AN UNEXPECTED FEATURE OF CLASSICAL PROPOSITIONAL LOGIC IN THE TRACTATUS

EINE UNVERMUTET BESONDERHEIT AUF KLASSISCHE AUSSAGENLOGIK IN DER TRACTATUS

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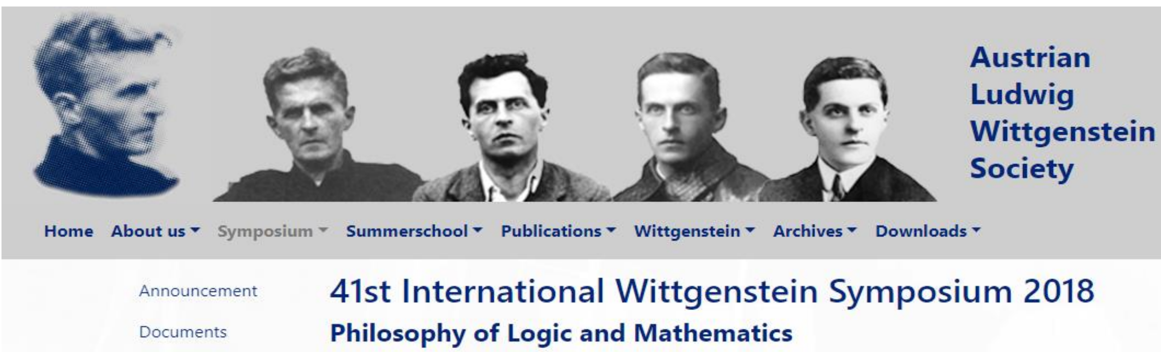


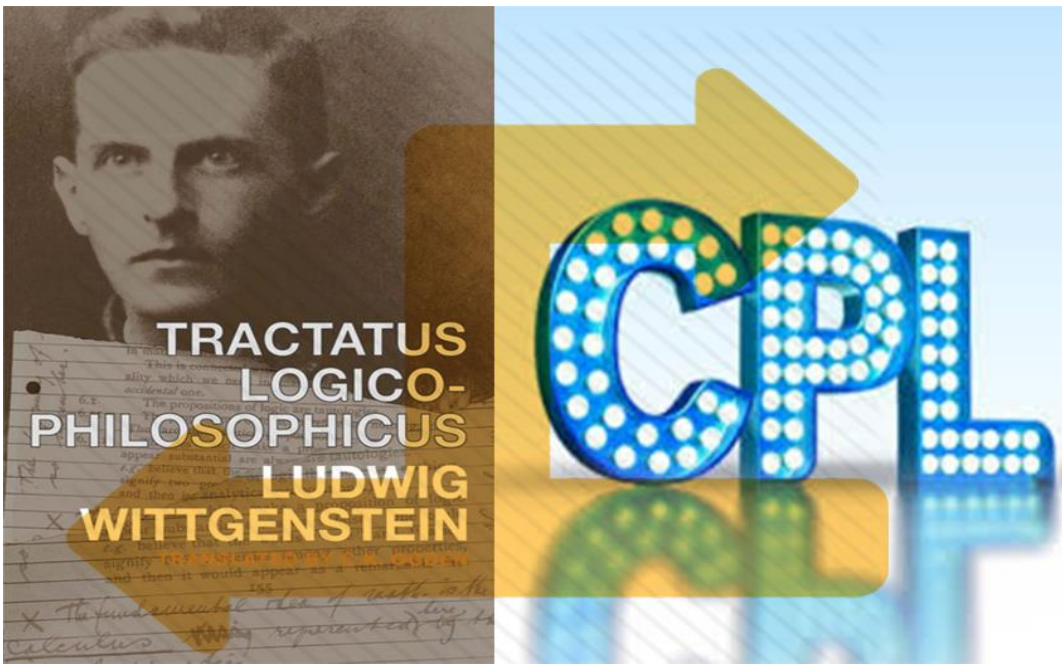
Abstract

We study the relation between classical propositional logic (CPL) as it is nowadays and how it appears in the *Tractatus* focusing on a specific feature expressed in the paragraph 5.141. In a first part we make some general considerations about CPL, pointing out that CPL is difficult to characterize and define, that there is no definitive final version of it presented in one given reference book. In a second part we analyze the network of concepts related to paragraph 5.141 of the *Tractatus* involving notions corresponding to what are nowadays called "semantical consequence", "distribution of truth values", "valuations" and "models". We make the link with Tarski's definition of logical consequence in his famous 1936 paper. This leads us to examine in a third part up to which point CPL is in the *Tractatus* considered as a Boolean algebra.

Introduction: the relation between two icons

On the one hand classical propositional logic (hereafter CPL) is the most famous logical system of modern logic, on the other hand the *Tractatus Logico-Philosophicus* (hereafter *Tractatus*) is one of the most famous books in the history of modern logic. It seems therefore worth to ask the following questions:

- How is CPL in the *Tractatus*?
- What are the differences and similarities of CPL in the *Tractatus* and other versions?
- What is the contribution of the *Tractatus* to CPL?



These three questions are interrelated. It is not that simple to give answers to them and the aim of the present paper is not to give full and final answers to these questions.

We will concentrate on the paragraph 5.141 of the *Tractatus*,[1] which is the following single sentence: If *p* follows from *q* and *q* from *p* then they are one and the same proposition.

We will of course not comment this proposition in isolation, artificially extracted from the *Tractatus*. We will deal with the related network of concepts presented in the *Tractatus*, and compare this framework to CPL. This involves all aspects of CPL: historical, mathematical and philosophical.

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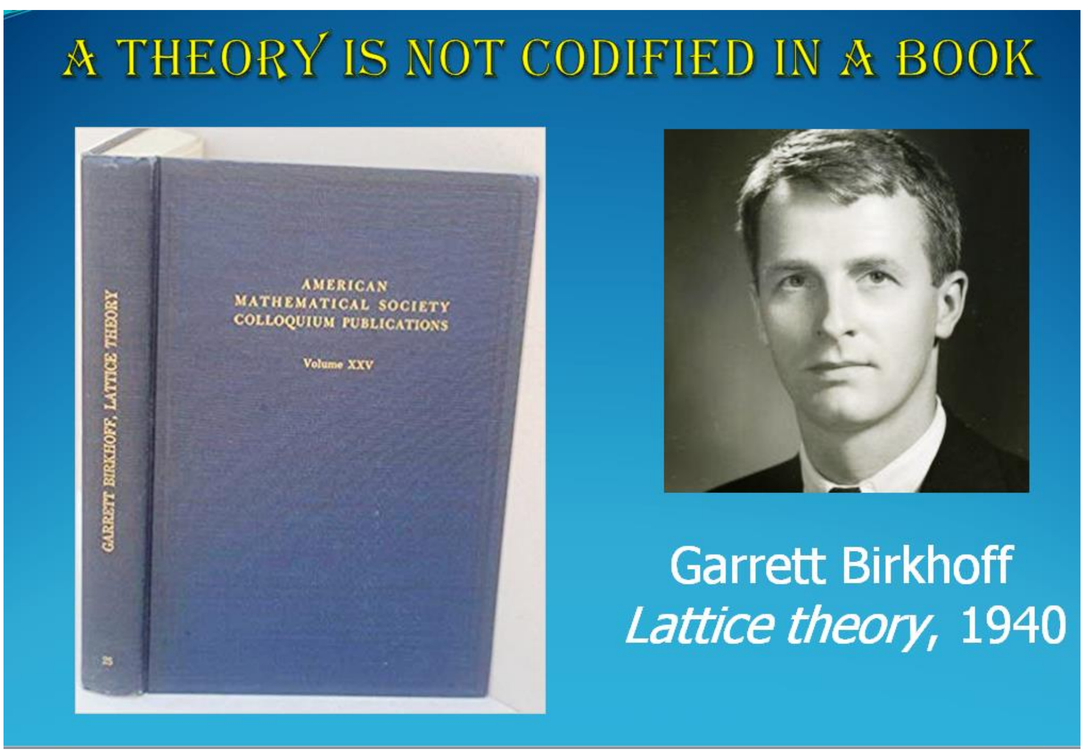
[1] We will call the numbered items of the *Tractatus* "paragraphs", following a certain tradition, although they are not always syntactically speaking such entities.

1. The inherent ambiguity of classical propositional logic

On the one hand we have a book, the *Tractatus*, on the other hand we have a logical system, CPL. One of the reasons why it is difficult to make a comparison between CPL and CPL as it is in the *Tractatus* is that, contrarily to what one may think, CPL is not something directly clear and obvious, precisely and univocally presented or defined.

Let us emphasize that this is the case of any scientific system or theory: the theory of evolution of course, but also the theory of relativity or to take an even simpler example, more directly related to CPL, lattice theory. It would be too naïve to believe that lattice theory reduces to a group of axioms. There are fairly different axiomatizations of lattice theory, moreover everything is in the axioms only potentially.

It is also important to stress that a scientific system or a theory is not codified in one given book. There is no Bible of lattice theory, although this case is the closet we can imagine because Garret Birkhoff's book is one of the most famous books of mathematics of the 20th century (see Birkhoff 1940 and Bennett 1973). But it is still quite different from the Bible … or the *Tractatus*. The *Tractatus* has a rigid and precise linguistic structure, allowing, like in the Bible case, never ending discussions and interpretations of each sentence.



Furthermore a scientific theory evolves forever, lattice theory as it is nowadays is not the same at it was at the time of Birkhoff. CPL has evolved quite a lot since the time of the *Tractatus*. The situation of CPL is much more complicated than the situation of lattice theory, because (1) it can be presented in even more different ways (proof theory vs. semantics) (2) it is surrounded by philosophical nebulosity and (3) there is not one specific reference book by a famous author devoted to it, like in the case of lattice theory with Birkhoff.

Many people have the idea that CPL is trivial and simple, logic for babies. But in fact it is not the case. The situation is similar with the one of natural numbers. It is only apparently simple. Number theory is not the simplest mathematical theory, as shown in quite different ways by Gödel and Bourbaki. The similarity is even stronger if we consider that the structure of the set of propositions in propositional logic is an absolutely free algebra, which, like Peano algebra (an absolutely free algebra with only one generator and one function), is not axiomatizable in first-order logic (for details, see Beziau, 1999).

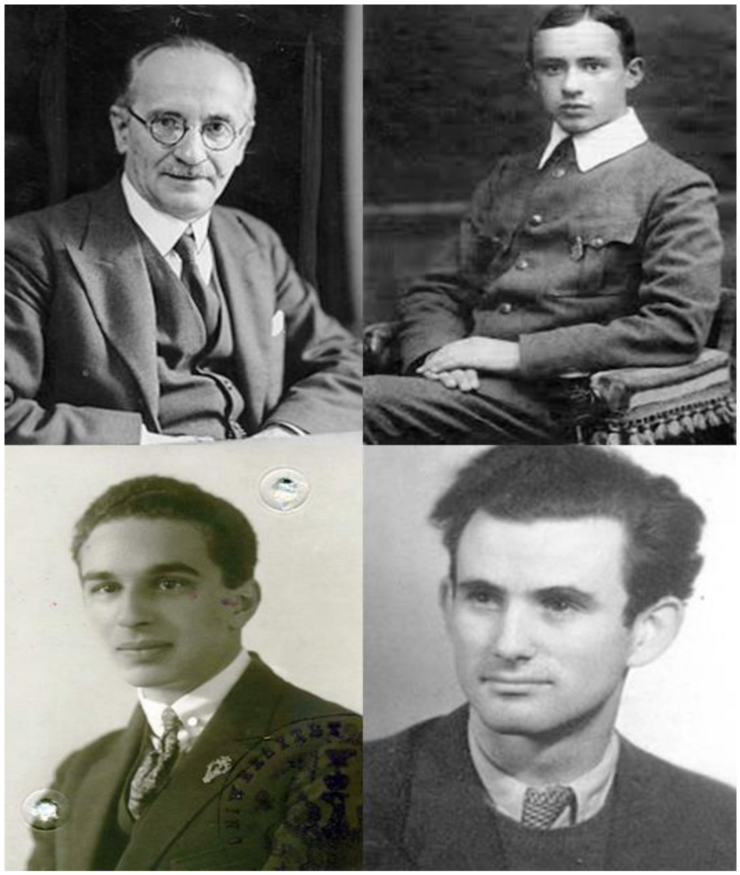
CPL was not born in one day, out of the spirit of someone. Before its definite version there were many drafts. People like Boole, Peirce, Frege, Russell, Wittgenstein made different contributions to it. And to speak of a definite version of it is quite misleading. However we can say that what we find in the work of Post in 1921 is something close to it (the case of first-order logic is more complicated).

Funny enough Post's work was published the same year as Wittgenstein's *Tractatus*: 1921. Post's work is a seminal work. After Peirce who proved that all the 16 connectives can be defined by only one (joint work with his student Christine Ladd-Franklin, 1882), it is the first work with important mathematical results: completeness, functional completeness and Post completeness. In mathematics results work together with conceptualization. In Post's paper we find for the first time a clear distinction between proof and truth in CPL, distinction on which basis the completeness theorem which is herein presented makes sense.

There are various philosophical interpretations of CPL and the philosophical view is interacting with the formal aspect of CPL. This is in particular the case in the *Tractatus* with the idea of elementary propositions, on the basis of which was promoted "logica atomism" by Russell, a terminology not used by Wittgenstein himself but already introduced in the preface of the *Tractatus* by Betrand Russell to describe Wittgenstein's theory.

Gödel showed that it is possible to prove the completeness theorem of CPL without considering that there are atomic formulas. Generally CPL is presented with atomic formulas, but it can also be presented without. This paper was commented by Quine (see Gödel 1932).

The same Quine wrote a famous paper in *Mind* in 1934, which was pivotal for the tendency to speak about "Sentential Logic" rather than "Propositional Logic", arguing that it is better to conceive CPL as dealing with sentences than propositions. At the same time, in Poland people were going in the other direction, in particular considering connectives as functions, so that on the one hand we have an algebra of propositions whose operators are connectives (idea due to Lindenbaum) and on the other hand logical matrices (theory developed by Łukasiewicz and Tarski) where there are some operators defined on truth-values corresponding to the connectives (see Beziau 2002). Using this correspondence Lindenbaum proves a famous theorem according to which any logic can be characterized by a matrix, result published by Jerzy Łoś after the war (see Łoś 1949).



In Poland was introduced the terminology "zero-order logic" to talk about CPL, which is quite neutral as the nature of the elements dealing with and establish a correspondence with first-order logic. Another important innovation in Poland was to consider a consequence operator or consequence relation, not only a set of tautologies. This approach of CPL is now quite standard, but few people know that if there is not restriction of finiteness, if we consider a consequence relation as a relation between on the one hand a set of fomulas (a theory) of any cardinality and on the other hand a formula (consequence of the theory), then CPL is not decidable, despite compactness (see Beziau 2001). This proof is presented in the book of Enderton (1972).

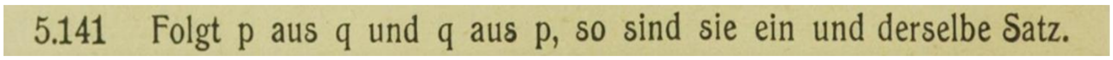
These remarks show that CPL as it is today is necessary quite different from as presented in the *Tractatus*.

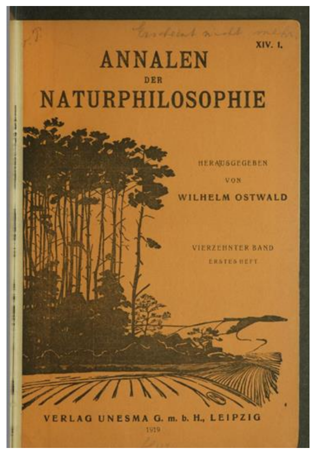
2. The paragraph 5.141 and related concepts

We will now focus on paragraph 5.141 of the *Tractatus*. It is as follows:

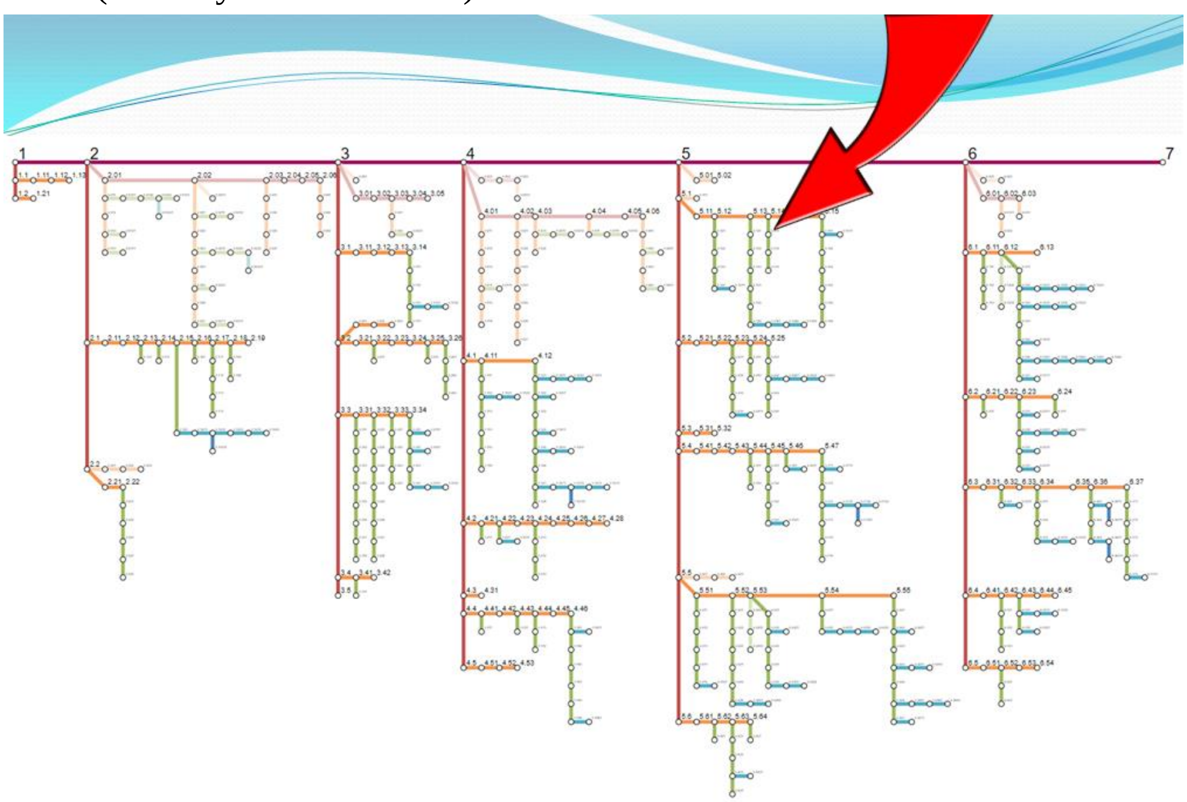
> 5.141 If *p* follows from *q* and *q* from *p* then they are one and the same proposition.

For the sake of precision and exactness, below is the German version as printed in the original publication:


5.141 Folgt p aus q und q aus p, so sind sie ein und derselbe Satz.



And here is the position of 5.141 in the *Tractatus*, explicitly presented as a tree (courtesy of David Stern).



We will not in general present the full German original of the sentences and paragraphs that we are commenting, but only the German originals of central notions of interest for us here. In general translations from German to English are quite straightforward for the topic of our paper.

The only, but essential, case which is tricky is the one of "Satz", which is very important for us here. Depending on the situation, it can be translated in English, as: sentence, proposition, statement, principle (cf. *Satz vom Grund*, corresponding to *Principle of Reason*). Let us note that on both English translations of the *Tractatus*[2], "Satz", has been translated by proposition and we will also follow here this translation. It seems reasonable to think that Wittgenstein uses "Satz" as corresponding to what is called a "proposition" in *Principia Mathematica*, and in fact he is using the letter "*p*" in "*q*" similarly to what is done in Whitehead and Russel's book (1910). The sign "*p*", which is the first letter of the word "proposition", is there used as a variable for propositions, due to the intended range of it, like "*n*" is used as a variable for numbers, and then "*q*" and "*m*" respectively follow.

There are two important notions in 5.141: *follow* and *proposition*. To properly understand 5.141 we need to have a correct understanding of these two notions, the two being interrelated: if we want to understand what a proposition in the *Tractatus* is, we need to understand the meaning of "follow". This is a technical notion depending on two other technical notions. The meaning of "follow" ("folgen", in Greman) is presented in the paragraph 5.11:

FOLLOW: If the truth-grounds which are common to a number of propositions are all also truth-grounds of some one proposition, we say that the truth of this proposition *follows (folge)* from the truth of those propositions. (5.11)

As we can see the meaning of "follow" depends on the notion of truth-ground (*Wahrheitsgrund*), which is defined in paragraph 5.101:

TRUTH-GROUNDS Those truth-possibilities of its truth-arguments, which verify the proposition, I shall call its *truth-grounds (Wahrheitsgründe)*. (5.101)

[2] We will generally follow Ogden's translation, which is less nice than Pears amd McGuinness's one, but closer to the German original.

And as we can see this notion depends on the notion of truth-possibilities. (*Wahrheitsmöglichkeiten*):

4.31 We can represent truth-possibilities by schemata of the following kind ('*T*' means 'true', '*F*' means 'false'; the rows of '*T*'s' and '*F*'s' under the row of elementary propositions symbolize their truth-possibilities in a way that can easily be understood):

p	q	r
T	T	T
F	T	T
T	F	T
T	T	F
F	F	T
F	T	F
T	F	F
F	F	F

,

p	q
T	T
F	T
T	F
F	F

,

p
T
F

.

This corresponds to what is called nowadays "distribution of truth-values". Some books follow Wittgenstein's presentation, using "T" and "F" (in the original text we have the German initials: "W" and "F"), other books instead use "1" and "0". Emil Post was using a notation which is rarely used: "+" and "-". The original terminology of Wittgenstein "truth-possibilties" has also not been followed, at least for CPL, but there is a connection with possible worlds in Kripke semantics for modal logic via Carnap (1947).

Let us note that nowadays there is no specific or/and standard word in CPL for what is called in the *Tractatus* a truth-ground ("ground" has recently became famous through Kit Fine but with another meaning, see e.g. Fine, 2012). However this notion is perfectly clear. This is what can be called a model, but a model in CPL is not a mathematical structure, like in first-order logic, it is a function from the set of propositions into {0,1}. Generally "model" is not used in CPL; but Chang and Keisler (1973) used it to make a uniform presentation of CPL and First-Order Logic.

Such a function is generally called a "valuation" by contrast to "distribution of truth-values" which are functions defined only on the set of atomic propositions. Wittgenstein is not making the distinction. The fact that valuations can be generated by distribution of truth-values and that a distribution of truth-value has a unique extension which is a valuation, is directly related to the concept of absolutely free algebra. These technicalities were made precisely clear in the Polish school, in particular by Łoś (see in particular Łoś and Suszko 1958).

Using a bit of symbolism, denoting a Truth-Ground as TG and a valuation as v, we can put the definition of Truth-Ground as follows:

TG $[p] = \{v; v(p)=1\}$

And if we replace the terminology "Truth-Ground" by "Model". We have:

mod $[p] = \{v; v(p)=1\}$

We can therefore reformulate 5.141. as follows:

$$\boxed{\text{if } \mathrm{mod}[p] = \mathrm{mod}[q], \text{ then } p = q}$$

It is worth noting that Tarski in 1936 used the same terminology, "folgen" (his paper was written in German, Tarski 1936a), and a definition similar to the one of Wittgenstein in the *Tractatus*, but more general in two aspects: it does not reduce to propositional logic, it is a relation between theories and propositions:

> The sentence *X* follows logically from the sentences of the class *K* if and only if every model of the class *K* is also a model of the sentence *X*. (Tarski 1936d)

And Tarski is the guy who proved that CPL is a Boolean algebra (1936e), but not on the basis of this notion, nowadays standardly called "semantical" consequence.

3. Is CPL in the *Tractatus* a Boolean Algebra?

First of all this question has not to be confused with the quite funny question "Is the *Tractatus* a Boolean Algebra?". The answer to the latter is: certainly not! This is what Donald Duck would reply, or any rational animal. The (structure of the) *Tractatus* is just a tree. However a nice tree with lots of flowers and fruits… But there is a connection between relations of order and Boolean algebra: as Marshall Stone discovered (1935), a distributed complemented lattice is a Boolean Ring, the two being two equivalent formulation of a Boolean algebra.

And this gives us a clue to the original question, because to answer it we need to have a clear idea of what is a Boolean Algebra. The simplest Boolean algebra is the Boolean algebra on $\{0,1\}$. And the simplest way to consider this algebra is to consider the two operations $+$ and $\times$ defined on these numbers by the following tables:

$\times$	0	1
0	0	0
1	0	1

$+$	0	1
0	0	1
1	1	1

We have then what is called an "Idempotent Ring". Something very simple despite this quite poetic name and that may look complicated or/and incomprehensible for non-mathematicians.

Someone may think that CPL in the *Tractatus* is not a Boolean Algebra because we cannot find "Idempotent Ring", "0" and "1" and such tables. But of course we have to go beyond appearances. We can rewrite these two tables as follows:

.	F	T
F	F	F
T	F	T

$\vee$	F	T
F	F	T
T	T	T

These are exactly the same tables, we just have changed the signs. All these signs are used in the *Tractatus*, but these tables themselves are not presented. Note however that Russell and Wittgenstein were drawing similar tables in the 1910s (before the *Tractatus*).

If we consider these tables are defining operations on {F,T}, i.e. with domain and co-domain {F,T}, then what we have is what is called the semantics of CPL. And the semantics of CPL is nothing else than the Boolean algebra on {0,1}. One could say that CPL is a Boolean algebra because its semantics is a Boolean algebra, but this would be a bit exaggerated not to say confusing. CPL is a Boolean algebra in another way which is different, in particular because it is a different Boolean algebra than this simplest one. And this is this second way which is connected to the paragraph 5.141.

Boole was considering that $x^2 = x$ (where *x* is a variable for a proposition, and using a notation mimicking arithmetic) is the fundamental law of thought, from which in particular it is possible to derive the law of contradiction (see Beziau 2018). Wittgenstein was less extravagant but nevertheless would have agreed with Boole that *p* and *p.p* are identical.

But in CPL *p* and *p.p* are considered as two *different* propositions. We are not writing "*p*" and "*p.p*", because we are considering the objects they refer to. Note also that Wittgenstein is not using quotation marks in the paragraph 5.141.

The two propositions *p* and *p.p* are different but they are considered as *logically equivalent*. What does this mean? According to 5.141 *p* and *p.p* are one and the same proposition because one follows from the other one and vice versa, because they have the same truth-grounds according to the definition given in 5.101. In CPL they are not the same, but the are equivalent. But considering that logical equivalence is a congruence relation we can "identify" them and this leads us to a Boolean algebra.

Wittgenstein does not make this detour, he is directly considering the algebra that we can get by factoring CPL with logical equivalence. This is generally called a Lindenbaum-Tarski algebra because this methodology can be applied to logics other than CPL, but in case of CPL the so-called Lindenbaum-Tarski algebra is in fact a Boolean algebra. It is not the Boolean algebra on {0,1}, it has in particular much more than two elements.

Leibniz is famous for the following definition:

> Two terms are the same (eadem) if one can be substituted for the other without altering the truth of any statement (salva veritate)." (ch 19, def 1).

Leibniz is talking here about terms and gives the following examples:

> For example, 'triangle' and 'trilateral', in every proposition demonstrated by Euclid concerning 'triangle', 'trilateral' can be substituted without loss of truth (salva veritate). (ch 20 def 1).

We can generalize this view applying this definition of identity to any objet including propositions. Now to claim that the two propositions *p* and *p.p* are the same, in this Leibnizian sense, because they can be substituted for the other without altering the truth of any statement, we have to prove the so-called replacement theorem. This is what Tarski did and therefore showed that CPL is a Boolean algebra. Wittgenstein did not prove this theorem, so the sameness he is talking about in 5.141 is ambiguous because there is no guarantee that it can work. And moreover Wittgenstein had no idea that this corresponds to what we now call a Boolean algebra.

What we can say is that the *Tractatus*, through 5.141, is aiming at conceiving CPL has a Boolean algebra.

9. References

Bennett, Mary Katherine, 1973, Review of *Lattice theory* by Garrett Birkhoff, 3rd edition and *Lattice theory: first Concepts and distributive lattices* by George Grätzer, Bulletin of the American Mathematical Society, 79, pp.1-5.

Beziau, Jean-Yves, 1999, "The mathematical structure of logical syntax", in *Advances in contemporary logic and computer science*, American Mathematical Society, Providence, 1999, pp.3-15.

Beziau, Jean-Yves, 2001, "What is classical propositional logic?", *Logical Investigations*, **8** , pp.266-277.

Beziau, Jean-Yves, 2002, "The philosophical import of Polish logic", in *Methodology and philosophy of science at Warsaw University*, M.Talasiewicz (ed.), Semper, Warsaw, pp.109-124.

Beziau, Jean-Yves, 2007, "Sentence, proposition and identity", *Synthese*, **154**, pp.371-382.

Beziau, Jean-Yves, 2018, "Is the Principle of Contradiction a Consequence of $x^2 = x$?", *Logica Universalis*, vol.12, pp.55-81.

Birkhoff, Garrett, 1940, *Lattice Theory*, American Mathematical Society, New York.

Boole, George, 1854, *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*, MacMilllan, London.

Carnap, Rudolf, 1947, *Meaning and necessity: a study in semantics and modal logic*, University of Chicago Press, Chicago.

Chang, Chen Chung and Keisler, Howard Jerome, 1973, *Model Theory*, North Holland, Amsterdam.

Enderton, Herbert, B., 1972, *A mathematical introduction to logic*, Academic Press, New York.

Fine, Kit, 2012, "The pure logic of ground", *Review of Symbolic Logic*, 12, pp.1-25.

Gödel, Kurt, 1932, "Eine Eigenschaft der Realisierungen des Aussagenkalküls", *Ergebnisse eines mathematischen Kolloquiums*, 2, pp.27.28, 1932. Presented by Quine in *Kurt Gödel Collected Works*, Volume 1, S.Feferman et al, (eds) pp.238-241, Oxford University Press, Oxford, 1986.

Ladd-Franklin, Christine, 1882, *On the Algebra of Logic*, PhD thesis directed by Charles Peirce, The Johns Hopkins University.

Leibniz, Gottfried Wilhelm, *On the Universal Science. Characteristic*.

Łoś, Jerzy, *O matrycach logicznych*, Prace Wrocławskiego Towarzystwa Naukowego, Seria B, No 19, 1949, pp. 1-41.

Łoś, Jerzy and Suszko, Roman, 1958, "Remarks on sentential logic", *Indigationes Mathematicae*, pp.177-183.

Quine, William van Orman, 1934, “Ontological. remarks on the propositional calculus”, *Mind*, vol.**43**, pp.472-476.

Post, Emil, 1921, “Introduction to a general theory of elemetary propositions”, *American Journal of Mathematics*, **43** (1921), 163-185.

Stone, Marshall, 1935, “Subsomption of the theory of Boolean algebras under the theory of rings”, *Proceedings of National Academy of Sciences*, 21 (1935), pp.103-105.

Tarski, Alfred, 1936a, “Über den Begriff der logischen Folgerung”, *Actes du Congrès International de Philosophie Scientifique, VII Logique*, Hermann & Cie, Paris, pp. 1–11.

Tarski, Alfred, 1936b, “O pojęciu wynikania logicznego”, *Przeglad Filozoficzny*, **39**, 58-68.

Tarski, Alfred, 1936c, “On the concept of logical consequence”, , in Logic, semantics, metamathematics

Tarski, Alfred, 1936d, “On the concept of following logically”, Translated from the Polish and German by Magda Stroińska and David Hitchcock, *History and Philosophy of Logic* **23**:155-196 (2003)

Tarski, Alfred, 1936e, “Grundzüge des Systemenkalküls, *Fundamenta Mathematicae*, Erster Teil 25 (1935), 503–526, Zweiter Teil 26 (1936), 283–301.

Whithead, Alfred North and Russell, Bertand, 1910, *Principa Mathematica*, Volume 1, Cambridge University Press, Cambridge.

Wittgenstein, Ludwig, 1921, “Logisch-Philosophische Abhandlung”, *Annalen der Naturphilosophie*, 14.

Wittgenstein, Ludwig, 1922, *Tractatus Logico-Philosophicus*, Translation of (Wittgenstein 1921) by C. K. Ogden, prepared with assistance from G. E. Moore, F. P. Ramsey, and L.Wittgenstein, Routledge & Kegan Paul, London.

Wittgenstein, Ludwig, 1961, *Tractatus Logico-Philosophicus*, Translation of (Wittgenstein 1921) by David Pears and Brian McGuinness, Routledge, London.