The expressions (8) and (17) are not very different. If the generator of ICT is considered to be the Hamiltonian, $G \equiv H$, then by comparison we find

$$\begin{aligned} \delta\, q_i &= \epsilon\, \{q_i,\, H\} \,=\, \epsilon\, \dot{q}_i \\ \delta\, p_i &= \epsilon\, \{p_i,\, H\} \,=\, \epsilon\, \dot{p}_i \end{aligned} \tag{21}$$

So the new point is phase space $(\bar{q}, \bar{p})$ is the point to which (q, p) would move in an infinitesimal time interval $\epsilon = dt$. In order to move over to finite transformation, we can consider sum of an infinite succession of infinitesimal canonical transformations. Take equation (17) expressing the change of variable ω varying continuously in parameter ϵ,

$$\delta\, \omega \,=\, \epsilon\, \{\omega,\, G\} \quad \Rightarrow \quad \frac{d\omega}{d\epsilon} \,=\, \{\omega,\, G\} \tag{22}$$

starting from the initial state $\epsilon = 0$. We can get $\omega(\epsilon)$ by integrating the above differential equation. A solution may be obtained by expanding $\omega(\epsilon)$ in a Taylor series about the initial condition:

$$\omega(\epsilon) \,=\, \omega_0 + \epsilon\, \left.\frac{d\omega}{d\epsilon}\right|_0 + \frac{\epsilon^2}{2!}\, \left.\frac{d^2\omega}{d\epsilon^2}\right|_0 + \frac{\epsilon^3}{3!}\, \left.\frac{d^3\omega}{d\epsilon^3}\right|_0 + \cdots \tag{23}$$

According to equation (22),

$$\left.\frac{d\omega}{d\epsilon}\right|_0 \,=\, \{\omega,\, G\}_0\,,$$

where zero subscript implies Poisson bracket evaluated at initial point $\epsilon = 0$. By repeated application of (22) we can have,

$$\frac{d^2\omega}{d\epsilon^2} \,=\, \{\{\omega,\, G\}\,,\, G\}\,, \quad \frac{d^3\omega}{d\epsilon^3} \,=\, \{\{\{\omega,\, G\}\,,\, G\}\,,\, G\} \quad \text{etc.} \tag{24}$$

Therefore, the Taylor series for $\omega(\epsilon)$ leads to the series solution,

$$\omega(\epsilon) \,=\, \omega_0 + \epsilon\, \{\omega,\, G\}_0 + \frac{\epsilon^2}{2!}\, \{\{\omega,\, G\}\,,\, G\}_0 + \frac{\epsilon^3}{3!}\, \{\{\{\omega,\, G\}\,,\, G\}\,,\, G\}_0 + \cdots \tag{25}$$

The series expansion shows directly that ICT can generate finite canonical transformation, depending on a parameter, and thus lead to solution to the equation of motion if $G = H$. The nest of Poisson brackets in the n-th term can be considered as the nth repeated application of the operator $\hat{G} = \{\quad, G\}$ from the right *i.e.* the n-th power of the operator. Hence, the equation (25) can symbolically be written as,

$$\omega(\epsilon) \,=\, \omega_0\, \left. e^{\hat{G}\epsilon}\right|_0 . \tag{26}$$

In this notation, the solution for finite translation, rotation and time evolution can be written as,

$$\begin{aligned} x + \epsilon \,&=\, x\, e^{\hat{p}\,\epsilon}, \qquad && p \,=\, \text{linear momentum} \\ \vec{r}' \,&=\, \vec{r}\, e^{\hat{\vec{J}}\cdot\hat{n}\,\theta}, && \vec{J} \,=\, \text{angular momentum} \\ u(t) \,&=\, u(0)\, e^{\hat{H}t}, && H \,=\, \text{Hamiltonian} \end{aligned} \tag{27}$$

Much of the above discussion on symmetries in classical mechanics are carried over to quantum mechanics. To draw a parallel, in an informal manner, let us use the association of Poisson bracket of classical mechanics with commutator bracket of quantum mechanics,

$$\{\cdots\} \,\longrightarrow\, \frac{1}{i\hbar}\, [\cdots] \tag{28}$$

and the correspondence of canonical transformation with the unitary transformation in terms of the generator of the transformation,

$$G \longrightarrow \frac{i}{\hbar}\,\hat{G}, \qquad \hat{G} \,=\, \text{q.m. hermitian operator.} \tag{29}$$

Then the equation of motion (8) for any arbitrary variable (observables in quantum mechanis) translates to,

$$\dot{\omega} \,=\, \{\omega,\, H\} \quad\longrightarrow\quad \dot{\omega} \,=\, \frac{1}{i\hbar}\,[\hat{\omega},\, H] \tag{30}$$

which is essentially the Heisenberg equation of motion. The generators of translation, rotation and time-evolution are the same linear and angular momentum (operators) and Hamiltonian operator,

$$\begin{aligned} x+\epsilon \,&=\, x\,e^{\hat{p}\,\epsilon} && \longrightarrow && |\mathbf{x}+d\mathbf{x}\rangle \,=\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}\,|\mathbf{x}\rangle \\ \vec{r}' \,&=\, \vec{r}\,e^{\hat{\vec{J}}\cdot\hat{n}\,\theta} && \longrightarrow && |\alpha\rangle_R \,=\, e^{-i\mathbf{J}\cdot\hat{\mathbf{n}}\theta/\hbar}\,|\alpha\rangle \\ u(t) \,&=\, u(0)\,e^{\hat{H}t} && \longrightarrow && |\psi(t)\rangle \,=\, e^{-iHt/\hbar}\,|\psi(0)\rangle \end{aligned} \tag{31}$$

And finally, the statement on generator of ICT being constant of motion corresponds to constant of motion are generators of unitary transformation that preserves Hamiltonian,

$$U \,=\, e^{-iG\epsilon/\hbar} \quad\rightarrow\quad U^\dagger\,H\,U \,=\, H \quad\Rightarrow\quad [G,\,H] \,=\, 0 \quad\Rightarrow\quad \frac{dG}{dt} \,=\, 0. \tag{32}$$

Translational and rotational symmetries in quantum mechanics

Here we address the symmetries in quantum mechanics somewhat more formally. Let us consider the translation first. Suppose a state is localized around x and a *translation operator*, $\mathcal{T}(\epsilon)$ changes this state into another which is localized around $x+\epsilon$, where ϵ is infinitesimally small displacement,

$$\mathcal{T}(\epsilon)\,|x\rangle \,=\, |x+\epsilon\rangle. \tag{33}$$

The kets $|x\rangle$ are postulated to form a complete set. Once the action of $\mathcal{T}(\epsilon)$ on such a complete basis is known, the action on any arbitrary ket $|\psi\rangle$ can be known as,

$$\begin{aligned} |\psi_\epsilon\rangle \,&=\, \mathcal{T}(\epsilon)\,|\psi\rangle \,=\, \mathcal{T}(\epsilon)\int_{-\infty}^{+\infty} dx\,|x\rangle\langle x|\psi\rangle \\ &=\, \int_{-\infty}^{+\infty} dx\,|x+\epsilon\rangle\langle x|\psi\rangle \,=\, \int_{-\infty}^{+\infty} dx'\,|x'\rangle\langle x'-\epsilon|\psi\rangle \\ \langle x|\psi_\epsilon\rangle \,&=\, \langle x|\mathcal{T}(\epsilon)|\psi\rangle \,=\, \psi(x-\epsilon) \end{aligned} \tag{34}$$

The properties we like to have of $\mathcal{T}(\epsilon)$ are:

1. To preserve the norm of the states, both initial and translated, $\mathcal{T}$ should be unitary:

$$\langle x|x\rangle \,=\, \langle x+\epsilon|x+\epsilon\rangle \,=\, \langle x|\mathcal{T}^\dagger(\epsilon)\mathcal{T}(\epsilon)|x\rangle \quad\Rightarrow\quad \mathcal{T}^\dagger\mathcal{T} \,=\, 1. \tag{35}$$

2. Successive translations by, say, ϵ_1 followed by ϵ_2 should be the same as single translation given by the vector sum $\epsilon_1+\epsilon_2$,

$$\mathcal{T}(\epsilon_1)\,\mathcal{T}(\epsilon_2) \,=\, \mathcal{T}(\epsilon_1+\epsilon_2) \tag{36}$$

3. The translation in reverse direction $-\epsilon$ is expected to be same as the inverse of original translation,
$$\mathcal{T}(-\epsilon) \,=\, \mathcal{T}^{-1}(\epsilon) \tag{37}$$

4. As $\epsilon \to 0$, the translation operation reduces to the identity operation,
$$\lim_{\epsilon \to 0} \mathcal{T}(\epsilon) \,=\, \mathbb{I} \tag{38}$$

This last requirement (38) suggests that $\mathcal{T}$ may be expanded in Taylor series to order ϵ as,
$$\mathcal{T} \,=\, 1 - \frac{i\,\epsilon}{\hbar}\,G, \tag{39}$$
where G is known as *generator of translation* and is Hermitian. It is trivial to see how properties 3 and 4 are satisfied. The properties 1 and 2 are satisfied as well,
$$\begin{aligned} \mathcal{T}^\dagger\mathcal{T} \,&=\, (1 + i\epsilon\, G/\hbar)\,(1 - i\epsilon\, G/\hbar) \,=\, 1 - i\epsilon\,(G - G)/\hbar + \mathcal{O}(\epsilon^2) \,\simeq\, 1 \\ \mathcal{T}(\epsilon_1)\,\mathcal{T}(\epsilon_2) \,&=\, (1 - i\epsilon_1\, G/\hbar)\,(1 - i\epsilon_2\, G/\hbar) \,\simeq\, 1 - i\,(\epsilon_1 + \epsilon_2)\,G/\hbar \,=\, \mathcal{T}(\epsilon_1 + \epsilon_2). \end{aligned}$$

To figure out what the explicit form of the operator G is, we turn to the equation (34) and try Taylor expansion of $\psi(x - \epsilon)$,
$$\begin{aligned} \langle x|\mathcal{T}(\epsilon)|\psi\rangle \,=\, \psi(x - \epsilon) \quad &\to \quad \langle x|1|\psi\rangle - \frac{i\,\epsilon}{\hbar}\,\langle x|G|\psi\rangle \,=\, \psi(x) - \epsilon\,\frac{d\psi}{dx} \\ &\Rightarrow \quad \langle x|G|\psi\rangle \,=\, -i\hbar\,\frac{d\psi}{dx} \quad \Rightarrow \quad G \equiv \hat{p} \\ \mathcal{T}(\epsilon) \quad &= \quad 1 - \frac{i\,\epsilon}{\hbar}\,\hat{p}. \end{aligned} \tag{40}$$

What would be the operator $\mathcal{T}(a)$ corresponding to the finite translation a? If the whole of translation is divided into N parts of size $\epsilon = a/N$ each, then as $N \to \infty$ a/N becomes infinitesimal and $\mathcal{T}$ in (40) can be written as,
$$\mathcal{T}(a/N) \,=\, 1 - \frac{i}{\hbar}\,\frac{a}{N}\,\hat{p}.$$

Since a translation by a equals N translations (from property 2) by a/N,
$$\mathcal{T}(a) \,=\, \lim_{N\to\infty} \left[\mathcal{T}(a/N)\right]^N \,=\, \lim_{N\to\infty} \left(1 - \frac{i\,\epsilon}{\hbar}\,\hat{p}\right)^N \,=\, e^{-i\,a\,\hat{p}/\hbar} \tag{41}$$

Another way to arrive at this is from (25), and using the Poisson bracket and commutator correspondence (28) and subsequently BCH formula.

It is now obvious that if the Hamiltonian remains invariant under translation,
$$e^{-i\,a\,\hat{p}/\hbar}\,H\,e^{i\,a\,\hat{p}/\hbar} \,=\, H \qquad \Rightarrow \quad [\hat{p},\, H] \,=\, 0 \tag{42}$$
then linear momentum, being generator of translation, is constant of motion.