4. Primitive Recursiveness

Having described (in §3.4) two ways of systematically forming new functions from existing ones, we introduce the class of ***initial functions***, and the concepts of ***primitive recursive (PR) closedness***, and ***primitive recursive functions***.

4.1 PR-closed classes

The three ***initial functions*** are

- the ***zero function*** $\boldsymbol{Z} = \lambda x \cdot 0$,
- the ***successor function*** $\boldsymbol{S} = \lambda x \cdot (x+1)$, and
- the ***projection functions*** $\boldsymbol{U}_i^n = \lambda x_1, \ldots, x_n \cdot x_i$ for $n \geq 1$, $1 \leq i \leq n$, of which the ***identity function*** $\boldsymbol{U}_1^1 = \lambda x \cdot x$ is a special case.

A class $\mathcal{C}$ of functions is ***PR-closed*** iff
(i) $\mathcal{C}$ contains the ***initial functions***, and
(ii) $\mathcal{C}$ is ***closed*** under ***composition*** and ***definition by primitive recursion***, i.e., any function obtained from functions in $\mathcal{C}$ by *composition* or *primitive recursion* is also in $\mathcal{C}$.

Example of a PR-closed class:

- FN (trivially).

Lemma 4.0. *The **intersection** of two PR-closed classes is PR-closed.*

Lemma 4.1. TFN *is PR-closed.*

Proof: By definition, the initial functions are total. From Lemmas 3.3, 3.6 and 3.8, totality is preserved by composition and prim. rec. □

Lemma 4.2. $\mathcal{G}$-COMP *is PR-closed.*

Proof: The $\mathcal{G}$-programs `skip`, $\boxed{\begin{array}{l} Y \leftarrow X \\ Y{+}{+} \end{array}}$, and $\boxed{Y \leftarrow X_i}$ compute the ***zero***, ***successor***, and ***projection*** functions respectively. By Thms 3.4, 3.7, and 3.9, $\mathcal{G}$-COMP is closed under comp. and prim. rec. □

Lemma 4.3. $\mathcal{G}$-TCOMP *is PR-closed.*

Proof: By Lemmas 4.1 and 4.2, the classes TFN and $\mathcal{G}$-COMP are PR-closed. Hence their intersection $\mathcal{G}$-TCOMP is PR-closed. □

4.2 Primitive recursive functions

A function f is ***primitive recursive*** (PR) iff it is obtained from the ***initial functions*** by a finite number of applications of ***composition*** and ***primitive recursion***. In other words, f is primitive recursive iff there is a ***finite sequence*** of functions $f_1, \ldots, f_n$ such that $f_n = f$, and for $i = 1, \ldots, n$, either f_i is an ***initial function***, or f_i is obtained from some f_j's, for $j < i$, by ***composition*** or ***primitive recursion***. Such a sequence is called a ***PR-derivation*** of f, of length n.

More formally, a ***PR-derivation*** of a function f is a sequence of labelled function symbols of the form:

$$
\begin{aligned}
f_1 &\leftarrow L_1 \\
f_2 &\leftarrow L_2 \\
&\vdots \\
f = f_n &\leftarrow L_n
\end{aligned}
$$

where for each $i = 1, \ldots, n$ one of the following cases applies:

- Case 1: f_i is an ***initial function***, and label L_i is (correspondingly) one of '$\boldsymbol{Z}$', '$\boldsymbol{S}$' or '$\boldsymbol{U}_j^n$'.
- Case 2: f_i is obtained from an ℓ-ary function f_j, and m-ary functions $f_{k_1}, \ldots, f_{k_\ell}$ by ***composition***, for $j, k_1, \ldots, k_\ell < i$, and the label L_i is '$f_j, f_{k_1}, \ldots, f_{k_\ell}$ $(\mathsf{compos} : \ell, m)$'.
- Case $3a$: f_i is obtained from f_j and f_k, for $j, k < i$, by ***recursion*** with m parameters $(m > 0)$, and the label L_i is 'f_j, f_k $(\mathsf{rec} : m)$'.
- Case $3b$: f_i is obtained from f_k, for $k < i$ by recursion without parameters, and initial value c, and the label L_i is 'c, f_k $(\mathsf{rec} : 0)$'.

(We are not distinguishing here between functions and their symbols).
The class of primitive recursive functions, and the class of n-ary primitive recursive functions are denoted by PR and $\mathrm{PR}^{(n)}$ respectively.

Lemma 4.4. PR *is PR-closed*

Proof: from the definition. □

Lemma 4.5. *Let* $\mathcal{C}$ *be any PR-closed class of functions. Then* PR $\subseteq \mathcal{C}$.

Proof: We can show that for all f,

$$f \in \text{PR} \implies f \in \mathcal{C}$$

by ***CV induction*** [*or*: by ***LNP***] on the length of a PR-derivation of f. There are three cases:

- Case 1: f is an ***initial function***. Then $f \in \mathcal{C}$, since $\mathcal{C}$ is PR-closed.
- Case 2: f is obtained from earlier functions $g_1, \ldots, g_k$ in the derivation by ***composition***. Then $g_1, \ldots, g_k$ have *shorter* PR-derivations (i.e. the initial parts of the PR-derivation of f ending with them), and so by the *induction hypothesis* they are in $\mathcal{C}$. Hence again, since $\mathcal{C}$ PR-closed, $f \in \mathcal{C}$.
- Case 3: f is obtained from earlier functions in the derivation by ***primitive recursion***. This is similar to Case 2. □

Theorem 4.6. PR *is the smallest PR-closed class. In other words:*
(i) PR *is PR-closed; and*
(ii) PR *is contained in every PR-closed class.*

Proof: By Lemmas 4.4 and 4.5. □

Corollary 4.7. PR $\subseteq$ TFN.

Proof: By Lemma 4.1, TFN is PR-closed, and so by Theorem 4.6, PR $\subseteq$ TFN. □

Corollary 4.8. PR $\subseteq \mathcal{G}$-COMP.

Proof: By Lemma 4.2, $\mathcal{G}$-COMP is PR-closed, and so by Theorem 4.6, PR $\subseteq \mathcal{G}$-COMP. □

Corollary 4.9. PR $\subseteq \mathcal{G}$-TCOMP.

Proof: By Corollaries 4.7 and 4.8, or since, by Lemma 4.3, $\mathcal{G}$-TCOMP is PR-closed. □

So clearly (cf. p. 3-9):

$$\boxed{\begin{array}{ccccccc} & & \mathcal{G}\text{-COMP} & \subseteq & \text{EFF} & \subseteq & \text{FN} \\ & & \cup & & \cup & & \cup \\ \text{PR} & \subseteq & \mathcal{G}\text{-TCOMP} & \subseteq & \text{TEFF} & \subseteq & \text{TFN} \end{array}}$$

Once again, the questions as to the properness of the various "$\subseteq$" inclusions still need to be answered.

Examples of PR functions:

- **Sum function** $f = \lambda x, y \cdot (x + y)$

  This has the well-known recursive definition:

$$\begin{cases} f(x,0) &= x \\ f(x,y+1) &= f(x,y)+1 \end{cases}$$

  However, we must put it in the form required by (3) on p. 3-8:

$$\begin{cases} f(x,0) &= g(x) \\ f(x,y+1) &= h(x,y,f(x,y)) \end{cases}$$

  where $g, h \in \text{PR}$ (with one parameter: x). So let us take $g(x) = x$, and $h(x,y,z) = z+1$. Putting $g(x) = \boldsymbol{U}_1^1(x)$ and $h(x,y,z) = \boldsymbol{S}(\boldsymbol{U}_3^3(x,y,z))$, a PR-derivation for f is

$$\begin{array}{r} f_1 \leftarrow \boldsymbol{U}_1^1 \\ f_2 \leftarrow \boldsymbol{S} \\ f_3 \leftarrow \boldsymbol{U}_3^3 \\ f_4 \leftarrow f_2, f_3 \ (\mathsf{compos} : 1,3) \\ f = f_5 \leftarrow f_1, f_4 \ (\mathsf{rec} : 1). \end{array}$$

- **Product function** $f = \lambda x, y \cdot (x * y)$

  Recursive definition:

$$\begin{cases} f(x,0) &= 0 \\ f(x,y+1) &= f(x,y)+x \end{cases}$$

Required form:

$$\begin{cases} f(x,0) &= g(x) \\ f(x,y+1) &= h(x,y,f(x,y)) \end{cases}$$

where g, $h \in$ PR (with one parameter: x). Put $g(x) = \boldsymbol{Z}(x)$, and

$$\begin{aligned} h(x,y,z) &= z + x \\ &= \boldsymbol{sum}\,(z,x) \\ &= \boldsymbol{sum}\,(\boldsymbol{U}^3_3(x,y,z),\, \boldsymbol{U}^3_1(x,y,z)). \end{aligned}$$

A PR-derivation for f is

$$\begin{aligned} &\vdots \\ \boldsymbol{sum} = f_5 &\leftarrow \cdots \\ f_6 &\leftarrow \boldsymbol{Z} \\ f_7 &\leftarrow \boldsymbol{U}^3_3 \\ f_8 &\leftarrow \boldsymbol{U}^3_1 \\ f_9 &\leftarrow f_5, f_7, f_8 \ (\mathsf{compos} : 2, 3) \\ f = f_{10} &\leftarrow f_6, f_9 \ (\boldsymbol{rec} : 1). \end{aligned}$$

- **Factorial** $f = \lambda x \cdot x!$

  Recursive definition:

$$\begin{cases} f(0) &= 1 \\ f(x+1) &= f(x) * (x+1) \end{cases}$$

  Required form:

$$\begin{cases} f(0) &= k \\ f(x+1) &= h(x, f(x)) \end{cases}$$

  where $h \in$ PR (with no parameters). Putting $k = 1$ and

$$\begin{aligned} h(x,y) &= y * (x+1) \\ &= \boldsymbol{prod}\,(y, \boldsymbol{S}(x)) \\ &= \boldsymbol{prod}\,(\boldsymbol{U}^2_2(x,y), \boldsymbol{S}(\boldsymbol{U}^2_1(x,y))), \end{aligned}$$

  we can obtain an appropriate PR-derivation, as before.

Clearly, we need an easier way to show that functions are PR!
We address this problem in §5.

4.3 Relative primitive recursiveness

Let $\vec{g} = g_1, \ldots, g_n$ be any functions. A function f is ***primitive recursive in*** $\vec{g}$ iff f is obtained from the ***initial functions*** and/or $g_1, \ldots, g_n$ by a finite number of applications of ***composition*** and ***primitive recursion***. Equivalently, f is ***PR in*** $\vec{g}$ iff there is a finite sequence of functions $f_1, \ldots, f_n$ such that $f_n = f$ and, for $i = 1, \ldots, n$, *either* f_i is an ***initial function***, *or* f_i is one of the g_j's, *or* f_i is obtained from some f_j's $(j < i)$ by ***composition*** or ***primitive recursion***. Such a sequence is called a ***PR-derivation*** of f from $\vec{g}$.

$\mathrm{PR}(\vec{g})$ is the class of functions PR in $\vec{g}$.

Lemma 4.10. [cf. Lemma 3.1, p. 3-6]

(a) $\mathrm{PR} \subseteq \mathrm{PR}(\vec{g})$

(b) $\mathrm{PR} = \mathrm{PR}(\langle\rangle)$

(c) *If* $\vec{g} \subseteq \vec{h}$, *then* $\mathrm{PR}(\vec{g}) \subseteq \mathrm{PR}(\vec{h})$.

Proof: Clear from the definition. □

Theorem 4.11 (Transitivity). [cf. Thm 3.2, p. 3-6]

(a) *If* $f \in \mathrm{PR}(\vec{g})$ *and* $g_1, \ldots, g_k \in \mathrm{PR}$, *then* $f \in \mathrm{PR}$.
More generally:

(b) *If* $f \in \mathrm{PR}(\vec{g})$ *and* $g_1, \ldots, g_k \in \mathrm{PR}(\vec{h})$, *then* $f \in \mathrm{PR}(\vec{h})$.

(c) *If* $f \in \mathrm{PR}(\vec{g}, \vec{h})$ *and* $g_1, \ldots, g_k \in \mathrm{PR}(\vec{h})$, *then* $f \in \mathrm{PR}(\vec{h})$.

Proof:
(a) *Prepend* PR-derivations of $g_1, \ldots, g_k$ to a PR-derivation of f from $\vec{g}$.
(b), (c) Similarly. □

A class $\mathcal{C}$ of functions is said to be $\mathrm{PR}(\vec{g})$***-closed*** iff
$\mathcal{C}$ is PR-closed and contains $\vec{g}$; i.e.,

(i) $\mathcal{C}$ contains the ***initial functions and*** $\vec{g}$, and
(ii) $\mathcal{C}$ is ***closed*** under ***composition*** and ***definition by PR***.

Q. Is FN $\mathrm{PR}(\vec{g})$-closed? Is TFN?

Lemma 4.12. [cf. Lemma 4.4, p. 4-3]
$\mathrm{PR}(\vec{g})$ *is* $\mathrm{PR}(\vec{g})$*-closed.*

Proof: from the definition. □

Lemma 4.13. [cf. Lemma 4.5, p. 4-3]
Let $\mathcal{C}$ *be any* $\mathrm{PR}(\vec{g})$*-closed class of functions. Then* $\mathrm{PR}(\vec{g}) \subseteq \mathcal{C}$*.*

Proof: We can show that

$$f \in \mathrm{PR}(\vec{g}) \implies f \in \mathcal{C}$$

by CV induction on the length of the PR-derivation of f from $\vec{g}$. □

Theorem 4.14. [cf. Theorem 4.6, p. 4-3]
$\mathrm{PR}(\vec{g})$ *is the* smallest $\mathrm{PR}(\vec{g})$*-closed class. In other words,*
(*i*) $\mathrm{PR}(\vec{g})$ *is* $\mathrm{PR}(\vec{g})$*-closed; and*
(*ii*) $\mathrm{PR}(\vec{g})$ *is contained in every* $\mathrm{PR}(\vec{g})$*-closed class.*

Proof: By Lemmas 4.12 and 4.13. □

Corollary 4.15. [cf. Cor. 4.9, p. 4-3]
$\mathrm{PR}(\vec{g}) \subseteq \mathcal{G}\text{-COMP}(\vec{g})$

Proof: Since $\mathcal{G}$-COMP$(\vec{g})$ contains $\vec{g}$ and is PR-closed. □

Note that $\mathrm{PR}(\vec{g})$ need not consist of total functions only, since the g_i might not be total! So if $\mathrm{TPR}(\vec{g})$ is the class of *total* functions PR in $\vec{g}$, then the *relativised* version of the diagram on page 4-4 is

$$\begin{array}{ccccccc}
\mathrm{PR}(\vec{g}) & \subseteq & \mathcal{G}\text{-COMP}(\vec{g}) & \subseteq & \mathrm{EFF}(\vec{g}) & \subseteq & \mathrm{FN} \\
\cup\!| & & \cup & & \cup & & \cup \\
\mathrm{TPR}(\vec{g}) & \subseteq & \mathcal{G}\text{-TCOMP}(\vec{g}) & \subseteq & \mathrm{TEFF}(\vec{g}) & \subseteq & \mathrm{TFN}
\end{array}$$

As before, the questions as to the properness of the various "⊆" inclusions need to be answered.