Contagion and Uninvadability in Local Interaction Games: The Bilingual Game and General Supermodular Games*

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Abstract

In a setting where an infinite population of players interact locally and repeatedly, we study the impacts of payoff structures and network structures on contagion of a convention beyond 2×2 coordination games. First, we consider the "bilingual game", where each player chooses one of two conventions or adopts both (i.e., chooses the "bilingual option") at an additional cost. For this game, we completely characterize when a convention spreads contagiously from a finite subset of players to the entire population in some network, and conversely, when a convention is never invaded by the other convention in any network. We show that the Pareto-dominant (risk-dominant, resp.) convention is contagious if the cost of bilingual option is low (high, resp.). Furthermore, if the cost is in a medium range, both conventions are each contagious in respective networks, and in particular, the Pareto-dominant convention is contagious only in some non-linear networks. Second, we consider general supermodular games, and compare networks in terms of their power of inducing contagion. We show that if there is a weight-preserving node identification from one network to another, then the latter is more contagion-inducing than the former in all supermodular games.

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1 Introduction

Behavior initiated by a small group of individuals, such as adoption of languages or technology standards, can spread in the long run over a large population through local interactions. Such a phenomenon is called contagion (also known as diffusion or epidemics) and has attracted much attention (Blume (1993, 1995), Ellison (1993, 2000), Morris (2000)).[1] In particular, focusing on the class of 2×2 coordination games, Morris (2000) analyzes how contagion is affected by the payoffs of the game played as well as the topology of the underlying network. This paper considers a larger class of supermodularity games, thereby enriching our understanding of the impacts of payoff structures and network structures on strategic behavior in local interaction games.

To be specific, consider an infinite population of players who are connected with each other through a network. Suppose that each player uses one of two computer programming languages, or two types of technologies in general, A and B. The payoffs from each interaction with his neighbors are given by the following 2×2 coordination game:

	A	B
A	a, a	b, c
B	c, b	d, d

where $a > c$ and $d > b$, so that (A, A) and (B, B) are strict Nash equilibria.[2] We assume that $a > d$, i.e., (A, A) Pareto-dominates (B, B), while $a - c < d - b$, i.e., (B, B) risk-dominates (A, A). We further assume that $d \geq c$, so that coordination on some action is always better than miscoordination. Morris (2000) shows that in 2×2 coordination games, the risk-dominant action B is always both contagious (i.e., in some network, there is some finite set of players such that if B is initially played by this set of players, then it is eventually played by the entire population) and uninvadable (i.e., in all networks, if B is initially played by almost all players, then it continues to be played by almost all players). Observe that B is a best response if at least a proportion $q = (a - c)/\{(a - c) + (d - b)\}$ of neighbors play B. Morris (2000) defines the contagion threshold of a given network to be the supremum of the payoff parameter q such that contagion occurs in that network. In particular, the linear network with nearest neighbor interactions, as depicted in Figure 1, has a contagion threshold $1/2$, which is the largest among all networks.

[1]See also Easley and Kleinberg (2010), Goyal (2007), Jackson (2008), Vega-Redondo (2007), Young (1998), among others.

[2]With only two actions, the model is a special case of the "threshold model" (Granovetter (1973)). For related studies in computer science, see, e.g., Easley and Kleinberg (2010) or Wortman (2008).

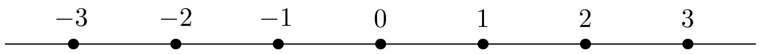

Figure 1: Nearest neighbor linear interaction

Now suppose that players can adopt a combination of the two actions, a "bilingual option" AB, with an additional cost $e > 0$. A player who plays AB receives a (gross) payoff a (d, resp.) from an interaction with an A-player (B-player, resp.). When two AB-players interact, they adopt the superior action A and receive a. This situation is described by

	A	AB	B
A	a, a	$a, a - e$	b, c
AB	$a - e, a$	$a - e, a - e$	$d - e, d$
B	c, b	$d, d - e$	d, d

where (A, A) and (B, B) are the only pure-strategy Nash equilibria.[3] One may expect that when the value of the cost parameter e is large, the action AB is not much relevant so that the situation is close to the previous 2×2 case where the risk-dominant action B survives; as e becomes smaller, AB becomes closer to dominating B so that eventually B will be abandoned and only the Pareto-dominant action A will survive.

For this class of 3×3 games, we completely characterize when an action is contagious and when it is uninvadable. Conforming to the conjecture in the previous paragraph, we show that if e is large, then the risk-dominant B is contagious and uninvadable; if e is small, then the Pareto-dominant A is contagious and uninvadable. In the latter case, the region of A-players, together with the "bilingual region" of AB-players, invades that of B-players in the following process: first, players at the boundary of the B-region interacting with AB-players switch to action AB as a "stepping stone"; then, the AB-players interacting with A-players switch to action A.[4] Combining the two cases, we show that generically at least one action is contagious. Moreover, in contrast to the 2×2 case, both actions are each contagious if e is in a medium range (which is nonempty and open under an additional

[3]This game has been studied by Galesloot and Goyal (1997), Goyal and Janssen (1997), Immorlica et al. (2007), and Easley and Kleinberg (2010).

[4]This process is reminiscent of, but different from, the selection of Pareto-efficient outcomes in evolutionary dynamics with pre-play communication (e.g., Matsui (1991)). In the latter, it incurs no cost to adopt a strategy switching between the Pareto-dominant and the Pareto-dominated actions contingent on the pre-play communication and the Pareto efficiency can prevail under global interactions, while in ours, adopting the bilingual option incurs a strictly positive cost and the Pareto efficiency can prevail only under local interactions.

condition on parameter values), i.e., A spreads contagiously in some network while B does in another. Indeed, for this range, our construction involves a "non-linear" network to induce the contagion of A.

The above construction motivates the question of how restrictions on the class of networks affect contagion. A class of networks is said to be *critical for contagion* if these networks induce all possible contagion, i.e., whenever an action can spread contagiously in some network, it does so within this class of networks. In the case of 2×2 coordination games, the network in Figure 1 constitutes a (singleton) critical class. For the bilingual game, however, we show that even if we extend our attention to all linear networks (networks whose interaction weights are translation invariant), only one action spreads contagiously for any generic parameter value. Combined with our complete characterization of contagion, this implies that the class of linear networks is not critical for contagion in the bilingual game.

The analysis described above is to fix a game and find a network (linear or non-linear) in which a given action is contagious, thereby highlighting the impact of payoff structures on contagion. We next consider a converse exercise to study the impact of network topologies on contagion: fix a network and find games in which a given action is contagious. Specifically, we ask the following question: for a pair of networks, which one has a larger set of games for which an action is contagious? We say that a network is *more contagion-inducing* in a class $\mathcal{G}$ of games than another network if for any game in $\mathcal{G}$, any action that is contagious in the latter network is also contagious in the former network. This notion defines a preorder over networks for general $\mathcal{G}$. The preorder is incomplete when $\mathcal{G}$ is our bilingual game with e being in a medium range; it is complete and represented by the contagion threshold of Morris (2000) when $\mathcal{G}$ is the class of 2×2 coordination games. We then introduce the concept of weight-preserving node identification between two networks, and prove that this concept provides a sufficient condition for a network to induce more contagion than another when $\mathcal{G}$ is the class of all supermodular games. For example, we can construct a weight-preserving node identification from a two-dimensional lattice network to the linear network in Figure 1, which implies that the latter is more contagion-inducing than the former in all supermodular games. We also show that our exercise based on 3×3 games provides a strictly finer analysis of network topologies than that by Morris (2000) based on 2×2 games; we find a pair of networks that are not differentiated in terms of contagion in 2×2 coordination games (i.e., have the same contagion threshold), but are strictly ordered in terms of their power of inducing contagion in the bilingual game.

In his series of papers, Morris (1997, 1999, 2000) defines general notions of contagion and uninvadability, and develops a method using potential functions to provide a sufficient condition on payoffs, independent of the underlying network structure, for uninvadability (and hence a necessary condition for contagion). He also gives an example of a symmetric 4×4 game to demonstrate that different actions are contagious in linear networks with different interaction weights. We generalize his potential method to strict monotone potential functions and characterize contagion and uninvadability for the bilingual game. In particular, we show that action A can be contagious in some non-linear network even if only action B is contagious in the class of linear networks.

Besides these papers by Morris, our paper is most closely related to Goyal and Janssen (1997), who analyze the local interaction game with the bilingual option in a circular network, and establish qualitatively the same results as ours: A is contagious if e is sufficiently small; B is contagious if e is sufficiently large. In contrast, we allow for all possible networks and identify the cutoff of e for contagion, which quantitatively differs from Goyal and Janssen's cutoff, i.e., for some payoff parameter values, some non-circular/non-linear networks induce an action to be contagious when the circular/linear networks do not. More fundamentally, we study how the network structure affects contagion, instead of fixing a particular network. In particular, we compare various networks in terms of their power of inducing contagion.[5]

This paper also makes a contribution to the literature on learning in games (e.g., Fudenberg and Levine (1998)). In general, long-run outcomes may depend on fine details of the underlying dynamics, such as simultaneity or sequentiality and order of action revisions. In our formulation, in contrast, contagion and uninvadability phenomena do not depend on such details. In particular, we prove (in Online Appendices B.1 and B.2) that in supermodular games, an action is contagious under sequential best responses if and only if it is contagious under simultaneous best responses.

Local interaction games and incomplete information games have formal connections, and both belong to a more general class of "interaction games" (Morris (1997, 1999), Morris and Shin (2005)). Accordingly, in Online Appendix B.7, we interpret our results on local interaction games in the context of incomplete information games, whereby we provide interesting implications on global games (Carlsson and van Damme (1993), Frankel et al. (2003)) and robustness to incomplete information (Kajii and Morris (1997)). We also discuss the common prior assumption translated in our context of local interactions (Oyama and Tercieux (2010, 2012)).[6]

The remainder of the paper is organized as follows. Section 2 formulates local interaction games. Section 3 provides a complete characterization of

[5] Immorlica et al. (2007) analyze contagion in the bilingual game for the case where there is no conflict between risk dominance and Pareto dominance in the original 2×2 game. See Online Appendix B.6.

[6] A class of dynamic games with Poisson action revisions due to Matsui and Matsuyama (1995) (perfect foresight dynamics) also belong to interaction games, where each revising player interacts with a set of past and future players and payoffs are given the discounted sum of flow payoffs from the interactions (Takahashi (2008)).

contagion and uninvadability for the bilingual game in the class of all unbounded networks, whereas Section 4 focuses on linear networks and establishes their non-criticality for contagion. Section 5 compares networks in terms of their power of inducing contagion in general supermodular games. Section 6 concludes with a discussion on incorporating randomness into the model.

2 Local Interaction Games

In this paper, we consider an infinite population of players connected through a network, where each player plays a given game with his neighbors.

2.1 Networks

Let X be a countably infinite set of players, and $P\colon X \times X \to \mathbb{R}_+$ a weighting function that satisfies

1. irreflexivity: $P(x,x) = 0$ for all $x \in X$,

2. symmetry: $P(x,y) = P(y,x)$ for all $x, y \in X$, and

3. bounded neighborhoods: $0 < \sum_{y \in X} P(x,y) < \infty$ for all $x \in X$.

A *network* (X, P) defines an undirected graph with vertices X and edges weighted by P. We will restrict our attention to *unbounded* networks, i.e., $\sum_{(x,y) \in X \times X} P(x,y) = \infty$. Write $\Gamma(x) = \{y \in X \mid P(x,y) > 0\}$ for the set of neighbors of player $x \in X$. Denote

$$P(y|x) = \frac{P(x,y)}{\sum_{y' \in \Gamma(x)} P(x,y')},$$

which is well defined due to property 3 above.

2.2 Pairwise Games

For each pairwise interaction between two neighbors, we consider a finite symmetric game (S, u), where S is the set of actions equipped with a total order, and $u\colon S \times S \to \mathbb{R}$ is the payoff function.[7] Let $\Delta(S)$ denote the set of probability distributions over S. Given payoff function u, we write $br(\pi)$ for the set of pure best responses to $\pi \in \Delta(S)$:

$$br(\pi) = \{h \in S \mid u(h,\pi) \geq u(h',\pi) \text{ for all } h' \in S\}, \tag{2.1}$$

where $u(h,\pi) = \sum_{k \in S} \pi_k u(h,k)$.

[7]For the sake of brevity, we will skip adjectives "finite" and "symmetric" when we refer to the pairwise game.

In Sections 3 and 4, we will focus on the bilingual game as described in the Introduction, where $S = \{A, AB, B\}$ and the payoff function $u\colon S \times S \to \mathbb{R}$ is given by

$$\begin{array}{c} \\ A \\ AB \\ B \end{array} \begin{array}{c} \begin{array}{ccc} A & AB & B \end{array} \\ \begin{pmatrix} a & a & b \\ a-e & a-e & d-e \\ c & d & d \end{pmatrix} \end{array}, \tag{2.2a}$$

where we assume

$$b < c \leq d < a, \;\; a - c < d - b, \;\; \text{and } e > 0. \tag{2.2b}$$

Note that action profiles (A, A) and (B, B) are the only pure-strategy Nash equilibria. By the assumption that $d < a$, (A, A) Pareto-dominates (B, B), while by $a - c < d - b$, (B, B) pairwise risk-dominates (A, A). By the additional assumption that $c \leq d$, this game is *supermodular* with respect to the order $A < AB < B$.

More generally, we say that the function $u\colon S \times S \to \mathbb{R}$ is supermodular if it satisfies $u(h', k) - u(h, k) \leq u(h', k') - u(h, k')$ for all $h, h', k, k' \in S$ with $h < h'$ and $k < k'$; the game (S, u) is supermodular if u is supermodular. We will exploit the property of supermodular games that the best response correspondence is nondecreasing in the stochastic dominance order. For $\pi, \pi' \in \Delta(S)$, we write $\pi \precsim \pi'$ (and $\pi' \succsim \pi$) if π' stochastically dominates π, i.e., if $\sum_{k \geq h} \pi_k \leq \sum_{k \geq h} \pi'_k$ for all $h \in S$. If u is supermodular, then it holds that $\max br(\pi) \leq \max br(\pi')$ and $\min br(\pi) \leq \min br(\pi')$ whenever $\pi \precsim \pi'$. In Section 5, we consider general supermodular games in comparing networks in terms of their power of inducing contagion.

2.3 Local Interaction Games

Given a network (X, P) and a pairwise game (S, u), we define the *local interaction game* (X, P, S, u), where at each point in time, each player is associated with an action and interacts with his neighbors by playing this action across all the interactions. An *action configuration* is a function $\sigma\colon X \to S$. For each action configuration σ, we denote by $\pi(\sigma|x) \in \Delta(S)$ the action distribution, weighted by $P(\cdot|x)$, over the actions of player x's neighbors: i.e.,

$$\pi_k(\sigma|x) = \sum_{y \in \Gamma(x)\colon \sigma(y) = k} P(y|x).$$

The payoff for player $x \in X$ playing action $h \in S$ is given by the weighted sum (with respect to $P(\cdot|x)$) of payoffs from the interactions with his neighbors:

$$U(h, \sigma|x) = \sum_{y \in \Gamma(x)} P(y|x) u(h, \sigma(y)),$$

which equals $u(h, \pi(\sigma|x))$. We write $BR(\sigma|x)$ for the set of pure best responses for player x to action configuration σ:

$$BR(\sigma|x) = \{h \in S \mid U(h, \sigma|x) \geq U(h', \sigma|x) \text{ for all } h' \in S\}, \tag{2.3}$$

which equals $br(\pi(\sigma|x))$.

2.4 Contagion and Uninvadability

For the adjustment process of players' actions, we consider the sequential best response dynamics as follows.

Definition 1. Given a local interaction game (X, P, S, u), a sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *best response sequence* if it satisfies the following properties: (i) for all $t \geq 1$, there is at most one $x \in X$ such that $\sigma^t(x) \neq \sigma^{t-1}(x)$; (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR(\sigma^{t-1}|x)$; and (iii) if $\lim_{t\to\infty} \sigma^t(x) = s$, then for all $T \geq 0$, $s \in BR(\sigma^t|x)$ for some $t \geq T$.

Property (i) requires that in each period at most one player revise his action,[8] while property (ii) requires that the revising player switch to a myopic best response to the current distribution of his neighbors' actions. Property (iii) requires that an action that is not a best response be abandoned eventually. Properties (i)–(iii) are satisfied with probability one, for example, in dynamics where in each period at most one player is randomly chosen (according to an i.i.d. full support distribution on X) to revise his action and the revising player switches to a myopic best response to the current distribution of his neighbors' actions. In particular, (ii) and (iii) imply that if there exists T such that $s \notin BR(\sigma^t|x)$ for all $t \geq T$, then there exists T' such that $\sigma^t(x) \neq s$ for all $t \geq T'$. Note that for a given initial action configuration, there are in general multiple best response sequences, as properties (i) and (iii) do not specify which player revises actions in which period.

We ask the following questions: is it possible in some unbounded network that if some finite group of players initially play action s^*, then the whole population will eventually play s^*? In this case, s^* is said to be contagious. Conversely, is it always the case in any unbounded network that if s^* is played by almost all players, it continues to be played by almost all players? If so, s^* is said to be uninvadable. Following Morris (1997, 1999), we define contagion and uninvadability as follows.

Definition 2. Given a pairwise game (S, u), action s^* is *contagious in network* (X, P) if there exists a finite subset Y of X such that every best response sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma^0(x) = s^*$ for all $x \in Y$ satisfies $\lim_{t\to\infty} \sigma^t(x) = s^*$ for each $x \in X$; action s^* is *contagious* if it is contagious in some unbounded network.

[8]All the results in this paper hold even if we allow for simultaneous best responses, under an additional assumption that $\Gamma(x)$ is finite for every player x; see Online Appendices B.1 and B.2.

Note that contagion of s^* in (X, P) requires that once a finite set Y of players initially play s^*, all players eventually play s^* along *any* best response sequence.[9] Note also that s^* is defined to be contagious if we can find *some* such network (X, P) and initial finite set Y.[10,11]

A game may have multiple contagious actions, and we will show in Section 3.1 that this is indeed the case for some (nonempty and open) set of payoff parameter values in our bilingual game, where we construct two different networks in which different actions respectively spread contagiously.[12]

For uninvadability, the notion of "almost all" is formalized by "except for a set of players whose weight with respect to P is finite". For an action configuration σ and a subset of actions $S' \subset S$, we write

$$\sigma_P(S') = \frac{1}{2} \sum_{(x,y)\colon \sigma(x)\in S' \text{ or } \sigma(y)\in S'} P(x,y).$$

In particular, for an action $s^* \in S$, $\sigma_P(S \setminus \{s^*\})$ is the total weight of pairs of players in which at least one of the pair members is not playing s^*.

Definition 3. Given a pairwise game (S, u), action s^* is *uninvadable in network* (X, P) if there exists no best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(S \setminus \{s^*\}) < \infty$ and $\lim_{t\to\infty} \sigma_P^t(S \setminus \{s^*\}) = \infty$; action s^* is *uninvadable* if it is uninvadable in all unbounded networks.

In particular, if s^* is uninvadable in (X, P), then there exists no best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = s^*$ for all but finitely many $x \in X$ and $\lim_{t\to\infty} \sigma^t(x) \neq s^*$ for all $x \in X$. Therefore, if s^* is uninvadable in (X, P), then actions other than s^* are not contagious in (X, P); if s^* is contagious in (X, P), then actions other than s^* are not uninvadable in (X, P).

Note that we call an action uninvadable if it is uninvadable in *all* unbounded networks. This strong notion of uninvadability is relevant when the analyst has no information about the underlying network structure but nonetheless wishes to predict the long-run behavior of the action distribution in the society. If an action s^* is uninvadable, then, once s^* is played

[9]In (generic) supermodular games, this definition of contagion is equivalent to the one that requires only *some* best response sequence to converge and to the one that allows for *simultaneous* best responses; see Online Appendix B.1.

[10]We will extensively discuss in Sections 4 and 5 how the choice of a network affects contagion. As for the choice of an initial set, generally speaking, our definition of contagion is weak in that the initial set may be very special. However, in the main construction for 3×3 supermodular games given in the proof of Lemma 1(1) in Section 3, for example, contagion of action s^* occurs from any finite set of s^*-players as long as it contains six consecutive players as in the proof.

[11]Our weak notion of contagion is the "mirror image" of our strong notion of robustness of a prediction as captured by uninvadability to be introduced later.

[12]In principle, two different actions may spread contagiously from different initial groups in the same network, but we are not aware of any such case.

by almost all players, the analyst should be confident that s^* will continue to be played by almost all players, whatever the actual network structure is and wherever the players who play actions other than s^* are located. Recall that we define a contagious action as an action that is contagious in *some* unbounded network (Definition 2). Thus, the notions of contagion and uninvadability are exclusive not only for a given network but also in the class of all unbounded networks. That is, if s^* is contagious, then actions other than s^* are not uninvadable; if s^* is uninvadable, then actions other than s^* are not contagious.

Our Theorem 1 will characterize contagion and uninvadability for the bilingual game. To study how the underlying network structure affects contagion and uninvadability, in Section 4, we will examine whether an action that is contagious (invaded, resp.) in some unbounded network remains contagious (becomes uninvadable, resp.) in the class of linear networks.

3 Contagion and Uninvadability for the Bilingual Game

In this section, we give a complete characterization of contagion and uninvadability for the bilingual game. In particular, we show that the Pareto-dominant action A prevails if the bilingual cost e is small, while the pairwise risk-dominant action B survives if e is large. The thresholds will be constructed based on two parameters:

$$e^* = \frac{(a-d)(d-b)}{2(c-b)},$$
$$e^{**} = \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)}.$$

It can be verified that $e^* \lesseqqgtr e^{**}$ if $c-b \lesseqqgtr a-c$ under the assumption (2.2b). The following result characterizes contagious and uninvadable actions in the bilingual game, quantifying our argument in the Introduction.

Theorem 1. *Let* (S,u) *be the bilingual game given by* (2.2). (i) A *is contagious if* $e < \max\{e^*, e^{**}\}$ *and uninvadable if* $e < e^*$. (ii) B *is contagious if* $e > e^*$ *and uninvadable if* $e > \max\{e^*, e^{**}\}$.

Figure 2 summarizes Theorem 1. Note that for any value of e (except for $e = e^*$ when $c - b \geq a - c$), at least one action is contagious, and therefore uninvadability implies contagion.[13] Moreover, if $e^* < e < e^{**}$ (which is possible if $c - b < a - c$), then the two actions A and B are each contagious in respective networks, and therefore neither action is uninvadable.

[13]More generally, one can show, by appropriately translating the contagion argument of Frankel et al. (2003) into our local interactions context, that any generic supermodular game has at least one contagious action. See Online Appendix B.7.2.

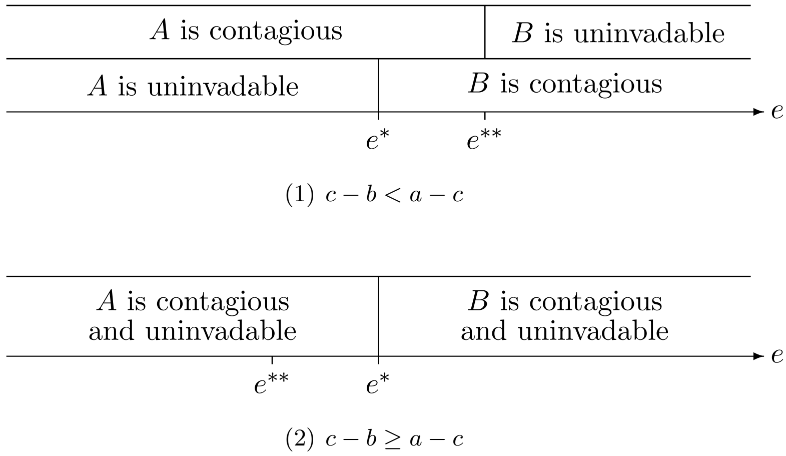

Figure 2: Contagion and uninvadability in the bilingual game

Note also that contagion and uninvadability are complementary: except for knife-edge cases, if A (B, resp.) is not contagious, then B (A, resp.) is uninvadable. This is not immediate from the definitions of contagion and uninvadability per se. Indeed, it implies that if $e > \max\{e^*, e^{**}\}$ ($e < e^*$, resp.), then starting from finitely many A-players (B-players, resp.), A (B, resp.) cannot spread not only to the *whole* population, but also to any *sub*-population with infinite sum of interaction weights.[14]

In Sections 3.1 and 3.2, we prove the contagion and the uninvadability parts of Theorem 1, respectively.

3.1 Contagion

We decompose the proof of the contagion part of Theorem 1 into two lemmas: Lemma 1 provides sufficient conditions for contagion in general 3×3 supermodular games; Lemma 2 then checks by direct computation when those conditions are satisfied in the bilingual game.

To better understand how contagion occurs for the bilingual game, recall the network in Figure 1, with a population of players indexed by integers $x \in X = \mathbb{Z}$, where player x interacts with players $x \pm 1$ with equal weights. Suppose that at time $t = 0$, all players play B except for players -1, 0, and 1 who play A, and assume that the bilingual cost e is small so that $e < (a-d)/2$ (where $(a-d)/2 \leq e^*$). We demonstrate that A spreads contagiously. (For concreteness, we here consider a particular best response sequence, while

[14]Therefore, even if we followed some of the recent papers (e.g., Acemoglu et al. (2011), Young (2011), and Kreindler and Young (2012)) to define contagion by requiring an action to spread to a substantial fraction of players, we would obtain the same characterization as in Theorem 1.

one can verify that contagion occurs for all best response sequences as the definition requires.) Note that since A is pairwise risk-dominated by B, no player is willing to switch from B to A. Suppose that player 2 adjusts his action at $t = 1$. With his two neighbors playing A and B, respectively, he abandons B and switches to AB since $e < (a - d)/2 \leq (a - c)/2$. Suppose next that player 3 revises his action at $t = 2$. Since he has one AB-neighbor and one B-neighbor, by $e < (a - d)/2$ he abandons B and switches to AB. Now let player 2 revise back again at $t = 3$. This time his neighbors are playing A and AB (instead of B), and hence he now switches to A. In this way, the region of A-players spreads, together with the "bilingual" region of AB-players between the A- and the B-regions; see Table 1.

	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$t = 0$	$\cdots$	B	A	A	A	B	B	B	$\cdots$
$t = 1$	$\cdots$	B	A	A	A	AB	B	B	$\cdots$
$t = 2$	$\cdots$	B	A	A	A	AB	AB	B	$\cdots$
$t = 3$	$\cdots$	B	A	A	A	A	AB	B	$\cdots$

Table 1: Contagion of action A

The above construction is extended to obtain contagion of A for $e < e^*$ (and symmetrically that of B for $e > e^*$) in Lemma 1(1), where we construct a "linear" network with four neighbors with appropriately chosen weights (Figure 3). To obtain contagion of A for $e^* \leq e < e^{**}$, however, we need to construct a "non-linear" network in Lemma 1(2), where different players may have different types of interacting neighborhoods (Figure 4).[15,16]

Consider a general 3×3 supermodular game with $S = \{0, 1, 2\}$. For $p \in (0, 1/2)$ and $q, r \in (0, 1)$ with $r \leq q$, let

$$\pi^a = \left(\tfrac{1}{2}, p, \tfrac{1}{2} - p\right), \quad \pi^b = \left(\tfrac{1}{2} - p, p, \tfrac{1}{2}\right),$$
$$\pi^c = \left(\tfrac{1+q}{2}, 0, \tfrac{1-q}{2}\right), \quad \pi^d = \left(\tfrac{1-r}{2}, 0, \tfrac{1+r}{2}\right), \quad \pi^e = \left(0, \tfrac{q+r}{2q}, \tfrac{q-r}{2q}\right),$$
$$\rho^c = \left(\tfrac{1-q}{2}, 0, \tfrac{1+q}{2}\right), \quad \rho^d = \left(\tfrac{1+r}{2}, 0, \tfrac{1-r}{2}\right), \quad \rho^e = \left(\tfrac{q-r}{2q}, \tfrac{q+r}{2q}, 0\right).$$

The conditions for contagion of actions 0 and 2 are stated in terms of best responses to the above mixed actions.

Lemma 1. *Let (S, u) be any 3×3 supermodular game with $S = \{0, 1, 2\}$.*

(1) (i) *If for some $p \in (0, 1/2)$,*

$$\max br(\pi^a) = 0 \text{ and } \max br(\pi^b) \leq 1, \tag{3.1}$$

[15] In Section 4, we prove the necessity of "non-linear" networks.

[16] Oyama and Takahashi (2011) employ an analogous construction of an incomplete information perturbation in the context of informational robustness. See Online Appendix B.7.1.

then 0 *is contagious.* (ii) *If for some* $p \in (0, 1/2)$,

$$\min br(\pi^a) \geq 1 \text{ and } \min br(\pi^b) = 2, \tag{3.2}$$

then 2 *is contagious.*

(2) (i) *If for some* $q, r \in (0, 1)$ *with* $r \leq q$,

$$\max br(\pi^c) = 0, \ \max br(\pi^d) \leq 1, \ \text{and} \ \max br(\pi^e) = 0, \tag{3.3}$$

then 0 *is contagious.* (ii) *If for some* $q, r \in (0, 1)$ *with* $r \leq q$,

$$\min br(\rho^c) = 2, \ \min br(\rho^d) \geq 1, \ \text{and} \ \min br(\rho^e) = 2, \tag{3.4}$$

then 2 *is contagious.*

Proof. See Appendix A.1. ▮

Here we only show how to construct a network and an initial group of players that trigger contagion in each case, while completing the proofs is relegated to the Appendix.

(1) Since cases (i) and (ii) are symmetric, we only consider case (i). Let $p \in (0, 1/2)$ satisfy (3.1). An unbounded network (X, P) and a finite set of players $Y \subset X$ for which action 0 spreads contagiously are constructed as follows.

Let $X = \mathbb{Z}$, and P be defined by

$$P(x, y) = \begin{cases} p & \text{if } |x - y| = 1 \\ \frac{1}{2} - p & \text{if } |x - y| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The network (X, P) is depicted in Figure 3.[17]

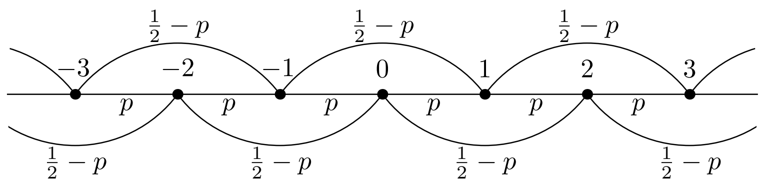

Figure 3: Linear interaction

Let $Y = \{-3, \ldots, 2\}$. Then for any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$, we have $\lim_{t \to \infty} \sigma^t(x) = 0$ for all $x \in X$, as shown in Appendix A.1.

[17] One can instead employ a constant-weight linear network with sufficiently many neighbors, i.e., a network (X, P) such that $X = \mathbb{Z}$ and $P(x, y) = 1$ if $1 \leq |x - y| \leq n$, where n is sufficiently large, and $P(x, y) = 0$ otherwise.

(2) Since cases (i) and (ii) are symmetric, we only consider case (i). Let $q, r \in (0,1)$, $r \leq q$, satisfy (3.3). An unbounded network (X, P) and a finite set of players $Y \subset X$ for which action 0 spreads contagiously are constructed as follows.

Let $X = \{\alpha, \beta\} \times \mathbb{Z}$, and P be defined by

$$P((\alpha,i),(\alpha,j)) = \begin{cases} 1-q & \text{if } |i-j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$P((\alpha,i),(\beta,j)) = P((\beta,j),(\alpha,i)) = \begin{cases} q+r & \text{if } i = j \\ q-r & \text{if } i = j+1 \text{ and } j \geq 0 \\ q-r & \text{if } i = j-1 \text{ and } j \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$P((\beta,i),(\beta,j)) = 0 \text{ for all } i, j.$$

The network (X, P) is depicted in Figure 4.

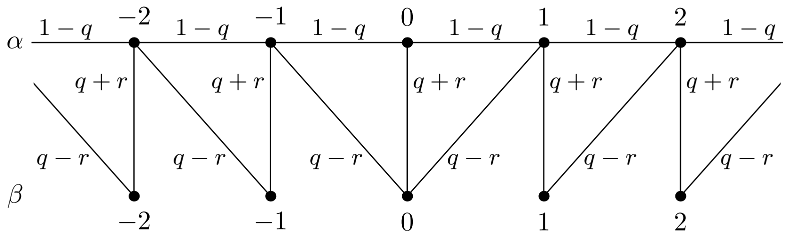

Figure 4: Non-linear interaction

Let $Y = \{(\alpha,i) \mid i = -1,0,1\} \cup \{(\beta,i) \mid i = -1,0,1\}$. Then for any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$, we have $\lim_{t\to\infty} \sigma^t(\alpha,i) = \lim_{t\to\infty} \sigma^t(\beta,i) = 0$ for all $i \in \mathbb{Z}$, as shown in Appendix A.1.

The following result characterizes when the hypotheses in Lemma 1 are satisfied in the bilingual game with $0 = A$, $1 = AB$, and $2 = B$. Denote

$$e^\sharp = \frac{(d-b)\{2(a-c)-(d-b)\}}{2(a-c)}.$$

Verify that $e^{**} \lesseqqgtr e^\sharp$ if $c - b \lesseqqgtr a - c$.

Lemma 2. *Let (S, u) be the bilingual game given by (2.2).*

(1) (i) *Condition (3.1) holds for some $p \in (0, 1/2)$ if $e < e^*$.* (ii) *Condition (3.2) holds for some $p \in (0, 1/2)$ if $e > e^*$.*

(2) *Condition (3.3) holds for some $0 < r \leq q < 1$ if $e < \min\{e^{**}, e^\sharp\}$.*

Proof. See Appendix A.2. ▌

Proof of the Contagion Part of Theorem 1. (i) Suppose that $e < \max\{e^*, e^{**}\}$. If $\max\{e^*, e^{**}\} = e^*$, then condition (3.1) holds for some $p \in (0, 1/2)$ by Lemma 2(1-i), and hence A is contagious by Lemma 1(1-i). If $\max\{e^*, e^{**}\} = e^{**}$, in which case $c - b < a - c$ and thus $\min\{e^{**}, e^\sharp\} = e^{**}$, then condition (3.3) holds for some $0 < r \leq q < 1$ by Lemma 2(2-i), and hence A is contagious by Lemma 1(2-i). In both cases, A is contagious.

(ii) Suppose that $e > e^*$. Then condition (3.2) holds for some $p \in (0, 1/2)$ by Lemma 2(1-ii), and hence B is contagious by Lemma 1(1-ii). ∎

3.2 Uninvadability

We first provide a sufficient condition for uninvadability by using the concept of strict monotone potential maximizer (strict MP-maximizer) due to Morris and Ui (2005) and Oyama et al. (2008). We then apply this concept to the bilingual game and characterize which action in the bilingual game is a strict MP-maximizer.

To avoid notational burden, we define strict MP-maximizer only for the smallest and the largest actions Consider a general finite action set S equipped with a total order. We denote $\underline{s} = \min S$ and $\overline{s} = \max S$. For a function $f \colon S \times S \to \mathbb{R}$ and a probability distribution $\pi \in \Delta(S)$, write

$$br_f(\pi) = \{h \in S \mid f(h, \pi) \geq f(h', \pi) \text{ for all } h' \in S\}.$$

(Thus the best response correspondence br for the game (S, u) as defined in (2.1) is now denoted br_u.) A function f is *symmetric* if $f(h, k) = f(k, h)$ for all $h, k \in S$. A game (S, u) is a *potential game* if there exists a symmetric function v such that $u(h, k) - u(h', k) = v(h, k) - v(h', k)$ for all $h, h', k \in S$; v is called a potential function, and s^* is called a *potential maximizer* of (S, u) if (s^*, s^*) uniquely maximizes v (Monderer and Shapley (1996)). Clearly, if (S, u) has a potential function v, then we have the *equality* $br_u(\pi) = br_v(\pi)$ for all $\pi \in \Delta(S)$. The following definition replaces this equality by an *inequality*.

Definition 4. (i) The smallest action $\underline{s}$ is a *strict MP-maximizer of game* (S, u) if there exists a symmetric function $v \colon S \times S \to \mathbb{R}$ such that

$$\max br_u(\pi) \leq \max br_v(\pi) \tag{3.5}$$

for all $\pi \in \Delta(S)$, and $v(\underline{s}, \underline{s}) > v(h, k)$ for all $(h, k) \neq (\underline{s}, \underline{s})$. Such a function v is called a *strict MP-function for* $\underline{s}$.

(ii) The largest action $\overline{s}$ is a *strict MP-maximizer of game* (S, u) if there exists a symmetric function $v \colon S \times S \to \mathbb{R}$ such that

$$\min br_u(\pi) \geq \min br_v(\pi) \tag{3.6}$$

for all $\pi \in \Delta(S)$, and $v(\overline{s}, \overline{s}) > v(h, k)$ for all $(h, k) \neq (\overline{s}, \overline{s})$. Such a function v is called a *strict MP-function for* $\overline{s}$.

If (S,u) has a potential function v, then the following function increases along any best response sequence $(\sigma^t)_{t=0}^{\infty}$ in any local interaction game (X,P,S,u):

$$V(t) = \frac{1}{2} \sum_{(x,y)\in X\times X} P(x,y)(v(\sigma^t(x),\sigma^t(y)) - v(s^*,s^*)).$$

Thus, if the potential maximizer s^* is initially played by almost all players (according to the interaction weights P) in the unbounded network (X,P) so that $V(0) > -\infty$, then it remains that $V(t) \geq V(0) > -\infty$, implying that s^* continues to be played by almost all players; that is, it is uninvadable (see Morris (1999, Proposition 6.1)). The following lemma shows a counterpart of this result for strict MP-maximizer; the proof is available in Online Appendix B.2.

Lemma 3. *Let (S,u) be any game with totally ordered action set S. If $s^* \in \{\underline{s},\overline{s}\}$ is a strict MP-maximizer of (S,u) with a strict MP-function v and if u or v is supermodular, then s^* is uninvadable.*

The next lemma establishes the existence of a strict MP-maximizer in the bilingual game, which, together with Lemma 3, implies the uninvadability part of Theorem 1.

Lemma 4. *Let (S,u) be the bilingual game given by* (2.2)*.* (i) *A is a strict MP-maximizer if $e < e^*$.* (ii) *B is a strict MP-maximizer if $e > \max\{e^*,e^{**}\}$.*

Proof. See Appendix A.3. ▮

In any 2×2 coordination game, the risk-dominant equilibrium is a potential maximizer. Beyond 2×2 coordination games, except for some special cases, no general method to find a strict MP-maximizer has been found. Morris (1999), Frankel et al. (2003), and Oyama and Takahashi (2009) show that a strict MP-maximizer exists generically in the class of symmetric 3×3 supermodular games with all three symmetric action profiles being Nash equilibria, but their proofs are by ad hoc construction of strict MP-functions.[18] Similarly, our proof of Lemma 4 is by ad hoc construction of strict MP-functions involving tedious computations that rely on the special payoff structure of our bilingual game. We do not know how to extend this result to other classes of games. For example, in the class of all symmetric 3×3 supermodular games with $S=\{0,1,2\}$ where only $(0,0)$ and $(2,2)$ are pure Nash equilibria, we do not know whether a strict MP-function exists generically for action 2 when Conditions (3.1) and (3.3) in Lemma 1 fail to hold.

[18] See also Honda (2011, Footnote 9).

4 Linear Networks and Their Non-Criticality

In the previous section, we considered all possible unbounded networks. In particular, in Lemma 1, we allowed ourselves to choose any unbounded network to induce contagion; in Lemma 3, a strict MP-maximizer was shown to be uninvadable in all unbounded networks. In this section, we restrict our attention to the class of linear networks and derive conditions for contagion and uninvadability in this subclass.[19]

In particular, we are interested in whether the class of linear networks is critical for contagion or uninvadability. For a given game (S, u), action s^* is *contagious* (*uninvadable*, resp.) *in a class* $\mathcal{C}$ *of networks* if it is contagious in some network in $\mathcal{C}$ (uninvadable in all networks in $\mathcal{C}$, resp.). We say that a class $\mathcal{C}$ of networks is *critical for contagion* if any action s^* that is contagious in some unbounded network is also contagious in $\mathcal{C}$. In that case, one can restrict attention to $\mathcal{C}$ to characterize contagion. Conversely, if $\mathcal{C}$ is not critical for contagion, then some action is contagious only in an unbounded network outside $\mathcal{C}$. For example, for any 2×2 coordination game, the risk-dominant equilibrium is contagious in the network in Figure 1, and hence this network forms a (singleton) critical class for contagion. On the other hand, if u is the bilingual game, it follows from our analysis in the previous section that the network in Figure 1 is not critical for some parameter values, while the union of two classes of networks given by Figures 3 and 4 is critical.

We say that a network (X, P) is *linear* if $X = \mathbb{Z}$ and interaction weights P are invariant up to translation: $P(x, y) = P(x+z, y+z)$ for any $x, y, z \in \mathbb{Z}$. (Note that any linear network is unbounded.) Clearly, both networks in Figures 1 and 3 are linear, whereas the network in Figure 4 is not.[20] Therefore, it follows from the proof of Lemma 1 that action A (B, resp.) is contagious in the class of linear networks if $e < e^*$ ($e > e^*$, resp.). The following theorem shows that these conditions are also sufficient for uninvadability in linear networks.

Theorem 2. *Let* (S, u) *be the bilingual game given by* (2.2). (i) A *is contagious and uninvadable in the class of linear networks if* $e < e^*$. (ii) B *is contagious and uninvadable in the class of linear networks if* $e > e^*$.

Proof. See Appendix A.4. ▮

Theorem 2 extends to "well-behaved" multidimensional lattice networks.[21] See Online Appendix B.3 for a formal statement and its proof. Goyal and

[19] See Online Appendix B.3 for the analysis of the larger class of multidimensional lattice networks.

[20] Even if we map $X = \{\alpha, \beta\} \times \mathbb{Z}$ to $\mathbb{Z}$ by relabeling (α, i) with $2i$ and (β, i) with $2i+1$, interaction weights do not satisfy translation invariance.

[21] In the context of incomplete information games, Oury (2013) analogously extends the noise-independent global game selection from one-dimensional to multidimensional signals. See Online Appendix B.7.2.

Janssen (1997, Theorem 2) obtain the same characterization for contagion in a circular network with translation invariant interactions in a setting with a continuum of players.

The characterization given in Theorem 2 differs from the one for the universal domain given in Theorem 1 when $c - b < a - c$, implying that, for the range of parameter values (e^*, e^{**}), the class of linear networks is *not* critical for contagion.

The non-criticality of linear networks is a novel phenomenon beyond 2×2 coordination games, and worth scrutinizing. Note that for the bilingual game, contagion in all linear networks occurs in more or less the same way: a contagious action spreads from middle to both sides as in Table 1. Differences in interaction weights may affect the size of the "bilingual" region of AB-players, but do not affect which action to be contagious.[22] In contrast, in the proof of Lemma 1(2), the non-linear network depicted in Figure 4 allows players (α, i) and (β, i) to face different distributions of neighbors' actions, and induces a fundamentally different form of contagion. It happens that for a 2×2 coordination game, the risk-dominant action is a potential maximizer, and hence it is not only contagious in linear networks, but also uninvadable in all unbounded networks. This is a peculiarity of 2×2 games. For a more general game, the fact that an action is contagious in linear networks does not guarantee that the action is an MP-maximizer, as the bilingual game with $e^* < e < e^{**}$ provides a concrete counterexample.

5 Comparison of Networks

The analysis in the previous two sections was to fix a game and find a network (linear or non-linear) in which a given action is contagious, thereby differentiating, whenever possible, among strict Nash equilibria. In this section, we consider a converse exercise: fix a network and identify the class of games in which a given action is contagious. More precisely, we ask the following question: For a pair of networks, which one has a larger set (with respect to set inclusion) of payoff parameter values for which a given action is contagious? Such comparison is formalized by the following preorder over the set of networks.

Definition 5. A network $(\hat{X}, \hat{P})$ is *more contagion-inducing than another network* (X, P) *in a class* $\mathcal{G}$ *of games* if for any game $(S, u) \in \mathcal{G}$, an action s^* is contagious in $(\hat{X}, \hat{P})$ whenever s^* is contagious in (X, P).[23]

[22] In the context of global games, Frankel et al. (2003) and Basteck and Daniëls (2011) prove analogous results for symmetric 3×3 supermodular games. See Online Appendix B.7.2.

[23] Recall that s^* is contagious in a network if there exists *some* finite set of players in that network from which s^* spreads contagiously. Thus, our preorder does not compare the "richness" of the sets of initial players who can trigger contagion in $(\hat{X}, \hat{P})$ and those

Morris' (2000) approach is in fact in the spirit of this converse exercise for the class of all 2×2 coordination games. He defines the *contagion threshold* of a network (X, P) to be the supremum of $q \in (0, 1)$ for which action 1 of the two-action game

$$\begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix} \end{array}$$

is contagious in (X, P). The contagion threshold naturally represents the preorder in Definition 5 with $\mathcal{G}$ being the class of 2×2 coordination games. That is, a network is more contagion-inducing than another network in the class of 2×2 coordination games if and only if the contagion threshold of the former is larger than that of the latter. In particular, the preorder in this case is complete.

We study this preorder with $\mathcal{G}$ being the class of all supermodular games. Note that our analysis so far implies that this preorder is incomplete.

Example 1 (Figure 3 versus Figure 4)**.** Consider the bilingual game with $e^* < e < e^{**}$. Recall the two networks in the proof of Lemma 1, where we choose p, q, and r that satisfy (3.2) and (3.3). Then action B is contagious in the network of Figure 3, whereas action A is contagious in the network of Figure 4. Thus neither network is more contagion-inducing than the other in the bilingual game.

In principle, comparison of networks based on games with more actions can provide at least a weakly finer analysis of network topologies. The next example shows that the bilingual game provides a strictly finer analysis than 2×2 coordination games.

Example 2 (Tree versus ladder)**.** Consider the "tree" depicted in Figure 5, where each player is indexed by a finite sequence of 0 and 1, $X = \bigcup_{n \geq 0} \{0, 1\}^n$. Player $x \in X$ interacts with $x0$, $x1$, and x^- with equal weights, where x^- is the immediate predecessor of x, i.e., the truncation of x that removes the last digit of x. Also consider the "ladder" depicted in Figure 6, where each player is indexed by a pair of α or β and an integer, $X = \{\alpha, \beta\} \times \mathbb{Z}$. With equal weights, player (α, i) interacts with $(\alpha, i \pm 1)$ and (β, i), and player (β, i) interacts with (α, i) and $(\beta, i \pm 1)$.

For these networks, we have the following:

- In 2×2 coordination games, the two networks are equally contagion-inducing. The contagion threshold of each network is $1/3$ (Examples 4 and 5 in Morris (2000) with $m = 2$).
- In the bilingual game, the ladder is more contagion-inducing than the tree. This will be proved in Example 3 by invoking Theorem 3 below.

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in (X, P).

- Moreover, in the bilingual game, the ladder is *strictly* more contagion-inducing than the tree.

To establish the third point, consider a set of parameter values of the bilingual game such that $br(2/3, 1/3, 0) = \{A\}$, $br(2/3, 0, 1/3) = \{AB\}$, and $br(1/3, 1/3, 1/3) = \{B\}$, that is, $2a - c < 2d - b$ and $(2a - c - d)/3 < e < \min\{(d - b)/3, 2(a - c)/3\}$, which are satisfied, for example, by $(a, b, c, d) = (11, 0, 3, 10)$ and $3 < e < 10/3$. For such parameter values, we claim that action B is contagious in the ladder, but not in the tree. First, in the ladder, suppose that initially players $(\alpha, -1)$, $(\alpha, 0)$, $(\beta, -1)$, and $(\beta, 0)$ play B, while all the others play A. Then players $(\alpha, 1)$ and $(\beta, 1)$ will switch from A to AB, and to B. In this way, all players subsequently switch from A to AB, and to B; thus B is contagious in the ladder. Second, in the tree, for any finite set Y of initial B-players, pick a maximal (longest) element x of Y and assume that all successors of x play A. Then players $x0$ and $x1$ switch from A to AB, but all the other successors of x continue to play A for any best response sequence; thus B is not contagious in the tree.

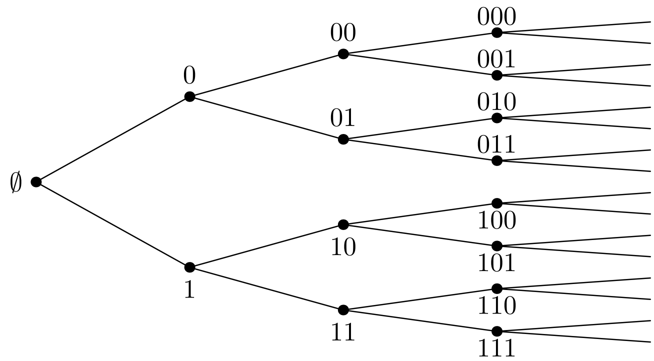

Figure 5: Tree

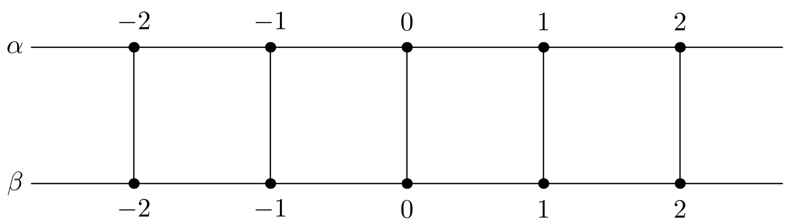

Figure 6: Ladder

In Example 2, note that the ladder is obtained by "bundling" (or "identifying") nodes of the tree in the following way: map $\emptyset$ in the tree to $(\alpha, 0)$

in the ladder, 0 to $(\alpha, 1)$, 1 to $(\beta, 0)$, 00 to $(\alpha, 2)$, 11 to $(\beta, -1)$, and "bundle" 01 and 10 and map them to $(\beta, 1)$, and so on. The existence of such a mapping is the key to proving the second point in Example 2. In fact, it implies that any best response sequence in the ladder can be replicated by a best response sequence in the tree, and thus if an action spreads along the latter sequence, so does it along the former. This in turn implies that whenever an action is contagious in the tree, so is it in the ladder.

The following notion generalizes the idea of "bundling" described above.

Definition 6. For two networks (X, P) and $(\hat{X}, \hat{P})$, a mapping $\varphi\colon X \to \hat{X}$ is a *weight-preserving node identification* from (X, P) to $(\hat{X}, \hat{P})$ if φ is onto and there exists a finite subset E of X such that for any $x \in X \setminus E$ and any $\hat{y} \in \hat{X}$,

$$\hat{P}(\varphi(x), \hat{y}) = \sum_{y \in \varphi^{-1}(\hat{y})} P(x, y).$$

A node in E is called an *exceptional node*.[24]

The next theorem shows that the existence of a weight-preserving node identification between two networks is sufficient for one network to be more contagion-inducing than the other in the class of supermodular games, including, but not limited to, all 2×2 coordination games and the bilingual game.

Theorem 3. *If there exists a weight-preserving node identification from (X, P) to $(\hat{X}, \hat{P})$, then $(\hat{X}, \hat{P})$ is more contagion-inducing than (X, P) in the class of all supermodular games.*

The proof proceeds roughly as follows. Suppose that s^* is contagious in (X, P), and φ is a weight-preserving node identification from (X, P) to $(\hat{X}, \hat{P})$. Take any best response sequence $(\hat{\sigma}^t)$ on $(\hat{X}, \hat{P})$. We construct a sequence (σ^t) on (X, P) by $\sigma^t = \hat{\sigma}^t \circ \varphi$ for any $t \geq 0$, which, by the definition of weight-preserving node identification, is almost (but not quite, as we explain below) a best response sequence. Since s^* is contagious in (X, P), $(\sigma^t(x))$ converges to s^* for any $x \in X$, and hence $(\hat{\sigma}^t(\hat{x}))$ also converges to s^* for any $\hat{x} \in \hat{X}$. This argument, however, has two issues: first, along the sequence (σ^t), players in each $\varphi^{-1}(\hat{x})$ change actions simultaneously, which violates property (i) in Definition 1; second, players at exceptional nodes may not play best responses. We resolve these issues in Online Appendix B.4.

Note that node identification is different from adding or subtracting edges. Even if we construct $(\hat{X}, \hat{P})$ by adding edges to (X, P), $(\hat{X}, \hat{P})$ may

[24] Allowing for exceptional nodes is essential for constructing a weight-preserving node identification from the tree to the ladder (see Example 3), but the reader may want to assume $E = \emptyset$ upon the first reading.

be more contagion-inducing than, less contagion-inducing than, or incomparable to (X, P).[25]

In the next two examples, we will construct weight-preserving node identifications and illustrate the implications of Theorem 3.

Example 3 (Tree versus ladder, continued)**.** There exists a weight-preserving node identification from the tree to the ladder. In fact, one can construct such a mapping recursively as follows: given that each node $x \in \bigcup_{k=0}^{n}\{0,1\}^k$ of the tree with depth at most n is assigned with a node $\varphi(x)$ of the ladder, for each $x \in \{0,1\}^n$, we assign $x0$ and $x1$ with two of the neighbors of $\varphi(x)$ in the ladder other than $\varphi(x^-)$. We can always do so since each node on the ladder has three neighbors. For example, let

$$\begin{aligned}&\varphi(\emptyset) = (\alpha, 0),\\ &\varphi(0) = (\alpha, 1),\ \varphi(1) = (\beta, 0),\\ &\varphi(00) = (\alpha, 2),\ \varphi(01) = \varphi(10) = (\beta, 1),\ \varphi(11) = (\beta, -1),\ \dots .\end{aligned}$$

Then φ preserves interaction weights except at the root $\emptyset$ (φ does not preserve interaction weights at the root $\emptyset$ because player $\emptyset$ has two neighbors in the tree while player $\varphi(\emptyset)$ has three neighbors in the ladder). Thus, by Theorem 3, the ladder is more contagion-inducing than the tree in the class of all supermodular games.

Example 4 (Line versus lattice)**.** Consider the line depicted in Figure 1, and the two-dimensional lattice depicted in Figure 7 where each player $x = (x_1, x_2) \in \mathbb{Z}^2$ interacts with $(x_1 \pm 1, x_2)$ and $(x_1, x_2 \pm 1)$ with equal weights. Then the mapping $\varphi(x_1, x_2) = x_1 + x_2$ is a weight-preserving node identification from the two-dimensional lattice to the line with no exceptional node. Thus, by Theorem 3, the line is more contagion-inducing (in fact, strictly more contagion-inducing) than the two-dimensional lattice in the class of all supermodular games.

Morris (2000) shows that the line is more contagion-inducing than the two-dimensional lattice in the class of 2×2 coordination games by computing their contagion thresholds explicitly. Our Theorem 3 gives an alternative proof to this result, which generalizes to other pairs of networks and to all supermodular games.

The next example shows that the converse of Theorem 3 does not hold. That is, the existence of a weight-preserving node identification is not necessary for two networks to be comparable.

[25] In contrast, Galeotti et al. (2010), for example, obtain the monotonicity of equilibrium actions with respect to adding or subtracting edges in their framework where the action space is a subset of the real line and the payoffs depend on the summation or maximum of neighbors' actions. See also Wolitzky (2013), who shows that the level of cooperation in a repeated game with a monitoring network is monotonic with respect to his notion of network centrality, where a network is more central than another network if the former is obtained by adding edges to the latter.

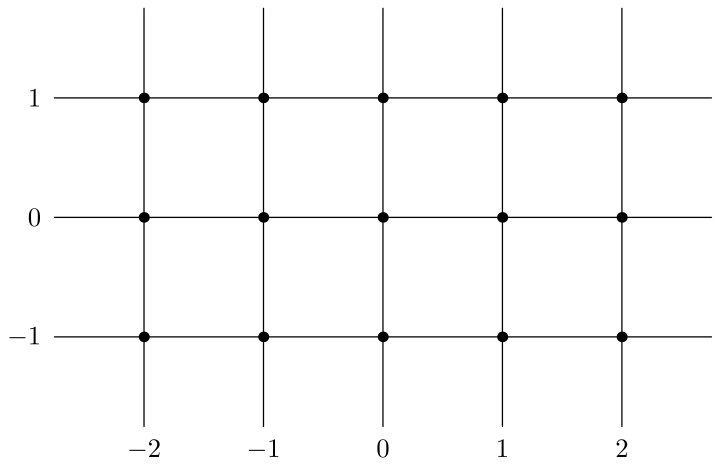

Figure 7: Two-dimensional lattice

Example 5 (Line versus line)**.** Consider two linear networks, one depicted in Figure 1 and the other in Figure 3 with any fixed $p \in (0, 1/2)$. Then one can verify that Figure 3 is (strictly) more contagion-inducing than Figure 1 in the class of all supermodular games. However, no node identification from Figure 1 to Figure 3 preserves interaction weights because any weight-preserving node identification is allowed to increase the number of neighbors only for exceptional nodes.

For other relevant examples, see Online Appendix B.5.

6 Conclusion

We have considered contagion and uninvadability beyond 2×2 coordination games, especially for the bilingual game. By incorporating a third action, our study refines Morris' (2000) approach that analyzes strategic behavior for two-action games. In principle, an analysis with more actions will reveal, at least weakly, finer structures of underlying networks than that with two actions, whereas conceivably the analysis would become intractable with too many actions. We have demonstrated that our three-action bilingual game is simple enough to obtain a complete characterization of contagion and uninvadability (Theorem 1), and yet rich enough to offer a strictly finer analysis than that based on two-action coordination games (Example 2). We have also developed the concept of weight-preserving node identification to provide a sufficient condition for comparing two networks in terms of their power of inducing contagion (Theorem 3). This concept may be of independent interest, and other contexts to which it applies are yet to be explored in future research.

We close with brief discussions on the robustness of our results to various forms of randomness (see also Morris (2000, Section 7)). First, introducing randomness in action revision opportunities will not change our results as we have already allowed for all best response sequences (Definition 1; see also Footnote 9 and Online Appendix B.1).

Second, randomness in the initial action configuration may foster more contagion. In fact, Lee and Valentinyi (2000) show that for 2×2 coordination games, the contagion threshold of the two-dimensional lattice is $1/2$ if actions in the initial configuration are given i.i.d. across agents, whereas the contagion threshold in Morris' (2000) sense is $1/4$. An extension to games with more than two actions, including the bilingual game, is left open.

Third, persistent randomness in best responses has been considered by Blume (1993), Ellison (1993), Montanari and Saberi (2010), Young (2011), and Kreindler and Young (2012), among others. For the bilingual game, stochastic stability in linear networks is characterized by $e \lessgtr e^*$, which is distinct from the characterization of stochastic stability under global interactions obtained by Galesloot and Goyal (1997). It remains open to characterize stochastic stability in non-linear networks.

Fourth, it would be interesting to incorporate heterogeneity in players' preferences. As in Young (2009) and Acemoglu et al. (2011) for two-action games, one may allow for player-specific bilingual cost e_x for each player x in the bilingual game. Theorem 1 obviously extends if the corresponding conditions are satisfied uniformly for all players; e.g., A is contagious if $\sup_{x \in X} e_x < \max\{e^*, e^{**}\}$. When e_x distributes below and above the cutoffs, coexistence of conventions may arise depending on the correlation between preferences and locations of the players. Following Manski (1993), one may also address the identification problem of local interactions and preference correlation in our model.

Fifth, we have applied our general results to various networks with some "recurrent" structures, but have been silent about more complex and less structured networks. As a tractable first step, it would be worthwhile to consider random networks, which generate such networks with high probability (see, e.g., Pastor-Satorras and Vespignani (2001), Watts (2002), López-Pintado (2006, 2008), Berninghaus and Haller (2010), Lelarge (2012)).

Appendix

A.1 Proof of Lemma 1

(1) Let (X, P) be as given in the main text. By (3.1) and the supermodularity of u, we have the following properties.

(a) If $\sigma(x-2) = \sigma(x-1) = 0$ and $\sigma(x+1) \leq 1$ (or symmetrically if $\sigma(x-1) \leq 1$ and $\sigma(x+1) = \sigma(x+2) = 0$), then $\max BR(\sigma|x) = 0$.

(b) If $\sigma(x-2)=0$ and $\sigma(x-1) \leq 1$ (or symmetrically if $\sigma(x+1) \leq 1$ and $\sigma(x+2)=0$), then $\max BR(\sigma|x) \leq 1$.

Let $Y = \{-3, \ldots, 2\} \subset X$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. We want to show that

$$\lim_{t \to \infty} \sigma^t(x) = 0 \qquad (\diamondsuit_x)$$

holds for all $x \in X$. We only consider players $x \geq 0$; the analogous argument applies to $x < 0$.

We show $(\diamondsuit_0)$ and $(\diamondsuit_1)$, or more strongly, that

$$\begin{aligned} &\sigma^t(x) = 0 \text{ for } x = -2, \ldots, 1 \\ &\sigma^t(x) \leq 1 \text{ for } x = -3, 2 \end{aligned}$$

for all $t \geq 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t-1$, then we have $\sigma^t(x) = 0$ for $x = -2, \ldots, 1$ and $\sigma^t(x) \leq 1$ for $x = -3, 2$ by properties (a) and (b), respectively.

Assume $(\diamondsuit_{x-2})$ and $(\diamondsuit_{x-1})$. Then, there exists T_0 such that $\sigma^t(x-2) = \sigma^t(x-1) = 0$ for all $t \geq T_0$. By property (b), this implies that there exists T_1 such that $\sigma^t(x) \leq 1$ for all $t \geq T_1$. By property (b) applied for $x+1$ in place of x, this implies that there exists T_2 such that $\sigma^t(x+1) \leq 1$ for all $t \geq T_2$. By property (a), this implies that there exists T_3 such that $\sigma^t(x) = 0$ for all $t \geq T_3$, meaning that $(\diamondsuit_x)$ holds.

(2) Let (X, P) be as given in the main text. By (3.3) and the supermodularity of u, we have the following properties.

(c) For $i \geq 1$, if $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = \sigma(\beta, i) = 0$ (or symmetrically for $i \leq -1$, if $\sigma(\alpha, i+1) = \sigma(\beta, i+1) = \sigma(\beta, i) = 0$), then $\max BR(\sigma|(\alpha, i)) = 0$.

(d) For $i \geq 1$, if $\sigma(\alpha, i-1) = \sigma(\beta, i-1) = 0$ (or symmetrically for $i \leq -1$, if $\sigma(\alpha, i+1) = \sigma(\beta, i+1) = 0$), then $\max BR(\sigma|(\alpha, i)) \leq 1$.

(e) If $\sigma(\alpha, i) \leq 1$, then $\max BR(\sigma|(\beta, i)) = 0$.

Let $Y = \{(\alpha, i) \mid i = -1, 0, 1\} \cup \{(\beta, i) \mid i = -1, 0, 1\} \subset X$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. We want to show that

$$\lim_{t \to \infty} \sigma^t(\alpha, i) = \lim_{t \to \infty} \sigma^t(\beta, i) = 0 \qquad (\heartsuit_i)$$

holds for all $i \in \mathbb{Z}$. We only consider $i \geq 0$; the analogous argument applies to $i < 0$.

We show $(\heartsuit_1)$, or more strongly, that

$$\sigma^t(\alpha, i) = \sigma^t(\beta, i) = 0 \text{ for } i = -1, 0, 1$$

for all $t \geq 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t-1$, then we have $\sigma^t(\alpha, i) = \sigma^t(\beta, i) = 0$, $i = -1, 0, 1$, by properties (c) and (e), respectively.

Assume $(\heartsuit_{i-1})$. Then, there exists T_0 such that $\sigma^t(\alpha, i-1) = \sigma^t(\beta, i-1) = 0$ for all $t \geq T_0$, By property (d), this implies that there exists T_1 such that $\sigma^t(\alpha, i) \leq 1$ for all $t \geq T_1$. By property (e), this implies that there exists T_2 such that $\sigma^t(\beta, i) = 0$ for all $t \geq T_2$. By property (c), this implies that there exists T_3 such that $\sigma^t(\alpha, i) = 0$ for all $t \geq T_3$. We thus obtain $(\heartsuit_i)$.

A.2 Proof of Lemma 2

(1) Note that for all $p \in (0, 1/2)$, we have $u(B, \pi^b) > u(A, \pi^b)$ and hence $A \notin br(\pi^b)$.

(α) If $e > (a-c)/2$, then condition (3.2) holds for some $p \in (0, 1/2)$ close to 0.

(β) If $e \leq (a-c)/2$, then for all $p \in (0, 1/2)$, $u(AB, \pi^a) > u(B, \pi^a)$ and hence $B \notin br(\pi^a)$. Therefore, $\max br(\pi^a) = A \Leftrightarrow u(A, \pi^a) > u(AB, \pi^a)$ and $\max br(\pi^b) \leq AB \Leftrightarrow u(AB, \pi^b) > u(B, \pi^b)$, while $\min br(\pi^a) \geq AB \Leftrightarrow u(AB, \pi^a) > u(A, \pi^a)$ and $\min br(\pi^b) = B \Leftrightarrow u(B, \pi^b) > u(AB, \pi^b)$.

Verify that $e^* \leq (a-c)/2$ with the equality holding if and only if $c = d$. If $e^* < (a-c)/2$ ($\Leftrightarrow c < d$), then, since

$$u(A, \pi^a) - u(AB, \pi^a) = (d-b)\left\{p - \frac{(d-b) - 2e}{2(d-b)}\right\},$$
$$u(AB, \pi^b) - u(B, \pi^b) = (d-c)\left\{\frac{(a-c) - 2e}{2(d-c)} - p\right\},$$

it follows that condition (3.1) holds for some $p \in (0, 1/2)$ if and only if

$$\frac{(d-b) - 2e}{2(d-b)} < \frac{(a-c) - 2e}{2(d-c)} \iff e < e^*,$$

while condition (3.2) holds for some $p \in (0, 1/2)$ if and only if

$$\frac{(a-c) - 2e}{2(d-c)} < \frac{(d-b) - 2e}{2(d-b)} \iff e > e^*.$$

If $e^* = (a-c)/2$ ($\Leftrightarrow c = d$), then $u(A, \pi^a) > u(AB, \pi^a)$ and $u(AB, \pi^b) > u(B, \pi^b)$ for some $p \in (0, 1/2)$ close to 1/2 whenever $e < e^*$. (Note that we never have $e > e^*$ in the current case of $e \leq (a-c)/2$ $(= e^*)$.)

(2) Note that $u(B, \pi^d) > u(A, \pi^d)$ and hence $A \notin br(\pi^d)$ for all $r \in (0, 1)$. Therefore,

$$\max br(\pi^d) \leq AB \iff u(AB, \pi^d) > u(B, \pi^d)$$

$$\iff r < \frac{(a-c)-2e}{a-c}. \tag{A.1}$$

For the last inequality to hold, it is necessary that $e < (a-c)/2$.

Under the condition that $e < (a-c)/2$, note that $u(AB, \pi^c) > u(B, \pi^c)$ and hence $B \notin br(\pi^c)$ for all $q \in (0,1)$. Therefore,

$$\begin{aligned} \max br(\pi^c) = A &\iff u(A, \pi^c) > u(AB, \pi^c) \\ &\iff q > \frac{(d-b)-2e}{d-b}. \end{aligned} \tag{A.2}$$

Finally,

$$\begin{aligned} \max br(\pi^e) = A &\iff u(A, \pi^e) > \max\{u(AB, \pi^e), u(B, \pi^e)\} \\ &\iff r > \max\left\{\frac{(d-b)-2e}{d-b}q, \frac{(d-b)-(a-d)}{a-b}q\right\}. \end{aligned} \tag{A.3}$$

From (A.1)–(A.3), it follows that condition (3.3) holds for some $0 < r \leq q < 1$ if and only if

$$\frac{(a-c)-2e}{a-c} > \frac{(d-b)-2e}{d-b} \max\left\{\frac{(d-b)-2e}{d-b}, \frac{(d-b)-(a-d)}{a-b}\right\},$$

which reduces to $e < \min\{e^\sharp, e^{**}\}$.

A.3 Proof of Lemma 4

For $f\colon S \times S \to \mathbb{R}$ and $h \in S$, let

$$\Pi_h(f) = \{\pi \in \Delta(S) \mid h \in br_f(\pi)\}.$$

Note that A is a strict MP-maximizer of (S, u) with MP-function v if and only if $\{(A, A)\} = \arg\max_{(h,k)\in S\times S} v(h, k)$, $\Pi_B(u) \subset \Pi_B(v)$, and $\Pi_{AB}(u) \subset \Pi_{AB}(v) \cup \Pi_B(v)$, while B is a strict MP-maximizer of (S, u) with MP-function v if and only if $\{(B, B)\} = \arg\max_{(h,k)\in S\times S} v(h, k)$, $\Pi_A(u) \subset \Pi_A(v)$, and $\Pi_{AB}(u) \subset \Pi_A(v) \cup \Pi_{AB}(v)$.

Denote

$$e^\flat = \frac{(a-d)(d-b)}{a-b}.$$

Verify that $e^\flat \lesseqqgtr e^* \lesseqqgtr e^{**}$ if $c - b \lesseqqgtr a - c$.

Lemma 4 is proved by Lemmas A.1–A.4 which follow. Lemma A.1 considers the case in which $c = d$ and $e < e^*$ $(= (a-c)/2)$; Lemma A.2 considers the cases of $e < e^*$ and $e^* < e \leq e^\flat$ under the assumption that $c \neq d$; and Lemmas A.3 and A.4 cover the cases of $\max\{e^{**}, e^\flat\} < e \leq (a-c)/2$ and $e > (a-c)/2$, respectively.

Lemma A.1. *Suppose that $c = d$. If $e < e^*$ $(= (a-c)/2 = (a-d)/2)$, then A is a strict MP-maximizer.*

Proof. Since $e < e^* = (a-c)/2 = (a-d)/2 < (d-b)/2$, we have

$$u(A,k) - u(AB,k) < u(AB,k) - u(B,k) \tag{A.4}$$

for all $k = A, AB, B$. Let v be defined by

$$\begin{array}{c} \\ A \\ AB \\ B \end{array} \begin{array}{c} \begin{array}{ccc} A & AB & B \end{array} \\ \begin{pmatrix} e & 0 & -\lambda e - (d-b) + e \\ 0 & -e & -\lambda e \\ -\lambda e - (d-b) + e & -\lambda e & 0 \end{pmatrix} \end{array},$$

where

$$\lambda = \frac{(d-b) - e}{(a-d) - 2e} > 0.$$

The function v is maximized at (A, A). Verify that

$$v(A,k) - v(AB,k) = u(A,k) - u(AB,k), \tag{A.5}$$

$$v(AB,k) - v(B,k) \leq \lambda\big(u(AB,k) - u(B,k)\big) \tag{A.6}$$

for all $k = A, AB, B$. Then, we have $\Pi_{AB}(u) \subset \Pi_{AB}(v) \cup \Pi_B(v)$ by (A.5), and $\Pi_B(u) \subset \Pi_B(v)$ by (A.4), (A.5), and (A.6). ▮

Lemma A.2. *Suppose that $c \neq d$.* (i) *If $e < e^*$, then A is a strict MP-maximizer.* (ii) *If $e^* < e \leq e^\flat$, then B is a strict MP-maximizer.*

Proof. Let v be defined by

$$\begin{array}{c} \\ A \\ AB \\ B \end{array} \begin{array}{c} \begin{array}{ccc} A & AB & B \end{array} \\ \begin{pmatrix} 2\lambda e & \lambda e & \lambda e - (a-c) + e \\ \lambda e & 0 & -(a-d) + e \\ \lambda e - (a-c) + e & -(a-d) + e & -(a-d) + 2e \end{pmatrix} \end{array},$$

where

$$\lambda = \frac{d-c}{d-b} > 0.$$

We show that the function v works as a strict MP-function if $e \leq \max\{e^*, e^\flat\}$ and $e \neq e^*$.

We have the following.

Claim 1. (i) $\{(A,A)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$ *if $e < e^*$.* (ii) $\{(B,B)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$ *if $e > e^*$.*

Verify that

$$v(A,k) - v(AB,k) = \lambda\big(u(A,k) - u(AB,k)\big), \tag{A.7}$$
$$v(AB,k) - v(B,k) = u(AB,k) - u(B,k) \tag{A.8}$$

for all $k = A, AB, B$. These immediately imply the following.

Claim 2. $\Pi_{AB}(u) = \Pi_{AB}(v)$.

For $\pi = (\pi_A, \pi_{AB}, \pi_B) \in \Delta(S)$, we have

$$u(A,\pi) - u(AB,\pi) = (d-b)\left(\frac{e}{d-b} - \pi_B\right), \tag{A.9}$$

$$\begin{aligned} u(AB,\pi) - u(B,\pi) = (d-c)&\left\{\pi_A - \frac{(a-b)e - (a-d)(d-b)}{(d-b)(d-c)}\right\} \\ &+ (a-d)\left(\frac{e}{d-b} - \pi_B\right), \end{aligned} \tag{A.10}$$

$$v(B,\pi) - v(A,\pi) = \big(u(B,\pi) - u(A,\pi)\big) + (c-b)\left(\frac{e}{d-b} - \pi_B\right). \tag{A.11}$$

These imply the following.

Claim 3. $\Pi_B(u) \subset \Pi_B(v)$.

Proof. Assume that $\pi = (\pi_A, \pi_{AB}, \pi_B) \in \Pi_B(u)$ ($\Leftrightarrow u(B,\pi) \geq u(A,\pi)$ and $u(B,\pi) \geq u(AB,\pi)$). First, by (A.8), $u(B,\pi) \geq u(AB,\pi)$ implies $v(B,\pi) \geq v(AB,\pi)$. Second, if $\pi_B \geq e/(d-b)$, then by (A.7) and (A.9), we have $v(AB,\pi) \geq v(A,\pi)$ and therefore $v(B,\pi) \geq v(A,\pi)$, while if $\pi_B < e/(d-b)$, then by (A.11), $u(B,\pi) \geq u(A,\pi)$ implies $v(B,\pi) > v(A,\pi)$. We thus have $\pi \in \Pi_B(v)$. ▮

Claim 4. *If $e \leq e^\flat$, then $br_u = br_v$.*

Proof. In light of Claim 2, we want to show that $\Pi_A(u) = \Pi_A(v)$ and $\Pi_B(u) = \Pi_B(v)$.

Note in (A.10) that $e \leq e^\flat$ implies $\{(a-b)e - (a-d)(d-b)\}/\{(d-b)(d-c)\} \leq 0$. By (A.9) and (A.10), we therefore have $u(A,\pi) \geq u(AB,\pi) \Rightarrow u(AB,\pi) \geq u(B,\pi)$ and $u(B,\pi) \geq u(AB,\pi) \Rightarrow u(AB,\pi) \geq u(A,\pi)$. By (A.7) and (A.8), it thus follows that $\pi \in \Pi_A(u) \Leftrightarrow u(A,\pi) \geq u(AB,\pi) \Leftrightarrow v(A,\pi) \geq v(AB,\pi) \Leftrightarrow \pi \in \Pi_A(v)$ and $\pi \in \Pi_B(u) \Leftrightarrow u(B,\pi) \geq u(AB,\pi) \Leftrightarrow v(B,\pi) \geq v(AB,\pi) \Leftrightarrow \pi \in \Pi_B(v)$. ▮

We now complete the proof of Lemma A.2. (i) If $e < e^*$, Claims 1, 2, and 3 imply that A is a strict MP-maximizer. (ii) If $e^* < e \leq e^\flat$, Claims 1 and 4 imply that B is a strict MP-maximizer. ▮

Lemma A.3. *If* $\max\{e^{**}, e^{\flat}\} < e \leq (a-c)/2$*, then* B *is a strict MP-maximizer.*

Proof. Let v be defined by

$$\begin{array}{c} \\ A \\ \\ AB \\ \\ B \end{array} \begin{array}{c} \begin{array}{ccc} A & AB & B \end{array} \\ \begin{pmatrix} 0 & -\lambda e & \begin{array}{l} -\lambda e \\ \quad -\{(a-c)-e\} \end{array} \\ -\lambda e & -2\lambda e & \begin{array}{l} \lambda\{(d-b)-2e\} \\ \quad -\{(a-c)-e\} \end{array} \\ \begin{array}{l} -\lambda e \\ \quad -\{(a-c)-e\} \end{array} & \begin{array}{l} \lambda\{(d-b)-2e\} \\ \quad -\{(a-c)-e\} \end{array} & \begin{array}{l} \lambda\{(d-b)-2e\} \\ \quad -\{(a-c)-2e\} \end{array} \end{pmatrix} \end{array},$$

where

$$\lambda = \frac{(a-c)(d-b)-(a-b)e}{(a-b)\{(d-b)-e\}} > 0.$$

The function v is maximized at (B, B). Verify that

$$v(A,k) - v(AB,k) = \lambda\big(u(A,k) - u(AB,k)\big), \tag{A.12}$$

$$v(A,k) - v(B,k) \geq \frac{a-c}{a-b}\big(u(A,k) - u(B,k)\big), \tag{A.13}$$

$$v(AB,k) - v(B,k) \geq u(AB,k) - u(B,k) \tag{A.14}$$

for all $k = A, AB, B$. Then, we have $\Pi_A(u) \subset \Pi_A(v)$ by (A.12) and (A.13), and $\Pi_{AB}(u) \subset \Pi_A(v) \cup \Pi_{AB}(v)$ by (A.14). ▮

Lemma A.4. *If* $e > (a-c)/2$*, then* B *is a strict MP-maximizer.*

Proof. (B, B) is a strictly p-dominant equilibrium with

$$p = \max\left\{\frac{a-c-e}{a-c}, \frac{a-c}{(a-c)+(d-b)}\right\} < \frac{1}{2}.$$

Thus, there exists a strict MP-function for B (see Morris and Ui (2005) and Oyama et al. (2008, Lemma 4.1)). ▮

A.4 Proof of Theorem 2

Since the network used in the proof of Lemma 1(1) is linear, combined with Lemma 2(1) it follows that if $e < e^*$ ($e > e^*$, resp.), then action A (B, resp.) is contagious in linear networks. Also, by Theorem 1(i), if $e < e^*$, then A is uninvadable, hence uninvadable in linear networks. Thus, what remains to be shown is that B is uninvadable in linear networks if $e > e^*$.

By Lemma 2(1-ii) and the upper semi-continuity of br, there exist $p \in (0, 1/2)$ and $\varepsilon \in (0, 1/2-p)$ such that $\min br(\tilde{\pi}^a) \geq AB$ and $\min br(\tilde{\pi}^b) = B$, where

$$\tilde{\pi}^a = \left(\tfrac{1}{2}+\varepsilon, p, \tfrac{1}{2}-p-\varepsilon\right), \quad \tilde{\pi}^b = \left(\tfrac{1}{2}-p+\varepsilon, p, \tfrac{1}{2}-\varepsilon\right).$$

Fix any linear network $(\mathbb{Z}, P)$. Since $P(0|0) = 0$ and $P(y|0) = P(-y|0)$ for all $y > 0$, we have $\sum_{y=1}^{\infty} P(y|0) = 1/2$. Let n_1 be the smallest natural number such that $\sum_{y=1}^{n_1} P(y|0) \geq p$, and n_2 be a sufficiently large natural number such that $\sum_{y>n_2} P(y|0) \leq \varepsilon$.

Consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(\{A, AB\}) < \infty$. Let K be the set of all $k \in \mathbb{Z}$ such that $\sigma^0(x) = B$ if $|x - k| \leq n_1 + n_2$. Then K is co-finite (i.e., $\mathbb{Z} \backslash K$ is finite), and so is $L = \bigcup_{k \in K} \{x \in \mathbb{Z} \mid |x-k| \leq n_2\}$. (Otherwise, $\sigma^0(x) \neq B$ for infinitely many x, which contradicts the finiteness of $\sigma_P^0(\{A, AB\})$.)

For each $k \in K$, we want to show that

$$\begin{aligned} &\sigma^t(x) = B \text{ if } |x - k| \leq n_2, \\ &\sigma^t(x) \geq AB \text{ if } n_2 + 1 \leq |x - k| \leq n_1 + n_2 \end{aligned}$$

for all $t \geq 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t - 1$, then for any player x such that $|x - k| \leq n_2$, we have

$$\begin{aligned} &\pi(\sigma^{t-1}|x)(B) \geq \sum_{y=1}^{n_2} P(y|0) \geq \frac{1}{2} - \varepsilon, \\ &\pi(\sigma^{t-1}|x)(AB) + \pi(\sigma^{t-1}|x)(B) \geq \sum_{y=1}^{n_1} P(y|0) + \sum_{y=1}^{n_2} P(y|0) \geq \frac{1}{2} + p - \varepsilon, \end{aligned}$$

which imply that $\pi(\sigma^{t-1}|x) \succsim \tilde{\pi}^b$ and hence $\sigma^t(x) = B$; for any player x such that $n_2 + 1 \leq |x - k| \leq n_1 + n_2$, we have

$$\begin{aligned} &\pi(\sigma^{t-1}|x)(B) \geq \sum_{y=1}^{n_2} P(y|0) - \sum_{y=1}^{n_1-1} P(y|0) > \frac{1}{2} - p - \varepsilon, \\ &\pi(\sigma^{t-1}|x)(AB) + \pi(\sigma^{t-1}|x)(B) \geq \sum_{y=1}^{n_1} P(y|0) \geq \frac{1}{2} - \varepsilon, \end{aligned}$$

which imply that $\pi(\sigma^{t-1}|x) \succsim \tilde{\pi}^a$ and hence $\sigma^t(x) \geq AB$. Therefore, $\{x \in \mathbb{Z} \mid \sigma^t(x) = B\} \supset L$, and hence $\sigma_P^t(\{A, AB\})$ is bounded from above.

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Online Appendix

B.1 Equivalent Definitions of Contagion in Supermodular Games

In this appendix, we discuss three other definitions of contagion, and show that all of them are equivalent to the original one for any (generic) symmetric supermodular game (S, u), where the smallest and the largest actions are denoted by $\underline{s}$ and $\overline{s}$, respectively. (None of the results here relies on the particular payoff structure of the bilingual game.) We use the partial order $\sigma \leq \sigma'$ whenever $\sigma(x) \leq \sigma'(x)$ for any $x \in X$.

Recall that in the main text, we consider the sequential best response dynamics, where at most one player revises his action in each period (property (i) in Definition 1). Instead, we can define the simultaneous (generalized, resp.) best response dynamics, where all (some, resp.) players revise their actions at a time.

Definition B.1. Given a local interaction game (X, P, S, u), a sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *simultaneous best response sequence* if $\sigma^t(x) \in BR(\sigma^{t-1}|x)$ for all $x \in X$ and $t \geq 1$. A sequence $(\sigma^t)_{t=0}^{\infty}$ is a *generalized best response sequence* if it satisfies the following properties: (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR(\sigma^{t-1}|x)$; and (iii) if $\lim_{t\to\infty} \sigma^t(x) = s$, then for all $T \geq 0$, $s \in BR(\sigma^t|x)$ for some $t \geq T$.

For clarity, we add adjective "sequential" to the original notion of best response sequences. Generalized best response sequences subsume both sequential and simultaneous best response sequences as special cases.

Using simultaneous or generalized best response sequences, we obtain two new definitions of contagion.[1]

Definition B.2. Given a pairwise game (S, u), action s^* is *contagious by simultaneous* (*generalized*, resp.) *best responses in network* (X, P) if there exists a finite subset Y of X such that every simultaneous (generalized, resp.) best response sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma^0(x) = s^*$ for all $x \in Y$ satisfies $\lim_{t\to\infty} \sigma^t(x) = s^*$ for each $x \in X$.

We refer to the notion of contagion in Definition 2 as "contagion by sequential best responses". By definition, contagion by generalized best responses implies both contagion by sequential best responses and by simultaneous best responses. Here we show the converse.

In the next lemma, we show that if s^* is contagious by sequential best responses, then there exist two sequential best response sequences that converge to s^* monotonically (one increasingly and the other decreasingly), and that any generalized best response sequence that starts between the two

[1] The notion of contagion used in Morris (2000) is similar to contagion by simultaneous best responses, but requires only that for each $x \in X$, $\sigma^t(x) = s^*$ for some $t \geq 0$.

sequences also converges to s^*. This lemma is used to prove both Proposition B.1 below and Theorem 3 in the main text.

Lemma B.1. *Fix a network (X, P) and a supermodular game (S, u). Suppose that s^* is contagious by sequential best responses in (X, P). Then there exist two sequential best response sequences $(\sigma_-^t)_{t=0}^\infty$ and $(\sigma_+^t)_{t=0}^\infty$ such that*

(1) *$\sigma_-^t(x) \leq s^* \leq \sigma_+^t(x)$ for all $x \in X$ and $t \geq 0$;*

(2) *$\sigma_-^0(x) = \underline{s}$ and $\sigma_+^0(x) = \overline{s}$ for all but finitely many $x \in X$;*

(3) *$\sigma_-^t(x) \in \{\sigma_-^{t-1}(x), \min BR(\sigma_-^{t-1}|x)\}$ and $\sigma_+^t(x) \in \{\sigma_+^{t-1}(x), \max BR(\sigma_+^{t-1}|x)\}$ for all $x \in X$ and $t \geq 1$;*

(4) *$\lim_{t\to\infty} \sigma_-^t(x) = \lim_{t\to\infty} \sigma_+^t(x) = s^*$ for all $x \in X$; and*

(5) *$\min BR(\sigma_-^0|x) \geq \sigma_-^0(x)$ and $\max BR(\sigma_+^0|x) \leq \sigma_+^0(x)$ for all $x \in X$.*

Moreover,

(6) *for any generalized best response sequence $(\tilde{\sigma}^t)_{t=0}^\infty$ with $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$, we have $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$ for all $x \in X$.*

Proof. Suppose that s^* is contagious by sequential best responses in (X, P) (and hence a strict Nash equilibrium of (S, u)). Let $Y \subset X$ be a finite set as in Definition 2, and let $(\phi_-^t)_{t=0}^\infty$ be the sequential best response sequence such that $\phi_-^0(x) = s^*$ for all $x \in Y$, $\phi_-^0(x) = \underline{s}$ for all $x \in X \setminus Y$, and $\phi_-^t(x) \in \{\phi_-^{t-1}(x), \min BR(\phi_-^{t-1}|x)\}$ for all $x \in X$ and $t \geq 1$. By definition, $\lim_{t\to\infty} \phi_-^t(x) = s^*$ for all $x \in X$.

The sequence $(\phi_-^t)_{t=0}^\infty$ satisfies properties (1)–(4), but not necessarily property (5). From $(\phi_-^t)_{t=0}^\infty$, we construct another sequence that satisfies property (5) as well. Let $\psi_-^0 = \phi_-^0$ and

$$\psi_-^t(x) = \begin{cases} \psi_-^{t-1}(x) & \text{if } \phi_-^t(x) \leq \psi_-^{t-1}(x), \\ \min BR(\psi_-^{t-1}|x) & \text{if } \phi_-^t(x) > \psi_-^{t-1}(x). \end{cases}$$

Clearly, $(\psi_-^t)_{t=0}^\infty$ is a sequential best response sequence. By the construction of $(\phi_-^t)_{t=0}^\infty$ and $(\psi_-^t)_{t=0}^\infty$ along with the supermodularity of u and s^* being a Nash equilibrium of (S, u), one can show by induction on t that $\phi_-^t(x) \leq \psi_-^t(x) \leq s^*$ for all $x \in X$ and $t \geq 0$. Thus for each $x \in X$, $(\psi_-^t(x))_{t=0}^\infty$ is weakly increasing and converges to s^*.

Since s^* is a strict Nash equilibrium of (S, u), we can take a finite but sufficiently large subset Z of $\bigcup_{x\in Y} \Gamma(x)$ such that for any $x \in Y$, the best response of player x is s^* if all players in Z play s^* (recall that $\Gamma(x)$ is the set of the neighbors of player x). Let T be sufficiently large so that $\psi_-^T(x) = s^*$ for all $x \in Z$.

We claim that $\min BR(\psi_-^T|x) \geq \psi_-^T(x)$ for all $x \in X$. For $x \in Y$, since all players in Z play s^* in period T, we have $\min BR(\psi_-^T|x) = s^* \geq \psi_-^T(x)$.

For $x \in X \setminus Y$, we first have $\min BR(\psi_-^0|x) \geq \underline{s} = \psi_-^0(x)$. Next, assume that $\min BR(\psi_-^{t-1}|x) \geq \psi_-^{t-1}(x)$. By the construction of $(\psi_-^t(x))_{t=0}^\infty$, $\psi_-^t(x)$ is equal to either $\psi_-^{t-1}(x)$ or $\min BR(\psi_-^{t-1}|x)$. In both cases, we have $\min BR(\psi_-^{t-1}|x) \geq \psi_-^t(x)$. Since $(\psi_-^t)_{t=0}^\infty$ is weakly increasing, we have $\min BR(\psi_-^t|x) \geq \min BR(\psi_-^{t-1}|x)$ by the supermodularity of u. Hence, $\min BR(\psi_-^t|x) \geq \psi_-^t(x)$.

Now let $\sigma_-^t = \psi_-^{t+T}$ for $t \geq 0$. Then $(\sigma_-^t)_{t=0}^\infty$ satisfies properties (1)–(5). In particular, along the sequential best response sequence $(\psi_-^t)_{t=0}^\infty$, at most T players change actions by period T, so that $\sigma_-^0(x) = \psi_-^T(x) = \underline{s}$ except for finitely many x. The construction of $(\sigma_+^t)_{t=0}^\infty$ is analogous.

For property (6), pick any generalized best response sequence $(\tilde{\sigma}^t)_{t=0}^\infty$ with $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$. For each $x \in X$, let $\underline{\tilde{\sigma}}^t(x) = \inf_{\tau \geq t} \tilde{\sigma}^\tau(x)$, and $\tilde{\sigma}_-(x) = \liminf_{t\to\infty} \tilde{\sigma}^t(x)$ $(= \lim_{t\to\infty} \underline{\tilde{\sigma}}^t(x))$.

Claim 1. $\liminf_{t\to\infty} \min BR(\tilde{\sigma}^t|x) \geq \min BR(\tilde{\sigma}_-|x)$ *for all* $x \in X$.

Proof. Fix any $x \in X$. By the supermodularity of u, we have $\min BR(\tilde{\sigma}^t|x) \geq \min BR(\underline{\tilde{\sigma}}^t|x)$ for all $t \geq 0$. Therefore, we have

$$\begin{aligned}\liminf_{t\to\infty} \min BR(\tilde{\sigma}^t|x) &\geq \liminf_{t\to\infty} \min BR(\underline{\tilde{\sigma}}^t|x) \\ &\geq \min BR\left(\lim_{t\to\infty} \underline{\tilde{\sigma}}^t \;\middle|\; x\right) = \min BR(\tilde{\sigma}_-|x),\end{aligned}$$

where the second inequality follows from the lower semicontinuity of $\min BR(\cdot|x)$ in the product topology on S^X. ▮

Claim 2. $\tilde{\sigma}_-(x) \geq \min BR(\tilde{\sigma}_-|x)$ *for all* $x \in X$.

Proof. Fix any $x \in X$. By Claim 1, there exists $T_1 \geq 0$ such that $\min BR(\tilde{\sigma}^t|x) \geq \min BR(\tilde{\sigma}_-|x)$ for all $t \geq T_1$. By (ii) and (iii) in Definition B.1, there exists $T_2 \geq T_1$ such that $\tilde{\sigma}^{T_2}(x) \geq \min BR(\tilde{\sigma}_-)$. By (ii) in Definition B.1, we also have $\tilde{\sigma}^t(x) \geq \tilde{\sigma}^{T_2}(x) \wedge \min_{T_2 \leq \tau < t} \min BR(\tilde{\sigma}^\tau|x)$ for all $t \geq T_2$. Therefore, by Claim 1 it follows that $\tilde{\sigma}^t(x) \geq \min BR(\tilde{\sigma}_-)$ for all $t \geq T_2$, and hence $\tilde{\sigma}_-(x) \geq \min BR(\tilde{\sigma}_-|x)$, as desired. ▮

Claim 3. $\sigma_-^t \leq \tilde{\sigma}_-$ *for all* $t \geq 0$.

Proof. We proceed by induction. First, we want to show $\sigma_-^0 \leq \tilde{\sigma}_-$. By assumption, $\sigma_-^0 \leq \tilde{\sigma}^0$. Assume that $\sigma_-^0 \leq \tilde{\sigma}^{t-1}$, and consider any $x \in X$ such that $\tilde{\sigma}^t(x) \neq \tilde{\sigma}^{t-1}(x)$. Then by the property (5) and the supermodularity of u, $\sigma_-^0(x) \leq \min BR(\sigma_-^0|x) \leq \min BR(\tilde{\sigma}^{t-1}|x) \leq \tilde{\sigma}^t(x)$. Therefore, we have $\sigma_-^0 \leq \tilde{\sigma}^t$ for all $t \geq 0$, and hence $\sigma_-^0 \leq \tilde{\sigma}_-$.

Next, assume that $\sigma_-^{t-1} \leq \tilde{\sigma}_-$, and let $x \in X$ be such that $\sigma_-^t(x) \neq \sigma_-^{t-1}(x)$. Then by the property (3), the induction hypothesis, the supermodularity of u, and Claim 2, we have $\sigma_-^t(x) = \min BR(\sigma_-^{t-1}|x) \leq \min BR(\tilde{\sigma}_-|x) \leq \tilde{\sigma}_-(x)$. Thus we have $\sigma_-^t \leq \tilde{\sigma}_-$. ▮

Symmetrically, defining $\tilde{\sigma}_+(x) = \limsup_{t\to\infty} \tilde{\sigma}^t(x)$, we can show that $\tilde{\sigma}_+ \leq \sigma_+^t$ for all $t \geq 0$. For each $x \in X$, since $\lim_{t\to\infty} \sigma_-^t(x) = \lim_{t\to\infty} \sigma_+^t(x) = s^*$, we have $\tilde{\sigma}_-(x) = \tilde{\sigma}_+(x) = s^*$, and hence $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$.

This completes the proof of Lemma B.1. ▮

Proposition B.1. *Fix a network (X, P) and a supermodular game (S, u). Then s^* is contagious by sequential best responses in (X, P) if and only if it is contagious by generalized best responses in (X, P).*

Proof. The "if" part holds by definition. To show the "only if" part, suppose that s^* is contagious by sequential best responses in (X, P). Let $(\sigma_-^t)_{t=0}^\infty$ and $(\sigma_+^t)_{t=0}^\infty$ be sequential best response sequences as in Lemma B.1. Let Y be a finite subset of X such that $\sigma_-^0(x) = \underline{s}$ and $\sigma_+^0(x) = \overline{s}$ for all $x \in X \setminus Y$. Then for any generalized best response sequence $(\tilde{\sigma}^t)_{t=0}^\infty$ with $\tilde{\sigma}^0(x) = s^*$ for all $x \in Y$, we have $\sigma_-^0 \leq \tilde{\sigma}^0 \leq \sigma_+^0$, and hence by Lemma B.1, $\lim_{t\to\infty} \tilde{\sigma}^t(x) = s^*$ for all $x \in X$. Thus s^* is contagious by generalized best responses in (X, P). ▮

Similarly, we can prove the equivalence between contagion by simultaneous best responses and contagion by generalized best responses. Here we assume that the set of neighbors $\Gamma(x)$ is finite for each player $x \in X$, which is satisfied in all of our examples.

Proposition B.2. *Fix a network (X, P) such that $\Gamma(x)$ is finite for each $x \in X$ and a supermodular game (S, u). Then s^* is contagious by simultaneous best responses in (X, P) if and only if it is contagious by generalized best responses in (X, P).*

Proof. The "if" part holds by definition. The proof of the "only if" part is to mimic the proofs of Lemma B.1 and the "only if" part of Proposition B.1. Indeed, as in the proof of Lemma B.1, we take a simultaneous best response sequence $(\phi_-^t)_{t=0}^\infty$, modify it to obtain a generalized (not necessarily simultaneous) best response sequence $(\psi_-^t)_{t=0}^\infty$, and then define $(\sigma_-^t)_{t=0}^\infty$ by $\sigma_-^t = \psi_-^{t+T}$ for sufficiently large T. The only difference lies here, where it is not the case in general that "at most T players change actions by period T". Instead, we assume without loss of generality that action $\underline{s}$ (as well as action $\overline{s}$) is a Nash equilibrium of (S, u), and resort to the finiteness of $\Gamma(x)$ to show that in each step of $(\psi_-^t)_{t=0}^T$, only finitely many players have minimum best responses other than action $\underline{s}$. ▮

Another definition is to only require *some* sequential best response sequence to converge.

Definition B.3. Given a pairwise game (S, u), action s^* is *weakly contagious in network* (X, P) if there exists a finite subset Y of X such that for any initial action configuration σ^0 such that $\sigma^0(x) = s^*$ for any $x \in Y$, there exists a sequential best response sequence (σ^t) such that $\lim_{t\to\infty} \sigma^t(x) = s^*$ for any $x \in X$.

By definition, contagion implies weak contagion. The converse does not always hold. A counterexample is given by the trivial payoff function $u \equiv 0$, where all actions are weakly contagious but none of them is contagious. Nevertheless, we can show that weak contagion is equivalent to contagion for generic supermodular games.

We say that a game (S, u) is *generic for* (X, P) if no player has multiple best responses to any action configuration on (X, P). If each player has finitely many neighbors, then genericity excludes at most countably many hyperplanes in the payoff parameter space.

Proposition B.3. *Fix a network* (X, P) *and a generic supermodular game* (S, u) *for* (X, P)*. Then* s^* *is weakly contagious in* (X, P) *if and only if it is contagious by generalized best responses in* (X, P)*.*

Proof. Once again, the proof is almost the same as the proofs of Lemma B.1 and Proposition B.1. We only need to make the following two changes.

First, in the first paragraph of the proof of Lemma B.1, given a finite set $Y \subset X$ as in Definition B.3, let $(\phi_-^t)_{t=0}^\infty$ be a sequential best response sequence such that $\phi_-^0(x) = s^*$ for all $x \in Y$, $\phi_-^0(x) = \underline{s}$ for all $x \in X \setminus Y$, and $\lim_{t\to\infty} \phi_-^t(x) = s^*$ for all $x \in X$. Here it follows from the genericity of (S, u) that we have $\phi_-^t(x) \in \{\phi_-^{t-1}(x), BR(\phi_-^{t-1}|x)\}$ for any $x \in X$ and $t \geq 1$, where with an abuse of notation, $BR(\phi_-^{t-1}|x)$ denotes the unique best response.

Second, a weakly contagious action is always a Nash equilibrium of (S, u), but may not be a *strict* Nash equilibrium. Here again, the genericity assumption guarantees that the weakly contagious action s^* is a strict Nash equilibrium. ∎

B.2 Proof of Lemma 3

Given a payoff function $f\colon S \times S \to \mathbb{R}$, we write BR_f for the best correspondence for the local interaction game (X, P, S, f):

$$BR_f(\sigma|x) = \bigl\{h \in S \bigm| \textstyle\sum_{y\in\Gamma(x)} P(y|x) f(h, \sigma(y)) \\ \geq \textstyle\sum_{y\in\Gamma(x)} P(y|x) f(h', \sigma(y)) \text{ for all } h' \in S\bigr\}.$$

(Thus the best response correspondence for the local interaction game (X, P, S, u) as defined in (2.3) is now denoted BR_u.) Recall that $BR_f(\sigma|x) = br_f(\pi(\sigma|x))$. We show a result stronger than Lemma 3, that a strict MP-maximizer is uninvadable by sequences that satisfy properties (i) and (ii) in Definition 1. Such sequences do not necessarily satisfy property (iii) in Definition 1, so that some players may have no opportunity to revise their suboptimal actions. We call those sequences partial best response sequences.

Definition B.4. Given a network (X, P) and for a payoff function $f\colon S \times S \to \mathbb{R}$, a sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *partial best response sequence* in local interaction game (X, P, S, f) if it satisfies the following properties: (i) for all $t \geq 1$, there is at most one $x \in X$ such that $\sigma^t(x) \neq \sigma^{t-1}(x)$; and (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR_f(\sigma^{t-1}|x)$.

The following result is due to Morris (1999, Proposition 6.1).

Lemma B.2. *Suppose that s^* is a potential maximizer of (S, u) with a potential function v. For any unbounded network (X, P) and any partial best response sequence $(\sigma^t)_{t=0}^{\infty}$ in local interaction game (X, P, S, u) with $\sigma_P^0(S \setminus \{s^*\}) < \infty$, there exists $M < \infty$ such that $\sigma_P^t(S \setminus \{s^*\}) \leq M$ for all $t \geq 0$.*

Lemma 3 is a direct corollary of the following.

Lemma B.3. *Suppose that $s^* \in \{\underline{s}, \overline{s}\}$ is a strict MP-maximizer of (S, u) with a strict MP-function v. If u or v is supermodular, then for any unbounded network (X, P) and any partial best response sequence $(\sigma^t)_{t=0}^{\infty}$ in local interaction game (X, P, S, u) with $\sigma_P^0(S \setminus \{s^*\}) < \infty$, there exists $M < \infty$ such that $\sigma_P^t(S \setminus \{s^*\}) \leq M$ for all $t \geq 0$.*

Proof. Let $s^* \in \{\underline{s}, \overline{s}\}$ be a strict MP-maximizer of (S, u) with a strict MP-function v. We only consider the case where $s^* = \underline{s}$. Fix any network (X, P). Let $(\sigma^t)_{t=0}^{\infty}$ be any partial best response sequence in (X, P, S, u). such that $\sigma_P^0(S \setminus \{\underline{s}\}) < \infty$.

Let $(\hat{\sigma}^t)_{t=0}^{\infty}$ be defined by $\hat{\sigma}^0 = \sigma^0$ and for $t \geq 1$,

$$\hat{\sigma}^t(x) = \begin{cases} \max BR_v(\hat{\sigma}^{t-1}|x) & \text{if } \sigma^t(x) \neq \sigma^{t-1}(x), \\ \hat{\sigma}^{t-1}(x) & \text{otherwise.} \end{cases}$$

Clearly, $(\hat{\sigma}^t)_{t=0}^{\infty}$ is a partial best response sequence in (X, P, S, v). Therefore, by Lemma B.2, there exists M such that $\hat{\sigma}_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show that if u or v is supermodular, then

$$\sigma^t \leq \hat{\sigma}^t \tag{$\star_t$}$$

for all $t \geq 0$. Then, $\sigma_P^t(S \setminus \{\underline{s}\}) \leq \hat{\sigma}_P^t(S \setminus \{\underline{s}\})$ for all t, and since $\hat{\sigma}_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t, it follows that $\sigma_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show by induction that $(\star_t)$ holds for all $t \geq 0$. First, $(\star_0)$ trivially holds by the definition of $\hat{\sigma}^0$. Next, assume $(\star_{t-1})$. It implies that for all $x \in X$, $\pi(\sigma^{t-1}|x) \precsim \pi(\hat{\sigma}^{t-1}|x)$. Let $x \in X$ be such that $\sigma^t(x) \neq \sigma^{t-1}(x)$, and hence $\hat{\sigma}^t(x^t) = \max br_v(\pi(\hat{\sigma}^{t-1}|x^t))$ by construction. If u is supermodular, then

$$\sigma^t(x^t) \leq \max br_u(\pi(\sigma^{t-1}|x^t))$$

$$\le \max br_u(\pi(\hat{\sigma}^{t-1}|x^t))$$
$$\le \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t),$$

where the second inequality follows from the supermodularity of u, and the third inequality follows from (3.5). If v is supermodular, then

$$\begin{aligned}\sigma^t(x^t) &\le \max br_u(\pi(\sigma^{t-1}|x^t)) \\ &\le \max br_v(\pi(\sigma^{t-1}|x^t)) \\ &\le \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t),\end{aligned}$$

where the second inequality follows from (3.5), and the third inequality follows from the supermodularity of v. Therefore, in each case, $(\star_t)$ holds. ∎

We show in passing that Lemma 3 extends to generalized best response sequences (Definition B.1) in any network where each player has finitely many neighbors.

Definition B.5. Given a pairwise game (S, u), action s^* is *uninvadable by generalized best response sequences in network* (X, P) if there exists no generalized best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(S \setminus \{s^*\}) < \infty$ and $\lim_{t\to\infty} \sigma_P^t(S \setminus \{s^*\}) = \infty$.

Proposition B.4. *Let (S, u) be any game with totally ordered action set S. If $s^* \in \{\underline{s}, \overline{s}\}$ is a strict MP-maximizer of (S, u) with a strict MP-function v and if u or v is supermodular, then s^* is uninvadable by generalized best responses in (X, P) such that $\Gamma(x)$ is finite for all $x \in X$.*

Proof. Let $s^* \in \{\underline{s}, \overline{s}\}$ be a strict MP-maximizer of u with a strict MP-function v. We only consider the case where $s^* = \underline{s}$. Fix any network (X, P) such that $\Gamma(x)$ is finite for all $x \in X$. Let $(\sigma^t)_{t=0}^{\infty}$ be any generalized best response sequence in (X, P, S, u) such that $\sigma_P^0(S \setminus \{\underline{s}\}) < \infty$. We will construct a nondecreasing partial best response sequence $(\hat{\sigma}^\tau)_{\tau=0}^{\infty}$ in (X, P, S, v) such that

$$\sigma^t \le \bar{\sigma} \qquad (\star\star_t)$$

for all $t \ge 0$, where $\bar{\sigma}$ is defined by $\bar{\sigma}(x) = \lim_{\tau\to\infty} \hat{\sigma}^\tau(x)$ for all $x \in X$. Then, we have, for all $t \ge 0$,

$$\sigma_P^t(S \setminus \{\underline{s}\}) \le \bar{\sigma}_P(S \setminus \{\underline{s}\}) = \lim_{\tau\to\infty} \hat{\sigma}_P^\tau(S \setminus \{\underline{s}\}) < \infty$$

as desired, where the last inequality (the finiteness of $\lim_{\tau\to\infty} \hat{\sigma}_P^\tau(S \setminus \{\underline{s}\})$) follows from Lemma B.2.

We construct such a sequence $(\hat{\sigma}^\tau)_{\tau=0}^{\infty}$ as follows. Pick a sequence $(x^\tau)_{\tau=1}^{\infty}$ in X such that $\{\tau \ge 1 \mid x^\tau = x\}$ is infinite for each $x \in X$.[2] Then, let $\hat{\sigma}^0 =$

[2] For example, enumerate $X = \{x_1, x_2, x_3, \ldots\}$, and for each $\tau \ge 1$, let $\ell(\tau)$ be the largest integer ℓ such that $\ell(\ell+1)/2 < \tau$, and let $k(\tau) = \tau - \ell(\tau)(\ell(\tau)+1)/2$ and $x^\tau = x_{k(\tau)}$.

σ^0, and for each $\tau \geq 1$, let $\hat{\sigma}^\tau(x^\tau) = \max\{\max BR_v(\hat{\sigma}^{\tau-1}|x^\tau), \hat{\sigma}^{\tau-1}(x^\tau)\}$ and $\hat{\sigma}^\tau(x) = \hat{\sigma}^{\tau-1}(x)$ for $x \neq x^\tau$. By construction, $(\hat{\sigma}^\tau)_{\tau=0}^\infty$ is a partial best response sequence in (X, P, S, v), and for each $x \in X$, $(\hat{\sigma}^\tau(x))_{t=0}^\infty$ is nondecreasing. Denote $\bar{\sigma}(x) = \lim_{\tau\to\infty} \hat{\sigma}^\tau(x)$. Note that $\bar{\sigma} \geq \hat{\sigma}^\tau$ for all $\tau \geq 0$.

Claim 1. $\max BR_v(\bar{\sigma}|x) \leq \bar{\sigma}(x)$ *for all* $x \in X$.

Proof. Fix any $x \in X$. By the finiteness of $\Gamma(x)$, there exists T such that $\hat{\sigma}^\tau(y) = \bar{\sigma}(y)$ for all $y \in \Gamma(x)$ and all $\tau \geq T$. By the construction of $(\hat{\sigma}^\tau)_{\tau=0}^\infty$, there exists $\tau' > T$ such that $x^{\tau'} = x$, and with such a τ' we have $\max BR_v(\bar{\sigma}|x) = \max BR_v(\hat{\sigma}^{\tau'-1}|x) \leq \hat{\sigma}^{\tau'}(x) \leq \bar{\sigma}(x)$. ▮

Now we show by induction that $(\star\star_t)$ holds for all $t \geq 0$. First, $(\star\star_0)$ holds by the construction of $(\hat{\sigma}^\tau)_{\tau=0}^\infty$. Next, assume $(\star\star_{t-1})$. It implies that for all $x \in X$, $\pi(\sigma^{t-1}|x) \precsim \pi(\bar{\sigma}|x)$. Let $x \in X$ be such that $\sigma^t(x) \neq \sigma^{t-1}(x)$, and hence $\sigma^t(x) \in br_u(\pi(\sigma^{t-1}|x))$. If u is supermodular, then $\sigma^t(x) \leq \max br_u(\pi(\sigma^{t-1}|x)) \leq \max br_u(\pi(\bar{\sigma}|x)) \leq \max br_v(\pi(\bar{\sigma}|x)) \leq \bar{\sigma}(x)$, where the second inequality follows from the supermodularity of u, the third from (3.5), and the fourth from Claim 1. If v is supermodular, then $\sigma^t(x) \leq \max br_u(\pi(\sigma^{t-1}|x)) \leq \max br_v(\pi(\sigma^{t-1}|x)) \leq \max br_v(\pi(\bar{\sigma}|x)) \leq \bar{\sigma}(x)$, where the second inequality follows from (3.5), the third from the supermodularity of v, and the fourth from Claim 1. Therefore, in each case, $(\star\star_t)$ holds. ▮

B.3 Multidimensional Lattice Networks

We fix the dimension m. A sequence $(P_n)_{n=0}^\infty$ of interaction weights on the m-dimensional lattice $\mathbb{Z}^m$ is *well-behaved* if the following conditions are satisfied.

- For each n, P_n is invariant up to translation, i.e., $P_n(x, y) = P_n(x+z, y+z)$ for $x, y, z \in \mathbb{Z}^m$.
- There exist a pair of nonnegative integrable functions $g, \bar{g} \colon \mathbb{R}^m \to \mathbb{R}_+$ such that for almost every $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{R}^m$, we have $n^m P_n([n\nu]|0) \to g(\nu)$ as $n \to \infty$ (pointwise convergence), and $n^m P_n([n\nu]|0) \leq \bar{g}(\nu)$ for every n.[3]
- The support of g is connected.

For example, consider n-max distance interactions P_n, where $P_n(x, y) = 1$ if $1 \leq \max_i |x_i - y_i| \leq n$ and $P_n(x, y) = 0$ otherwise. Then $(P_n)_{n=0}^\infty$ is well-behaved since $n^m P_n([n\nu]|0)$ converges to 2^{-m} times the indicator function of $\{\nu \in \mathbb{R}^m \mid \max_i |\nu_i| \leq 1\}$.

[3]For $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$, $[\eta] = ([\eta_1], \ldots, [\eta_m])$ denotes the profiles of the largest integers that do not exceed η_i.

The next theorem characterizes contagion and uninvadability in the limit of any well-behaved sequence of multidimensional lattice networks. The core of the proof is similar to that of Lemma 1, but we take $n \to \infty$ in order to mitigate the "lumpiness" of interaction weights.

Theorem B.1. *Let (S, u) be the bilingual game given by (2.2). Fix the dimension m and a well-behaved sequence $(P_n)_{n=0}^{\infty}$ of interaction weights on $\mathbb{Z}^m$.* (i) *If $e < e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$, A is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.* (ii) *If $e > e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$, B is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.*

Proof. We will show (i) only. The proof for (ii) is analogous.

By Lemma 2(i-1) and the upper semi-continuity of br, there exist $p \in (0, 1/2)$ and $\varepsilon \in (0, 1/2-p)$ such that $\max br(\hat{\pi}^a) = A$ and $\max br(\hat{\pi}^b) \leq AB$, where

$$\hat{\pi}^a = \left(\tfrac{1}{2} - \varepsilon, p, \tfrac{1}{2} - p + \varepsilon\right), \quad \hat{\pi}^b = \left(\tfrac{1}{2} - p - \varepsilon, p, \tfrac{1}{2} + \varepsilon\right).$$

Let $g(\nu)$ be the pointwise limit of $n^m P_n([n\nu]|0)$ as $n \to \infty$. Since P_n is symmetric and translation invariant, g is symmetric, i.e., $g(\nu) = g(-\nu)$ for almost all ν. We also have $\int_{\mathbb{R}^m} g(\nu) d\nu = 1$.

Since g is symmetric and has a connected support, for each $\lambda \in \mathbb{R}^m$ whose Euclidean norm $\|\lambda\|$ is 1, there exists a unique $\delta = \delta(\lambda) > 0$ that satisfies

$$\int_{0 \leq \lambda \cdot x \leq \delta} g(x) dx = p$$

and $\delta(\lambda)$ is continuous in λ, since the left hand side is continuous in λ and δ and strictly increasing in δ (whenever the left hand side is less than 1/2).

For each $r > 0$, let D_r be a disk $\{\nu \in \mathbb{R}^m \mid \|\nu\| \leq r\}$ and R_r be a ring-shaped object $\{\nu \in \mathbb{R}^m \mid r < \|\nu\| \leq r + \delta(\nu/\|\nu\|)\}$. Note that for large r and any boundary point ν of D_r, we have $\lambda \cdot \xi \approx r$ for any boundary point ξ of D_r near ν. By the continuity of $\delta(\cdot)$, the same is true for the boundary of R_r; i.e., for large r and any boundary point ν of D_r, we have $\lambda \cdot \xi \approx r + \delta(\nu/\|\nu\|)$ for any boundary point ξ of R_r near ν. Thus, there exists r_1 such that for any $r \geq r_1$,

$$\begin{aligned} \nu \in D_r &\Longrightarrow \int_{D_r} g(\xi - \nu) d\xi \geq \frac{1}{2} - \frac{\varepsilon}{3}, \int_{D_r \cup R_r} g(\xi - \nu) d\xi \geq \frac{1}{2} + p - \frac{\varepsilon}{3}, \\ \nu \in R_r &\Longrightarrow \int_{D_r} g(\xi - \nu) d\xi \geq \frac{1}{2} - p - \frac{\varepsilon}{3}, \int_{D_r \cup R_r} g(\xi - \nu) d\xi \geq \frac{1}{2} - \frac{\varepsilon}{3}. \end{aligned}$$

For each $k \in \mathbb{N}$, let $\hat{D}_k = \{x \in \mathbb{Z}^m \mid \|x\| \leq k\}$ and $\hat{R}_{k,n} = \{x \in \mathbb{Z}^m \mid k < \|x\| \leq k + n\delta(x/\|x\|)\}$. Since $(P_n)_{n=0}^{\infty}$ is well-behaved, one can apply the dominated convergence theorem to show that there exists n_1 such that for any $n \geq n_1$, any $x \in \mathbb{Z}^m$, and any $k \in \mathbb{N}$,

$$\left| \sum_{y \in \hat{D}_k} P_n(y-x|0) - \int_{D_{k/n}} g(\xi - x/n) d\xi \right| \leq \frac{\varepsilon}{3},$$

$$\left| \sum_{y \in \hat{D}_k \cup \hat{R}_{k,n}} P_n(y-x|0) - \int_{D_{k/n} \cup R_{k/n}} g(\xi - x/n) d\xi \right| \leq \frac{\varepsilon}{3}.$$

Therefore, there exists $n_2 \geq n_1$ such that for any $n \geq n_2$ and any $k \geq r_1 n$,

$$x \in \hat{D}_{k+1} \Longrightarrow \sum_{y \in \hat{D}_k} P_n(y|x) \geq \frac{1}{2} - \varepsilon, \quad \sum_{y \in \hat{D}_k \cup \hat{R}_{k,n}} P_n(y|x) \geq \frac{1}{2} + p - \varepsilon,$$

$$x \in \hat{R}_{k+1,n} \Longrightarrow \sum_{y \in \hat{D}_k} P_n(y|x) \geq \frac{1}{2} - p - \varepsilon, \quad \sum_{y \in \hat{D}_k \cup \hat{R}_{k,n}} P_n(y|x) \geq \frac{1}{2} - \varepsilon.$$

Now let $n \geq n_2$. We show that A is contagious in $(\mathbb{Z}^m, P_n)$. The proof is similar to that of Lemma 1(1). Pick a natural number $K \geq r_1 n$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = A$ for all $x \in \hat{D}_K \cup \hat{R}_{K,n}$. Then one can show by induction on k that for any $k \geq K$, there exists T_k such that for any $T \geq T_k$, we have $\sigma^t(x) = A$ for all $x \in \hat{D}_k$ and $\sigma^0(x) \leq AB$ for all $x \in \hat{R}_{k,n}$.

This argument also shows that A is uninvadable in $(\mathbb{Z}^m, P_n)$ because for any initial configuration that satisfies $\sigma^0_{P_n}(\{AB, B\}) < \infty$, there exists a translation Y of $\hat{D}_K \cup \hat{R}_{K,n}$ such that $\sigma^0(x) = A$ for all $x \in Y$. ▌

B.4 Proof of Theorem 3

We denote by $\underline{s}$ and $\overline{s}$ the smallest and the largest actions, respectively. We use the partial order $\sigma \leq \sigma'$ whenever $\sigma(x) \leq \sigma'(x)$ for any $x \in X$.

Let φ be a weight-preserving node identification from (X, P) to $(\hat{X}, \hat{P})$ with a finite set E of exceptional nodes. Fix a supermodular game (S, u), and assume that s^* is contagious in (X, P). We show that s^* is contagious in $(\hat{X}, \hat{P})$.

Since s^* is a strict Nash equilibrium of (S, u), there exists a finite subset $F \subset X$ such that $F \supset E$ and s^* is the unique best response for any $\hat{x} \in \varphi(E)$ if all players in $\varphi(F)$ play s^*.

Let $(\sigma_-^t)_{t=0}^{\infty}$ and $(\sigma_+^t)_{t=0}^{\infty}$ be sequential best response sequences in (X, P) that satisfy properties (1)–(5) in Lemma B.1. Pick a $T \geq 0$ such that $\sigma_-^T(x) = \sigma_+^T(x) = s^*$ for all $x \in F$, and let $Y = \{x \in X \mid \sigma_-^T(x) \neq \underline{s} \text{ or } \sigma_+^T(x) \neq \overline{s}\}$. Note that $Y \supset F$ and Y is finite.

Define action configurations $\hat{\sigma}_-$ and $\hat{\sigma}_+$ in $(\hat{X}, \hat{P})$ by

$$\hat{\sigma}_-(\hat{x}) = \max_{x \in \varphi^{-1}(\hat{x})} \sigma_-^T(x) \text{ and } \hat{\sigma}_+(\hat{x}) = \min_{x \in \varphi^{-1}(\hat{x})} \sigma_+^T(x)$$

for all $\hat{x} \in \hat{X}$. Note that $\hat{\sigma}_-(\hat{x}) = \hat{\sigma}_+(\hat{x}) = s^*$ for all $\hat{x} \in \varphi(F)$, and $\hat{\sigma}_-(\hat{x}) = \underline{s}$ and $\hat{\sigma}_+(\hat{x}) = \overline{s}$ for all $\hat{x} \in \hat{X} \setminus \varphi(Y)$. Denote by $\widehat{BR}$ the set of best responses defined in $(\hat{X}, \hat{P})$.

Claim 1. $\min \widehat{BR}(\hat{\sigma}_-|\hat{x}) \geq \hat{\sigma}_-(\hat{x})$ *and* $\hat{\sigma}_+(\hat{x}) \leq \max \widehat{BR}(\hat{\sigma}_+|\hat{x})$ *for all* $\hat{x} \in \hat{X}$.

Proof. We only show the first inequality; the proof of the second is analogous. For any $\hat{x} \in \varphi(E)$, since $\hat{\sigma}_-(\hat{y}) = s^*$ for all $\hat{y} \in \varphi(F)$, we have $\widehat{BR}(\hat{\sigma}_-|\hat{x}) = \{s^*\}$ by the construction of F. For any $\hat{x} \in X \setminus \varphi(E)$, let $\bar{\sigma}_-^T = \hat{\sigma}_- \circ \varphi$, and let $\bar{x} \in \arg\max_{x \in \varphi^{-1}(\hat{x})} \sigma_-^T(x)$. Then we have $\min \widehat{BR}(\hat{\sigma}_-|\hat{x}) = \min BR(\bar{\sigma}_-^T|\bar{x}) \geq BR(\sigma_-^T|\bar{x}) \geq \sigma_-^T(\bar{x}) = \hat{\sigma}_-(\hat{x})$, where the first equality follows from the weight-preserving property of φ, the first inequality from the supermodularity of u, and the second inequality from property (5) in Lemma B.1. ∎

Let $\hat{Y} = \varphi(Y)$, which is finite. Pick any sequential best response sequence $(\hat{\sigma}^t)$ in $(\hat{X}, \hat{P})$ such that $\hat{\sigma}^0(\hat{x}) = s^*$ for all $\hat{x} \in \hat{Y}$. We want to show that $\lim_{t\to\infty} \hat{\sigma}^t(\hat{x}) = s^*$ for all $\hat{x} \in \hat{X}$.

Claim 2. $\hat{\sigma}_- \leq \hat{\sigma}^t \leq \hat{\sigma}_+$ *for all* $t \geq 0$.

Proof. We only show the first inequality; the proof of the second is analogous. First, we have $\hat{\sigma}^0 \geq \hat{\sigma}_-$ by construction. Next, assume $\hat{\sigma}^{t-1} \geq \hat{\sigma}_-$. If $\hat{\sigma}^t(\hat{x}) \neq \hat{\sigma}^{t-1}(\hat{x})$, then we have $\hat{\sigma}^t(\hat{x}) \geq \min \widehat{BR}(\hat{\sigma}^{t-1}|\hat{x}) \geq \min \widehat{BR}(\hat{\sigma}_-|\hat{x}) \geq \hat{\sigma}_-(\hat{x})$, where the first inequality follows from the definition of sequential best response sequence, the second follows from the supermodularity of u, and the third from Claim 1. ∎

Claim 2 implies in particular that $\hat{\sigma}^t(\hat{x}) = s^*$ for all $\hat{x} \in \varphi(F)$ and all $t \geq 0$.

Given the sequence $(\hat{\sigma}^t)_{t=0}^{\infty}$ in $(\hat{X}, \hat{P})$, let $(\tilde{\sigma}^t)_{t=0}^{\infty}$ be the corresponding sequence in (X, P) defined by $\tilde{\sigma}^t = \hat{\sigma}^t \circ \varphi$ for all $t \geq 0$. First, by Claim 2, we have $\sigma_-^0 \leq \sigma_-^T \leq \hat{\sigma}_- \circ \varphi \leq \tilde{\sigma}^0 \leq \hat{\sigma}_+ \circ \varphi \leq \sigma_+^T \leq \sigma_+^0$. Second, $(\tilde{\sigma}^t)_{t=0}^{\infty}$ is a generalized best response sequence in (X, P) as defined in Definition B.1. (Notice that players in $\varphi^{-1}(\hat{x})$ change actions simultaneously.) Indeed, for $x \in X \setminus E$, we have $BR(\tilde{\sigma}^t|x) = \widehat{BR}(\hat{\sigma}^t|\varphi(x))$ for all $t \geq 0$ by the weight-preserving property of φ, while for $x \in E$, we have $\tilde{\sigma}^t(x) = s^*$ and $BR(\tilde{\sigma}^t|x) = \{s^*\}$ for all $t \geq 0$ by construction. Thus, by Lemma B.1(6), $(\sigma^t(x))_{t=0}^{\infty}$ converges to s^* for all $x \in X$, and hence $(\hat{\sigma}^t(\hat{x}))_{t=0}^{\infty}$ also converges to s^* for all $\hat{x} \in \hat{X}$. ∎

B.5 Examples

Example 6 (Line versus replicated lines). Let $(\{1, \ldots, m\} \times \mathbb{Z}, P)$ be a *replicated linear network*, where for $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in$

$\{1,\dots,m\}\times\mathbb{Z}$, we have $P(x,y) = P(x+z, y+z)$ (sums in the first coordinate are defined modulo m) and $P(x,y) = 0$ whenever $x_2 = y_2$.[4] An example of replicated linear network with $m = 3$ is depicted in Figure B.1. The mapping

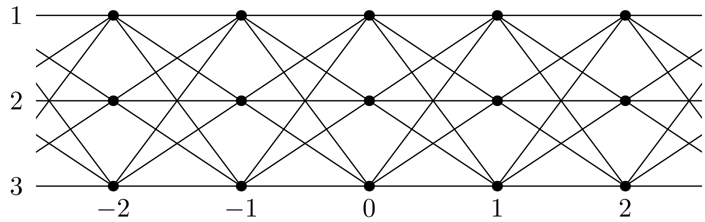

Figure B.1: Replicated linear network

$\varphi\colon \{1,\dots,m\}\times\mathbb{Z}\to\mathbb{Z}$ defined by $\varphi(k,i) = i$ is a weight-preserving node identification (with no exceptional node) from this network to the linear network $(\mathbb{Z},\hat{P})$ with $\hat{P}(i,j) = \sum_{k=1}^{m} P((1,i),(k,j))$. In fact, one can show that the two networks are equally contagion-inducing in the class of all supermodular games. In particular, Theorem 2 extends to replicated linear networks.

Example 7 (Line versus max distance)**.** Consider the m-dimensional lattice with n-max distance interactions, i.e., the network $(\mathbb{Z}^m, P)$ where $P(x,y) = 1$ if $1 \le \max_i |x_i - y_i| \le n$ and $P(x,y) = 0$ otherwise. Define the mapping $\varphi\colon \mathbb{Z}^m \to \mathbb{Z}$ by

$$\varphi(x_1,\dots,x_m) = x_1 + (n+1)x_2 + \cdots + (n+1)^{m-1}x_m$$

for any $(x_1,\dots,x_m)\in\mathbb{Z}^m$. Then φ is a weight-preserving node identification (with no exceptional node) from this network to the linear network $(\mathbb{Z},\hat{P})$ with $\hat{P}(x,y) = \#(\varphi^{-1}(y-x)\cap[-n,n]^m)$ for any $x,y\in\mathbb{Z}$ with $x\ne y$.[5] Thus, by Theorem 3, the n-max distance interaction network is less contagion-inducing than some linear network. Combined with Theorem 2, this implies that for the bilingual game, action A is not contagious in the n-max distance interaction network if $e > e^*$.

Example 8 (Regions versus lattice)**.** Consider the network depicted in Figure B.2, where the players are divided into infinitely many "regions", and each region consists of three players: $X = \{1,2,3\}\times\mathbb{Z}$, and with equal weights, player (k,i) interacts with players (ℓ,j) such that $\ell\ne k$ and $j = i$, or $\ell = k$ and $j = i\pm 1$. Then the mapping $\varphi\colon \mathbb{Z}^2\to\{1,2,3\}\times\mathbb{Z}$ defined by $\varphi(x_1,x_2) = (k,x_2)$ such that $k\equiv x_1 \pmod 3$ is a weight-preserving node

[4] The "thick line graph" in Immorlica et al. (2007, Figure 2) is a special case of replicated linear network.

[5] $\#X$ denotes the cardinality of X.

identification from the two-dimensional lattice to the regions network (with no exceptional node). Thus, by Theorem 3 and in a similar manner as in Example 2, one can show that the regions network is strictly more contagion-inducing than the two-dimensional lattice in the class of all supermodular games. This is in contrast to the class of 2×2 coordination games, where the two networks have the same contagion threshold $1/4$ (Examples 2 and 4 in Morris (2000)).

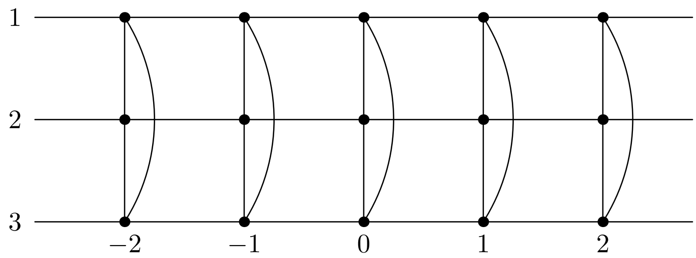

Figure B.2: Regions

Example 9 (Line versus Figure 4)**.** Theorems 2 and 3 imply that there exists no weight-preserving node identification from the network in Figure 4 to any linear network.

Example 10 (Line versus regions)**.** Consider the regions network as depicted in Figure B.2. In 2×2 coordination games, the regions network has contagion threshold $1/4$, whereas the linear network in Figure 1 has contagion threshold $1/2$. Also, in a similar manner as in Example 2, one can show that there is a set of parameter values of the bilingual game such that B is contagious in the regions network, but not in the linear network. Therefore, the regions network is incomparable to the linear network in the class of all supermodular games.

B.6 The Case Where Pareto Dominance and Risk Dominance Coincide

For completeness, we report the contagion and uninvadability result also for the case where action A is both Pareto-dominant and pairwise risk-dominant. The bilingual game (S, u) now satisfies

$$c \leq d < a, \;\; d - b < a - c, \;\; \text{and } e > 0. \tag{B.1}$$

Theorem B.2. *Let* (S, u) *be the bilingual game given by* (2.2a) *and* (B.1)*. A is always contagious and uninvadable.*

Proof. In light of Lemma 1(1-i) and Lemma 3, it suffices to show that condition (3.1) holds for some p and that A is a strict MP-maximizer. If

$e \leq (d-b)/2$, we have $(c-b)e < (a-d)(d-b)/2$. Therefore, these follow from the argument in case (α) in the proof of Lemma 2(1) and Claims 1–3 in the proof of Lemma A.2. If $e > (d-b)/2$, they follow from the symmetric arguments for A in place of B as in case (β) in the proof of Lemma 2(1) and Lemma A.4. ▮

Goyal and Janssen (1997, Theorem 3) show the contagion part of this theorem in their circular network.

Immorlica et al. (2007) consider the current case with a payoff parameter restriction $a = 1-q$, $b = c = 0$, and $d = q$, so the game is given by

$$\begin{array}{c} \\ A \\ AB \\ B \end{array} \begin{array}{c} \begin{array}{ccc} A & AB & B \end{array} \\ \begin{pmatrix} 1-q & 1-q & 0 \\ 1-q-e & 1-q-e & q-e \\ 0 & q & q \end{pmatrix} \end{array}, \qquad 0 < q < \frac{1}{2}.$$

This game is a potential game with action A being the potential maximizer (more generally, the bilingual game is a potential game whenever $b = c$). Immorlica et al. (2007) focus on the class $\mathcal{N}_\Delta$ of Δ-regular networks; for each $\Delta \in \mathbb{N}$, a Δ-regular network is a network where each player has Δ neighbors with constant weights. They consider the "epidemic region" $\Omega(X,P) \subset (0,1/2) \times \mathbb{R}_{++}$, the set of parameter values (q,e) for which action A spreads contagiously in network (X,P), and show that for any fixed Δ, there exists a point $(q,e) \notin \Omega_\Delta := \bigcup_{(X,P)\in\mathcal{N}_\Delta} \Omega(X,P)$, and in particular, Ω_Δ is not convex. On the other hand, since contagion in Lemma 1(1-i) can be induced by a Δ-regular network with some Δ (see Footnote 17 in the main text), our Theorem B.2 implies that $\bigcup_{\Delta\in\mathbb{N}} \Omega_\Delta = (0,1/2) \times \mathbb{R}_{++}$, which is convex.

B.7 Interpretations in Incomplete Information Games

Local interaction games and incomplete information games, though capturing different economic or social situations, share the same formal structures and thus belong to a more general class of "interaction games" (Morris (1997, 1999), Morris and Shin (2005)): in local interaction games, each node interacts with a set of neighbors and payoffs are given by the weighted sum of those from the interactions; in incomplete information games, each type interacts with a subset of types and payoffs are given by the expectation of those from the interactions.[6] Indeed, Morris (1997, 1999) demonstrates, in spite of some technical differences, that several tools and results in the context of incomplete information games can be utilized also in the context of local interaction games, and vice versa.[7] In this section, we interpret our

[6] For example, with the incomplete information interpretation, the linear network in Figure 1 is essentially equivalent to the information structure of the email game of Rubinstein (1989).

[7] For example, the contagion threshold of a network due to Morris (2000) is essentially equivalent to the belief potential of an information system due to Morris et al. (1995).

results in the language of incomplete information games, thereby shedding new light on two existing lines of literature, robustness to incomplete information and global games. We also discuss our symmetry assumption of interaction weights in relation to the common prior assumption in incomplete information games.

B.7.1 Robustness to Incomplete Information

A Nash equilibrium (s_1^*, s_2^*) of a two-player game (S, u) is said to be *robust to incomplete information* if any ε-incomplete information perturbation of (S, u) with ε sufficiently small has a Bayesian Nash equilibrium that plays (s_1^*, s_2^*) with high probability, where an ε-incomplete information perturbation of (S, u) refers to an incomplete information game in which the set T^u of type profiles whose payoffs are given by u has ex ante probability $1 - \varepsilon$ while types outside T^u ("crazy types") may have very different payoff functions (Kajii and Morris (1997)).[8] Robustness to incomplete information corresponds to uninvadability in networks in that both notions require that a small amount of "crazy types" should not affect the aggregate behavior.

Indeed, they have the same characterizations in many classes of games. For example, in parallel with Lemma 3, an MP-maximizer of a game (S, u) with MP-function v is robust to incomplete information if u or v is supermodular (Morris and Ui (2005)). Combining this result with Lemma 4, we obtain a sufficient condition for robustness in the bilingual game.

Conversely, a necessary condition for robustness is obtained by constructing ε-incomplete information perturbations in which a given action profile is contagious, where an action s^* is said to be contagious in an ε-incomplete information perturbation if s^* is a dominant action for types outside T^u and playing s^* everywhere is a unique rationalizable strategy. Specifically, in any symmetric 3×3 supermodular game (S, u), adjusting the proof of Lemma 1, for any $\varepsilon > 0$ one can construct ε-incomplete information perturbations in which 0 (2, resp.) is contagious if (3.1) ((3.2), resp.) holds for some $p \in (0, 1/2)$, or (3.3) holds for some $q, r \in (0, 1)$ with $r \leq q$ (Oyama and Takahashi (2011)). The necessary condition thus follows by applying this result to the bilingual game combined with Lemma 2.

These arguments characterize, exactly as in Theorem 1, when an equilibrium in the bilingual game is robust to incomplete information.

Proposition B.5. *Let* (S, u) *be the bilingual game given by* (2.2). (i) (A, A) *is a unique robust equilibrium if* $e < e^*$. (ii) (B, B) *is a unique robust equilibrium if* $e > \max\{e^*, e^{**}\}$. (iii) *No equilibrium is robust if* $e^* < e < \max\{e^*, e^{**}\}$.

[8] Kajii and Morris (1997) consider games with any finite number of players.

B.7.2 Global Games

Global games constitute a subclass of incomplete information games, where the underlying state θ is drawn from the real line, and each player i receives a noisy signal $x_i = \theta + \nu\varepsilon_i$ with ε_i being a noise error independent across players and from θ. Under supermodularity and state-monotonicity in payoffs, it has been shown by a contagion argument that an essentially unique equilibrium survives iterative deletion of dominated strategies as $\nu \to 0$, while the limit equilibrium may depend on the distribution of noise terms ε_i (Frankel et al. (2003)).

Global game perturbations in the class of all incomplete information perturbations can be viewed as linear networks in the class of all networks. In global games, the distribution of the opponent's signal x_j conditional on x_i is (approximately) invariant up to translation (for small $\nu > 0$) due to the assumption of state-independent noise errors, which parallels the translation invariance in linear networks. In fact, in the context of local interactions, by adopting the argument of Frankel et al. (2003), one can show that a generic supermodular game has at least one contagious action, and hence if an action is uninvadable, then it is also contagious and no other action is uninvadable.[9]

Basteck and Daniëls (2011) prove that in any global game, independently of the noise distribution, action profile $(0, 0)$ ($(2, 2)$, resp.) is played at θ as $\nu \to 0$ if the game at that state θ is a symmetric 3×3 supermodular game that satisfies (3.1) ((3.2), resp.) for some $p \in (0, 1/2)$. Together with Lemma 2(1), this leads to the following characterization of global-game noise-independent selection in the bilingual game, the same characterization as in Theorem 2.

Proposition B.6. *Let (S, u) be the bilingual game given by* (2.2). (i) (A, A) *is a noise-independent global game selection if* $e < e^*$. (ii) (B, B) *is a noise-independent global game selection if* $e > e^*$.

Since this characterization is different from that in Proposition B.5, global games are not a critical class of incomplete information games that determines whether or not an action profile is robust to incomplete information. See Oyama and Takahashi (2011) for further discussions.

Global games have been extended to multidimensional states and signals while maintaining the assumption of state-independent noise errors. (Indeed, multidimensional states and signals are already accommodated in Carlsson and van Damme (1993).) Recently, Oury (2013) shows that if an action is played in some one-dimensional global game of supermodular games independently of the noise distribution, then it is also played in any multidimensional global game. This result, combined with that of Oyama and

[9] For the bilingual game, these results also follow from our Theorem 1.

Takahashi (2011), implies that Proposition B.6 extends to multidimensional global games.

B.7.3 Non-Common Priors and Asymmetric Interaction Weights

All results reported in Sections B.7.1 and B.7.2 rely on the implicit assumption that in incomplete information perturbations the players share a common prior probability distribution, from which each player derives his conditional beliefs based on the information he has. This common prior assumption corresponds in our local interaction context to the assumption that the weight function P on interactions is symmetric, i.e., $P(x, y) = P(y, x)$ for all $x, y \in X$. The symmetry of the weight function naturally arises when the value $P(x, y)$ represents the duration (within a period) or intimacy of the interaction between x and y. Alternatively, if asymmetric weights are allowed, the situation corresponds to one of non-common priors.

Oyama and Tercieux (2010, 2012) study contagion and robustness under non-common priors, where players may have heterogeneous priors in ε-incomplete information perturbations and the probability of crazy types is no larger than ε with respect to all the players' priors. They show that under non-common priors, any strict Nash equilibrium of a complete information game is contagious in some ε-perturbations, and that generically, a game has a robust equilibrium if and only if it is dominance solvable, in which case the unique surviving action profile is robust.

Their results have a direct translation in our local interactions context: under asymmetric weights, any strict Nash equilibrium of a pairwise game is contagious, and generically, a game has an uninvadable action if and only if it is dominance solvable, in which case the unique surviving action is uninvadable.