdominant of the dominant. But as argued in subsection 1.3, it is not our intention to reconstruct the rich contexts of diatonism or functional harmony solely on the basis of *pure apperception*.

Another way of approaching the question is the implied structural analogy between virtual semitone apperception and virtual fifth apperception. The following similar pair of examples from Felix Salzer's and Carl Schachter's book ([24], p. 134/135) invites us to interpret the ascending fourths in the bass of the left example as fourth alterations.

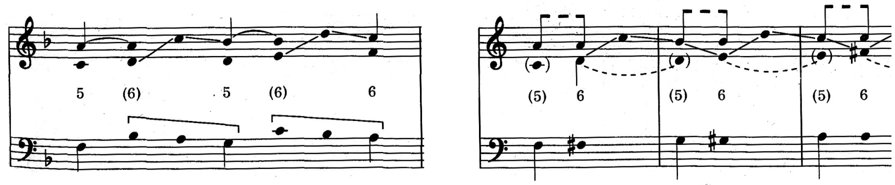

Figure 9: 'Fourth Alteration' and Chromatic Alteration in simlilar prolongational contexts cf. ([24])

3.3 The Active Tone System

So far we have introduced the two lattices $\mathbb{D}_s$ and $\mathbb{D}_q$ verbally and through figures and therefore we have not yet made explicit use of coordinates. Now we focus on the direct sum of both lattices $\mathbb{D} = \mathbb{D}_s \times \mathbb{D}_q$ as embedded in a four-dimensional real vector space $\mathbb{V}$. In order to be able to make calculations we now distinguish between two coordinate systems as we did in subsection 2.3. We will use the same symbols $\mathbb{V}_{diasyn}$ and $\mathbb{V}_{mono}$ as in subsection 2.3 to denote the four-dimensional real linear space of tone vectors $\mathbb{V}$ with respect to the *diasyntonic* and the *monotonic* bases respectively. The diasyntonic basis consists of the four vectors $\{\delta_0, \sigma_0, \delta_1, \sigma_1\}$ and the monotonic basis consists of the chromatic and quintic step- and alteration vectors: $\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3\}$ with

$$\epsilon_0 = \frac{\delta_0 + \sigma_0}{2}, \quad \epsilon_3 = \frac{\delta_0 - \sigma_0}{2}, \quad \epsilon_1 = \frac{\delta_1 + \sigma_1}{2}, \quad \epsilon_2 = \frac{-\delta_1 + \sigma_1}{2}.$$

We mention that the monotonic vector ϵ_2 does not represent the fifth alteration $\frac{\delta_1 - \sigma_1}{2}$, but instead its negative counterpart $\frac{-\delta_1 + \sigma_1}{2}$,i.e. the *fourth alteration.*[12] Vectors with respect to the diasyntonic basis are written as quadruples

$$(d_0, s_0, d_1, s_1) \in \mathbb{R}^4.$$

Vectors with respect to the monotonic basis are written as 2×2-matrices

$$\begin{pmatrix} x_0 & x_3 \\ x_1 & x_2 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}).$$

[12] The reason for this particular choice becomes evident in subsection **??**, when we investigate the diasyntonic norm which is then expressed by the determinant of the corresponding matrix in the monotonic basis.

We consider the corresponding pair

$$\Phi : \mathbb{V}_{diasyn} \to \mathbb{V}_{mono} \quad \text{and} \quad \Psi : \mathbb{V}_{mono} \to \mathbb{V}_{diasyn}$$

of mutually inverse coordinate transformations which are defined as:

$$\begin{array}{lcl} \Phi((d_0, s_0, d_1, s_1)) & := & \begin{pmatrix} d_0 + s_0 & d_1 + s_1 \\ -d_1 + s_1 & d_0 - s_0 \end{pmatrix} \\ \Psi(\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix}) & := & (\frac{x_0 + x_3}{2}, \frac{x_0 - x_3}{2}, \frac{x_1 - x_2}{2}, \frac{x_1 + x_2}{2}) \end{array}.$$

The following table displays the diasyntonic and monotonic basis vectors with respect to both coordinate systems.

Vector	Name	Diasyntonic coordinates	Monotonic coordinates
δ_0	Diachromatic Height Unit	$(1, 0, 0, 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
σ_0	Synchromatic Width Unit	$(0, 1, 0, 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
δ_1	Diaquintic Heigth Unit	$(0, 0, 1, 0)$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
σ_1	Synquintic Width Unit	$(0, 0, 0, 1)$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
ϵ_0	Semitone Step	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
ϵ_1	Fifth Step	$(0, 0, \frac{1}{2}, \frac{1}{2})$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
ϵ_2	Fourth Alteration	$(0, 0, \frac{1}{2}, -\frac{1}{2})$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
ϵ_3	Semitone Alteration	$(\frac{1}{2}, -\frac{1}{2}, 0, 0)$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

We now specify the discrete lattice $\mathbb{D} \subset \mathbb{V}$ of *integral tone vectors* with respect to both bases. With respect to the monotonic basis we obtain the lattice of all integral 2×2-matrices, i.e. matrices having integral coefficients.

$$\mathbb{D}_{mono} := \mathcal{M}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} z_0 & z_3 \\ z_1 & z_2 \end{pmatrix} \mid z_0, z_1, z_2, z_3 \in \mathbb{Z} \right\}.$$

The image $\mathbb{D}_{diasyn} := \Psi(\mathbb{D}_{mono})$ contains the integral points of $\mathbb{V}_{diasyn}$ as a proper sublattice on the one hand, and is embedded into the half-integral points of $\mathbb{V}_{diasyn}$ as a proper sublattice on the other.

$$\mathbb{D}_{diasyn} := \left\{(u_0, v_0, u_1, v_1) \mid \quad u_0, v_0, u_1, v_1 \in \frac{1}{2}\mathbb{Z} \quad \text{and} \quad u_0 + v_0, u_1 + v_1 \in \mathbb{Z}\right\}.$$

This completes the basic definitions we need for the investigation of virtual apperception. In the next step we must define the exponential function $exp : \mathbb{V} \to \mathbb{A}$. This can suitably be done with respect to the monotonic basis, i.e. we define a map $exp_{mono} : \mathbb{V}_{mono} \to \mathbb{A}_{mono}$, where

$$\mathbb{A}_{mono} := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \mid \quad ad - bc > 0\right\}$$

denotes the multiplicative group of all real 2×2-matrices with positive determinant.
The four-dimensional coordinate space $\mathcal{M}_2(\mathbb{R})$ serves as an ambient space for both $\mathbb{A}_{mono}$ and $\mathbb{V}_{mono}$. Therefore we must always specify whether a given matrix represents a vector $v \in \mathbb{V}$ or a group element $X \in \mathbb{A}$. According to our considerations in subsection 2.1 we furthermore must distinguish between the two interpretations of group elements, namely as tone loci and as (aquisitative) acts. In the latter case we speak of *active tones*. With regard to the apperception formula active tones $A = exp(v)$ are considered as exponentials of tone vectors v. The exponential function for 2×2-matrices[13] is defined in the same way as for complex or real numbers numbers, namely in terms of the power series

$$exp_{mono} : \mathbb{V}_{mono} \to \mathbb{A}_{mono} \quad \text{with} \quad exp_{mono}(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!}$$

In order to define the exponential function with respect to the diasyntonic basis we simply transport exp_{mono} via coordinate transformation to $\mathbb{V}_{diasyn}$, i.e. we consider $\mathbb{R}^4$ as an ambient space for both $\mathbb{V}_{diasyn}$ and $\mathbb{A}_{diasyn} = \Psi(\mathbb{A}_{mono})$ and define

$$exp_{diasyn} := \Psi \circ exp_{mono} \circ \Phi : \mathbb{V}_{diasyn} \to \mathbb{A}_{diasyn}.$$

The restriction of the exponential exp_{mono} to the lattice $\mathbb{D}_{mono}$ of integral tone vectors leads to the *discrete active tone system*, namely the group $\mathbb{G}_{mono} = \langle exp_{mono}(\mathbb{D}_{mono})\rangle$ generated by all exponentials of integral tone vectors.
We conclude this subsection by showing that the two-dimensional situations discussed in subsections 2.2 and 2.3 are special cases of the present four-dimensional situation and by providing a general formula for the calculation of the exponential function exp_{mono}.

[13] With respect to the standard nomenclature of matrix groups we deal with the exponential $exp : gl_2(\mathbb{R}) \to GL_2(\mathbb{R})^+)$ with Lie-Algebra $gl_2(\mathbb{R})$ as its domain and Lie-Group $GL_2(\mathbb{R})^+$ as its codomain.

- The *diatonic plane* corresponding to real and imaginary height axes is embedded into $\mathcal{M}_2(\mathbb{R})$ by virtue of

$$dia : \mathbb{C} \to \mathcal{M}_2(\mathbb{R}) \quad \text{with} \quad dia(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

  and the complex exponential function in this embedding reads as

$$exp_{mono}(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}) = exp(a) \cdot \begin{pmatrix} cos(b) & sin(b) \\ -sin(b) & cos(b) \end{pmatrix}.$$

  This active tone includes the rectilinear apperception $exp(a)$ and the circular apperception corresponding to a planar rotation along the angle b.

- The *semitone plane* corresponds to the diagonal matrices $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ and the exponential function reads as

$$exp_{mono}(\begin{pmatrix} x_0 & 0 \\ 0 & x_3 \end{pmatrix}) = \begin{pmatrix} exp(x_0) & 0 \\ 0 & exp(x_3) \end{pmatrix}.$$

The exponential function exp_{mono} is well defined and can furthermore be explicitly calculated. First recall that the determinant and the trace of a matrix define maps $det, tr : \mathbb{V}_{mono} = \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}$,

$$det \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = x_0x_3 - x_1x_2, \qquad tr \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = x_0 + x_3.$$

Due to the fact that scalar matrices $diag(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ multiplicatively commute with any other matrix we can suitably write

$$\xi = diag(tr(\xi)) + \xi_0 = \begin{pmatrix} \frac{x_0+x_3}{2} & 0 \\ 0 & \frac{x_0+x_3}{2} \end{pmatrix} + \begin{pmatrix} \frac{x_0-x_3}{2} & x_1 \\ x_2 & \frac{-x_0+x_3}{2} \end{pmatrix}$$

with $tr(\xi_0) = 0$ and obtain

$$\begin{aligned} exp_{mono}(\xi) &= exp_{mono}(diag(tr(\xi)) + \xi_0) \\ &= diag(exp(tr(\xi))) \cdot exp_{mono}(\xi_0). \end{aligned}$$

In order to express $exp_{mono}(\xi_0)$, set $l = \sqrt{|\det(\xi_0)|}$. Then[14]

$$exp_{mono}(\xi_0) = \begin{cases} diag(cos(l)) + \frac{sin(l)}{l} \cdot \xi_0 & , \text{ if } det(\xi_0) > 0, \\ diag(1) + \xi_0 & , \text{ if } det(\xi_0) = 0, \\ diag(cosh(l)) + \frac{sinh(l)}{l} \cdot \xi_0, & \text{ if } det(\xi_0) < 0. \end{cases}$$

[14]We present these formulas without proof and refer to [8].

With respect to the common embedding of tone vectors and active tones into the same ambient space $\mathcal{M}_2(R)$ we have observed that the exponential exp_{mono} maps the diatonic as well as the semitone planes into themselves respectively. Me emphasize that this is not the case for the fifth plane. The apperception of fifths has many interesting properties some of which are studied in the following section 4.

We conclude this subsection by introducing conventions for the denotation and description of apperceptive processes. Recall that apperceptive processes are sequences $(X_0, ..., X_n)$ of tone loci, so that $X_{i+1} = App(X_i, v_i)$ for suitable tone vectors v_i. We denote such a process by $App(X_0, (v_1, ...v_n))$. It involves four sequences of interest:

1. The *actual apperceptive path* $(X_0, ..., X_n)$,
2. The *acquired tone vectors* $(v_1, ..., v_n)$,
3. The *acquired active tones* $(A_1, ..., A_n)$ with $A_i = exp_{mono} v_i$,
4. The *virtual apperceptive path* $(x_1, ..., x_n)$ with $x_i = \sum_{j=1}^{i} v_j$.

The last vector of the virtual apperceptive path which is the sum $\sum_{i=1}^{n} v_i$ of all accumulated tone vectors is called the *virtual result* and the product $\Pi_{i=1}^{n} A_i$ of all accumulated active tones is called the *actual result* of this apperceptive process. We say that an apperceptive process has a *virtually unnoticable result*, if its virtual result is the zero-vector $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and we speak of an *actually unnoticable result* if the actual result is the identical active tone $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

4 Effects of Enharmonicity

While the entire model is a more or less self-evident elaboration of Wilhelm Wundt's idea (including our further prolegomena), we are now entering another type of speculative thought which is motivated by the inner mathematics of the model on the one hand and unsolved music-theoretical problems on the other. In the present section we will show how one can elegantly solve some music-theoretical problems. A proper formulation of these problems raises several preliminary questions (cf. section 1 as well as [20], [21]) We avoid these questions by directly replying to a recent proposal on the modeling of enharmonicity in Edward Gollin's Dissertation [5], who reviews Lewin's GIS-model and several Neo-Riemannian approaches in the language of group presentations.

4.1 Enharmonic Relators

The problem of enharmonic identification is typically discussed as a subject matter of intonation and/or tuning practice and one is then concerned with the investigation of occasional and/or systematic neglect of small pitch intervals (syntonic comma, enharmonic diesis, pythagorean comma). Among the controversial issues behind this discussion we mention the celebrated question as to whether musical intervals (harmonic as well as melodic) should ideally be intoned as just intervals. From the apperceptive point of view there is a very interesting problem related to this question, namely the implied occurrence of identity conflicts between tones in musical pieces. This specific question is related to a more general and perhaps more important question, namely as to whether tone conflicts are essential to musical apperception or not. Compared to that it seems to be a rather secondary although empirically useful question how particular conflicts in tone apperception can be supported or avoided through intonation and tuning.[15] With some restrictions to be mentioned later we try to characterise *enharmonicity* as a modality to support virtually or actually unnoticable transitions between tone apperceptions which typically create or solve identity conflicts. Edward Gollin's approach to musical spaces is based on David Lewin's transformational generalisation of the GIS model which is formulated in the second half of Lewin's book and which is mainly motivated through the anti-isomorphism between $IVLS$ and $STRANS$. While no longer concerned with the interval group $IVLS$ Gollin studies a space S and a group G acting on S. He introduces another distinction which resembles to some extent our distinction between actual and virtual apperception as we will see below, but we prefer not to apply these concepts here. Gollin considers proper *group elements* of a given transformation group G on the one hand and *words* composed of suitably chosen group generators presenting these elements on the other. He therefore attributes music-theoretical or even appperceptive meaning to two levels of a finite group presentation

$$G = F/N = (g_1, ..., g_n; r_1, ..., r_m).$$

First recall that in such a group presentation

- $F = \langle g_1, ..., g_n \rangle$ denotes the free group generated by a finite set $\{g_1, ..., g_n\}$ of elements, i.e. F consists of all[16] words, whose 'letters' are $g_1, ..., g_n$ and their inverses $g_1^{-1}, ..., g_n^{-1}$, including the empty word which represents the group identity.

[15] Empirically it seems reasonable to study intonation and tuning as embedded into other performance parameters and to investigate performance from a rhetorical point of view, which presupposes a theory of apperception.

[16] Words containing redundant subwords $g^{-1}g$ have to be omitted. Otherwise one would work with the free semigroup generated by $g_1, ...g_n, g_1^{-1}, g_n^{-n}$ which Gollin actually does in some of his examples. This third interpretative layer to Gollin's model corresponds to what we call processes, or more precisely to the accumulative denotation of such processes through sequences of active tones.

- $N = \bigcup_{g \in F} g \langle r_1, ..., r_m \rangle g^{-1}$ denotes the smallest normal subgroup of F, containing the finite set $\{r_1, ..., r_m\} \subset F$ of relators.
- G = F/N denotes the factor group of F with respect to the normal subgroup N, i.e. G consists of equivalence classes of elements of $F \, mod \, N$ which are said to present the same element of G. All elements of N present the identity of G.

Gollin interprets a word w in two ways, namely as an element of F and as (presenting) an element of G. In some cases and especially in our particular case he considers an intermediate level F_{ab} between F and G being constituted by a special type of relators, namely *commutators* of generators which then leads to the free commutative group $F_{ab} = F/[F,F]$ being isomorphic to $\mathbb{Z}g_1 \times ... \times \mathbb{Z}g_n$.

The following example is a central one in Gollin's investigations. He introduces the free group $F =< Q, T >$ generated by two elements Q and T abreviating the transformations QUINT and TERZ. Its elements are considered as particular pathways on the Riemann Tone-Net (displayed in the figure below). The free commutative group $F_{ab} = (Q, T; Com)$ with the commutation relator $Com = QTQ^{-1}T^{-1}$ acts simply and transitively on this space. If one looks at the Riemann-Tone-Net as a graph it coincides with the Cayley-Graph associated with this group presentation.

D	A	E	B	$F_{\#}$	$C_{\#}$	$G_{\#}$
B_b	F	C	G	D	A	E
G_b	D_b	A_b	E_b	B_b	F_b	C

Figure 10: Riemann-Tone-Net

In a further step Gollin introduces the enharmonic relators $Syn = Q^4T^{-1}$, $Enh_1 = T^3$ and $Enh_2 = Q^{12}$ and studies the group presentation

$$\mathcal{G}_{12} = (Q, T; Com, Syn, Enh_1, Enh_2)$$

The relator Syn expresses the equivalence of a TERZ transformation T with a sequence $QQQQ$ of four QUINT transformations. Within the just tuning context this would be related to the neglect of the syntonic comma. The relator Enh_1 expresses the identification of a sequence TTT of three TERZ transformations with the identity element of the group. The relators Syn and Enh_1 already imply the relator Enh_2, expressing the identification of a sequence $QQQQQQQQQQQQ$ of twelve QUINT transformations with the identity element. For reasons of redundance one could therefore omit one of the two relators Enh_1 or Enh_2, but Gollin takes both into account for music-theoretical reasons. The presented group $\mathcal{G}_{12}$ denotes the translations within a 12-tone space. The idea of modelling enharmonic identifications in terms of group factorisation takes a major point into account, namely that concatenations of negligible intervals have to be neglected as well. This might seem problematic from a psycho-accoustic point of view, where a sufficient accumulation of negligible quantities does not remain negligible. But from an apperceptive point of view *homogeneity* is a reasonable condition for the configuration space. Gollin avoids total neglect of these relators in terms of his distinction of the layers F, F_{ab} and $\mathcal{G}_{12}$. The very choice of the particular relators Syn and Enh_1 or Enh_2 out of an infinite number of possibilities is somewhat arbitrary from a mathematical point of view, which then would imply further cognitive questions. According to our distinction between actual apperception (not necessarily being commutative) and virtual apperception (being commutative by assumption) we split Gollin's successive introduction of the relators Com and then Syn and Enh_1 into two group presentations:

- a multiplicative non-commutative group $\mathcal{G} = (Q, T; Syn, Enh_1)$ whose elements might be called *Gollin Acts*, and.
- an additive commutative group $\mathcal{D} = (q, t; q + t - q - t)$ whose elements might be called *Riemann Vectors*, if one interprets them as intervals in the Riemann-Tone-Net.

We inspect these two groups $\mathcal{G}$ and $\mathcal{D}$ and their music-theoretical interpretations at the end of the next subsection and compare them to the quintic subgroup of the active tone system and the quintic lattice respectively.

4.2 Relations in the Quintic Group

We consider two particular elements of the discrete active tone system $\mathbb{G}_{mono}$, namely the *active fifth step* $Q_1 = exp_{mono}(\epsilon_1)$ and the *active fourth alteration* $Q_2 = exp_{mono}(\epsilon_2)$. Let $\mathbb{G}_q := \langle Q_1, Q_2 \rangle \subset \mathbb{G}_{mono}$ denote the *quintic subgroup* of the discrete active tone system generated by the two active tones

$$\begin{aligned} Q_1 &= exp_{mono}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ Q_2 &= exp_{mono}(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned}.$$

The quintic group $\mathbb{G}_q$ is a proper subgroup of the group $\langle exp_{mono}(\mathbb{D}_q) \rangle$ generated by all(!) exponentials of vectors from the quintic lattice $\mathbb{D}_q$. The quintic group contains only those active tones composed from monotonic fifth apperceptions. Two interesting relations hold between Q_1 and Q_2^{-1}, i.e. between active fifth step and active fifth alteration:

- *Hallucination Relation:* $Q_1 \cdot Q_2^{-1} \cdot Q_1 = Q_2^{-1} \cdot Q_1 \cdot Q_2^{-1}$
- *Escher-Staircase Relation:* $(Q_1 \cdot Q_2^{-1})^6 = Id$

These relations are easily verified. As to the Hallucination relation we have

$$\begin{aligned} Q_1 \cdot Q_2^{-1} \cdot Q_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ Q_2^{-1} \cdot Q_1 \cdot Q_2^{-1} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

Concerning the Escher-Staircase relation we have

$$\begin{aligned} (Q_1 \cdot Q_2^{-1})^6 &= \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right)^6 \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^{3 \cdot 2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

The names *Hallucination* and *Escher-Staircase* are motivated by the specific discrepancies between actual and virtual apperceptions in these cases. The picture below reproduces a drawing of the famous graphic artist Maurits Cornelis Escher depicting a staircase which exemplifies such a discrepancy visually.

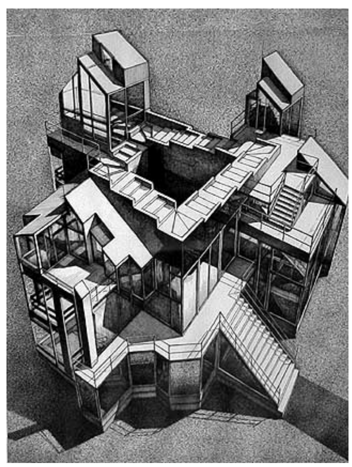

Figure 11: Maurits Cornelis Escher: Staircase

We interpret $Q_1 \cdot Q_2^{-1} \cdot Q_1 = Q_2^{-1} \cdot Q_1 \cdot Q_2^{-1}$ as a common final tone locus of two apperceptive processes starting from the same tone locus X, namely

$$\begin{aligned} X \cdot Q_1 \cdot Q_2^{-1} \cdot Q_1 &= App(X, (\epsilon_1, -\epsilon_2, \epsilon_1) \\ X \cdot Q_2^{-1} \cdot Q_1 \cdot Q_2^{-1} &= App(X, (-\epsilon_2, \epsilon_1, -\epsilon_2) \end{aligned}$$

The corresponding virtual apperceptions differ from one another. Both final tone vectors in the corresponding virtual apperceptive paths share the same amount of reason $\frac{3}{2}\delta_1$ but differ in the sign of their decision part $\frac{1}{2}\sigma_1$ vs. $-\frac{1}{2}\sigma_1$:

$$\begin{aligned} \epsilon_1 - \epsilon_2 + \epsilon_1 &= 2\epsilon_1 - \epsilon_2 &= \tfrac{3}{2}\delta_1 + \tfrac{1}{2}\sigma_1 \\ -\epsilon_2 + \epsilon_1 - \epsilon_2 &= \epsilon_1 - 2\epsilon_2 &= \tfrac{3}{2}\delta_1 - \tfrac{1}{2}\sigma_1 \end{aligned}$$

We choose the term 'hallucination' because virtually there is a confusion about the sign of the nontrivial decision. In the Escher-Staircase we have a process of 12 elementary apperceptions which leads back to the start tone locus X.

$$X = X \cdot (Q_1 \cdot Q_2^{-1})^6 = App(X, (\epsilon_1, -\epsilon_2, \epsilon_1, -\epsilon_2, \epsilon_1, -\epsilon_2, \epsilon_1, -\epsilon_2, \epsilon_1, -\epsilon_2, \epsilon_1, -\epsilon_2)$$

The corresponding virtual apperception process finally leads to a sum vector of six imaginary whole-tones $6\epsilon_1 - 6\epsilon_2 = 6\delta_1$. The figure below displays the virtual processes involved in the Hallucination relation (left) and the Escher-Staircase relation with a 6-fold $(\epsilon_1, -\epsilon_2)$-zigzag (right).

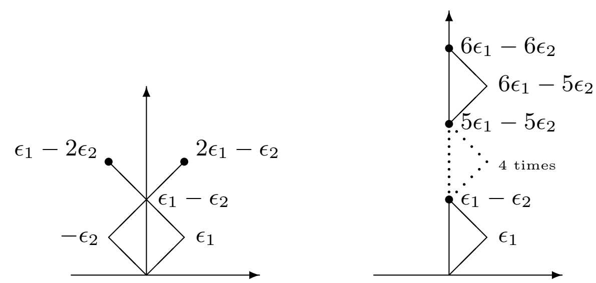

Figure 12: Hallucination and Escher-Staircase as virtual processes

It is essential to mention that all involved matrices - those representing the vectors as well as those representing their exponentials, i.e. the active tones, - are integral. An iterated apperception of the imaginary whole tone vector $\delta_1 = \epsilon_1 - \epsilon_2$ cutting the corners in each of the zigzags leads instead to an infinite sequence of tone loci on a circle

$$(X, X \cdot \begin{pmatrix} cos(1) & sin(1) \\ -sin(1) & cos(1) \end{pmatrix}, X \cdot \begin{pmatrix} cos(2) & sin(2) \\ -sin(2) & cos(2) \end{pmatrix}, ...)$$

which never returns to the tone locus X. To reach a finite sequence of length 6 on a circle one must apperceive the vector $\frac{2\pi}{6}\delta_1$ with transcendental length. Taking into account both relations one can easily conclude that $Q_1 \cdot Q_2^{-1} \cdot Q_1$ is of finite order 4.

$$\begin{aligned} Id = (Q_1 \cdot Q_2^{-1})^6 &= ((Q_1 \cdot Q_2^{-1}) \cdot (Q_1 \cdot Q_2^{-1}) \cdot (Q_1 \cdot Q_2^{-1}))^2 \\ &= ((Q_1 \cdot Q_2^{-1} \cdot Q_1) \cdot (Q_2^{-1} \cdot Q_1 \cdot Q_2^{-1}))^2 \\ &= ((Q_1 \cdot Q_2^{-1} \cdot Q_1) \cdot (Q_1 \cdot Q_2^{-1} \cdot Q_1))^2 \\ &= (Q_1 \cdot Q_2^{-1} \cdot Q_1)^4 \end{aligned}$$

This leads to the observation that an iterated circular apperception of the transcendental vector $\frac{\pi}{2}\delta_1$ can be refined into an apperception process with integral 'zig-zag-zig'-processes instead of the single transcendental ones.

It is practical to introduce abbreviations as basic letters of a kind of 'cuneiform writing' for appercetive processes.

$$\begin{array}{ll} \text{active fifth step} & Q_1 = \triangleright \\ \text{active fourth step} & Q_1^{-1} = \triangleleft \\ \text{active fourth alteration} & Q_2 = \bigtriangledown \\ \text{active fifth alteration} & Q_2^{-1} = \bigtriangleup \end{array}$$

Any further relation in the quintic group is already implied by the hallucination $\triangleright \bigtriangleup \triangleright = \bigtriangleup \triangleright \bigtriangleup$ and the Escher-Staircase $(\triangleright\bigtriangleup)^6 = Id$. The following two group presentations for the quintic group are equivalent[17] The generators S, T of the second presentation correspond to QP^{-1} and $QP^{-1}Q$ in the first.

$$\begin{array}{l} \mathbb{G}_q := (Q, P; QP^{-1}QPQ^{-1}P, (QP^{-1})^6) \\ \mathbb{G}_q := (S, T; S^6, T^4, S^3T^2) \end{array}$$

We emphasise that these presentations of the quintic group can be suitably compared with Gollin's group presentation. But we do not(!) intend to introduce further ontological interpretations of these meta level structures in addition to our processual ontology of tone loci and active tones. After these more or less technical preparations we now set to work on investigating the phenomena of enharmonicity. How does the first of these two presentations relate to Gollin's group presentation of $\mathcal{G}_{12}$ and the two ramifications $\mathcal{G}$ and $\mathcal{D}$?

4.2.1 Quintic Lattice and Riemann Vectors

The quintic lattice $\mathbb{D}_q$ is generated by two quintic vectors, namely virtual fifth step ϵ_1 and virtual fourth alteration ϵ_2. In contrast, the space $\mathcal{D}$ of Riemann-Vectors is generated from a fifth vector q and a third vector t. The music-theoretical comparability of both spaces is therefore restricted through different assumptions about the basic intervals. While taking this into account we can nevertheless investigate suitable mappings between $\mathcal{D}$ and $\mathbb{D}_q$. We define three isomorphisms $\phi_{3,1}, \phi_{2,2}$ and $\phi_{1,3}$ sending q to ϵ_1 and sending t to compound third vectors $t_{3,1}, t_{2,2}$ and $t_{1,3}$, composed of virtual fifth steps and alterations in three different ways.

$$\begin{array}{lll} \phi_{3,1} : \mathcal{D} \to \mathbb{D}_q & \text{with} & \phi_{3,1}(q) := \epsilon_1, \quad \phi_{3,1}(t) = t_{3,1} := 3\epsilon_1 - \epsilon_2 \\ \phi_{2,2} : \mathcal{D} \to \mathbb{D}_q & \text{with} & \phi_{2,2}(q) := \epsilon_1, \quad \phi_{2,2}(t) = t_{2,2} := 2\epsilon_1 - 2\epsilon_2 \\ \phi_{1,3} : \mathcal{D} \to \mathbb{D}_q & \text{with} & \phi_{1,3}(q) := \epsilon_1, \quad \phi_{1,3}(t) = t_{1,3} := \epsilon_1 - 3\epsilon_2 \end{array}$$

The inverse map $\psi_{3,1} : \mathbb{D}_q \to \mathcal{D}$ to $\phi_{3,1}$ sends the fourth alteration ϵ_2 to the Rieman vector $p := \psi_{3,1}(\epsilon_2) = 3q - t$, because $\phi_{3,1}(3q-t) = 3\epsilon_1 - (3\epsilon_1 - \epsilon_2) = \epsilon_2$. This vector is called the *syntonic fourth*, because it differs from a fourth $-q$ by

[17] The quintic group coincides with the classical group $SL(2, \mathbb{R})$ which is known to be isomorphic to the amalgamated product $\mathbb{Z}_6 \times_{\mathbb{Z}_2} \mathbb{Z}_4$ with $\mathbb{Z}_6 = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle$, $\mathbb{Z}_4 = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle$ and $\mathbb{Z}_2 = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$.

a *syntonic vector* $s = 4q - t$. Along with the isomorphism $\psi_{3,1}$ one would identify the syntonic fourth with the virtual fourth alteration. In contrast, the isomorphism $\psi_{2,2}$ supports an interpretation where the third t is identified with two imaginary whole tone vectors $t_{2,2} = 2\delta_1$. This latter interpretation is interesting in connection with a quintic zig-zag model of the diatonic scale. However, according to the hallucination relation $\triangleright\triangle\triangleright = \triangle\triangleright\triangle$ there is actually no serious conflict between these two alternatives:

$$App(X, (\epsilon_1, \epsilon_1, -\epsilon_2, \epsilon_1)) = X \triangleright \triangleright \triangle \triangleright = X \triangleright \triangle \triangleright \triangle = App(X, (\epsilon_1, -\epsilon_2, \epsilon_1, -\epsilon_2))$$

From the actual point of view it does not matter whether the Riemann third vector t is interpreted as $t_{3,1}$ or as $t_{2,2}$ as long as the elementary fifth steps and alterations are apperceived in a suitable order. Virtually, of course, both interpretations differ from one another.

4.2.2 Hallucination and Synquintic Identification

The syntonic relator $Syn = QQQQT^{-1} \in \mathcal{F}$ corresponds to the syntonic Riemann vector $s = 4q - t \in \mathcal{D}$. Under the isomorphism $\phi_{3,1}$ it is mapped onto the synquintic width unit vector $\epsilon_2 + \epsilon_1 = \sigma_1$. A direct apperception $App(X, \sigma_1)$ of this tone vector would violate the economy principle of apperception. But of course it is possible to split this into a process $App(X, (\epsilon_2, \epsilon_1))$ of length two, which we call a *mediated synquintic identification*, composed of a fourth alteration and a fifth step. We therefore compare the relator Syn with the composed active tone

$$\triangleright\triangledown = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{G}_q.$$

This matrix differs from the unit matrix and hence the situation in $\mathbb{G}_q$ is different from the group $\mathcal{G}$ of Gollin acts, where Syn represents the identical act. But surprisingly the situation is not essentially different. We convert the hallucination relation into the following equivalent forms:

$$\begin{array}{lcl} \triangleright \triangle \triangleright & = & \triangle \triangleright \triangle \\ \triangleleft \triangleright \triangle \triangleright \triangledown & = & \triangleleft \triangle \triangleright \triangle \triangledown \\ \triangle \triangleright \triangledown & = & \triangleleft \triangle \triangleright \\ \triangleright\triangledown & = & \triangledown \triangleleft \triangle\triangleright \end{array}$$

The last equation shows that mediated synquintic identification $App(X, (\epsilon_1, \epsilon_2))$ and a *fifth commutation process* $App(X, (\epsilon_2, -\epsilon_1, -\epsilon_2, \epsilon_1))$ actually lead to the same tone locus. But virtually the former yields a nontrivial synquintic width unit while the latter yields a zero vector because

$$\epsilon_2 - \epsilon_1 - \epsilon_2 + \epsilon_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

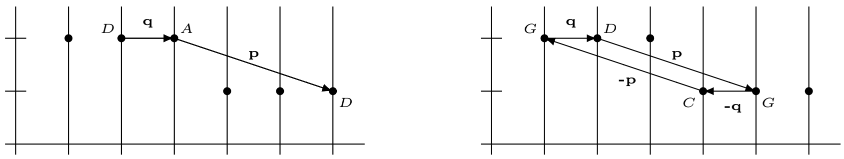

Figure 13: Mediated synquintic identification and the fifth commutation

The figures above display the virtual apperception processes corresponding to the mediated synquintic identification (left) and the fifth commutation (right) in terms of Riemann vectors, i.e. as vector sequences (q, p) and $(p, q, -p, -q)$.

There are 8 possible *fifth commutations*, i.e. concatenations $A \cdot B \cdot A^{-1} \cdot B^{-1}$ where $A, B \in \{\triangleright \triangleleft \bigtriangledown, \triangle\}$ and $B \neq A, A^{-1}$. We distinguish between two types according to the virtual result of the first two elementary apperceptions $X \cdot A \cdot B = App(X, (a, b))$ in such a process. If $a + b = \delta_1$ or $-\delta_1$ we speak of a *diaquintic fifth commutation* and if $a + b = \sigma_1$ or $-\sigma_1$ we speak of a *synquintic fifth commutation*. With regard to this terminology we may paraphrase our result as follows: A mediated synquintic identification has the same result as a suitable diaquintic commutation. We mention that a similar statement is not(!) true for a mediated imaginary wholetone apperception $\triangleright\triangle$ because it is has not have the same actual result as any commutation process.

4.2.3 Escher-Staircases and Enharmonic Identification

Through a similar consideration as in the previous subsection we may compare the enharmonic relator $Enh_1 = TTT$ either with

1. a zig-zag-Escher-Staircase

$$(\triangleright \triangle \triangleright \triangle) \cdot (\triangleright \triangle \triangleright \triangle) \cdot (\triangleright \triangle \triangleright \triangle),$$

   which means that we interpret the third as being composed of two fifth steps and two fifth alterations or

2. a zig-zig-zag-zig-Escher-Staircase

$$(\triangleright \triangleright \triangle \triangleright) \cdot (\triangleright \triangleright \triangle \triangleright) \cdot (\triangleright \triangleright \triangle \triangleright),$$

   which means that we interpret the third as being composed of three fifth steps and one fifth alteration.

With respect to these comparisons we find similar relations in $\mathcal{G}$ and $\mathbb{G}_q$. [18] However, Gollins relator $Enh_2 = Q^{12}$ has no correspondence in the quintic group because $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{12} = \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix}$ obviously differs from the identity

[18] Likewise one may compose the third into one step and three alterations.

matrix. More generally, one may investigate all $2^{12} = 4096$ possible apperception processes of length 12 consisting of either a fifth step or a fifth alteration at each tone locus. Closer examination shows there are exactly 196 generalised Escher-Staircases among these 4096 possibilities whose product is the identity element. The following 10 examples classify these 196 possibilities up to 12-cyclic permutations, reversal of the symbols and exchange of $\triangle$ and $\triangleright$.

sequence of active tones	virtual result
$\triangle\ \triangle\ \triangle\ \triangleright\ \triangle\ \triangle\ \triangleright\ \triangleright\ \triangleright\ \triangle\ \triangleright\triangleright$	$6\epsilon_1 - 6\epsilon_2$
$\triangle\ \triangle\ \triangleright\ \triangle\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangle\ \triangleright\triangleright$	$6\epsilon_1 - 6\epsilon_2$
$\triangle\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangle\ \triangleright\ \triangle\triangleright$	$6\epsilon_1 - 6\epsilon_2$
$\triangle\ \triangleright\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangle\triangleright$	$6\epsilon_1 - 6\epsilon_2$
$\triangle\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangleright\ \triangle\ \triangleright\triangleright$	$7\epsilon_1 - 5\epsilon_2$
$\triangle\ \triangleright\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangle\ \triangleright\triangleright$	$7\epsilon_1 - 5\epsilon_2$
$\triangle\ \triangle\ \triangleright\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangleright\ \triangleright\ \triangle\ \triangleright\triangleright$	$8\epsilon_1 - 4\epsilon_2$
$\triangle\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangleright\ \triangle\ \triangleright\triangleright$	$8\epsilon_1 - 4\epsilon_2$
$\triangle\ \triangleright\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangle\ \triangleright\triangleright$	$8\epsilon_1 - 4\epsilon_2$
$\triangle\ \triangleright\ \triangleright\ \triangleright\ \triangle\ \triangleright\ \triangleright\ \triangleright\ \triangle\ \triangleright\ \triangleright\triangleright$	$9\epsilon_1 - 3\epsilon_2$

4.2.4 Twelve-Tone System and Abstract Process Classes

We conclude our investigations of the quintic group by introducing the quintic commutation processes such as $\triangleright \triangle \triangleleft \bigtriangledown$ and $\triangleright \bigtriangledown \triangleleft \triangle$ as relators. Recall that the mediated synquintic identification $\bigtriangledown\triangleright$ differed from the identity element (i.e. from the role of *Syn* as a relator) exactly by a commutation process. This motivates the consideration of the factor group $\mathbb{G}_q/[\mathbb{G}_q, \mathbb{G}_q]$ of *process classes* which - surprisingly - is a finite cyclic group of order 12.[19] From the mathematical point of view the construction of this group is much less arbitrary than Edward Gollin's presentation of $\mathcal{G}_{12}$ and may be interpreted in the following way: if one considers all apperception processes which are composed from fifth and fourth steps and fifth and fourth alterations and classifies them according to the occurrences of these elementary active tones $\triangleright$, $\triangleleft$, $\triangle$ and $\bigtriangledown$ disregarding their concrete order, one obtains exactly 12 equivalence classes of such processes forming a cyclic group with respect to the composition of active tones. Ontologically, we do not interpret these abstract classes of active tones as something 'active'. The group factorisation yields interesting information about the deep structure of the quintic group. But a concept of 'active identification' of two processes of the same process class would imply a rather strange quality of apperceptive abstraction. According to our present processual ontology of single(!) processes of tone apperception such identifications would imply the capability of the apperceiving mind to rewrite its own active process history. Hence, in a next theoretical step one must develop an idea of multiple processes co-occuring in the same configurational space[20]. Our definition of enharmonicity as a modality

[19]For a proof of this fact cf. [19].
[20]or alternatively in suitable tensor products

to support virtually or actually unnoticable transitions between tone apperceptions therefore remains a preliminary one and must to be further elaborated in terms of incidence relations between processes. But we review the two cases studied so far in the context of quintic group $\mathbb{G}_q$ and the quintic lattice $\mathbb{D}_q$.

1. The virtual and actual results of a mediated synquintic identification are both noticable, but the actual result coincides with the actual result of a quintic commutation whose virtual result is unnoticable. In other words, synquintic identification is based on a transition of virtual unnoticability via actual result coincidence.

2. The actual result of an Escher-Staircase is unnoticable, but the virtual result is noticable. In other words, enharmonic identification is based on actual unnoticability which can be interpreted as a transfer of the identical apperception to a virtually noticable one.

4.3 Synchromatic Identification

Now we direct our attention from the quintic group to similar questions concerning semitone apperception. First of all, there is a oddity to be mentioned which appears in the music-theoretical interpretation of the four-dimensional apperception model. The model offers two alternatives in order to express enharmonic conflicts.

1. A proper *enharmonic conflict* is expressed by a tone vector $n_1\epsilon_1 - n_2\epsilon_2 \in \mathbb{D}_q$ with $n_1 + n_2 = 12$ (cf. 4.2.3).

2. A *synchromatic conflict* is expressed by the synchromatic width unit vector $\sigma_0 = \epsilon_0 - \epsilon_3 \in \mathbb{D}_s$ or its inverse $-\sigma_0$.

In the first case one refers to a conflict between twelve virtual fifth steps or alterations and a zero vector. Especially (via $\psi_{3,1}$) such a conflict occurs between the Riemann vector $3t$ of three thirds and the octave/prime. In the second case one refers to a conflict between a virtual semitone step and semitone alteration. From the music-theoretical point of view it is interesting to discuss these two alternatives independently of the present geometrical model. In extrapolation of a traditional line of argument there is a simple conclusion which at least suggests taking the possibility of such an alternative between enharmonic and synchromatic interpretations of enharmonic conflicts seriously. Traditionally one distinguishes between *passing tones* on the one hand and harmonically interpreted *inserted tones* on the other. When both interpretations are possible, attention is nevertheless paid to distinguish between them (c.f. [14] paragr. 60). This traditional distinction is still applied to chromatically altered passing tones. In other words, the property of a tone as an alteration of an underlying scale tone does not exclude the possibility of a passing tone interpretation. In extrapolation of this traditional distinction one must be prepared to distinguish between conflicts between competing harmonic interpretations and conflicts that involve passing-tone interpretations. In the beginning of Chopin's prelude in E minor (Op. 28 No. 4, bars 1-3) there is a conflict between a chromatic passing tone $E \rightarrow E_b \rightarrow D$ and the local resolution $E \rightarrow D_\#$ of a 7-6 suspension (bar 2) in the upper voice of the left hand accompaniment. Orthographically Chopin supports the first reading E_b. The succession $E \rightarrow D_\#$ is associated with a harmonic interpretation of $D_\#$ as a major dominant agent, but this association does not automatically imply a harmonic interpretation of the conflict between $D_\#$ and E_b. We suggest speaking of a synchromatic conflict between the two semitone steps $E \rightarrow D_\#$ and $E_b \rightarrow D$, because the real whole tone $E \rightarrow D$ must be split into an alteration and a step.

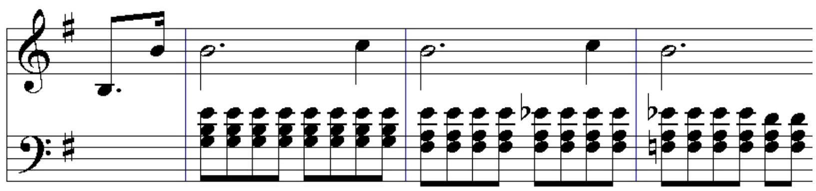

Figure 14: Chopin Op.28 No.4

There are at least two possible objections to be mentioned. One argument would directly oppose the above statement saying that in this and similar situations a harmonic interpretation is associated with E_b as well, so that the conflict appears as a proper harmonic one. Below we discuss a sophistication of this argument in the context of our mathematical model. Another objection would question the relevance of these 'conflicts' at all. We formulate two alternative working hypotheses that may guide a comparative study of the Chopin prelude(s).

1. These preludes are 'dia-chromatic' compositions. Non-diatonic tones (with respect to the main tonality or to local surface tonalities) must be interpreted as chromatic alterations. Conflicts between alternative interpretations are highly relevant for the aesthesis of these pieces.

2. These preludes are written for well-tempered piano and hence[21] it is superflous to inquire about enharmonic differentiation on the level of tones. The interpretation of notational decisions is a matter of connotation and performance. All contextual constructs must be defined on the basis of the twelve-tone-system.

A discussion of these arguments with reference to several authors is subject of a separate study. Under the preliminary assumption that the concept *synchromatic conflict* is an interesting issue we conclude this subsection with further investigations into the active tone system.

We addressed synchromatic conflicts without formulating the problem of *synchromatic identification* as another possible case of enharmonicity. How could

[21]The 'naive' formulation of this hypothesis is not in accordance with our ontological preliminaries and can - of course - be improved.

an unnoticable transfer between semitone steps and semitone alterations work? Recall that a direct apperception $App(X, \sigma_0)$ of the synchromatic width unit vector σ_0 violates the economy principle of apperception. A mediated apperception of this vector $App(X, (-\epsilon_3, \epsilon_0)$ is actually and virtually noticable. Let $S_0 = exp_{mono}(\epsilon_0)$ and $S_3 = exp_{mono}(\epsilon_3)$ denote the active semitone step and semitone alteration respectively and let $\mathbb{G}_s := \langle S_0, S_3 \rangle$ denote the *semitone group*, generated by these two elements. According to the commutativity of semitone apperception and in contrast to the situation of fifth apperception we have

$$\mathbb{G}_s = \langle exp_{mono}(\mathbb{D}_s)\rangle = \left\{ \begin{pmatrix} exp(a) & 0 \\ 0 & exp(b) \end{pmatrix} \mid \quad a, b \in \mathbb{Z} \right\}.$$

Recall that the quintic group $\mathbb{G}_q$ is a proper(!) subgroup of the full group $\langle exp_{mono}(\mathbb{D}_q)\rangle$ generated by the exponential of the quintic lattice $\mathbb{D}_q$.

Consequently there is no way in which to find similar relations in the semitone group $\mathbb{G}_s$ in order to model synchromatic identifications as unnoticable transfers via processes with common actual or virtual results. In other words, at first sight the mathematical model provides a way in which to describe synchromatic conflicts but it offers no explanatory power as to how they can be solved apperceptively. The following rather sophisticated proposal for a solution is motivated by purely inner-mathematical properties of the model and must be dealt cautiously. The attraction of this proposal is to music-theoretically interpret the additional Lie-Algebra structure of the space $\mathbb{D}$ of discrete tone vectors. Without taking this structure into account $\mathbb{D} = \mathbb{D}_s \times \mathbb{D}_q$ is the direct sum of two independent two-dimensional lattices, i.e virtual semitone apperception and virtual fifth apperception are independent from one another.

As said in section 2, we do not presuppose the reader's familiarity with the theory of Lie Groups and Lie Algebras. Therefore we provide a heuristic but instructive calculation with respect to the continuous ambient space $\mathcal{M}_2(\mathbb{R})$ whose elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be interpreted either as tone vectors $v \in \mathbb{V}_{mono}$ or as active tones $A \in \mathbb{A}_{mono}$, if $ad - bc > 0$. The calculation is heuristic in the sense that we do not have music-theoretical interpretations for continuous families of processes and their partial derivatives. Instead of an integral diaquintic fifth commutation process $\bigtriangledown \lhd \bigtriangleup \rhd$ we consider a two-parameter familiy

$$\{\bigtriangledown_s \lhd_t \bigtriangleup_s \rhd_t \mid \quad s, t \in \mathbb{R}\}$$

of such processes with variable active tones

$$\begin{aligned} \bigtriangledown_s &= exp_{mono}(s \cdot \epsilon_2) &= exp_{mono}(\begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}) &= \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \\ \rhd_t &= exp_{mono}(t \cdot \epsilon_1) &= exp_{mono}(\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}) &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and inverses $\bigtriangleup_s = \bigtriangledown_s^{-1}$ and $\lhd_t = \rhd_t^{-1}$. We calculate the partial derivative

$$\frac{\partial^2}{\partial s \partial t} \bigtriangledown_s \lhd_t \bigtriangleup_s \rhd_t = \frac{\partial^2}{\partial s \partial t} \begin{pmatrix} 1 + st & st^2 \\ s^2 t & 1 + -st + s^2 t^2 \end{pmatrix} = \begin{pmatrix} 1 & 2t \\ 2s & -1 + 4st \end{pmatrix}$$

Its value for $s = 0$ and $t = 0$ is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which is the synchromatic width unit vector σ_0. How is one to interpret this calculation heuristically? The apperception process $App(X, (s \cdot \epsilon_2, -t \cdot \epsilon_1, -s \cdot \epsilon_2, t \cdot \epsilon_1))$ always has an actual result different from X with the one and only exception for $s = t = 0$. The partial derivative $\frac{\partial^2}{\partial s \partial t} \bigtriangledown_s \lhd_t \bigtriangleup_s \rhd_t \big|_{s=0,t=0}$ at this point therefore expresses the (virtual) direction of the actual result of an infinitesimal commutation process. Surprisingly this direction is not(!) a linear combination of ϵ_1 and ϵ_2, and belongs to the semitone lattice instead. The calculation of the partial derivative of the family of active commutation processes can be related to a virtual commutation of ϵ_2 and $-\epsilon_1$ within $\mathbb{V}$. This operation is called the Lie-bracket:

$$[\,,\,] : \mathbb{V} \times \mathbb{V} \to \mathbb{V} \quad \text{with}[x, y] := \quad x \cdot y - y \cdot x.$$

In our special case we calculate the Lie-bracket $[\epsilon_2, -\epsilon_1]$ of ϵ_2 and $-\epsilon_1$

$$\begin{aligned} [\epsilon_2, -\epsilon_1] &= -\epsilon_2 \epsilon_1 + \epsilon_1 \epsilon_2 \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_0 \end{aligned}$$

The Lie-Brackett can be defined also in the discrete situation of integral matrices and does not require the methods of infinitesimal calculus as in the heuristics.

Music-theoretically we may interpret these aspects of the mathematical model in terms of a refined interdependence of quintic and semitone apperception. A detailed discussion of possible specifications of this interpretation is subject to further study. But to conclude our considerations on synchromatic identification we recall the objection against a rigid distinction between synchromatic and proper enharmonic identification, stating that a conflict such as in the Chopin example always induces a harmonic interpretation on the passing tones as well. In a slight modification of this argument the mathematical model suggests searching for interactions between semitone constellations and suitable fifth commutations. With regard to the conflict between E_b and $D_\#$ in the example there is little motivation for an Escher-Staircase of length 12. But in search of a suitable fifth commutation the seventh $F_\# - E$ in bar two gains interest because of the additional presence of B in the melody. As an imaginary quintic height unit (inversion of the wholetone $E - F_\#$) it can either be divided by B or $E_\uparrow$ into a fifth step followed by a fifth alteration $E - B - B_\uparrow$ or vice versa $E - E_\uparrow - B_\uparrow$. The active commutation of both possibilities yields a diaquintic fifth commutation $App(E, (-\epsilon_2, \epsilon_1, \epsilon_2, -\epsilon_1))$ with E as a tone locus of departure. The corresponding Lie-bracket is $[-\epsilon_2, \epsilon_1] = \sigma_0$. Thus we may state that the fifth commutation creates an impulse to synchromatic identification at the tone locus E. The identification is governed by the same tone locus from which the fifth commutation departs. It is promising to study the *leading tone* phenomenon in the context of this approach. Recall our interpretation of a $I \quad IV \quad I \quad V \quad I$ cadence (cf. subsection 3.2) with the virtual apperception

$$c \quad \to \quad c_\downarrow \quad \to \quad g_\downarrow \quad \to \quad g \quad \to \quad c.$$

The active counterpart to this virtual process is a synquintic fifth commutation and the corresponding Lie-bracket is $[\epsilon_2, \epsilon_1] = -\sigma_0$. We hypothesise, that the soprano clause in suitable contexts is an interesting candidate for a hidden synchromatic identification, which is not evident from score notation. The conflict appears if we interpret the downward semitone as an alteration and the following upward semitone as a step. The conflict and its solution is of course not evident from the melody, which returns to the same note. It must be created by the context. But as a prerequisite for further elaboration and justification or falsification of this hypothesis we need systematic way of attributing apperceptive pathways including diaquintic and synquintic commutations to given situations. The crux of developing such a systematic method is the interpretation of temporal structures, because time is not yet an apperception parameter in our model.

5 Metrical Aspects

There is a long tradition to describe tone relations through concepts of *distance*. Among the mathematical approaches we mention Leonhard Euler's Gradus function, Guerino Mazzola's third distance on the Terztorus (cf. [17]), Roger Shepard's pitch spaces (cf. [27]). Our present proposal shares several aspects with some of these approaches but differs in some respect significantly. In subsection 5.1 we introduce a pseudo-Euclidean structure to the space of tone vectors $\mathbb{V}_{diasyn}$ in order to complete our formulation of the economy principle of apperception (cf. subsection 2.3). In subsection 5.2 we go further by introducing a pseudo-Riemannian metrics to the space $\mathbb{A}_{diasyn}$ such that apperceptive acts become isometries.

5.1 Relativity, Economy Principle and Artin Four-Space

Our way to characterise the economy principle geometrically has a formal analogy in the treatment of spacetime and causality in special relativity. It would therefore be promising to revisit the early psychological discussions on relativity on the background of those mathematical ideas which are associated with the concepts of special and general relativity in physics. Recall, that Wilhelm Wundt postulates a relativity principle for apperception, stating that the consciousness accesses its own states only through relative measures. Carl Stumpf objects this assumption on the first pages of his *Tonpsychologie* cf. [28] with quite debatable arguments. On the one hand Wundt's relativity postulate implicitely questions our investigations of isolated apperceptive acts and processes insofar single acts should not 'matter' as such but only in relation to other single acts. On the other hand, a theory of relativity of apperception has to be developed on the basis of single apperceptive acts. The *diasyntonic pairing* provides a suitable point of departure for further elaboration of such a theory. In the present state of the theory it is motivated by the economy principle of apperception. Recall the dialogic principle of apperception assigning external dimensions to the reason vectors as well as to the decision vectors (cf. subsection 2.3). The figure below displays the three domains corresponding to our typology of dialog-like, monolog-like and synlog-like vectors. The monolog-like vectors form two diagonal axes. These *monologic axes* separate the *dialogic domain* from the *synlogic one*. Geometrically this can be characterised as an *an-isotropy*.

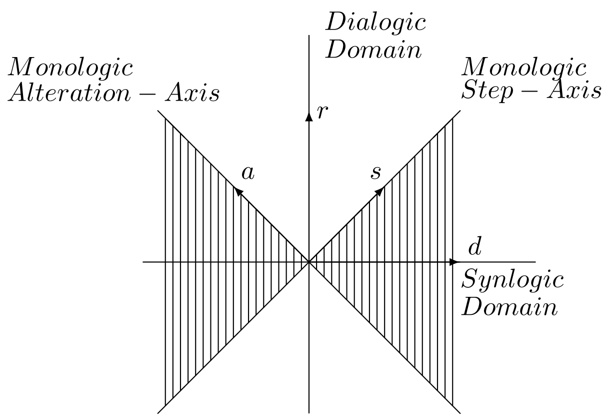

Figure 15: An-isotropy according to the economy principle. Vectors in the synlogic domain (hatched) are not apperceivable.

The diasyntonic pairing $\langle\, .\, ,\, .\,\rangle_{diasyn}$ is an indefinite non-degenerate scalar product on the vector space $\mathbb{V}_{diasyn}$, which is defined as

$$\begin{aligned}
&\langle\, .\, ,\, .\,\rangle_{diasyn} : \mathbb{V}_{diasyn} \times \mathbb{V}_{diasyn} \to \mathbb{R}, \quad \text{with} \\
&\langle (d_0, s_0, d_1, s_1), (d_0', s_0', d_1', s_1') \rangle := d_0 d_0' - s_0 s_0' + d_1 d_1' - s_1 s_1' \\
&= \begin{pmatrix} d_0 & s_0 & d_1 & s_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} d_0' \\ s_0' \\ d_1' \\ s_1' \end{pmatrix}
\end{aligned}$$

The diasyntonic pairing induces a norm $\|.\| : \mathbb{V}_{diasyn} \to \mathbb{R}$ with

$$\|(d_0, s_0, d_1, s_1)\|_{diasyn} = d_0^2 - s_0^2 + d_1^2 - s_1^2.$$

The typology of tone vectors according to the economy principle has the following geometric characterisation. A tone vector v is:

$$\begin{array}{lll}
diatone-like & \text{if} & \|v\|_{diasyn} > 0, \\
montone-like & \text{if} & \|v\|_{diasyn} = 0, \\
syntone-like & \text{if} & \|v\|_{diasyn} < 0.
\end{array}$$

The four-dimensional real vector space $\mathbb{R}^4$ with this pairing is called *Artin four-space* (cf. [4], p.137). The restriction of this norm to the two-dimensional subspaces $\mathbb{V}_s$ and $\mathbb{V}_q$ yields Minkowski-planes[22] having signature (1,1).
We calculate the same pairing $\langle\, . \,, \,.\, \rangle_{mono} : \mathbb{V}_{mono} \times \mathbb{V}_{mono} \to \mathbb{R}$ and norm $\|.\| : \mathbb{V}_{mono} \to \mathbb{R}$ with respect to the monotonic basis as well. The corresponding formulas are determined through the following the conditions

$$\begin{array}{ll} \langle \Psi(x), \Psi(y) \rangle_{diasyn} = \langle x, y \rangle_{mono} & \text{for all } x, y \in \mathbb{V}_{mono} \\ \|\Psi(x)\|_{diasyn} = \|x\|_{mono} & \text{for all } x \in \mathbb{V}_{mono}. \end{array}$$

Choose $x = \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix}, y = \begin{pmatrix} y_0 & y_1 \\ y_2 & y_3 \end{pmatrix} \in \mathbb{V}_{mono}$ and set $\widetilde{y} = \begin{pmatrix} y_3 & -y_1 \\ -y_2 & y_0 \end{pmatrix}$.
As to the pairing we have

$$\begin{aligned} \langle x, y \rangle_{mono} &= \langle \Psi(x), \Psi(y) \rangle_{diasyn} \\ &= \langle (\frac{x_0 + x_3}{2}, \frac{x_0 - x_3}{2}, \frac{x_1 - x_2}{2}, \frac{x_1 + x_2}{2}), \\ &\quad\; (\frac{y_0 + y_3}{2}, \frac{y_0 - y_3}{2}, \frac{y_1 - y_2}{2}, \frac{y_1 + y_2}{2}) \rangle_{diasyn} \\ &= \frac{x_0 y_3 + x_3 y_0 - x_1 y_2 - x_2 y_1}{2} \\ &= \frac{1}{2} \cdot tr \begin{pmatrix} x_0 y_3 - x_1 y_2 & -x_0 y_1 + x_1 y_0 \\ x_2 y_3 - x_3 y_2 & -x_2 y_1 + x_3 y_0 \end{pmatrix} \\ &= \frac{1}{2} \cdot tr(x \cdot \widetilde{y}). \end{aligned}$$

As to the norm we have

$$\|x\|_{mono} = \frac{1}{2} \cdot tr(x \cdot \widetilde{x}) = det(x).$$

Independently of the basis we will speak of the *diasyntonic pairing* because the pairing takes a diagonal form with respect to the diasyntonic basis.

5.2 Apperceptions as Isometries

The embeddings of the discrete lattice $\mathbb{D} \subset \mathbb{V}$ and the discrete group $\mathbb{G} \subset \mathbb{A}$ into continuous ambient spaces are so far illustrative and practical extentions of our discrete model. But up to this point we did not make serious use of concepts from differential geometry. In the present subsection we now investigate a suitable *pseudo-Riemannian metric* g of signature (2,2) on the Lie-group $\mathbb{A}$. This is motivated by the combination of group theory on the one hand and geometry on the other. If the active tone system as a group has an inner geometrical structure it is interesting so investigate the behaviour of this structure under the action $act : \mathbb{A} \times \mathbb{A}$ of the active tone system on itself. The metric g is defined

[22] Recall that the 4-dimensional Minkowski space has signature (3,1) while the Artin-Four- has signature (1,1)

in such a way, that the active tones are isometries, i.e. structure preserving transformations of the geometrical space. For each tone locus $X \in \mathbb{A}$ we equip the tangent space $T_X\mathbb{A}$ with a diasyntonic pairing $\langle\,.\,,\,.\,\rangle_X : T_X\mathbb{A} \times T_X\mathbb{A} \to \mathbb{R}$. This can be done in such a way that the pairing of two tone vectors at the identity element coincides with the pairings of all their translates to X:

$$\langle v_1, v_2\rangle_{Id} = \langle X \cdot v_1, X \cdot v_2\rangle_X \quad \text{for all} \quad v_1, v_2 \in \mathbb{V} \quad \text{and for all} \quad X \in \mathbb{A}$$

Terminologically, we have to distinguish between *tone vectors* $v \in \mathbb{V}$ in the narrow sense and *tone vectors at tone locus* X. We define the *diasyntonic pairing at tone locus* X with respect to the monotonic basis. One obtains the tangential space at X through matrix multiplication of all $x \in \mathbb{V}_{mono}$ by X from the left:

$$T_X\mathbb{A}_{mono} = \{X \cdot x \,|\, x \in x \in \mathbb{V}_{mono}\}$$

Let $x_1' = X \cdot x_1$ and $x_2' = X \cdot x_2$ denote tone vectors at tone locus X. We define

$$\langle x_1', x_2'\rangle_{X,mono} := \frac{1}{2det(X)} tr(x_1' \cdot \widetilde{x_2'})$$

The following calculation shows that this is equal to $\langle x_1, x_2\rangle_{mono}$:

$$\begin{aligned}
\langle x_1', x_2'\rangle_{X,mono} &= \frac{1}{2det(X)} tr(x_1' \cdot \widetilde{x_2'}) \\
&= \frac{1}{2det(X)} tr(X \cdot x_1 \cdot \widetilde{X \cdot x_2}) \\
&= \frac{1}{2det(X)} tr(X \cdot x_1 \cdot \widetilde{x_2} \cdot \widetilde{X}) \\
&= \frac{1}{2det(X)} tr(X \cdot x_1 \cdot \widetilde{x_2} \cdot \begin{pmatrix} det(X) & 0 \\ 0 & det(X) \end{pmatrix} \cdot X^{-1}) \\
&= \frac{1}{2} tr(x_1 \cdot \widetilde{x_2}) \\
&= \langle x_1, x_2\rangle_{mono}
\end{aligned}$$

Hence, the pseudo-Riemannian metric g is a scalar variation of the Artin four-space. We call it the *diasyntonic metrical tensor*. In contrast to the flat Artin four-space we now obtain a nontrivial curvature. In order to investigate the space $(\mathbb{A}_{diasyn}, g)$ with respect to the diasyntonic basis we parametrise the open subset $\mathbb{A}_{diasyn} \subset \mathbb{R}^4$ with local coordinates $\xi = (\xi^1, \xi^2, \xi^3, \xi^4)$, such that the tone vectors of the diasyntonic basis coincide with the partial derivatives $\partial_i = \frac{\partial}{\partial \xi^i}$ at the identity element $Id = (1, 0, 0, 0)$:

$$\delta_0 = \tfrac{\partial}{\partial \xi^1} \quad \sigma_0 = \tfrac{\partial}{\partial \xi^2} \quad \delta_1 = \tfrac{\partial}{\partial \xi^3} \quad \sigma_1 = \tfrac{\partial}{\partial \xi^4}$$

The diasyntonic metrical tensor $g(\xi)$ at tone locus $\xi = (\xi^1, \xi^2, \xi^3, \xi^4)$ is associated with the matrix

$$\frac{1}{\|\xi\|} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where $\|\xi\| := (\xi^1)^2 - (\xi^2)^2 + (\xi^3)^2 - (\xi^4)^2$. Recall that the Riemann curvature tensor R has 256 components

$$R^l_{ijk} = \partial_j(\Gamma^l_{ik}) - \partial_i(\Gamma^l_{jk}) + \sum_{m=1}^{4}(\Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}),$$

where

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{4} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

denote the Christoffel-symbols and the g^{ij} denote the coefficients of the inverse matrix of the metrical tensor g_{ij}. According to the diagonal form of the metrical tensor we have $\Gamma^k_{ij} = \frac{1}{2}g^{kk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$. Further we have $g_{kk} = (-1)^{k+1}\|\xi\|^{-1}$, $g^{jj} = (-1)^{j+1}\|\xi\|$ and $\partial_i g_{kk} = (-1)^{i+k+1}\frac{2\xi^i}{\|\xi\|^2}$ und hence

$$\frac{1}{2}g^{jj}\partial_i g_{kk} = (-1)^{i+k+j}\frac{\xi^i}{\|\xi\|}.$$

We obtain the following three cases:

$$\begin{array}{rcll} \Gamma^k_{ik} = \Gamma^k_{ki} & = & (-1)^i\frac{\xi^i}{2\|\xi\|} & \text{for} \quad i,k = 1,...,4 \\ \Gamma^k_{ii} & = & (-1)^{i+1}\frac{\xi^k}{2\|\xi\|} & \text{for} \quad i \neq k \\ \Gamma^i_{jk} & = & 0 & \text{for} \quad i \neq j \neq k \end{array}$$

The *Ricci curvature tensor Ric* has 16 components and is defined as

$$Ric_{ik} := \sum_{l=1}^{4} R^l_{ilk}.$$

A straight forward calculation leads to the following result

$$Ric(\xi) = \frac{1}{2\|\xi\|^2}\begin{pmatrix} 3(\|\xi\| - (\xi^1)^2) & -\xi^1\xi^2 & \xi^1\xi^3 & -\xi^1\xi^4 \\ -\xi^2\xi^1 & -3((\|\xi\| + (\xi^2)^2) & -\xi^2\xi^3 & \xi^2\xi^4 \\ \xi^3\xi^1 & -\xi^3\xi^2 & 3((\|\xi\| - (\xi^3)^2) & -\xi^3\xi^4 \\ -\xi^4\xi^1 & \xi^4\xi^2 & -\xi^4\xi^3 & -3((\|\xi\| + (\xi^4)^2) \end{pmatrix}$$

The *scalar curvature* $S = \sum_{i.k=1}^{4} g^{ik}Ric_{ik}$ is constant, namely $S(\xi) = \frac{9}{2}$.

With respect to the monotonic basis $\mathbb{A}_{mono} = GL_2(\mathbb{R})^+$ consists of all real 2×2-matrices $X = \Phi(\xi)$ with positive determinant $det(X) > 0$. One has a multiplicative decomposition $GL_2(\mathbb{R})^+ = \mathbb{R}^*_+ \times SL_2(\mathbb{R})$ of these matrices into scalar ones $\begin{pmatrix} det(X) & 0 \\ 0 & det(X) \end{pmatrix}$ and matrices $det(X)^{-1} \cdot X$ of determinant 1. The diasyntonic metrical tensor varies only with the first factor of this decomposition, i.e. with the determinants $det(X)$. Therefore it is interesting to study the second factor $\mathbb{S}_{mono} = SL_2(\mathbb{R})$ of *special active tones* as a 3-dimensional subspace of the active tone system separately. Geometrically it is known as the 3-dimensional *anti-de Sitter space* adS_3 with a Lorentz metric of constant sectional curvature equal to -1, and group-theoretically as a Lie group with its bi-invariant pseudo-Riemannian metric defined by the Ad-invariant Killing form. The restriction of Ψ to $\mathbb{S}_{mono}$ transforms the latter viewpoint to the former.[23]

5.3 Summary

The decomposition $\mathbb{A} = \mathbb{R}^*_+ \times \mathbb{S}$ suits well to draw a balance of this article. The first factor $\mathbb{R}^*_+$ directly corresponds to Wundt's apperceptive interpretation of Weber-Fechner's law and is embedded into our model through the apperception of real height. It yields the ontological interface between apperception and perception of pitch height. The second factor $\mathbb{S}$ is more or less new with respect to Wundt's idea. But with respect to the Riemann-tone-net and Neo-Riemannian approaches to enharmonicity we were able find interesting correspondences (cf. subsections 4.1 and 4.1). A rather new and very speculative idea is the assumption of a refined dependency between the basic quintic vectors ϵ_1, ϵ_2 and the synchromatic unit vector σ_0. A further elaboration of this idea is essential to the music-theoretical interpretation of the full four-dimensional model. Recall, that the virtual apperception model is basically decomposed into semitone- and quintic apperception, i.e. into two two-dimensional Minkowski-planes $\mathbb{V}_s$ and $\mathbb{V}_q$. The effects of synchromatic identification would in addition provide a music-theoretical support of the decomposition $\mathbb{A} = \mathbb{R}^*_+ \times \mathbb{S}$. In terms of mathematical physics one may characterize the space $\mathbb{S}$ as a *vacuum* according to its constant sectional curvature. In accordance to our notion of *pure apperception* it can be viewed as an idealized homogeneous apperception space. We conclude with a remark on Carol Krumhansl's discussion of geometrical models of musical pitch (cf. [11] p. 119). She objects the study of homogeneous[24] pitch spaces arguing that listener's judgments of tone distances depend on a given tonal context as well as on the temporal order of their appearance. Her solution to this problem is the representation of context-dependent distances in a Euclidean ambient space via multi-dimensional scaling methods. But instead of shifting isolated points in a rigid absolute parameter space one may alternatively deform its metrical structure. Therefore we paraphrase Krumhansl's objection: Associative tone apperception is characterized through inhomogeneities of the configuration space. This formulation can be metaphorically related to Johann Friedrich Herbart's (cf. [9], p. 32) concept of *apperceptive masses* on the one hand and to the description of mass and inertia in general relativity on the other. To be honest, this is only a metaphorical link since our model exemplifies just a 'vacuum' case. But taken seriously it may stimulate further research strategies.

[23] See also [20].

[24] Krumhansl uses the term 'regular'

Acknowledgements: First of all I want to thank my collegue and co-author Andreas Nestke for many fruitful contributions and critical discussions on the subject of this paper (see our joint papers [20] and [21]). I am grateful to the organizers and participants of the *MaMuPhi*[25] and the *MaMuX*[26] seminars at IRCAM (Paris) for their interest in our research and for critical comments. Further I'm grateful to Guerino Mazzola (Univerity of Zurich) for reviewing my earlier approach to enharmonicity (cf. [19]) in his book [18] and in personal conversations, to Prof. Nicolas Meeus (Sorbonne, Paris) for a detailed critical comment on a manuscript of the paper ([21]), to Anja Fleischer and Oliver Schwab-Felisch for fruitful discussions on related music-theoretical issues and to Nicola Heine for her assistance in the English proof-reading. Finally, I thank the Volkswagen-Foundation for the funding of our research group KIT-MaMuTh[27].

References

[1] Ebeling, Martin (1999) *Tonhöhe physikalisch - musikalisch - psychologisch - mathematisch* Frankfurt a. M.: Peter Lang.

[2] Fechner, Gustav Theodor (1860) *Elemente der Psychophysik*

[3] Fechner, Gustav Theodor (1877) *In Sachen der Psychophysik*, Leipzig: Breitkopf und Härtel.

[4] Goldblatt, Robert (1987): *Orthogonality and Spacetime Geometry*, Springer, New York 1987

[5] Gollin, Edward (2000): *Representations of Space and Conceptions of Distance in Transformational Music Theories*, Havard University, Cambridge 2000.

[6] Handschin, Jacques (1948): *Der Toncharakter*. Atlantis, Zürich.

[7] Harrison, Daniel (1994): *Harmonic Function in Chromatic Music*, The University of Chicago Press, Chicago and London.

[8] Helgason, S.: *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York 1978.

[9] Herbart, Johann Friedrich (1816, 1834 = 1887): Lehrbuch zur Psychologie, edited by G. Hartenstein. Voss, Hamburg and Leipzig, 1887.

[10] Iversen, B.: *Hyperbolic Geometry*, London Mathematical Society Student Texts 25, Cambridge 1992.

[11] Krumhansl, Carol L. (1990): *Cognitive Foundations of Musical Pitch*, Oxford University Press, New York and Oxford.

[25] cf. http://www.entretemps.asso.fr/Seminaire/mamuphi.html
[26] cf. http://www.ircam.fr/equipes/repmus/mamux/
[27] cf. http://www.mamuth.de

[12] LERDAHL, FRED (2001): *Tonal pitch space*, Oxford University Press, 2001.

[13] LEYTON, MICHAEL (1992): *Symmetry, Causality, Mind*, MIT Press, Massachusetts 1992.

[14] LOUIS, RUDOLF UND LUDWIG THUILLE(1907 = 1913): *Harmonielehre*, Verlag von Carl Grüninger Nachf. Ernst Klett, Stuttgart.

[15] LEWIN, DAVID(1987): *Generalized Musical Intervals and Transformations*, Yale University Press, New Haven.

[16] MAZZOLA, GUERINO (1985): *Gruppen und Kategorien in der Musik*, Heldermann, Berlin.

[17] MAZZOLA, GUERINO (1990): *Geometrie der Töne*, Birkhäuser,

[18] MAZZOLA, GUERINO (2002): *The Topos of Music*, Birkhäuser, Basel.

[19] NOLL, THOMAS: *Morphologische Grundlagen der abendländischen Hamonik*, Brockmeyer, Bochum 1997.

[20] NOLL, THOMAS UND ANDREAS NESTKE (2001): *Die Apperzeption von Tönen*, Electronic Journal of the GMTH, www.gmth.org.

[21] NOLL, THOMAS UND ANDREAS NESTKE (2002): *L'Apperception d' Hauter*, In ASSAYAG, GERARD (ed.) *Mathematique, Musique et Philosophie*, IRCAM, Paris 2002.

[22] PFROGNER HERMANN (1953): *Die Zwölfordnung der Töne*, Amalthea-Verlag, Zürich.

[23] RIEMANN, HUGO: "Ideen zu einer Lehre von den Tonvorstellungen" in: *Jahrbuch der Musikbibliothek Peters 21/22*, Leipzig 1914/15. 1-26.

[24] SALZER, FELIX UND CARL SCHACHTER (1969): *Counterpoint in Composition* Columbia University Press 1969 = 1989.

[25] SCHENKER, HEINRICH (1935 = 1956): *Der Freie Satz*, Universal Edition, Wien.

[26] SCHROEDER, ULRICH E. (2001): *Gravitation - Einführung in die Allgemeine Relativitätstheorie*, Frankfurt am Main: Deutsch.

[27] SHEPARD, ROGER N.(1982): *Structural Representations of Musical Pitch* In: DEUTSCH, DIANA: *The Psychology of Music*. Orlando: Academic Press 1982, 343 - 390.

[28] STUMPF, CARL (1883 = 1965): *Tonpsychology*, Hilversum.

[29] VOGEL, MARTIN: *Die Lehre von den Tonbeziehungen*, Verlag für systematische Musikwissenschaft, Bonn 1975.

[30] WUNDT, WILHELM (1887) *Grundzge der Physiologischen Psychologi* Bd. I, Leipzig: Verlag von Wilhelm Engelmann, 1887