Infinitesimal paraholomorphically projective transformations on tangent bundles with diagonal lift connection

M. Iscan and A. Magden

Abstract. Let (M_n, g) be a Riemannian manifold and $T(M_n)$ its tangent bundle with diagonal lift connection and adapted almost paracomplex structure. We determine the infinitesimal paraholomorphically projective transformation on $T(M_n)$. Furthermore, if $T(M_n)$ admits a non-affine infinitesimal paraholomorphically projective transformation, then M_n and $T(M_n)$ are locally flat.

M.S.C. 2000: 53C20, 53C15.
Key words: infinitesimal paraholomorphically projective transformation, diagonal lift connection, adapted almost paracomplex structure.

1 Introduction

Let M_n be an n-dimensional manifold and $T(M_n)$ its tangent bundle. We denote by $\Im_q^p(M_n)$ the set of all tensor fields of type (p, q) on M_n. Similarly, we denote by $\Im_q^p(T(M_n))$ the corresponding set on $T(M_n)$.

Let ∇ be an affine connection on M_n. A vector field V on M_n is called an *infinitesimal projective transformation* if there exists a 1-form Ω on M_n such that

$$(L_V\nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X$$

for any $X, Y \in \Im_0^1(M_n)$, where L_V is the Lie derivation with respect to V. In this case Ω is called the *associated 1-form* of V. Especially, if $\Omega = 0$, then V is called an *infinitesimal affine transformation.*

An almost paracomplex manifold is an almost product manifold (M_n, φ), $\varphi^2 = I$, such that the two eigenbundles T^+M_n and T^-M_n associated to the two eigenvalues +1 and -1 of φ, respectively, have the same rank [1], [4]. An integrable almost product manifold is usually called a locally product manifold. Note that the dimension of an almost paracomplex manifold is necessarily even.

Next Let (M_n, φ) be an almost paracomplex manifold with affine connection ∇. A vector field V on $M_n$is called an *infinitesimal paraholomorphically projective transformation* if there exists a 1-form Ω on M_n such that

Differential Geometry - Dynamical Systems, Vol.10, 2008, pp. 170-177.

$$(L_V\nabla)(X,Y)=\Omega(X)Y+\Omega(Y)X+\Omega(\varphi X)\varphi Y+\Omega(\varphi Y)\varphi X$$

for any $X,Y\in\Im_0^1(M_n)$. In this case Ω is also called the *associated 1-form* of V[2], [3]. Especially, if $\Omega=0$, then V is the infinitesimal affine transformation.

It is well-known that there are several lift connections of ∇ on $T(M_n)$[5], [6]. In this paper, we study the infinitesimal paraholomorphically projective transformation on $T(M_n)$ with diagonal lift connection.

2 Preliminaries

In this section we shall give some definitions and formulae on $T(M_n)$ for later use (for details, see [5] , [6]). Let (M_n,g) be a Riemannian manifold, ∇ the Riemannian connection of g and Γ_{ji}^h the coefficients of ∇, i.e., $\Gamma_{ji}^a\partial_a=\nabla_{\partial_j}\partial_i$, where $\partial_h=\frac{\partial}{\partial x^h}$ and (x^h) is the local coordinates of M_n.

2.1 Adapted frame of $T(M_n)$

We define a local frame $\{E_i,E_{\bar{i}}\}$ of $T(M_n)$ as follows:

$E_i=\partial_i-y^b\Gamma_{ib}^a\partial_{\bar{a}}$ and $E_{\bar{i}}=\partial_{\bar{i}}$,

where (x^h,y^h) is the induced coordinates of $T(M_n)$ derived from the local coordinates (x^h) of M_n and $\partial_{\bar{i}}=\frac{\partial}{\partial y^i}$. This frame $\{E_i,E_{\bar{i}}\}$ is called the *adapted frame* of $T(M_n)$. Then $\left\{dx^h,\delta y^h\right\}$ is the dual frame of $\{E_i,E_{\bar{i}}\}$, where $\delta y^h=dy^h+y^b\Gamma_{ab}^h dx^a$.

By the definition of the adapted frame, we have the following

Lemma 1. *The Lie brackets of the adapted frame of $T(M_n)$ satisfy the following identities:*

$$[E_j,E_i]=y^bR_{ijb}^aE_{\bar{a}},\tag{2.1}$$

$$[E_j,E_{\bar{i}}]=\Gamma_{ji}^aE_{\bar{a}},\tag{2.2}$$

$$[E_{\bar{j}},E_{\bar{i}}]=0.\tag{2.3}$$

2.2 Diagonal lift connection of ∇

A tensor field of type $(0,q)$ on $T(M_n)$ completely determined by its action on all vector fields $\tilde{X}_i,\ i=1,2,...,q$ which are of the form ${}^VX$ (vertical lift) or ${}^{\mathrm{H}}X$ (horizontal lift)[6, p.101]:

$${}^VX=X^i\frac{\partial}{\partial x^{\bar{i}}},\ {}^{\mathrm{H}}X=X^i\frac{\partial}{\partial x^i}-y^s\Gamma_{sh}^iX^h\frac{\partial}{\partial x^{\bar{i}}}$$

Therefore, we define the Sasakian metric ${}^Dg$ on $T(M_n)$ by

$$(2.4)\qquad \begin{cases} {}^Dg({}^HX,{}^HY)={}^V(g(X,Y)),\\ {}^Dg({}^VX,{}^VY)={}^V(g(X,Y)),\\ {}^Dg({}^VX,{}^HY)=0, \end{cases}$$

for any $X,Y\in\Im^1_0(M_n)$. ${}^Dg$ has local components

$${}^Dg=\begin{pmatrix} g_{ji}+g_{ts}y^ky^l\Gamma^t_{kj}\Gamma^s_{li} & y^k\Gamma^s_{kj}g_{si}\\ y^k\Gamma^s_{ki}g_{js} & g_{ji}\end{pmatrix}$$

with respect to the induced coordinates (x^h,y^h) in $T(M_n)$, where Γ^k_{ij} are components of Levi-Civita connection ∇_g in M_n. The metric ${}^Dg$ has components

$$(2.5)\qquad {}^Dg=\begin{pmatrix} g_{ji} & 0\\ 0 & g_{ji}\end{pmatrix}$$

with respect to the adapted frame in $T(M_n)$.

Let ${}^D\nabla$ be a Levi-Civita connection of ${}^Dg$, then

$${}^D\nabla_{E_j}E_i=\Gamma^a_{ji}E_a-\frac{1}{2}y^bR^a_{jib}E_{\bar a},$$

$${}^D\nabla_{E_j}E_{\bar i}=\frac{1}{2}y^bR^a_{bij}E_a+\Gamma^a_{ji}E_{\bar a},$$

$${}^D\nabla_{E_{\bar j}}E_i=\frac{1}{2}y^bR^a_{bji}E_a,\quad {}^D\nabla_{E_{\bar j}}E_{\bar i}=0.$$

2.3 Adapted almost paracomplex structure on $T(M_n)$

The diagonal lift ${}^D\varphi$ in $T(M_n)$ is defined by

$$(2.6)\qquad \begin{cases} {}^D\varphi^HX={}^H(\varphi X),\\ {}^D\varphi^VX=-{}^V(\varphi X),\end{cases}$$

for any $X\in\Im^1_0(M_n)$ and $\varphi\in\Im^1_1(M_n)$. The diagonal lift ${}^DI$ of the identity tensor field $I\in\Im^1_1(M_n)$ has the components

$${}^DI=\begin{pmatrix} \delta^j_i & 0\\ -2y^t\Gamma^j_{ti} & -\delta^j_i\end{pmatrix}$$

with respect to the induced coordinates and satisfies ${}^DI^VX=-{}^VX$, ${}^DI^HX={}^HX$ and $({}^DI)^2=I_{T(M_n)}$ for any $X\in\Im^1_0(M_n)$, i.e.,

${}^DIE_{\bar i}=-E_{\bar i}$ and ${}^DIE_i=E_i$.

Therefore ${}^DI$ is an almost paracomplex structure on $T(M_n)$. ${}^DI$ has components

$${}^DI=\begin{pmatrix} \delta^j_i & 0\\ 0 & -\delta^j_i\end{pmatrix}$$

with respect to the adapted frame in $T(M_n)$. This almost paracomplex structure is called *adapted almost paracomplex structure*. Its know that ${}^DI$ is integrable if and only if M_n is locally flat.

3 Infinitesimal paraholomorphically projective transformation

Theorem 1. *Let (M_n, g) be a Riemannian manifold and $T(M_n)$ its tangent bundle with diagonal lift connection and adapted almost paracomplex structure. A vector field $\tilde{V}$ is an infinitesimal paraholomorphically projective transformation with associated 1-form $\tilde{\Omega}$ on $T(M_n)$ if and only if there exist $\psi \in \Im_0^0(M_n)$, $B = (B^h)$, $D = (D^h) \in \Im_0^1(M_n)$, $A = (A_i^h)$, $C = (C_i^h) \in \Im_1^1(M_n)$ satisfying*

$$(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (B^h + y^a A_a^h,\ \ D^h + y^a C_a^h + 4\psi y^h + 2y^h y^a \Phi_a) \tag{3.1}$$

$$\tilde{\Omega}_{\bar{i}} = \partial_{\bar{i}}\tilde{\varphi} = \Phi_i \tag{3.2}$$

$$\nabla_j \Phi_i = 0, \nabla_j(\partial_i \psi) = 0 \tag{3.3}$$

$$\nabla_i A_j^a = -\frac{1}{2} D^h R_{hji}^a, \nabla_i C_j^a = -B^c R_{cij}^a - 4\partial_i \psi \delta_j^a \tag{3.4}$$

$$L_B \Gamma_{ji}^a = \nabla_j \nabla_i B^a + R_{hji}^a B^h = 2\tilde{\Omega}_j \delta_i^a + 2\tilde{\Omega}_i \delta_j^a \tag{3.5}$$

$$\nabla_j \nabla_i D^a = \frac{1}{2} R_{jih}^a D^h \tag{3.6}$$

$$\nabla_j \nabla_i A_b^a = -\frac{1}{2} A_h^a R_{jib}^h, \nabla_j \nabla_i C_b^a = -\frac{1}{2} C_h^a R_{jib}^h \tag{3.7}$$

$$A_b^h R_{hij}^a = 0, C_b^h R_{jih}^a = 0, C_b^h R_{hji}^a = -4\psi R_{bji}^a \tag{3.8}$$

$$\Phi_l R_{kji}^h = 0, R_{bcj}^a \partial_i \psi = 0 \tag{3.9}$$

$$B^h \nabla_h R_{bji}^a = R_{bji}^h \nabla_h B^a + R_{jbh}^a \nabla_i B^h, A_c^h \nabla_h R_{bji}^a = -R_{bjh}^a \nabla_i A_c^h \tag{3.10}$$

$$R_{bji}^h \nabla_h D^a = 0 \tag{3.11}$$

$$B^h \nabla_h R^a_{jib} = R^a_{hib} \nabla_j B^h + R^a_{hjb} \nabla_i B^h \tag{3.12}$$

$$A^h_c \nabla_j R^a_{ihb} = -\frac{1}{2} A^h_c \nabla_h R^a_{jib} - \frac{1}{2} R^a_{jhb} \nabla_i A^h_c + \frac{1}{2} R^a_{hib} \nabla_j A^h_c \tag{3.13}$$

where $\tilde{V} = (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^a E_a + \tilde{V}^{\bar{a}} E_{\bar{a}}$ *and* $\tilde{\Omega} = (\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = \tilde{\Omega}_a dx^a + \tilde{\Omega}_{\bar{a}} \delta y^a$.

Proof. Here we prove only the necessary condition because it is easy to prove the sufficient condition. Let $\tilde{V}$ be an infinitesimal paraholomorphically projective transformation with the associated 1-form $\tilde{\Omega}$ on $T(M_n)$.

$$(L_{\tilde{V}} \tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X} + \tilde{\Omega}(\varphi\tilde{X})\varphi\tilde{Y} + \tilde{\Omega}(\varphi\tilde{Y})\varphi\tilde{X} \tag{3.14}$$

for any $\tilde{X},\ \tilde{Y} \in \Im^1_0(T(M_n))$.

From $(L^D_{\tilde{V}} \nabla)(E_{\bar{j}}, E_{\bar{i}}) = 2\tilde{\Omega}_{\bar{j}} E_{\bar{i}} + 2\tilde{\Omega}_{\bar{i}} E_{\bar{j}}$, we obtain

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^h = 0 \tag{3.15}$$

and

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{V}^{\bar{h}} = 2\tilde{\Omega}_{\bar{j}} \delta^h_i + 2\tilde{\Omega}_{\bar{i}} \delta^h_j. \tag{3.16}$$

From (3.15), there exist $A = (A^h_i) \in \Im^1_1(M_n)$ and $B = (B^h) \in \Im^1_0(M_n)$ satisfying

$$\tilde{V}^h = B^h + y^a A^h_a. \tag{3.17}$$

From (3.16), there exist $\psi \in \Im^0_0(M_n)$, $\Phi = (\Phi_i) \in \Im^0_1(M_n)$, $D = (D^h) \in \Im^1_0(M_n)$ and $C = (C^h_i) \in \Im^1_1(M_n)$ satisfying

$$\tilde{\varphi} = \psi + y^a \Phi_a, \tag{3.18}$$

$$\tilde{\Omega}_{\bar{i}} = \partial_{\bar{i}} \tilde{\varphi} = \Phi_i \tag{3.19}$$

and

$$\tilde{V}^{\bar{h}} = D^h + y^a C^h_a + 4\psi y^h + 2y^h y^a \Phi_a, \tag{3.20}$$

where $\tilde{\varphi} = \frac{1}{2(n+1)} \partial_{\bar{a}} \tilde{V}^{\bar{a}}$.

Next, from (3.14) we have

$$(L_{\tilde{V}}^{D}\nabla)(E_{\bar{j}},E_i)=0 \tag{3.21}$$

or

$$(L_{\tilde{V}}^{D}\nabla)(E_j,E_{\bar{i}})=0$$

from which, we get

$$0=\Big\{(\nabla_i A_j^a+\frac{1}{2}D^h R_{hji}^a)+y^b(\frac{1}{2}B^h\nabla_h R_{bji}^a+\frac{1}{2}C_b^h R_{hji}^a+4\psi R_{bji}^a+\frac{1}{2}C_j^h R_{bhi}^a$$

$$-\frac{1}{2}R_{bji}^h\nabla_h B^a+\frac{1}{2}R_{bjh}^a\nabla_i B^h)+y^by^c(\frac{1}{2}R_{bjh}^a\nabla_i A_c^h+\frac{1}{2}A_c^h\nabla_h R_{bji}^a+2\Phi_c R_{bji}^a$$

$$+\Phi_j R_{bci}^a)\Big\}E_a+\big\{(\nabla_i C_j^a+4\partial_i\psi\delta_j^a+B^hR_{hij}^a)+y^b(2\delta_b^a\nabla_i\Phi_j+2\delta_j^a\nabla_i\Phi_b$$

$$+A_b^hR_{hij}^a+\frac{1}{2}A_j^hR_{hib}^a-\frac{1}{2}R_{bji}^h\nabla_h D^a)+y^by^cy^hR_{jbi}^a\nabla_h\Phi_c\big\}E_{\bar{a}}.$$

From the above equation, we obtain

$$\nabla_i A_j^a=-\frac{1}{2}D^hR_{hji}^a,\nabla_i C_j^a=-B^hR_{hij}^a-4\partial_i\psi\delta_j^a,$$

$$B^h\nabla_h R_{bji}^a=R_{bji}^h\nabla_h B^a+R_{jbh}^a\nabla_i B^h,A_c^h\nabla_h R_{bji}^a=-R_{bjh}^a\nabla_i A_c^h, \tag{3.22}$$

$$C_b^hR_{hji}^a=-4\psi R_{bji}^a,\Phi_l R_{kji}^h=0,\nabla_i\Phi_j=0,A_b^hR_{hij}^a=0,R_{bji}^h\nabla_h D^a=0.$$

Lastly, from $(L_{\tilde{V}}^{D}\nabla)(E_j,E_i)=2(\tilde{\Omega}_j\delta_i^a+\tilde{\Omega}_i\delta_j^a)E_a$, we obtain

$$2(\tilde{\Omega}_j\delta_i^a+\tilde{\Omega}_i\delta_j^a)E_a$$

$$=\big\{L_B\Gamma_{ji}^a+y^b(A_b^hR_{hji}^a+\nabla_j\nabla_i A_b^a+\frac{1}{2}A_h^aR_{jib}^h-\frac{1}{2}R_{bhj}^a\nabla_i D^h$$

$$-\frac{1}{2}R_{bhi}^a\nabla_j D^h)+y^by^c(2R_{bcj}^a\partial_i\psi+2R_{bci}^a\partial_j\psi)+y^by^cy^h(R_{bhj}^a\nabla_i\Phi_c+R_{bhi}^a\nabla_j\Phi_c)\big\}E_a$$

$$+\Big\{(\nabla_j\nabla_i D^a-\frac{1}{2}D^hR_{jih}^a)+y^b(-\frac{1}{2}B^h\nabla_h R_{jib}^a+\frac{1}{2}R_{hib}^a\nabla_j B^h+\frac{1}{2}R_{hjb}^a\nabla_i B^h-\frac{1}{2}C_b^hR_{jih}^a$$

$$+\frac{1}{2}C^a_h R^h_{jib} + 4\delta^a_b \nabla_j(\partial_i\psi) + \nabla_j\nabla_i C^a_b) + y^b y^c(\Phi_c R^a_{jib} - \Phi_b R^a_{jic} + \Phi_h \delta^a_c R^h_{jib}$$

$$+2\delta^a_b \nabla_j \nabla_i \Phi_c - \frac{1}{2}A^h_c \nabla_h R^a_{jib} + \frac{1}{2}R^a_{hib}\nabla_j A^h_c - \frac{1}{2}R^a_{jhb}\nabla_i A^h_c - A^h_c \nabla_j R^a_{ihb})\} E_{\bar{a}}$$

from which, we get the following important information:

$$L_B \Gamma^a_{ji} = \nabla_j \nabla_i B^a + R^a_{hji} B^h = 2\tilde{\Omega}_j \delta^a_i + 2\tilde{\Omega}_i \delta^a_j, \tag{3.23}$$

(That is, B is infinitesimal projective transformation on M_n)

$$\nabla_j \nabla_i A^a_b = -\frac{1}{2}A^a_h R^h_{jib}, \nabla_j \nabla_i C^a_b = -\frac{1}{2}C^a_h R^h_{jib},$$

$$\nabla_j \nabla_i D^a = \frac{1}{2}R^a_{jih} D^h, C^h_b R^a_{jih} = 0, R^a_{bcj}\partial_i \psi = 0, \tag{3.24}$$

$$\nabla_j(\partial_i \psi) = 0, B^h \nabla_h R^a_{jib} = R^a_{hib}\nabla_j B^h + R^a_{hjb}\nabla_i B^h,$$

$$A^h_c \nabla_j R^a_{ihb} = -\frac{1}{2}A^h_c \nabla_h R^a_{jib} - \frac{1}{2}R^a_{jhb}\nabla_i A^h_c + \frac{1}{2}R^a_{hib}\nabla_j A^h_c.$$

This completes the proof. □

Theorem 2. *Let (M_n, g) be a Riemannian manifold and $T(M_n)$ its tangent bundle with diagonal lift connection and adapted almost paracomplex structure. If $T(M_n)$ admits non-affine infinitesimal paraholomorphically projective transformation, then M_n and $T(M_n)$are locally flat.*

Proof. Let $\tilde{V}$ be non-affine infinitesimal paraholomorphically projective transformation on $T(M_n)$. Using (3.3) in Theorem 1, we have $\nabla_i \|\Phi\|^2 = \nabla_j \|\partial\psi\|^2 = 0$. Hence, $\|\Phi\|$ and $\|\partial\psi\|$ are constant on M_n. Suppose that M_n is not locally flat, then $\Phi = \partial\psi = 0$ by virtue of (3.9) in Theorem metricconverterProductID1. In1. In addition to this, using the equations (3.4) and (3.8) in Theorem 1, we have $B = 0$. that is, $\tilde{V}$ is an infinitesimal affine transformation. This is a contradiction. Therefore, M_n is locally flat. In this case, $T(M_n)$ is locally flat.[5], [6]. □

Acknowledgement. This is paper suppoted by The Scientific and Technological Research Council of Turkey.

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