Correlation functions in optics and quantum optics.

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1. Some history

Some history:

- Auguste Bravais (1811-63), French, physicist, also worked on metheorology.
- Francis Galton (1822-1911), English, statistician, sociologist, psicologist, proto-genetisist, eugenist.
- Norbert Wiener (1894-1964), United States, mathematician interested in noise....

Correlation of a set of data without noise:

- Think of data as two column vectors, such that their mean is zero.
- Each point is (x_i, y_i)
- The correlation function *C* is the internal product of the two vectors normalized by the norm of the vectors.

$$C = \frac{\sum_i x_i y_i}{\left(\sum_i x_i^2 \sum_i y_i^2\right)^{1/2}} = \frac{\vec{x} \bullet \vec{y}}{|\vec{x}||\vec{y}|} = \vec{X} \bullet \vec{Y}$$

If $y_i = mx_i$ (assuming the mean of x is zero and the mean of y is zero)

$$C = \frac{\sum_i x_i y_i}{\left(\sum_i x_i^2 \sum_i y_i^2\right)^{1/2}} = \frac{\sum_i x_i m x_i}{\left(\sum_i x_i^2 \sum_i m^2 x_i^2\right)^{1/2}}$$

$$C = \frac{m \sum_i x_i^2}{\left(m^2 \sum_i x_i^2 \sum_i x_i^2\right)^{1/2}} = \frac{m\vec{x} \bullet \vec{x}}{|m||\vec{x}||\vec{x}|} = \pm 1$$

C acquires the extreme values

given $\vec{y}, \vec{x}$

there may be a function (fit) such that $y_i = f(x_i)$

$$\bar{y} = \frac{1}{n}\sum_i y_i,\ \bar{x} = \frac{1}{n}\sum_i x_i$$

$$Var(y) = \frac{1}{(n-1)}\sum_i (y_i - \bar{y})^2,\ Var(x) = \frac{1}{(n-1)}\sum_i (x_i - \bar{x})^2$$

$$Var\ Exp(y) = \frac{1}{(n-1)}\sum_i (f(x_i) - \bar{y})^2$$

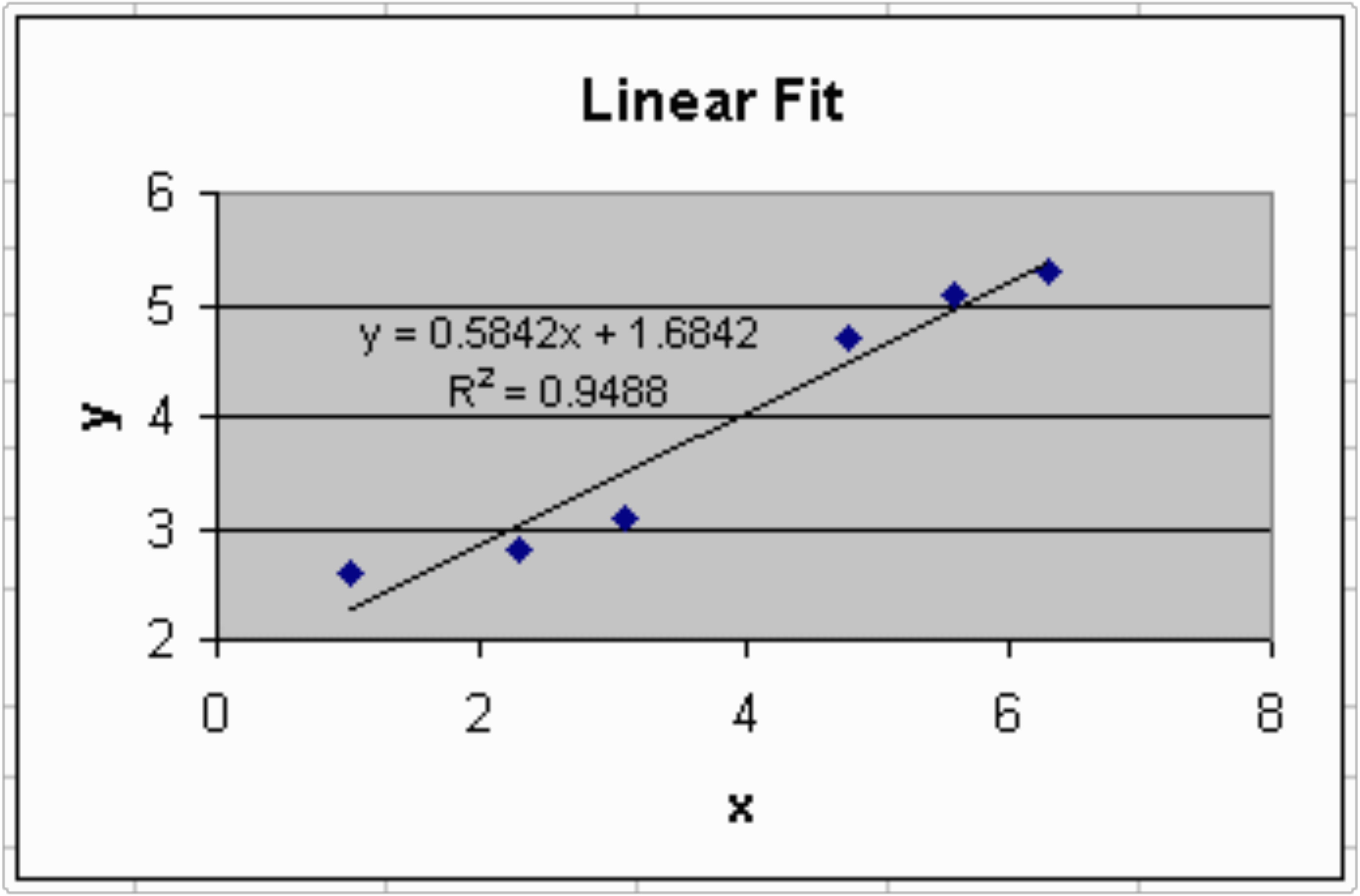
$$Var\ no\ Exp(y) = \frac{1}{(n-1)}\sum_i (f(x_i) - y_i)^2$$

$$C = \left(1 - \frac{Var\ no\ Exp}{Var}\right)^{1/2}$$

C is the internal product between the output data vector (y) and the value of the expected (fit) function (f(x)) with the input data (x)

$$C = \frac{\sum_i f(x_i) y_i}{\left(\sum_i f(x_i)^2 \sum_i y_i^2 \right)^{1/2}} = \frac{\vec{f} \bullet \vec{y}}{|\vec{f}||\vec{y}|} = \vec{F} \bullet \vec{Y}$$

Note that there is no reference to error bars or uncertainties in the data points

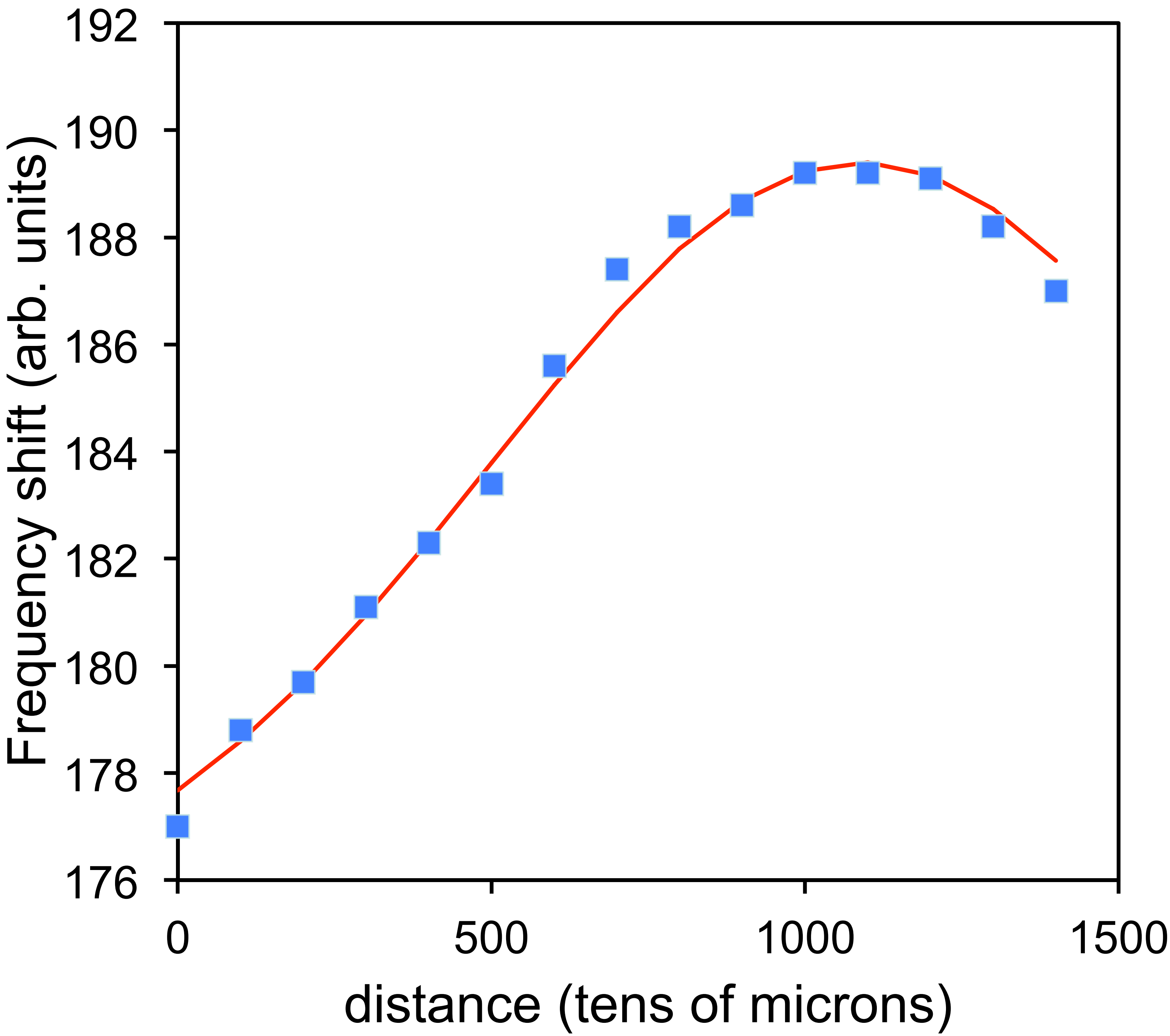


Correlation coefficient in this case C=R

This correlation coefficient can be between two measurements or a measurement and a prediction…

- C is bounded : $-1<C+1$
- C it is *cos*(ϕ) where ϕ is in some abstract space.
- Correlation does not imply causality!

Think of your data as vectors, it can be very useful.



192
190
188
186
184
182
180
178
176
Frequency shift (arb. units)
0
500
1000
1500
distance (tens of microns)

Data Vectors

X	Y	Y calc	(Y- Y calc)		XY	
0	177	177.67	-0.6736		31448.2	
100	178.8	178.59	0.20984		31931.9	
200	179.7	179.69	0.01373		32289.6	
300	181.1	180.95	0.15392		32769.3	
400	182.3	182.33	-0.0337		33239.4	
500	183.4	183.79	-0.3924		33707.5	
600	185.6	185.25	0.35335		34381.8	
700	187.4	186.61	0.79232		34970.3	
800	188.2	187.78	0.41799		35340.6	
900	188.6	188.68	-0.0818		35585.4	
1000	189.2	189.24	-0.0356		35803.4	
1100	189.2	189.4	-0.198		35834.1	
1200	189.1	189.16	-0.0552		35769.2	
1300	188.2	188.53	-0.3275		35480.9	
1400	187	187.57	-0.5665		35074.9	
			Least Squares			Correlation
∑Y^2	513552	513703	2.08244	∑YYc	513627	0.99999798

Correlations are not limited to a single spatial or temporal point.

In continuous functions, such as a time series, the correlation depends on the difference between the two comparing times.

The correlation can depend on real distance, angular distance or on any other parameter that characterizes a function or series.

Beyond equal indices (time, position, …)

Cross correlation (two functions)
Autocorrelation (same function)

C(n)= $$(f \star g)[n] \stackrel{\mathrm{def}}{=} \sum_{m=-\infty}^{\infty} f^*[m]\, g[m+n].$$

C(τ)= $$(f \star g)(\tau) \stackrel{\mathrm{def}}{=} \int_{-\infty}^{\infty} f^*(t)\, g(t+\tau)\, dt,$$

Resembles the convolution between two functions

Some mathematical properties (there are generalizations to matrices)

The correlation of f(t) and g(t) is equivalente to the convolution of f*(-t) and g(t)

$$f \star g = f^*(-t) * g = f^* * g(-t).$$

If f is Hermitic, then the correlation and the convolution are the same.

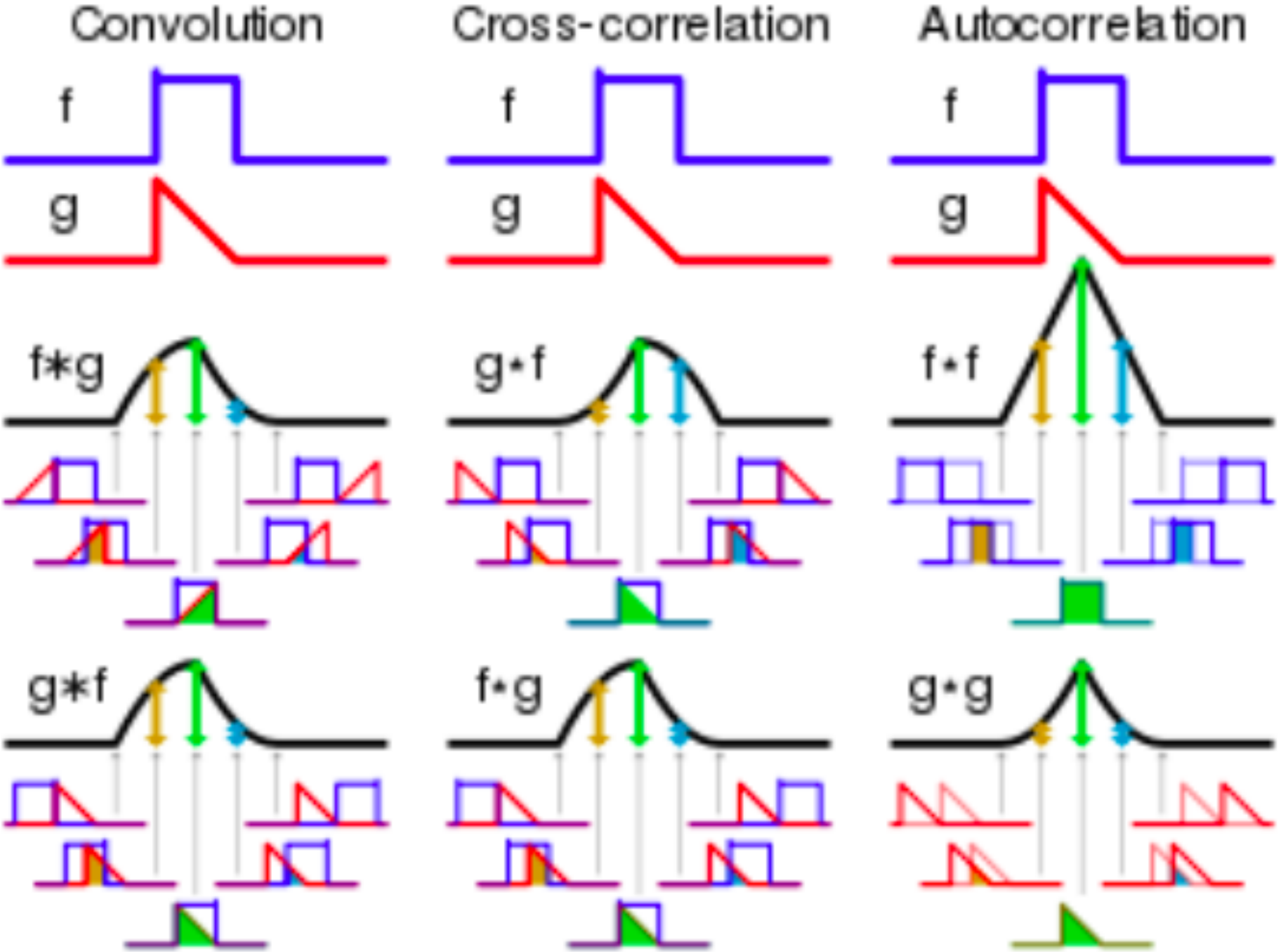
$$f \star g = f * g.$$

If both are Hermitic then: so

$$f \star g = g \star f.$$

Just as the convolution, the Fourier Transform:

$$\mathcal{F}\{f \star g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}^*$$


Convolution
Cross-correlation
Autocorrelation
f
g
f∗g
g∗f
g⋆f
f⋆g
f⋆f
g⋆g

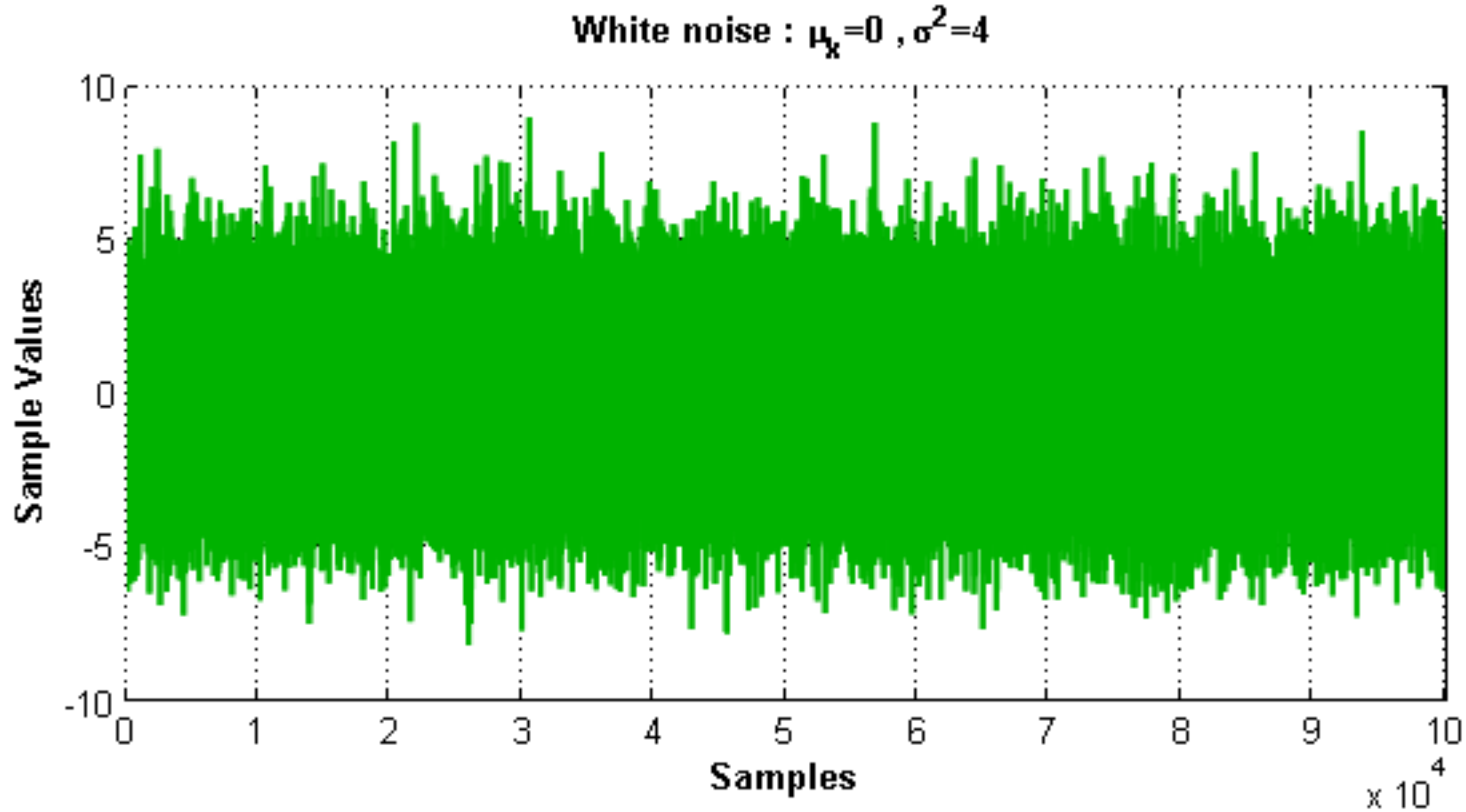
The correlation function contains averaging, and you could think of it as some monent over a distribution:

$$C(\tau)=<x(t)x(t+\tau)>$$

Where the probability density has to satisfy the properties of a positivity, integral equal to one…

Now let us think on what happens when the measurement has signal and noise.

If you only have noise, there are formal problems to find the power spectral density, it is not a simple Fourier transform.



The Wiener–Khinchin-Kolmogorov theorem says that the power spectral density of noise is the Fourier transform of its autocorrelation.

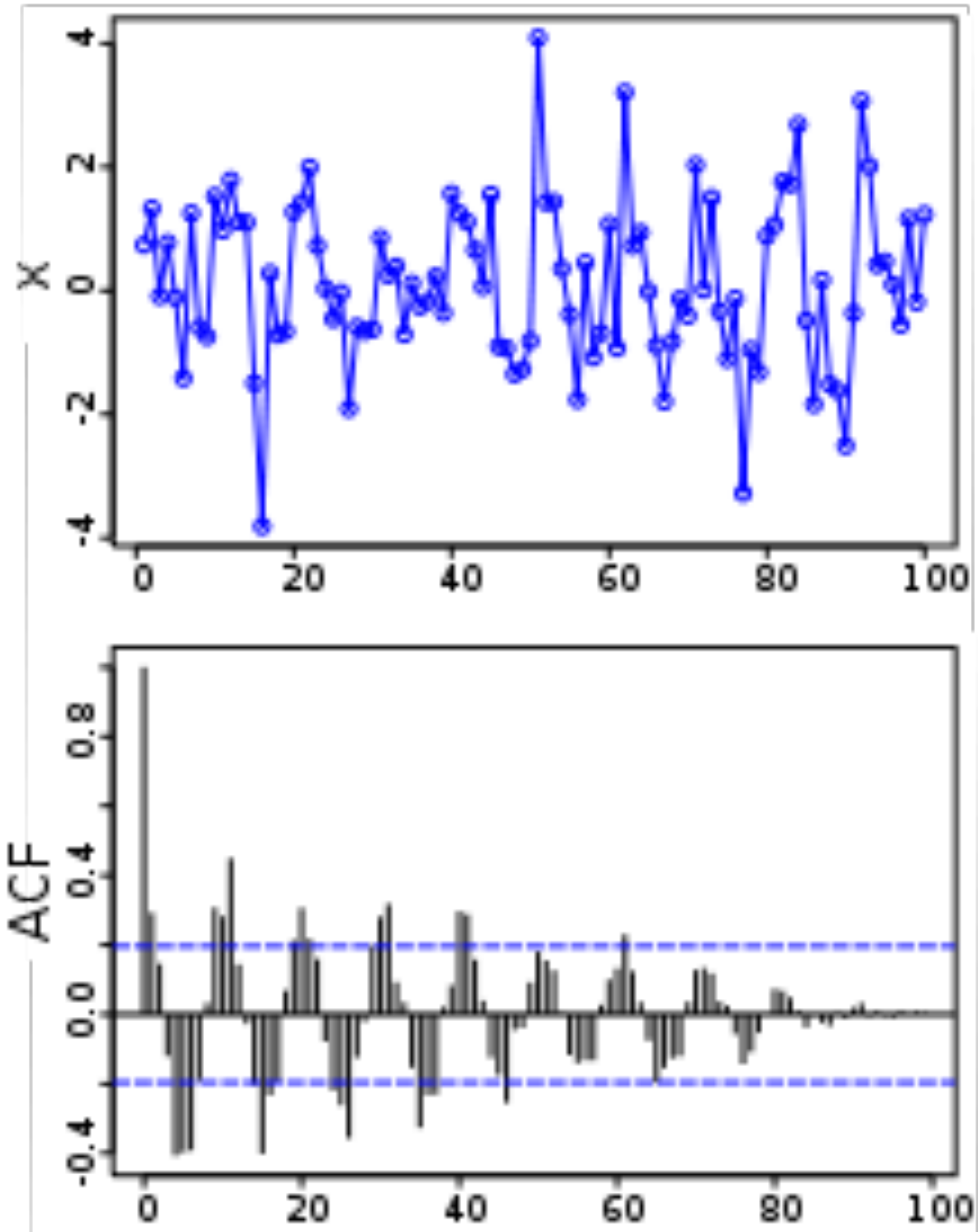
The Fourier transform of *f(t)*

$$g(\omega) = \int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t}$$

The Fourier transform of $|g(\omega)|^2$

$$\begin{aligned}
\int \frac{d\omega}{2\pi} |g(\omega)|^2 e^{-i\omega t} &= \int \frac{d\omega}{2\pi} g^*(\omega) e^{-i\omega t} \int dt'\, f(t') e^{i\omega t'} \\
&= \int dt'\, f(t') \int \frac{d\omega}{2\pi} g^*(\omega) e^{i\omega t'} e^{-i\omega t} \\
&= \int dt'\, f(t') \left[\int \frac{d\omega}{2\pi} g(\omega) e^{-i\omega(t'-t)} \right]^* \\
&= \int dt'\, f(t') f(t'-t)^*
\end{aligned}$$

The blue signal has a sinusoidal only visible in the autocrrelation (black)



Correlation functions in Optics (Wolf 1954)

- M. Born and E. Wolf, *Principles of Optics* Cambridge University Press, Cambridge, 1999, 7th expanded.

- L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* Cambridge University Press, Cambridge, 1995.

- E. Wolf, *Introduction to the Theory of Coherence and Polarization of Light* Cambridge University Press, Cambridge, 2007.

Wave equation for a scalar wave field $V(\mathbf{r},t)$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V(\mathbf{r},t) = 0,$$

$$U(\mathbf{r},\omega) = \int_{-\infty}^{\infty} V(\mathbf{r},t)e^{i\omega t}\mathrm{d}t,$$

$$(\nabla^2 + k^2)U(\mathbf{r},\omega) = 0,$$

Mutual coherence function and in normalized form:

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2;\tau) = \langle V^*(\mathbf{r}_1,t)V(\mathbf{r}_2,t+\tau)\rangle. \quad \gamma(\mathbf{r}_1,\mathbf{r}_2;\tau) = \frac{\Gamma(\mathbf{r}_1,\mathbf{r}_2;\tau)}{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}},$$

The intensity is the mutual coherence at equal t and $\mathbf{r}$

$$I(\mathbf{r}) = \Gamma(\mathbf{r},\mathbf{r};0) = \langle V^*(\mathbf{r},t)V(\mathbf{r},t)\rangle$$

The cross spectral density is the Fourier Transform of the correlation (normalized as spectral degree of coherence).

$$W(\mathbf{r}_1,\mathbf{r}_2;\omega)=\int_{-\infty}^{\infty}\Gamma(\mathbf{r}_1,\mathbf{r}_2;\tau)e^{i\omega\tau}\mathrm{d}\tau.$$

$$W(\mathbf{r}_1,\mathbf{r}_2;\omega)=\langle U^*(\mathbf{r}_1,\omega)U(\mathbf{r}_2,\omega)\rangle,$$

Spectral degree of coherence and its bounds are:

$$\mu(\mathbf{r}_1,\mathbf{r}_2;\omega)=\frac{W(\mathbf{r}_1,\mathbf{r}_2;\omega)}{\sqrt{S(\mathbf{r}_1,\omega)S(\mathbf{r}_2,\omega)}} \qquad 0\le|\mu(\mathbf{r}_1,\mathbf{r}_2;\omega)|\le 1$$

Spectral density (intensity at frequency ω)

$$S(\mathbf{r},\omega)=W(\mathbf{r},\mathbf{r};\omega)$$

Modern coherence theory began in 1954 when Wolf found that the mutual coherence function in free space satisfies the wave equations

$$\left(\nabla_1^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Gamma(\mathbf{r}_1,\mathbf{r}_2;t) = 0,$$

$$\left(\nabla_2^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Gamma(\mathbf{r}_1,\mathbf{r}_2;t) = 0,$$

The cross spectral density (the Fourier Transform of the correlation) also satisfies the Helmhotz equation:

$$(\nabla_1^2 + k^2) W(\mathbf{r}_1, \mathbf{r}_2; \omega) = 0,$$

$$(\nabla_2^2 + k^2) W(\mathbf{r}_1, \mathbf{r}_2; \omega) = 0,$$

With the associated diffraction integrals

Then, knowledge of $W(0)$ the cross spectral density in the source plane allows in principle the calculation of the cross-spectral density function everywhere in the halfspace $z>0$.

For vectorial beams the *electric cross-spectral density matrix*, which may be used to characterize the state of coherence and the state of polarization of the beam in the source plane *z*=0 is (ρ_i is a point on the source plane):

$$\mathbf{W}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega)=\begin{pmatrix} W_{xx}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega) & W_{xy}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega) \\ W_{yx}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega) & W_{yy}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega) \end{pmatrix},$$

$$W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega)=\langle E_i^*(\boldsymbol{\rho}_1,\omega)E_j(\boldsymbol{\rho}_2,\omega)\rangle, \quad (i,j=x,y).$$

Then at the observation point for two points r_i *E* is linked to the source through a Green Funciton or a diffraction integral gives:

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\langle E_i^*(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle, \quad (i,j=x,y).$$

As we are dealing now with vectorial fields, the *spectral degree of polarization* of the field at any point is defined as the ratio of the intensity of the polarized part of the beam to the total beam intensity and is given by:

$$\mathcal{P}(\mathbf{r},\omega) = \sqrt{1 - \frac{4 \det \mathbf{W}(\mathbf{r},\mathbf{r};\omega)}{[\mathrm{Tr}\,\mathbf{W}(\mathbf{r},\mathbf{r};\omega)]^2}},$$

With now the spectral density of the field (within a prefactor):

$$S(\mathbf{r},\omega) = \mathrm{Tr}\,\mathbf{W}(\mathbf{r},\mathbf{r};\omega).$$

The degree of polarization is bounded from unpolarized (0) to completely polarized (1):

$$0 \leq \mathcal{P}(\mathbf{r},\omega) \leq 1$$

The spectral stokes parameters can be expressed as linear combinations of the elements of the electric cross-spectral density matrix:

$$s_0(\mathbf{r},\omega) = W_{xx}(\mathbf{r},\mathbf{r};\omega) + W_{yy}(\mathbf{r},\mathbf{r};\omega),$$

$$s_1(\mathbf{r},\omega) = W_{xx}(\mathbf{r},\mathbf{r};\omega) - W_{yy}(\mathbf{r},\mathbf{r};\omega),$$

$$s_2(\mathbf{r},\omega) = W_{xy}(\mathbf{r},\mathbf{r};\omega) + W_{yx}(\mathbf{r},\mathbf{r};\omega),$$

$$s_3(\mathbf{r},\omega) = i[W_{yx}(\mathbf{r},\mathbf{r};\omega) - W_{xy}(\mathbf{r},\mathbf{r};\omega)].$$

Now you see that the degree of polarization is related to the correlation function and it can change as a function of position as the wave propagates.

The Polarization coherence theorem, a recent corollary:

$$P^2 = V^2 + D^2, \text{ and } V^2 + D^2 \leq 1$$

The degree of polarization *P* is equal to the visibility *V* plus the distinguishability *D*, all of them related to the elements of the *electric cross-spectral density matrix* .

J. H. Eberly, X.-F. Qian, and A. N. Vamivakas, "Polarization coherence theorem," Optica **4**, 1113 (2017).

A. F. Abouraddy, "What is the maximum attainable visibility by a partially coherent electromagnetic field in Young's double-slit interference?" Opt. Express **15**, 18320 (2017).

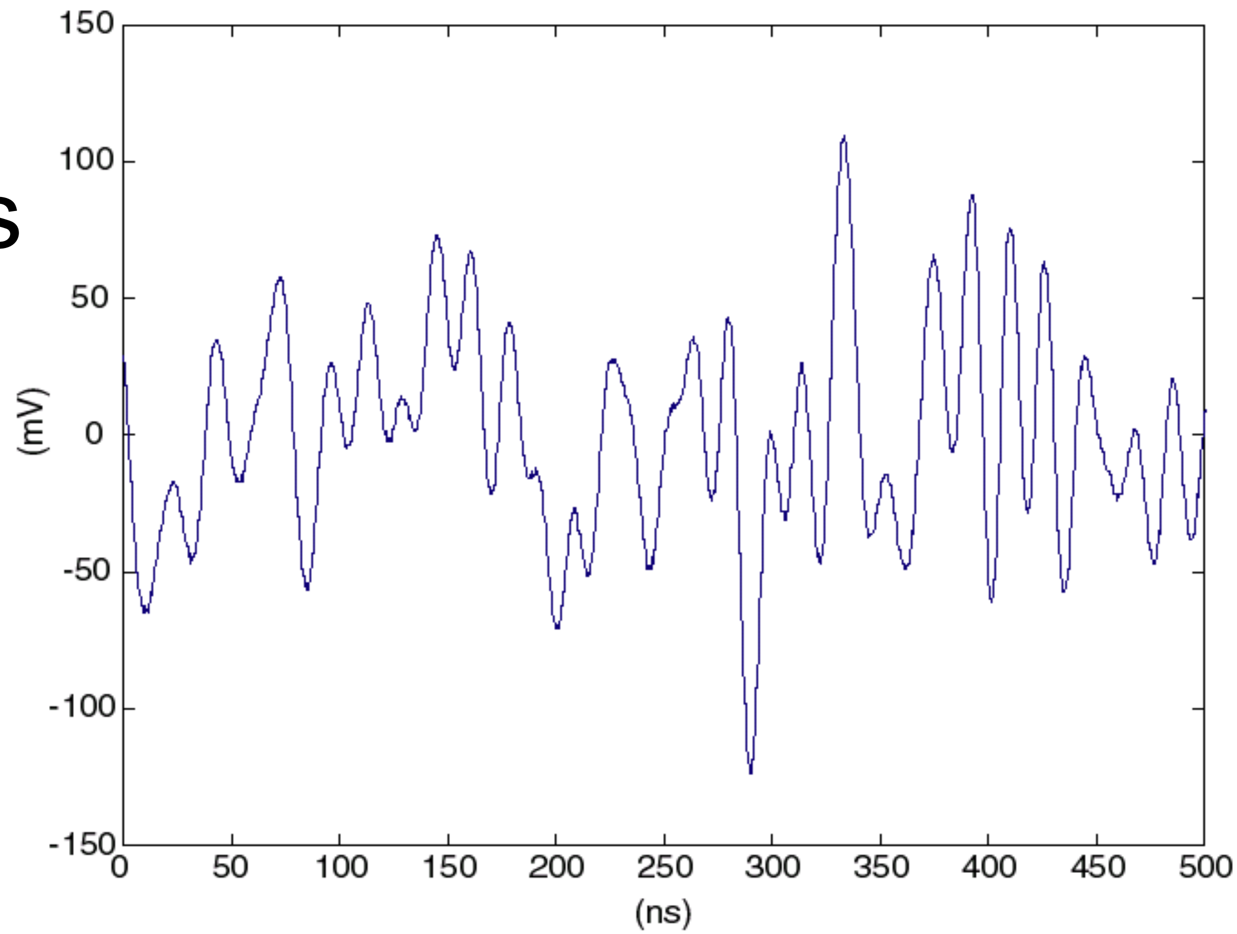
Correlation measurements

The study of optical noisy signals uses correlation functions.

$$\langle F(t)\, F(t+\tau) \rangle$$
$$\langle F(t)\, G(t+\tau) \rangle$$

For optical signals the variables usually are: Field and Intensity, but they can be cross correlations as well.

Photocurrent with noise:



How do we measure these functions?

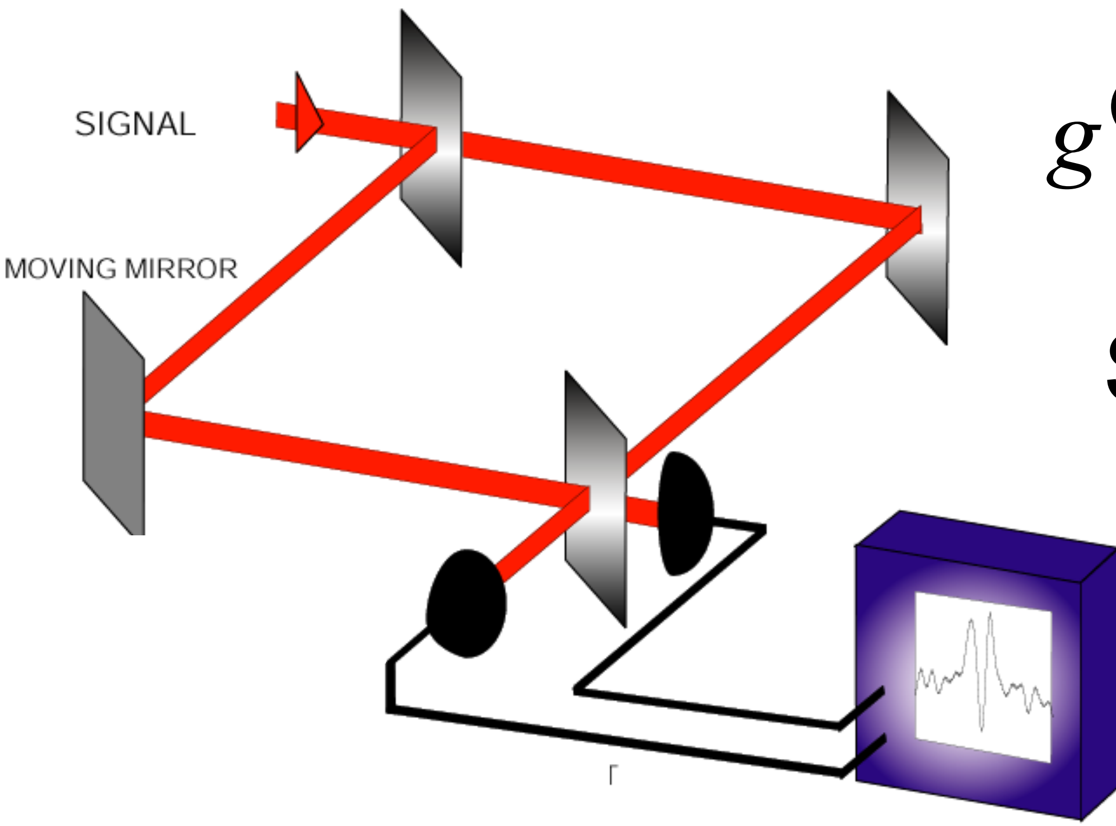
$G^{(1)}(\tau) = \langle E(t)^* E(t+\tau) \rangle$ field-field

$G^{(2)}(\tau) = \langle I(t) I(t+\tau) \rangle$ intensity-intensity

$H(\tau) = \langle I(t) E(t+\tau) \rangle$ intensity-field

- Correlation functions tell us something about fluctuations.
- The correlation functions have classical limits.
- They are related to conditional measurements. They give the probability of an event given that something has happened.

Mach Zehnder or Michelson Interferometer
Field –Field Correlation



$$g^{(1)}(\tau) = \frac{\left\langle E^*(t)E(t+\tau)\right\rangle}{\left\langle I(t)\right\rangle}$$

Spectrum:

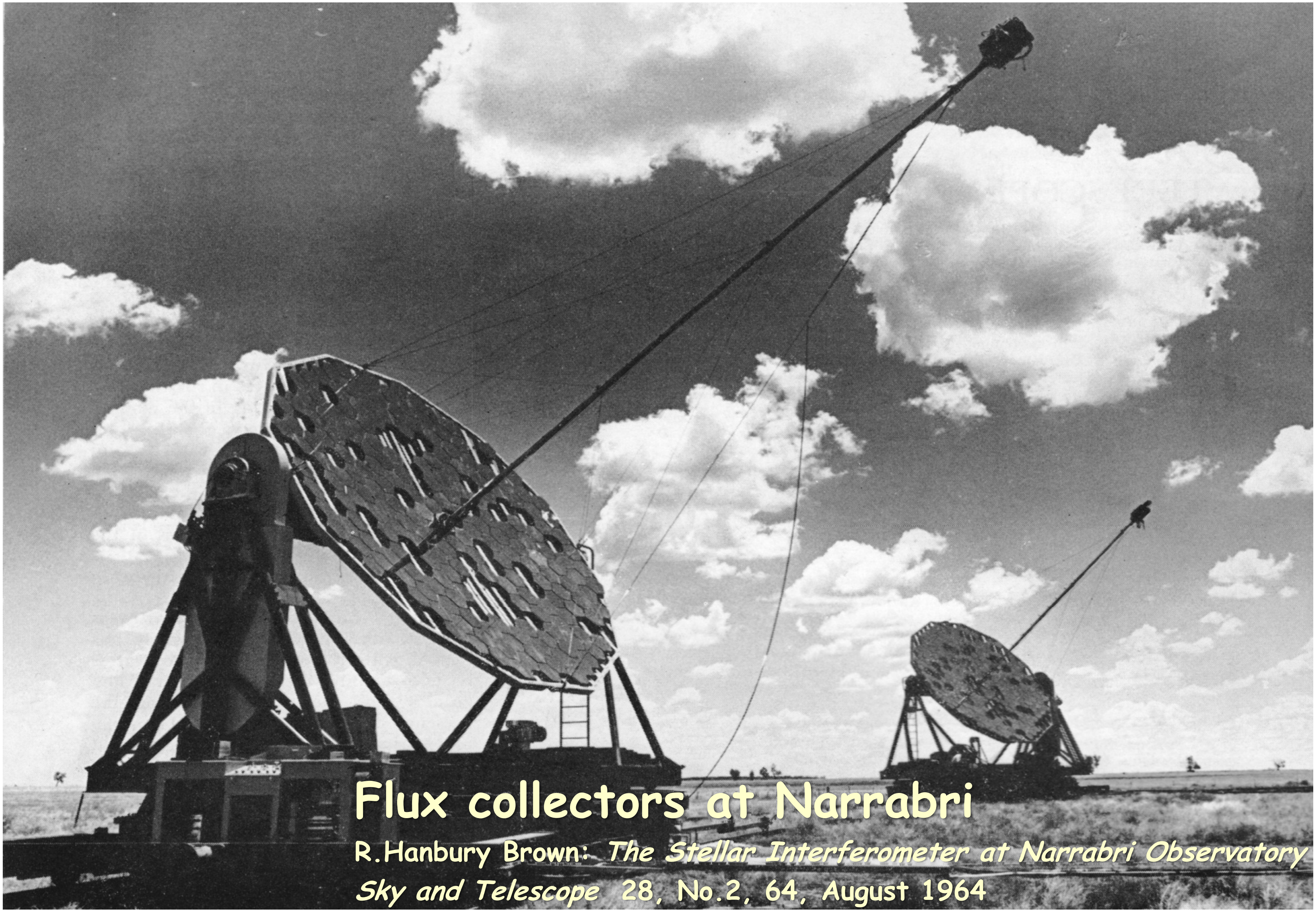
$$F(\omega) = \frac{1}{2\pi}\int exp(i\omega\tau)g^{(1)}(\tau)d\tau$$

This is the basis of Fourier Spectroscopy

Handbury Brown and Twiss

Can we use intensity fluctuations, noise, to measre the size of a star? They were radio astronomers and had done it around 1952,

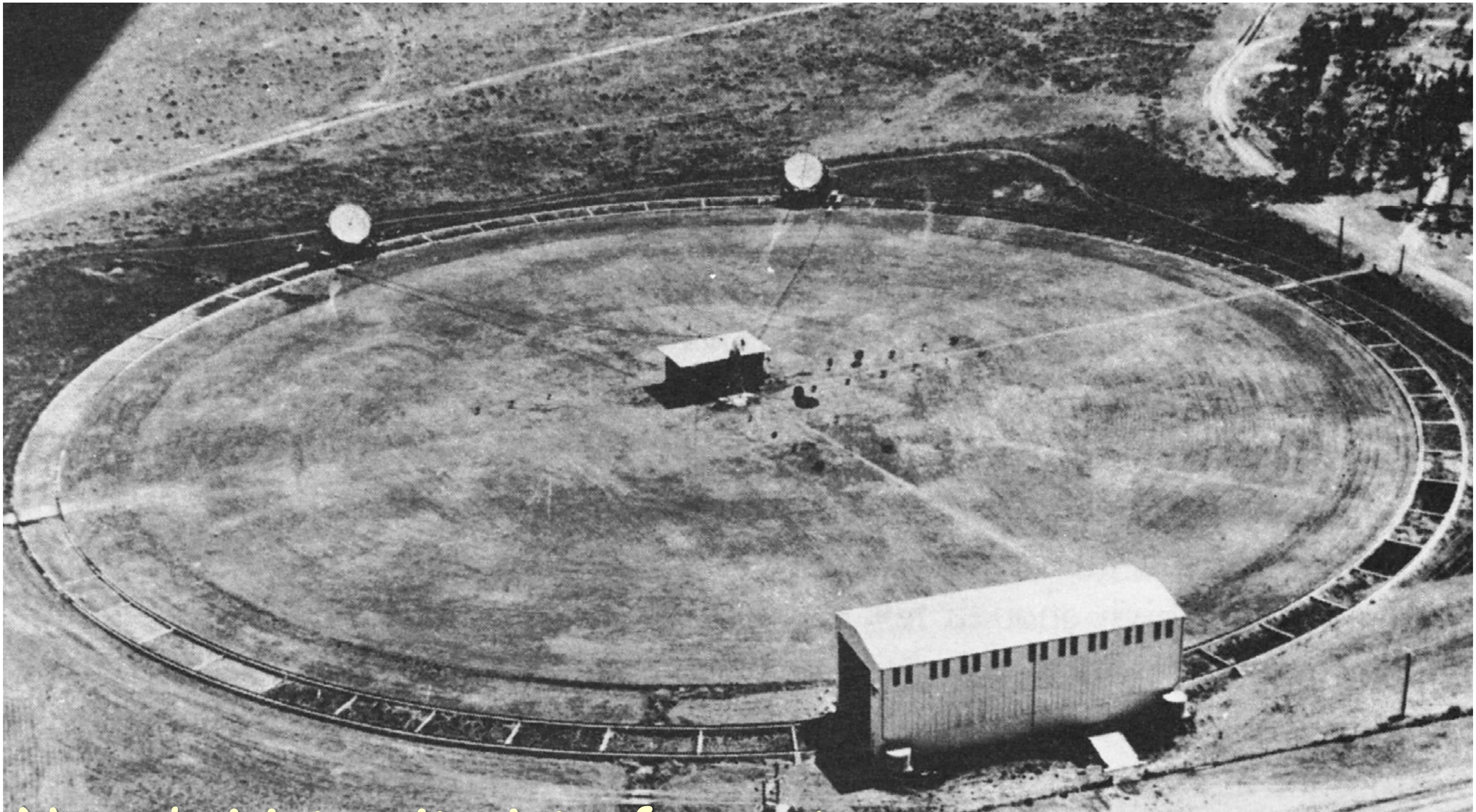
R. Hanbury Brown and R.Q. Twiss, “A New Type of Interferometer for Use in Radio Astronomy,” Phil. Mag. **46**, 663 (1954).



Flux collectors at Narrabri

R.Hanbury Brown: *The Stellar Interferometer at Narrabri Observatory*

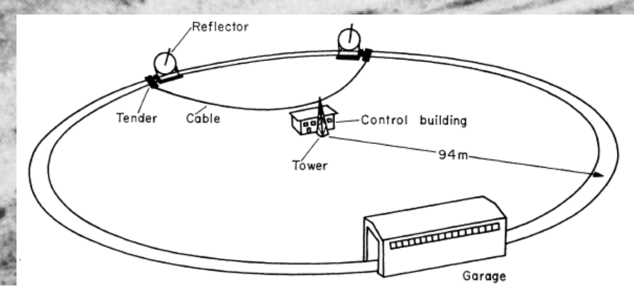
Sky and Telescope 28, No.2, 64, August 1964



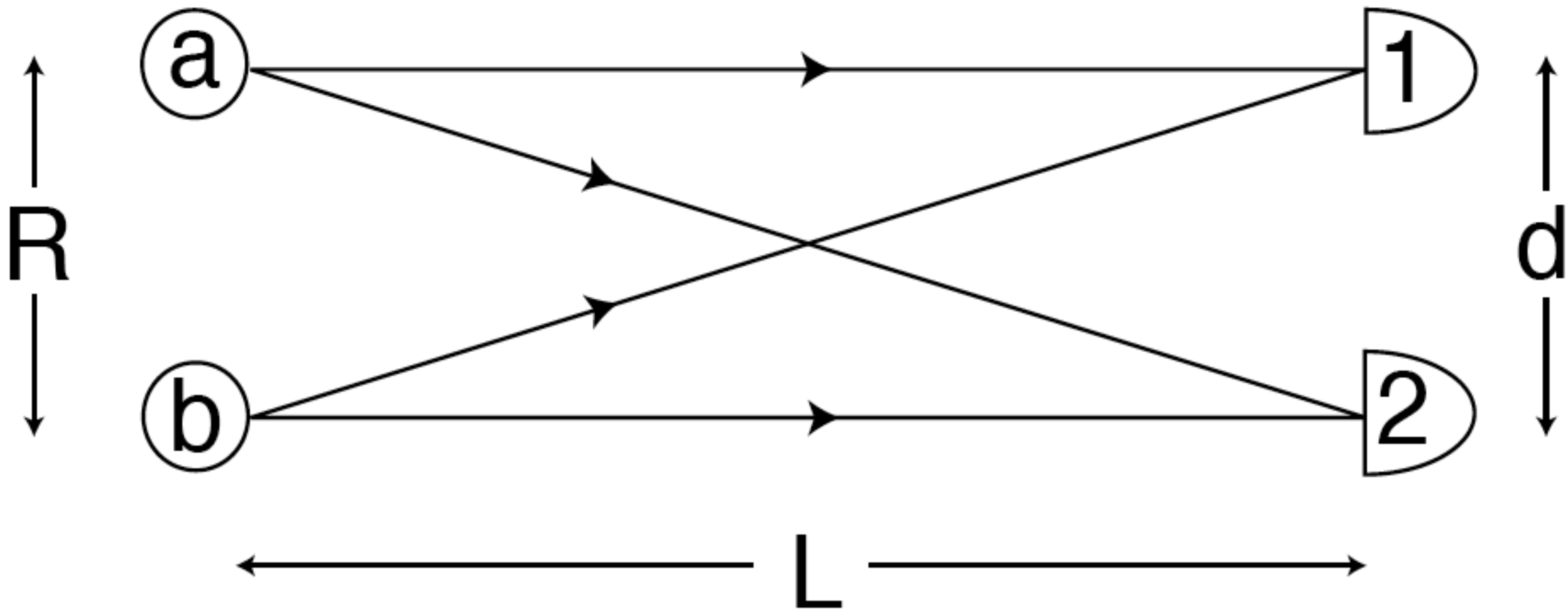
Narrabri intensity interferometer

with its circular railway track

R.Hanbury Brown: *BOFFIN. A Personal Story of the Early Days of Radar, Radio Astronomy and Quantum Optics* (1991)



The HBT controversy



The physics of Hanbury Brown–Twiss intensity interferometry: from stars to nuclear collisions.*

Gordon Baym

Presented at the XXXVII Cracow School of Theoretical Physics, Zakopane, Poland. May 30 - June 10, 1997.

Source a and b are within a Star. Can we measure the angular distance R/L~θ so that we could know the diameter?

Source a:

$$\alpha e^{ik|\vec{r}-\vec{r}_a|+i\phi_a}/|\vec{r}-\vec{r}_a|$$

Source b:

$$\beta e^{ik|\vec{r}-\vec{r}_b|+i\phi_b}/|\vec{r}-\vec{r}_b|$$

The amplitude at detector 1 from sources a and b is:

$$A_1 = \frac{1}{L}\left(\alpha e^{ikr_{1a}+i\phi_a} + \beta e^{ikr_{1b}+i\phi_b}\right)$$

And the intensity I_1

$$= \frac{1}{L^2}\left(|\alpha|^2 + |\beta|^2 + \alpha^* \beta e^{i(k(r_{1b}-r_{1a})+\phi_b-\phi_a)} + \alpha\beta^* e^{-i(k(r_{1b}-r_{1a})+\phi_b-\phi_a)}\right)$$

The average over the random phases ϕ_a and ϕ_b gives zero

$$\langle I_1\rangle = \langle I_2\rangle = \frac{1}{L^2}\left(\langle|\alpha|^2\rangle + \langle|\beta|^2\rangle\right)$$

And the product of the intensity of each of the detectors $\langle I_1\rangle\langle I_2\rangle$ is independent of the separation of the detectors.

Multiply the two intensities and then average.

$$\langle I_1 I_2 \rangle = \langle I_1 \rangle \langle I_2 \rangle + \frac{2}{L^4}|\alpha|^2|\beta|^2 \cos\left(k(r_{1a} - r_{2a} - r_{1b} + r_{2b})\right)$$

$$= \frac{1}{L^4}\left[(|\alpha|^4 + |\beta|^4) + 2|\alpha|^2|\beta|^2(1 + \cos\left(k(r_{1a} - r_{2a} - r_{1b} + r_{2b})\right)\right].$$

$$g^{(2)} = \frac{\langle I_1 I_2 \rangle}{\langle I_1 \rangle \langle I_2 \rangle}$$

$$= 1 + 2\frac{\langle|\alpha|^2\rangle\langle|\beta|^2\rangle}{(\langle|\alpha|^2\rangle + \langle|\beta|^2\rangle)^2} \cos\left(k(r_{1a} - r_{2a} - r_{1b} + r_{2b})\right).$$

$$(L \gg R),\ k(r_{1a} - r_{2a} - r_{1b} + r_{2b}) \rightarrow k(\vec{r}_a - \vec{r}_b) \cdot (\hat{r}_2 - \hat{r}_1) = \vec{R} \cdot (\vec{k}_2 - \vec{k}_1)$$

This function changes as a function of the separation between the detectors.

$d = \lambda/\theta$ with $\theta = R/L$

Relation to the Michelson Interferometer

$$|A_1 + A_2|^2 = |A_1|^2 + |A_1|^2 + (A_1^* A_2 + A_1 A_2^*)$$

The term in parenthesis is the associated to the fringe Visibility (first order coherence) if we now take the square of the fringe visibility and average it:

$$\langle V^2 \rangle = 2\langle |A_1|^2 |A_2|^2 \rangle + \langle A_1^{*2} A_2^2 \rangle + \langle A_1^2 A_2^{*2} \rangle$$

$$\langle V^2 \rangle \rightarrow 2\langle I_1 I_2 \rangle$$

Can we use intensity fluctuations, noise, to measre the size of a star in the visible.

R. Hanbury Brown and R. Q. Twiss do it with an isotopically enriched source of Hg showing that it is possible.

The solution of E. M. Purcell, *Nature* **178**, 1449 (1956).

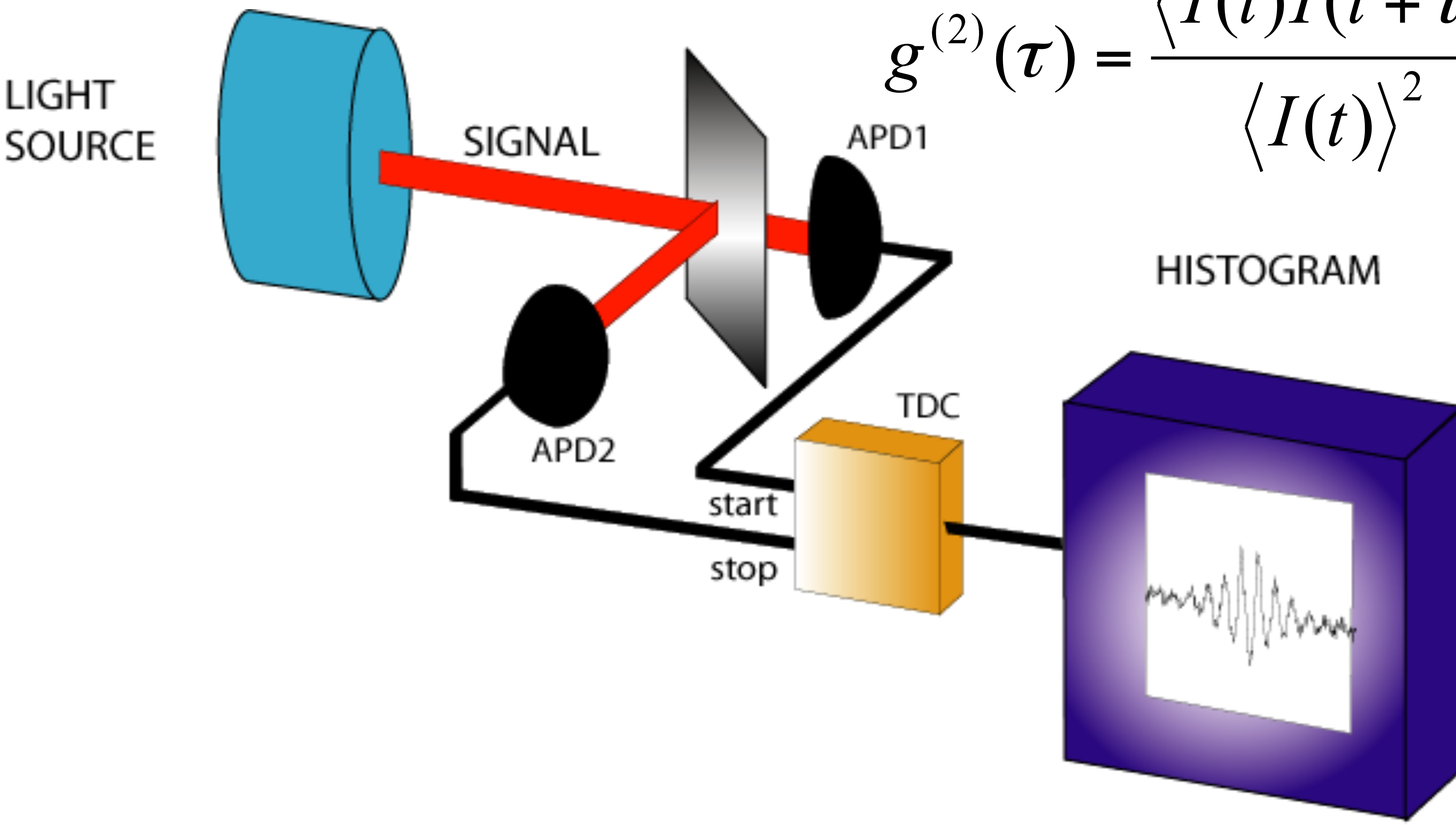
Mentions the work A. T. Forrester, R. A. Gudmundsen and P. O. Johnson, "*Photoelectric Mixing of Incoherent Light,*" Phys. Rev. **99**, 1691 (1955).

Mentions that bosons tend to appear together

Does the calculation and relates it to the first order coherence.

Hanbury Brown and Twiss; Intensity Intensity correlation

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2}$$



R. Hanbury Brown and R.Q. Twiss, Correlation between Photons in Two Coherent Beams of Light, Nature 177, 27 (1956).

Correlations of the intensity at τ=0

$$g^{(2)}(0) = \frac{\langle I(t)^2 \rangle}{\langle I(t) \rangle^2}$$

$$= \frac{\langle (I_0 + \delta(t))^2 \rangle}{\langle I_0 + \delta(t) \rangle^2}$$

$$= 1 + \frac{\langle \delta(t)^2 \rangle}{I_0^2}$$

It is proportional to the variance

Intensity correlations (bounds)

$$g^{(2)}(0) = 1 + \frac{\langle \delta(t)^2 \rangle}{I_0^2}$$

$$g^{(2)}(0) - 1 \geq 0$$

Cauchy-Schwarz

$$2I(t)I(t+\tau) \leq I^2(t) + I^2(t+\tau)$$

$$|g^{(2)}(\tau) - 1| \leq |g^{(2)}(0) - 1|$$

The correlation is maximal at equal times (τ=0) and it can not increase.

How do we measure them?

Build a "Periodogram". The photocurrent is proportional to the intensity I(t)

$$I(t) \rightarrow I_i$$

$$I(t+\tau) \rightarrow I_j$$

$$\langle I(t)I(t+\tau)\rangle \rightarrow \sum_{i=0}^{M}\sum_{n=0}^{N} I_i I_{i+n}$$

- Discretize the time series.
- Apply the algorithm on the vector.
- Careful with the normalization.

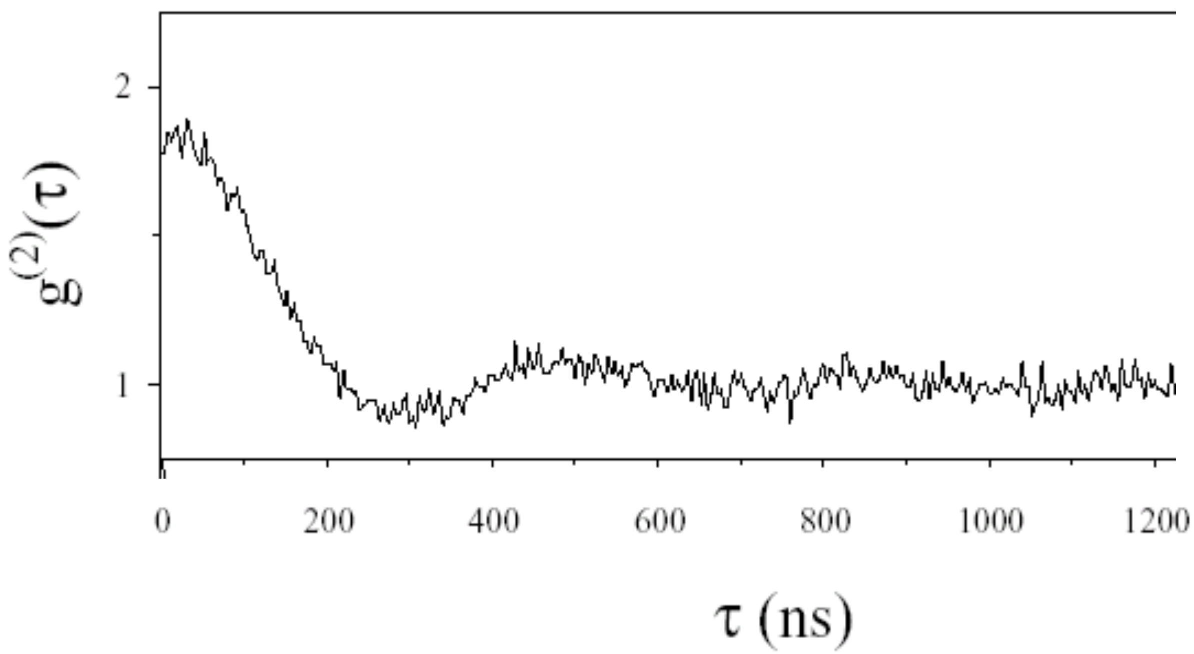
Discretize:

I_i

Multiply with displacement :

I_{i+n}

Add and average:



Another form to measure the correlation with with the waiting time distribution of the photons. The minimum size of the variance of the electromagnetic field.

- Store the time separation between two consecutive pulses (start and stop).
- Histogram the separations
- If the fluctuations are few you get after normalization $g^{(2)}(\tau)$.
- Work at low intensities (low counting rates).

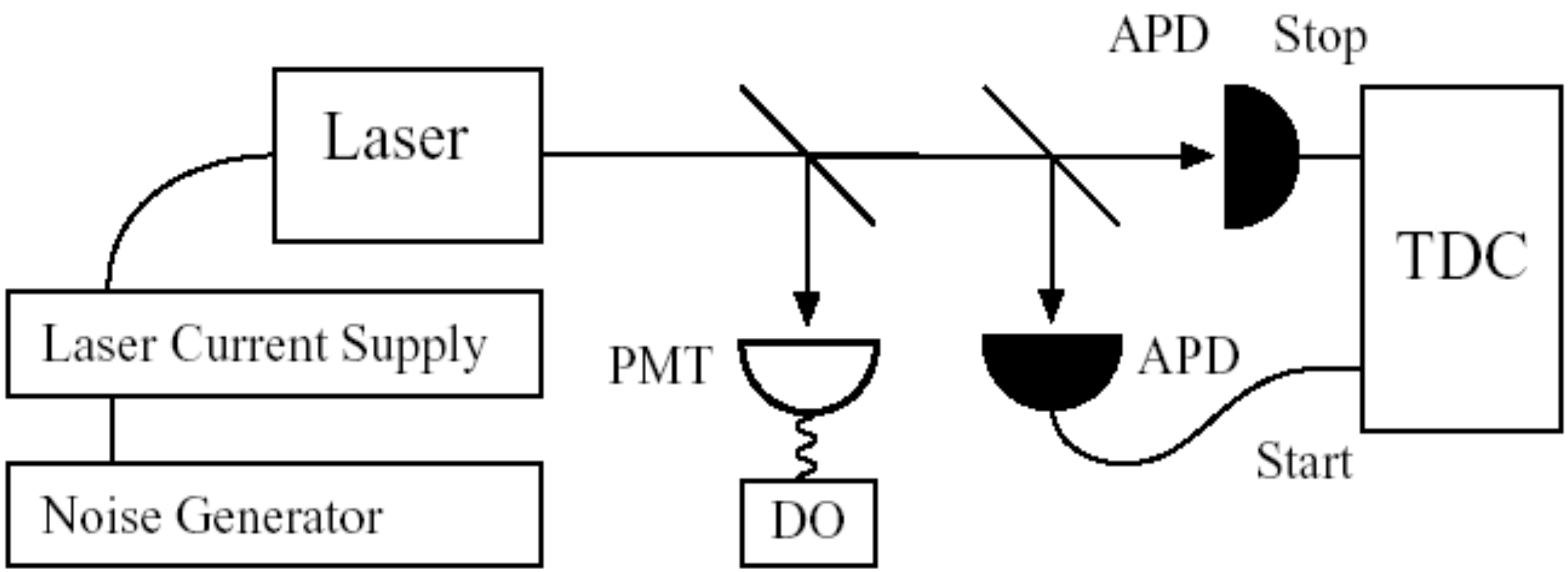
Intensity (photons)

time

Use a time stamp card. Later process the data, then you can calculate all sorts of correlations.

Digitize the full signal (important to identify the nature of the event *e.g.* particle physics).

Simultaneous measurement of $g^{(2)}(\tau)$.



Digital storage oscilloscope (DO) captures the photocurrent from the photomultiplier tube (PMT). The correlation is calculated later

Waiting time distribution with avalanche photodiodes (APD). The time to digital converter (TDC) stores the intervals for later histogramming.

Intensity correlations of a diode laser driven by noise.

$$I(t) = \alpha i(t) \qquad\qquad i(t) = i_0 + i_{noise}(t)$$

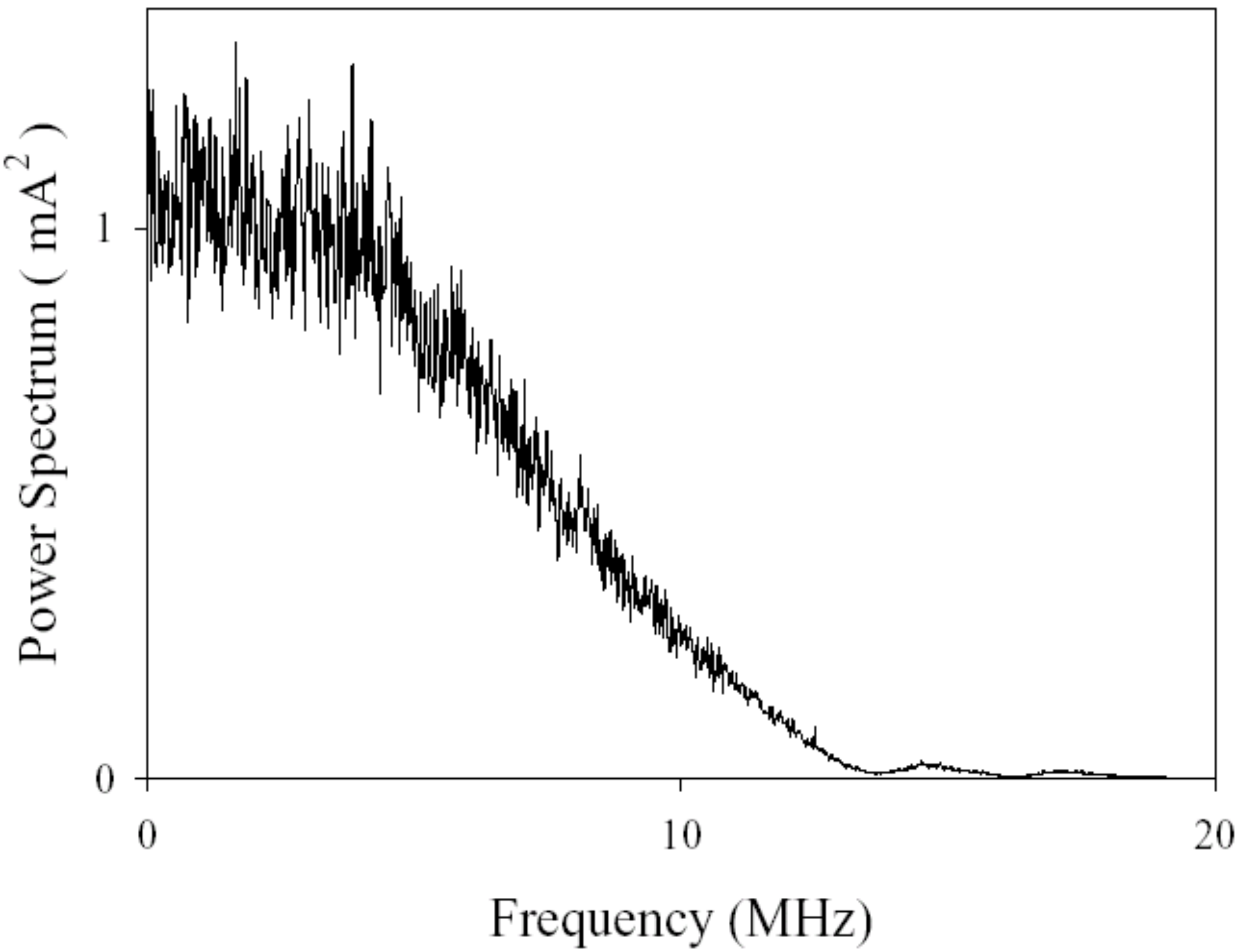
The noise term averages to zero.

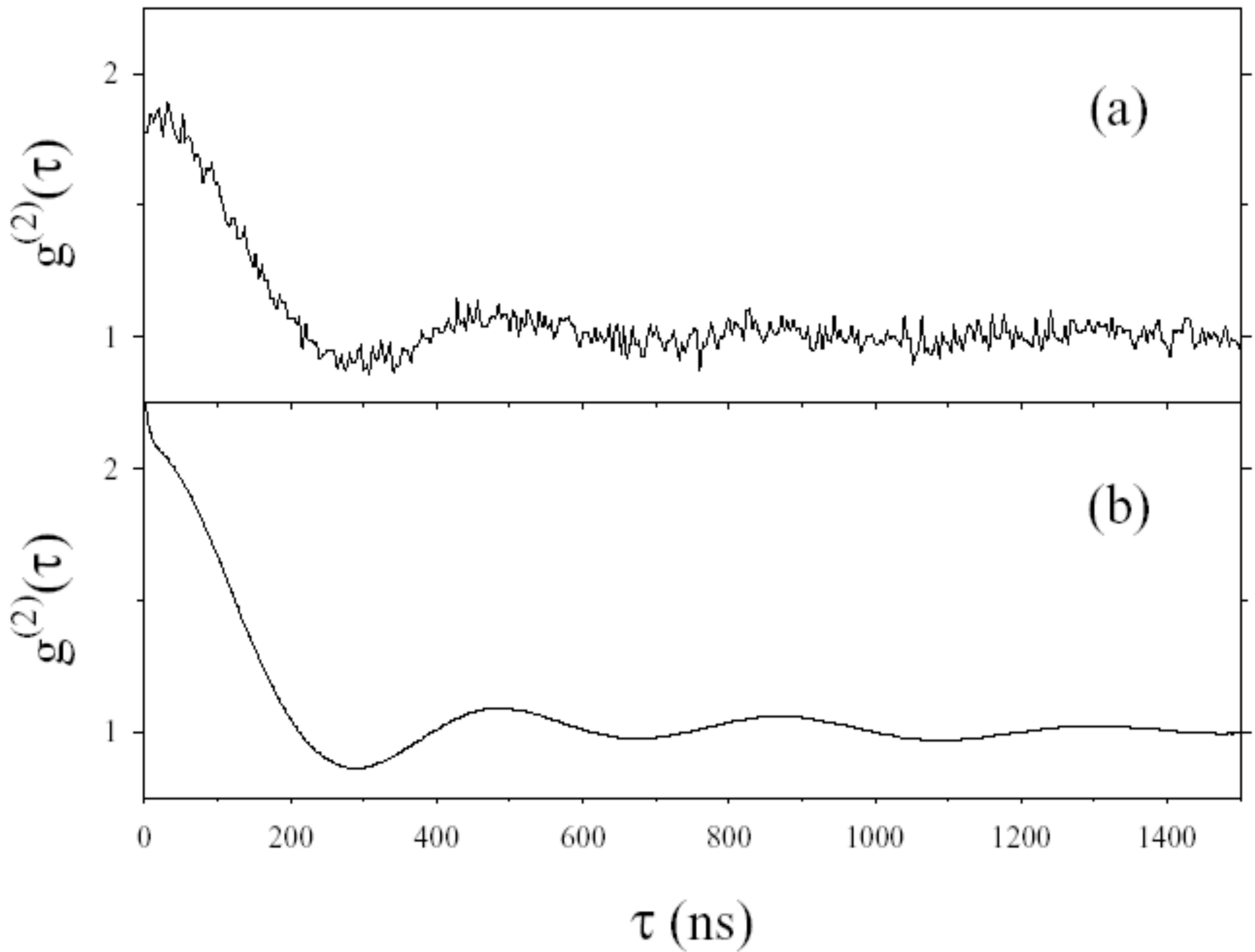
$$g^{(2)}(\tau) = 1 + \frac{\langle i_{noise}(t) i_{noise}(t+\tau)\rangle}{i_0^2}$$

The Wiener-Khintchine-Kolmogorov gives the unnormalized correlation function $G^{(2)}(t)$ from the power spectral density of the noise.

$$G(\tau) = \frac{1}{\pi}\int_0^\infty F(\omega) e^{-i\omega\tau} dw = \langle i_{noise}(t) i_{noise}(t+\tau)\rangle$$

Power spectrum of the noise source.





Comparison of $g^{(2)}(\tau)$: (a) waiting time distribution of photon counts, (b) time series of photocurrent.

Quantum optics

- The photon is the smallest fluctuation of the intensity of the intensity of the electromagnetic field, its variance.
- The photon is the quantum of energy of the electromagnetic field. With energy $\hbar\omega$ at frequency ω.

An important point about the quantum calculation $g^{(2)}(\tau)$

Quantum Correlations (Glauber):

$$g^{(2)}(\tau) = \frac{\langle T : \hat{I}(t)\hat{I}(t+\tau) : \rangle}{\langle \hat{I}(t) \rangle^2}$$

The intensity operator *I* is proportional to the number of photons, but the operators have to be normal (:) and time (*T*) ordered. All the creation operators do the left and the annihilation operators to the right (just as a photodetector works). The operators act in temporal order.

R. Glauber, *"The Quantum Theory of Optical Coherence,"* Phys. Rev. **130**, 2529 (1963).

At equal times (normal order) :

$$g^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2}.$$

Conmutator : $\hat{a}^{+}\hat{a} = \hat{a}\,\hat{a}^{+} - 1$

$$\left\langle \hat{a}^{+}\hat{a}^{+}\hat{a}\,\hat{a} \right\rangle = \left\langle \hat{a}^{+}(\hat{a}\,\hat{a}^{+} - 1)\hat{a} \right\rangle = \left\langle \hat{a}^{+}\hat{a}\;\hat{a}^{+}\hat{a} \right\rangle - \left\langle \hat{a}^{+}\hat{a} \right\rangle$$

$$\left\langle \hat{a}^{+}\hat{a}^{+}\hat{a}\,\hat{a} \right\rangle = \left\langle \hat{n}^{2} \right\rangle - \left\langle \hat{n} \right\rangle \quad \text{where} \quad \hat{n} = \hat{a}^{+}\hat{a}$$

The correlation requires detecting two photons, so if we detect one, we have to take that into consideration in the accounting.

In terms of the variance of the photon number:

$$\sigma^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$$

$$g^{(2)}(0) = 1 + \frac{\sigma^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2}$$

The classical result says:

$$= 1 + \frac{\langle \delta(t)^2 \rangle}{I_0^2}$$

The quantum correlation function can be zero, as the detection changes the number of photons in the field. This is related to the variance properties: is the variance larger or smaller than the mean (Poissonian, Super-Poissonian or Sub-Poissonian).

$$g^{(2)}(0) = 1 + \frac{\sigma^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2}.$$

At equal times the value gives:

$g^{(2)}(0)=1$ Poissonian

$g^{(2)}(0)>1$ Super-Poissonian

$g^{(2)}(0)<1$ Sub-Poissonian

The slope at equal times:

$g^{(2)}(0)>g^{(2)}(0^{+})$ Bunched

$g^{(2)}(0)<g^{(2)}(0^{+})$ Antibunched

Classically we can not have Sub-Poissonian nor Antibunched.

Quantum Correlations (Glauber):

$$g^{(2)}(\tau) = \frac{\left\langle : \hat{I}(t)\, \hat{I}(t+\tau) : \right\rangle}{\left\langle \hat{I}(t) \right\rangle^2}$$

$$g^{(2)}(\tau) = \frac{\left\langle : \hat{I}(\tau) : \right\rangle_c}{\left\langle : \hat{I} : \right\rangle}$$

If we detect a photon at time *t* , $g^{(2)}(\tau)$ gives the probability of detecting a second photon after a time τ .

Correlation functions as conditional measurements in quantum optics.

- The detection of the first photon gives the initial state that is going to evolve in time.
- This may sound as Bayesian probabities.
- $g^{(1)}(t)$ Interferograms.
- $g^{(2)}(t)$ Hanbury-Brown and Twiss.
- Can be used in the process of quantum feedback.

Quantum regression theorem

The correlation functions can be calculated using the master equation with the appropriate initial and boundary conditions (Lax 1968).

This is somehow reminiscent of the propagation of the correlations using the wave equation for the electromagnetic (Wolf 1954, 55)

Are there quantum effects in optical cavity QED?

Look at the intensity fluctuations

Optical Cavity QED

Quantum electrodynamics for pedestrians. No need for renormalization. One or a finite number of modes from the cavity.

ATOMS + CAVITY

Regimes:

Perturbative: Coupling<< Dissipation. Atomic decay suppressed or enhanced (cavity smaller than $\lambda/2$), changes in the energy levels.

Non Perturbative: Coupling>>Dissipation

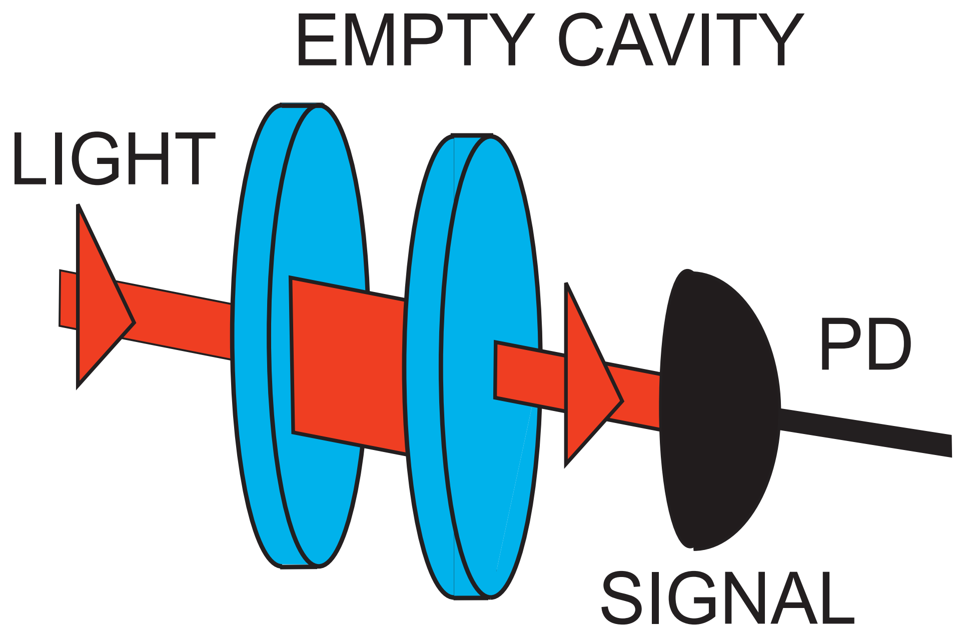
Vacuum Rabi Splittings. Conditional dynamics.

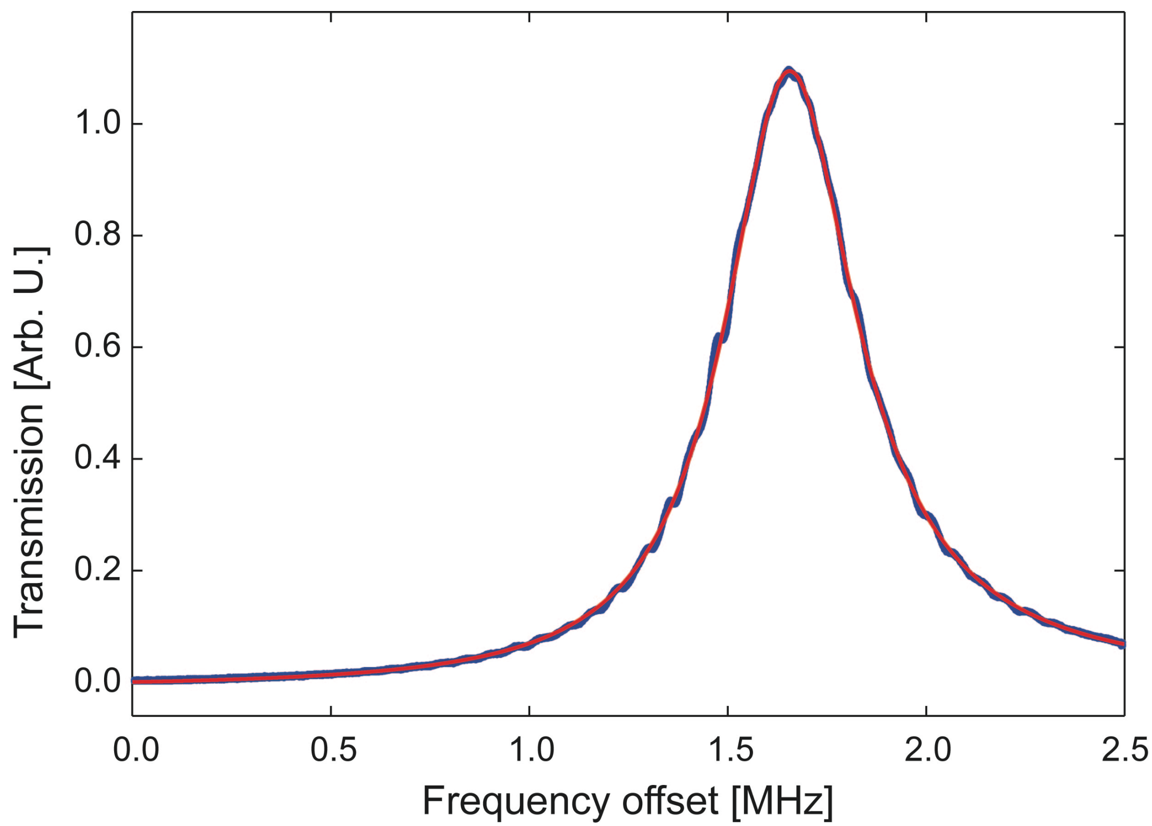
Dipolar coupling between the atom and the mode of the cavity:

$$g = \frac{d \cdot E_v}{\hbar}$$

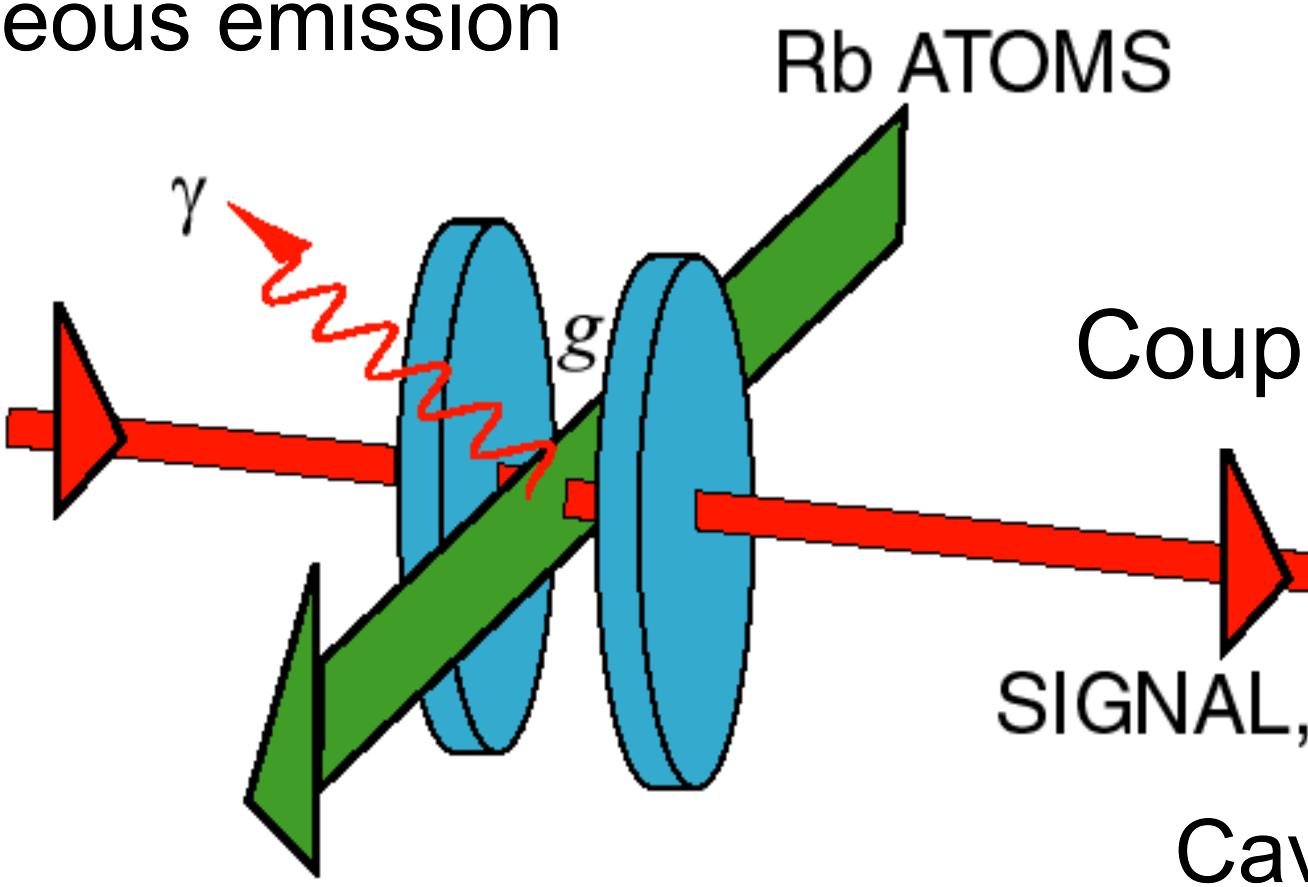
El electric field associated with one photon on average in the cavity with volume: V_{eff} is:

$$E_v = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V_{eff}}}$$


EMPTY CAVITY
LIGHT
PD
SIGNAL


Transmission [Arb. U.]
Frequency offset [MHz]
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
2.5

Spontaneous emission



Coupling

Cavity decay

Cooperativity for one atom: C_1

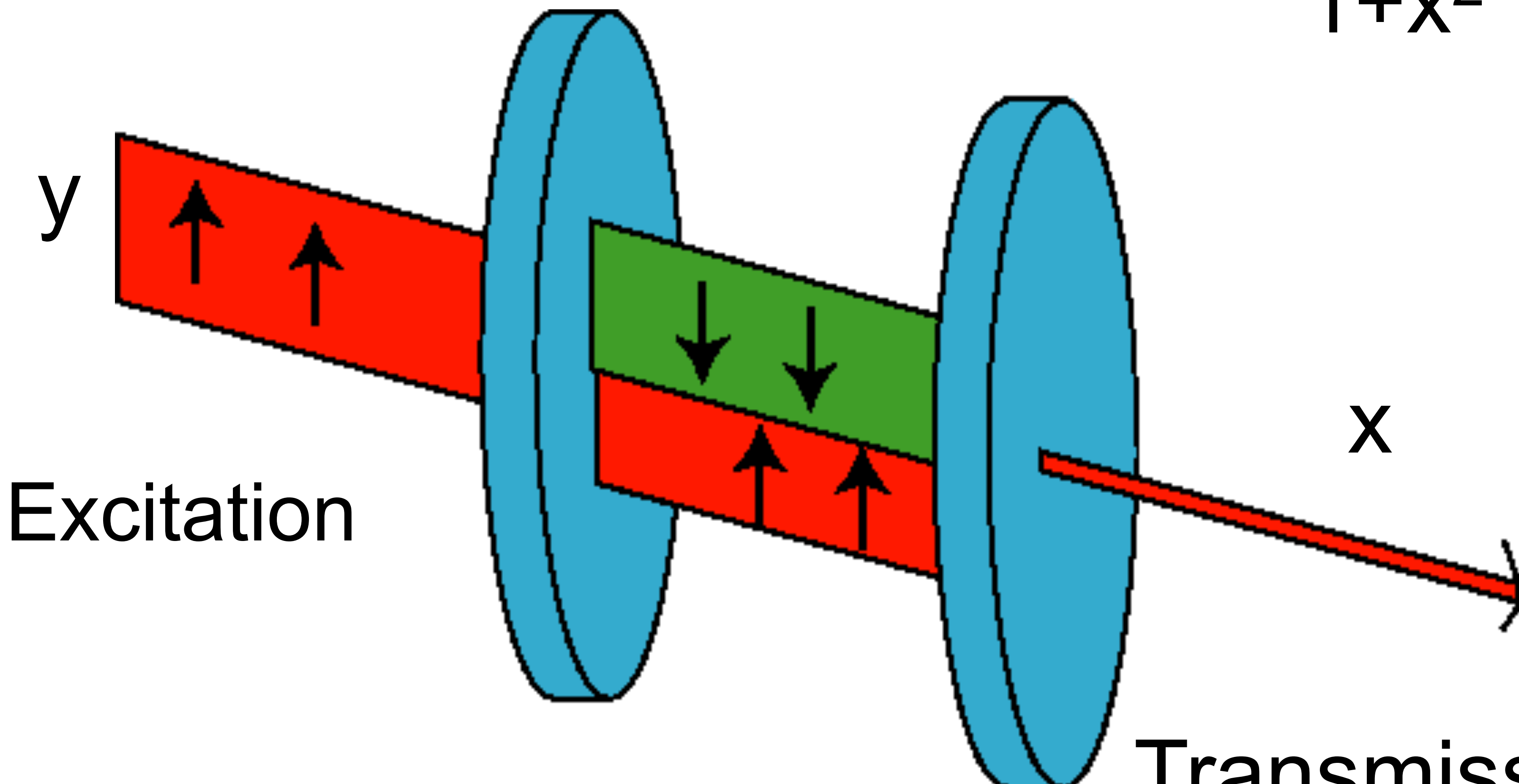
$$C_1 = \frac{g^2}{\kappa\gamma} \qquad C = C_1 N$$

Cooperativity for N atoms: C

$$g \approx \kappa \approx \gamma$$

Steady State

Atomic polarization: $\dfrac{-2Cx}{1+x^2}$

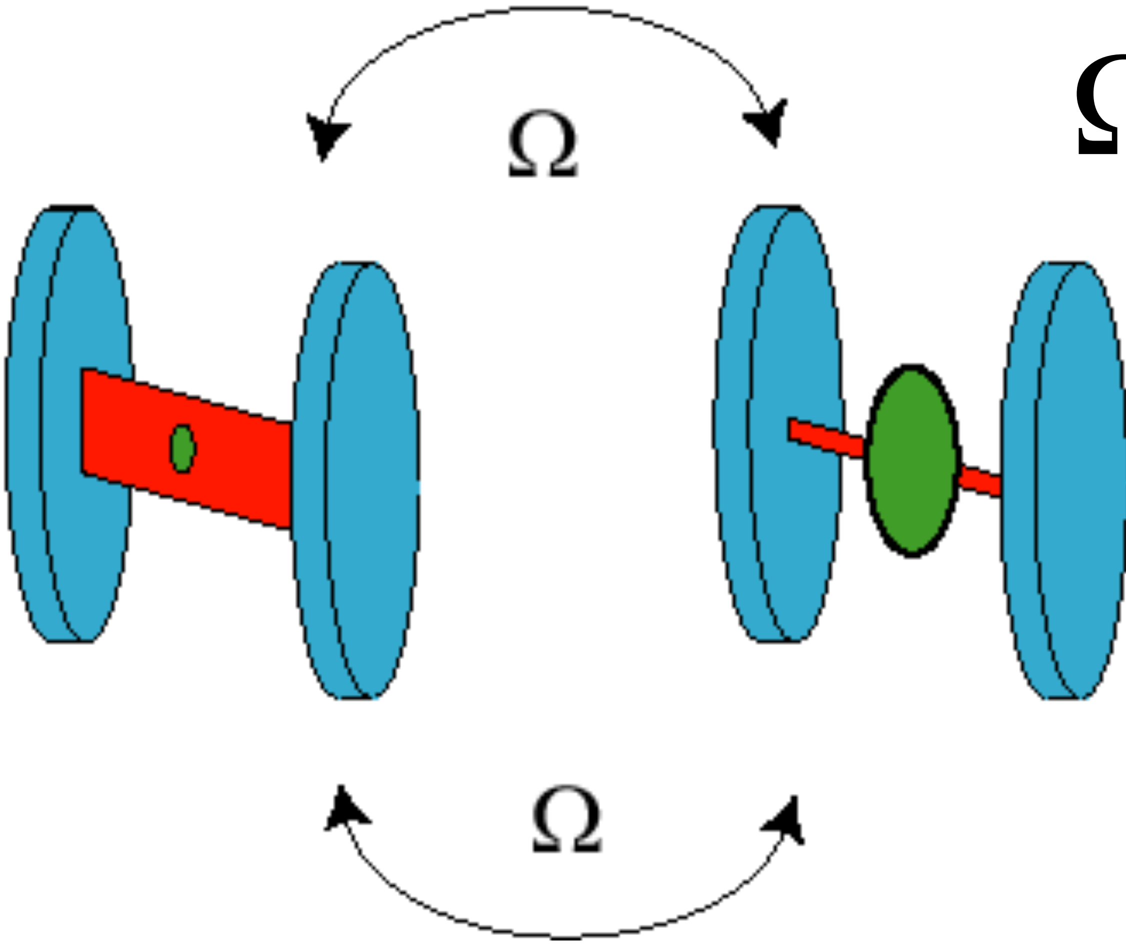


Transmission
$x/y = 1/(1+2C)$

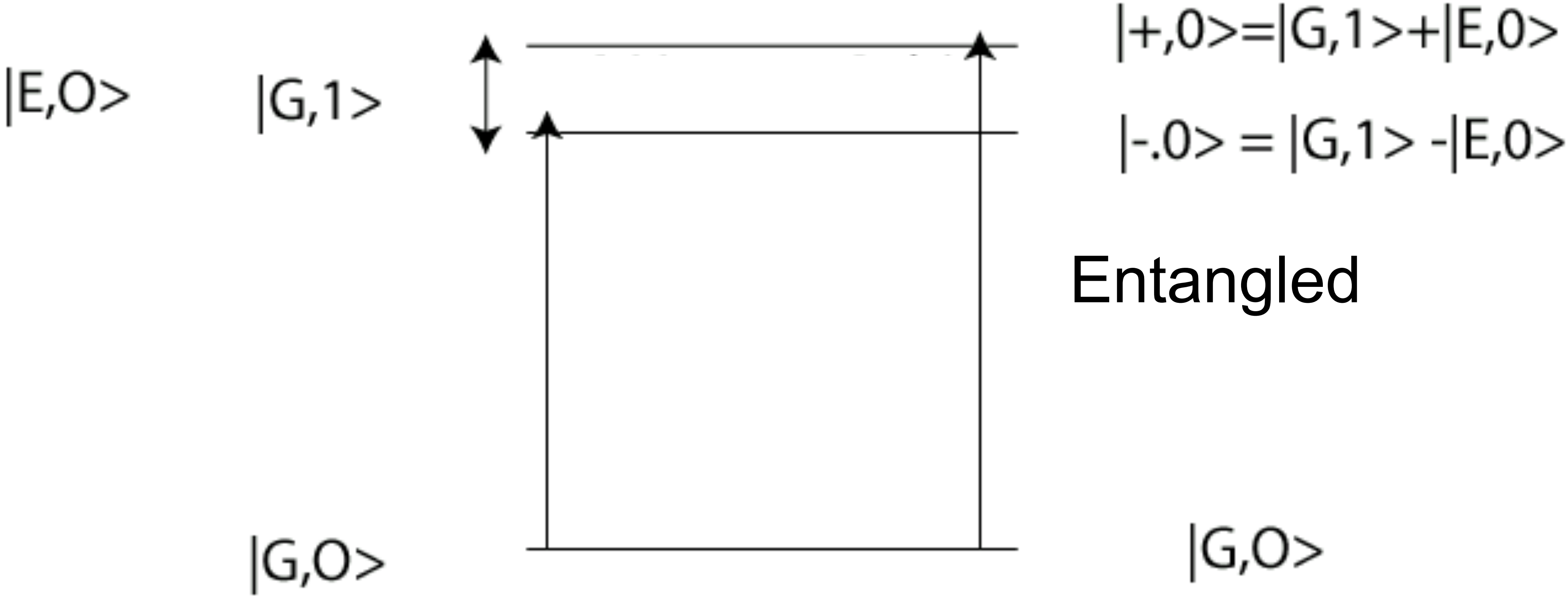
Jaynes Cummings Dynamics
Rabi Oscillations

Exchange of excitation for *N* atoms:

$$\Omega \approx g\sqrt{N}$$



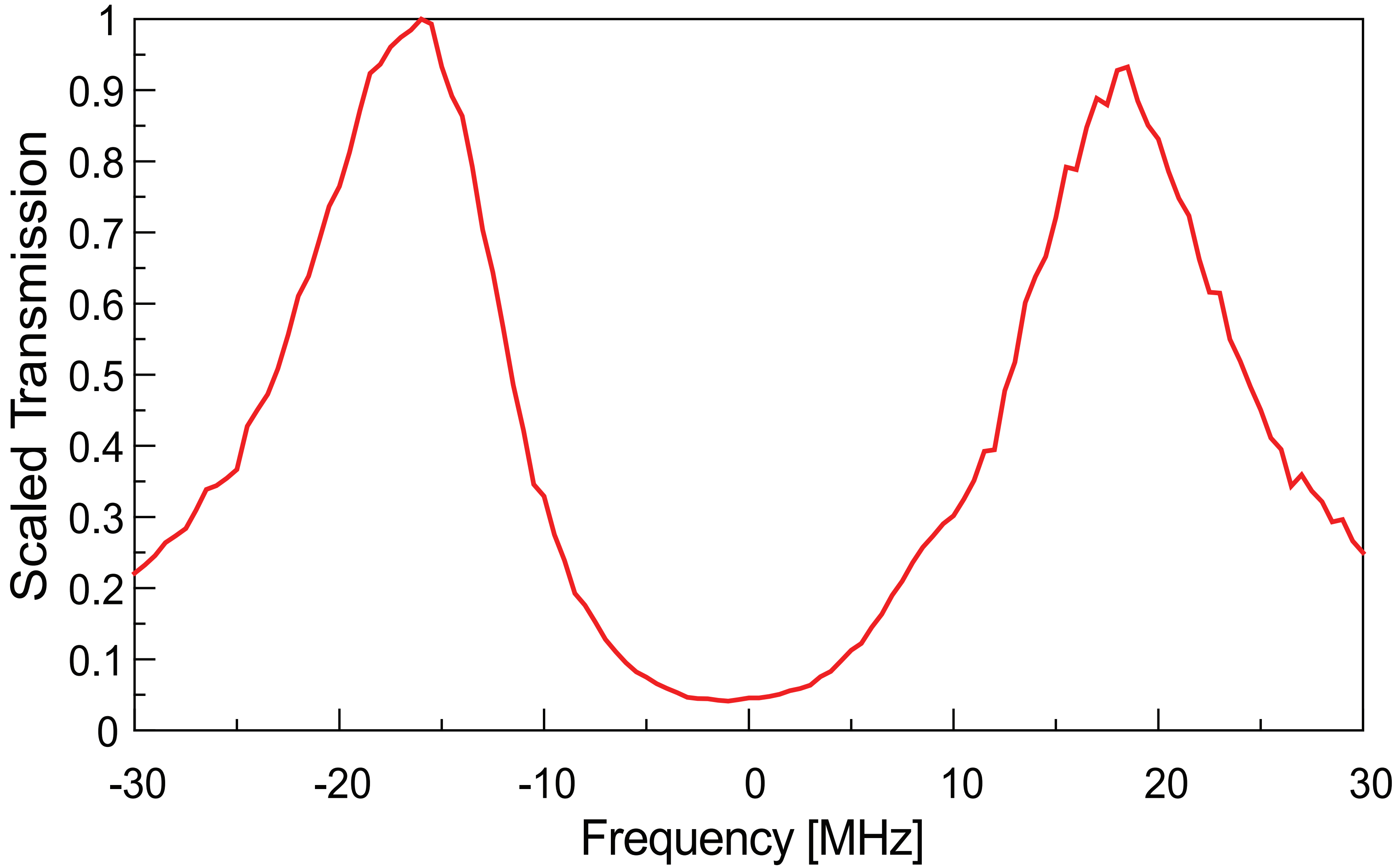
2g Vacuum Rabi Splitting

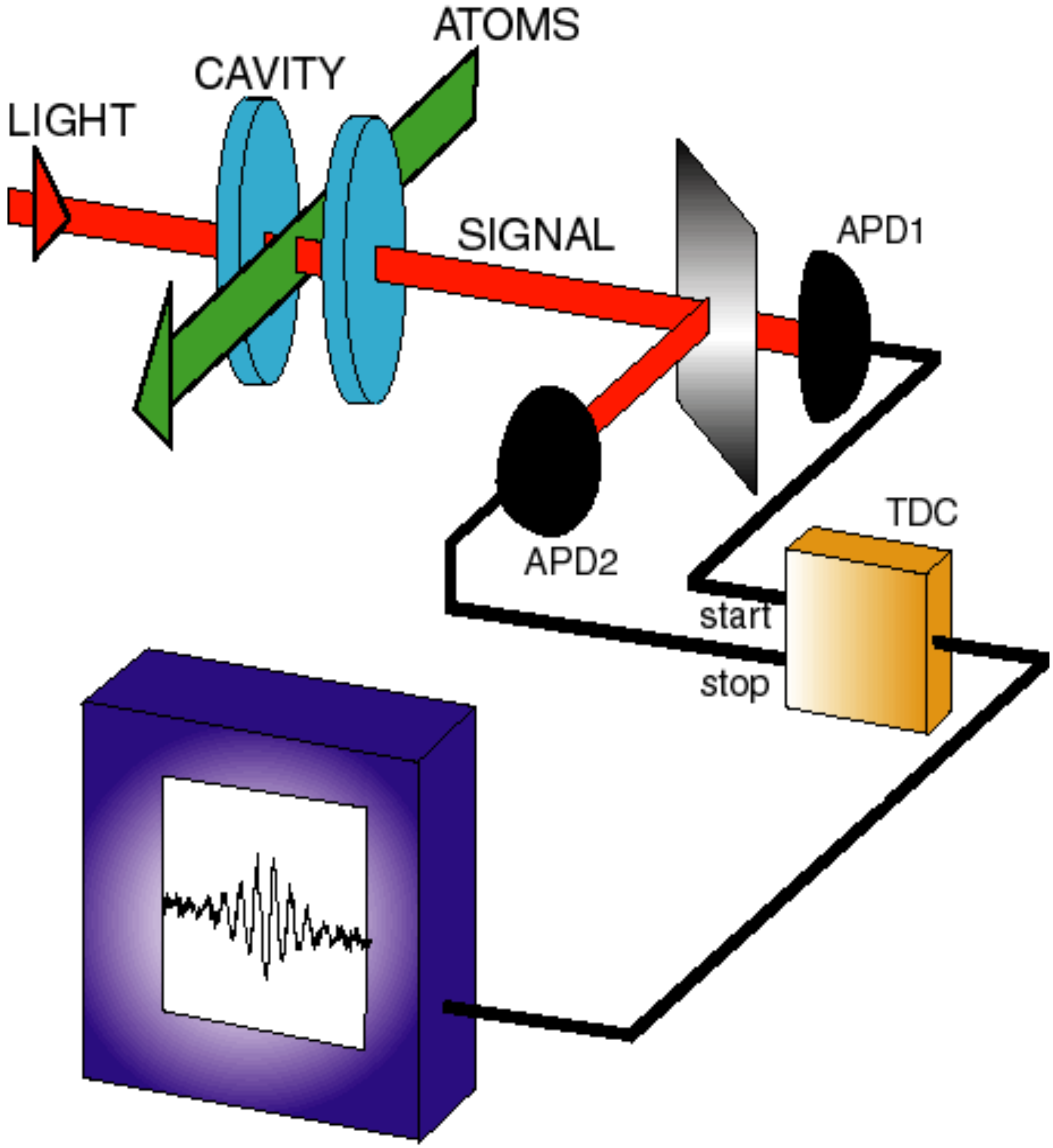


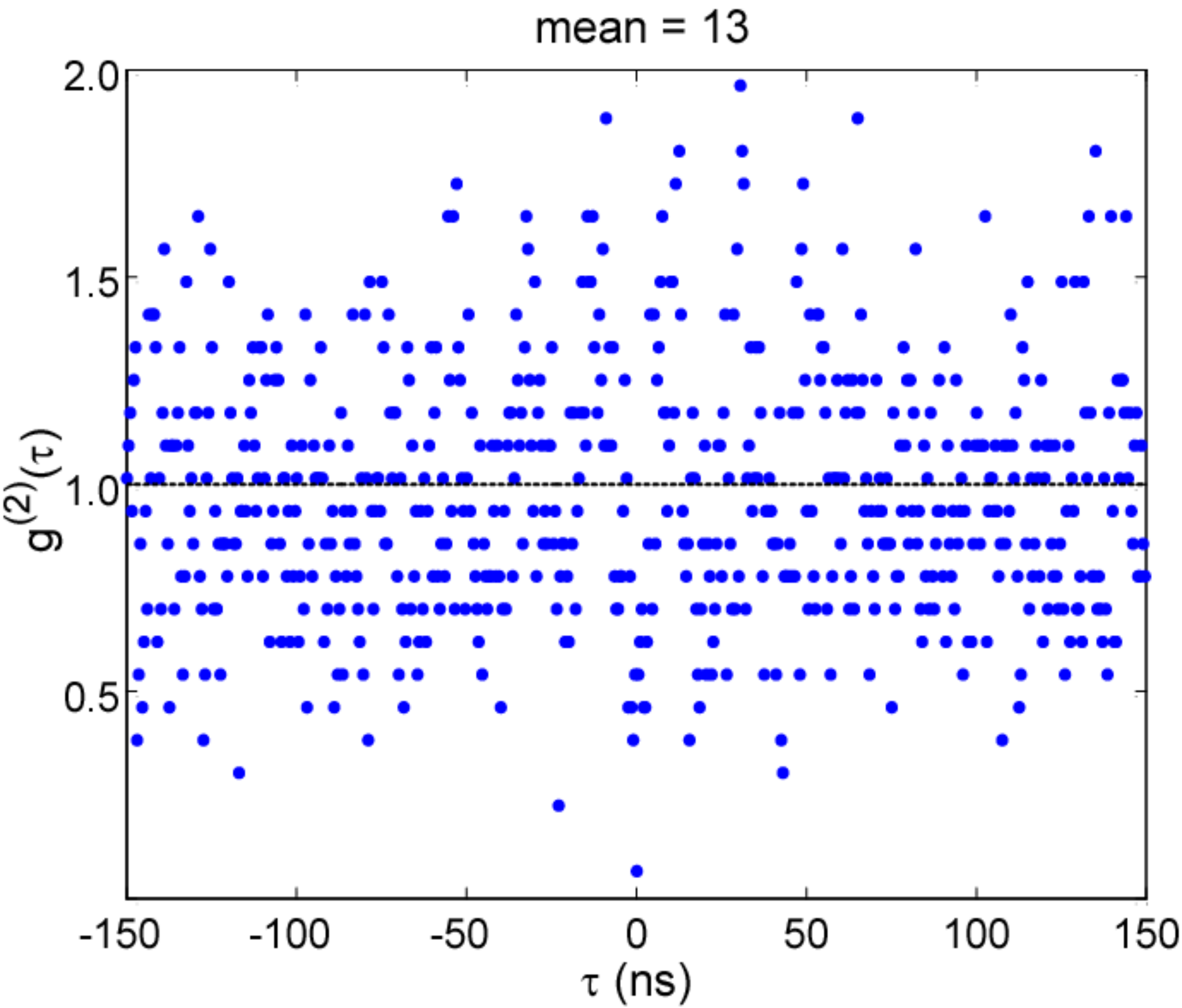
Not coupled

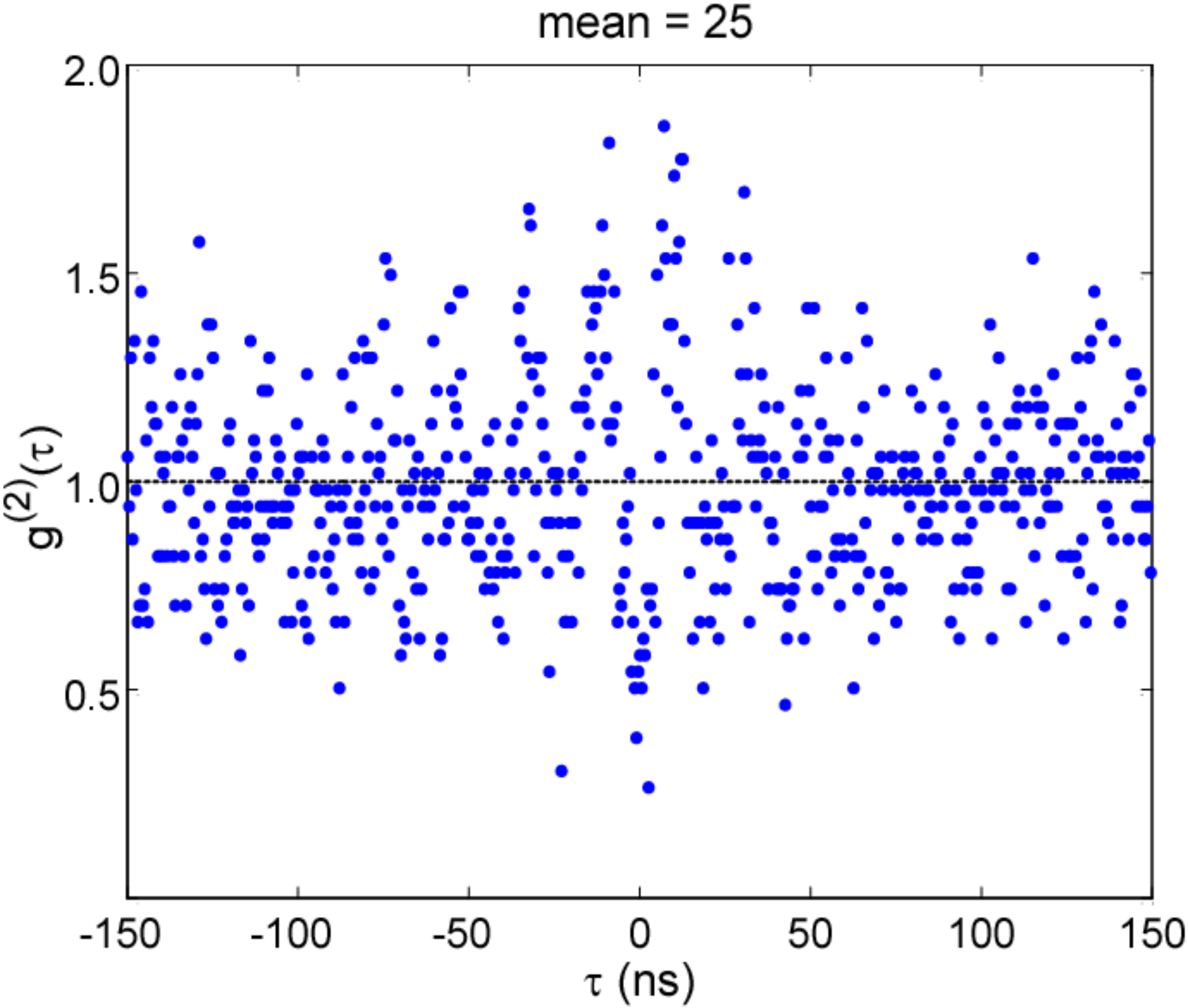
Two normal modes

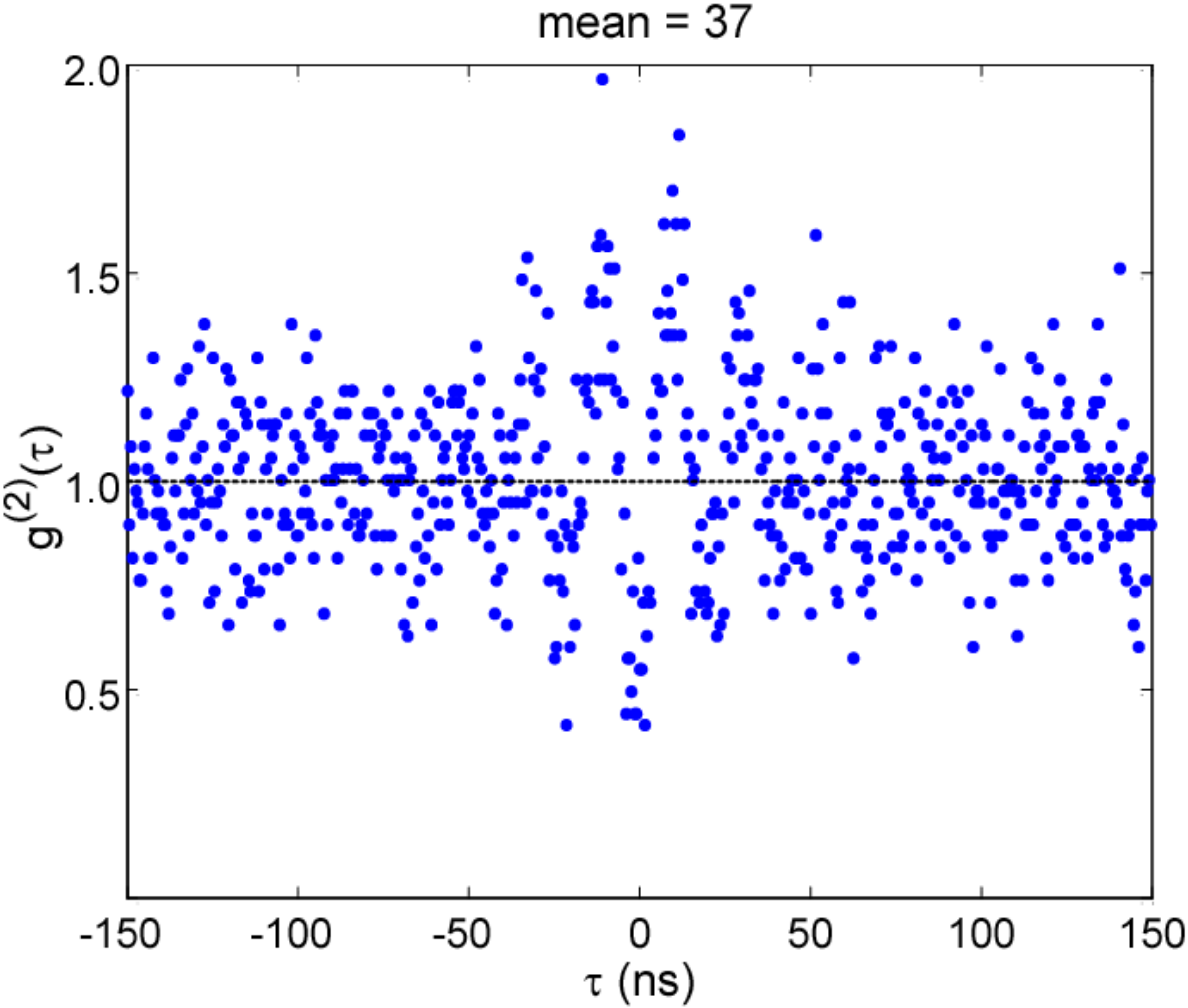
Transmission doublet different from the Fabry Perot resonance

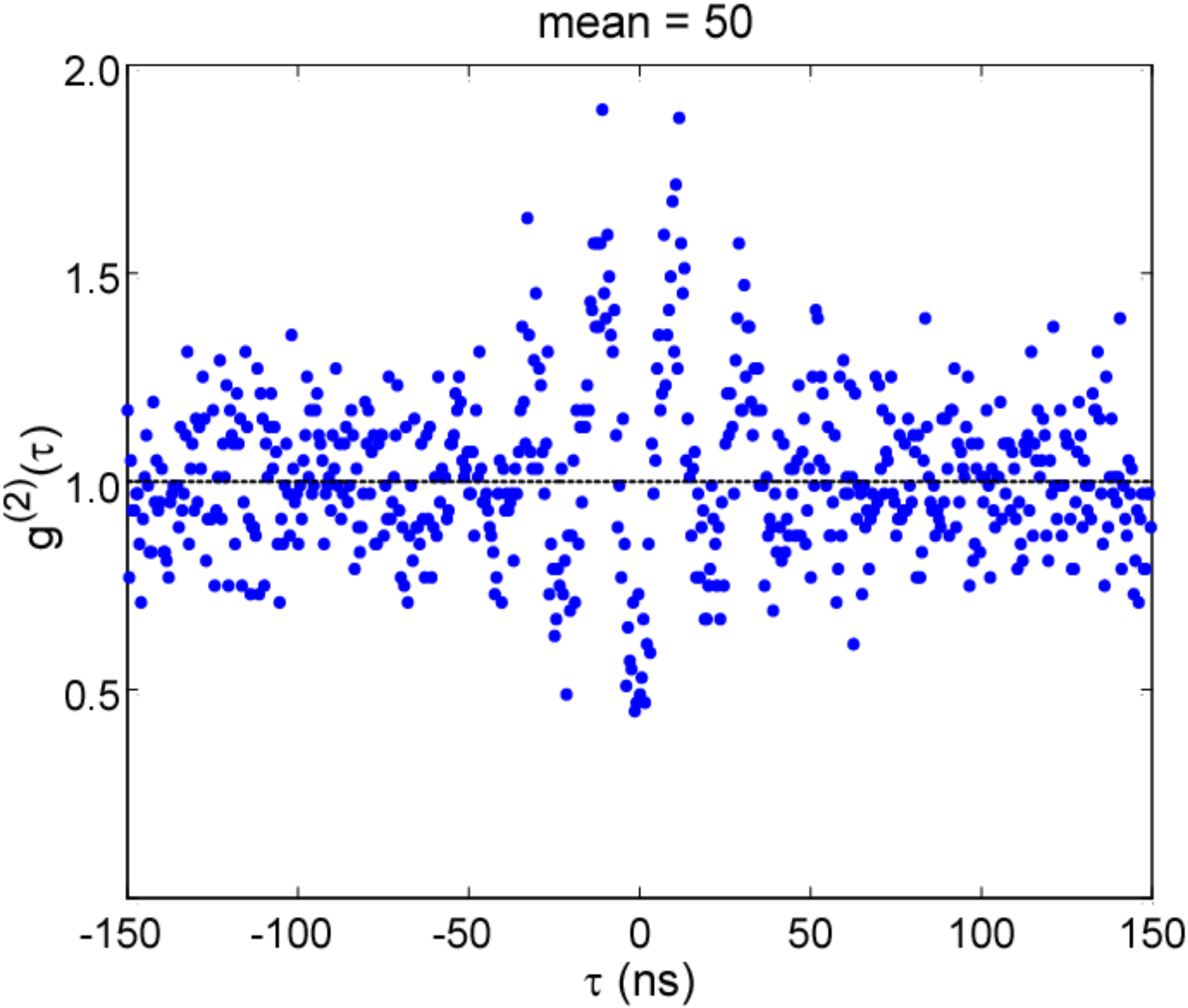



ATOMS
CAVITY
LIGHT
SIGNAL
APD1
APD2
TDC
start
stop

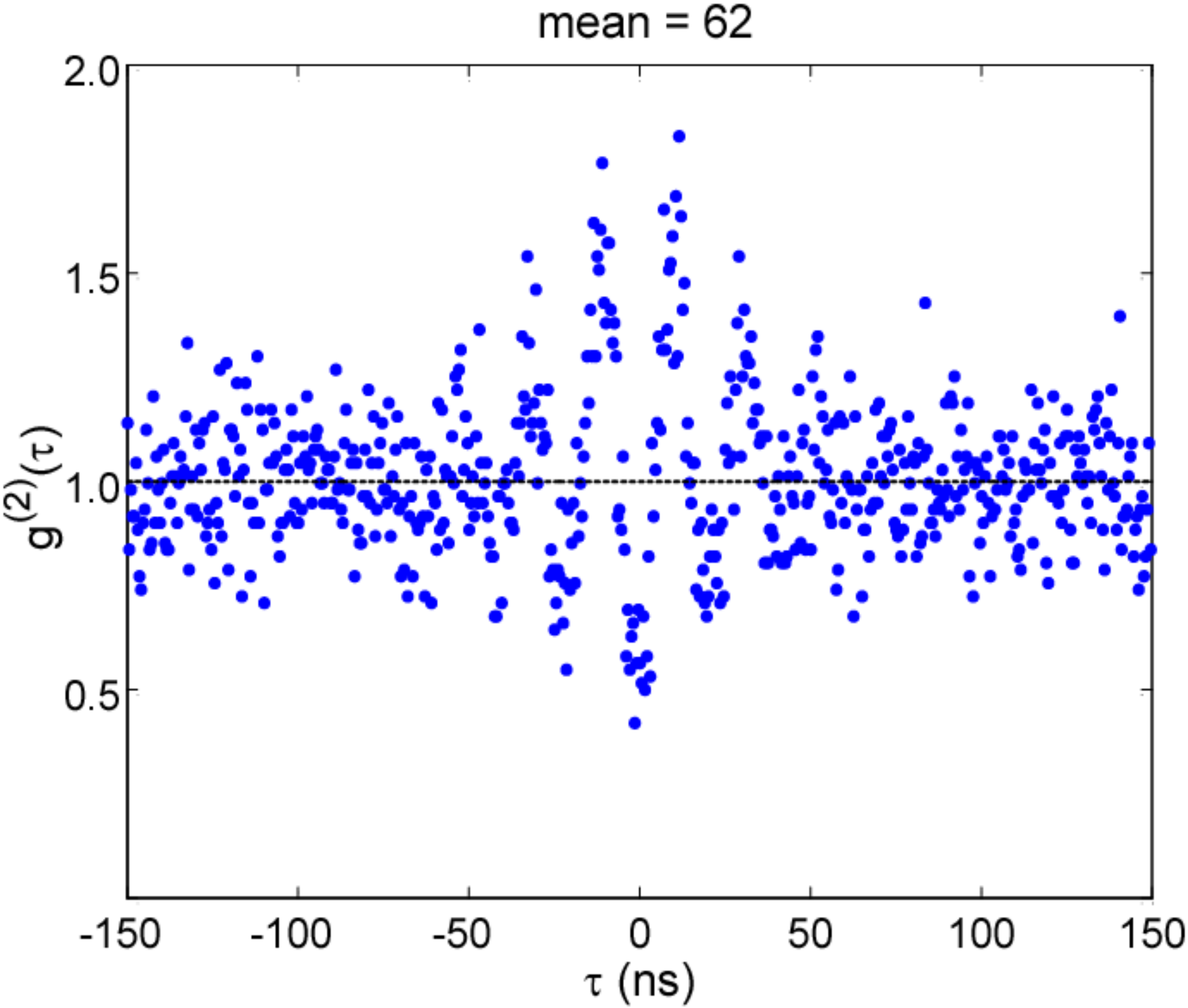

mean = 13
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

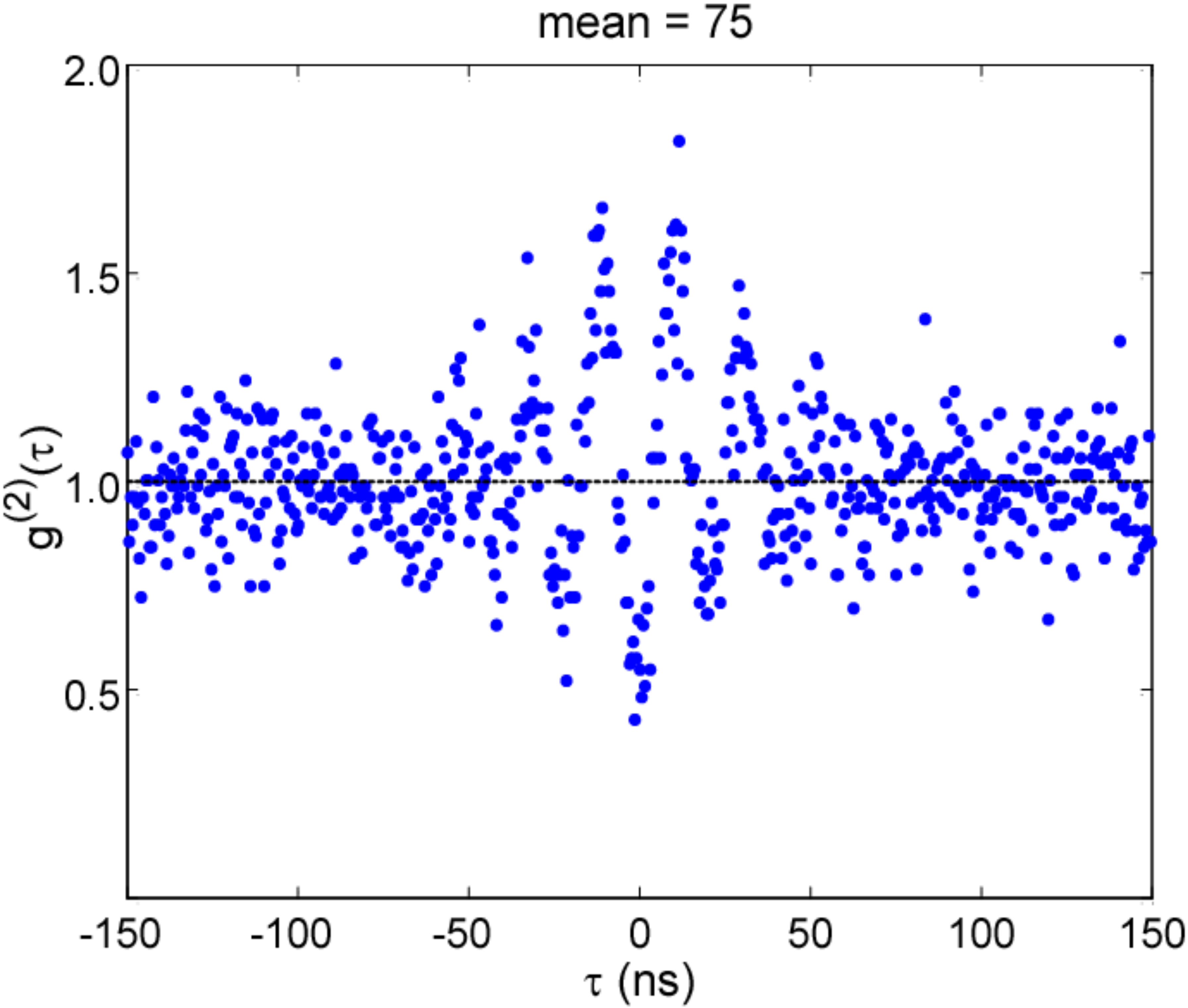

mean = 25
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)


mean = 37
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

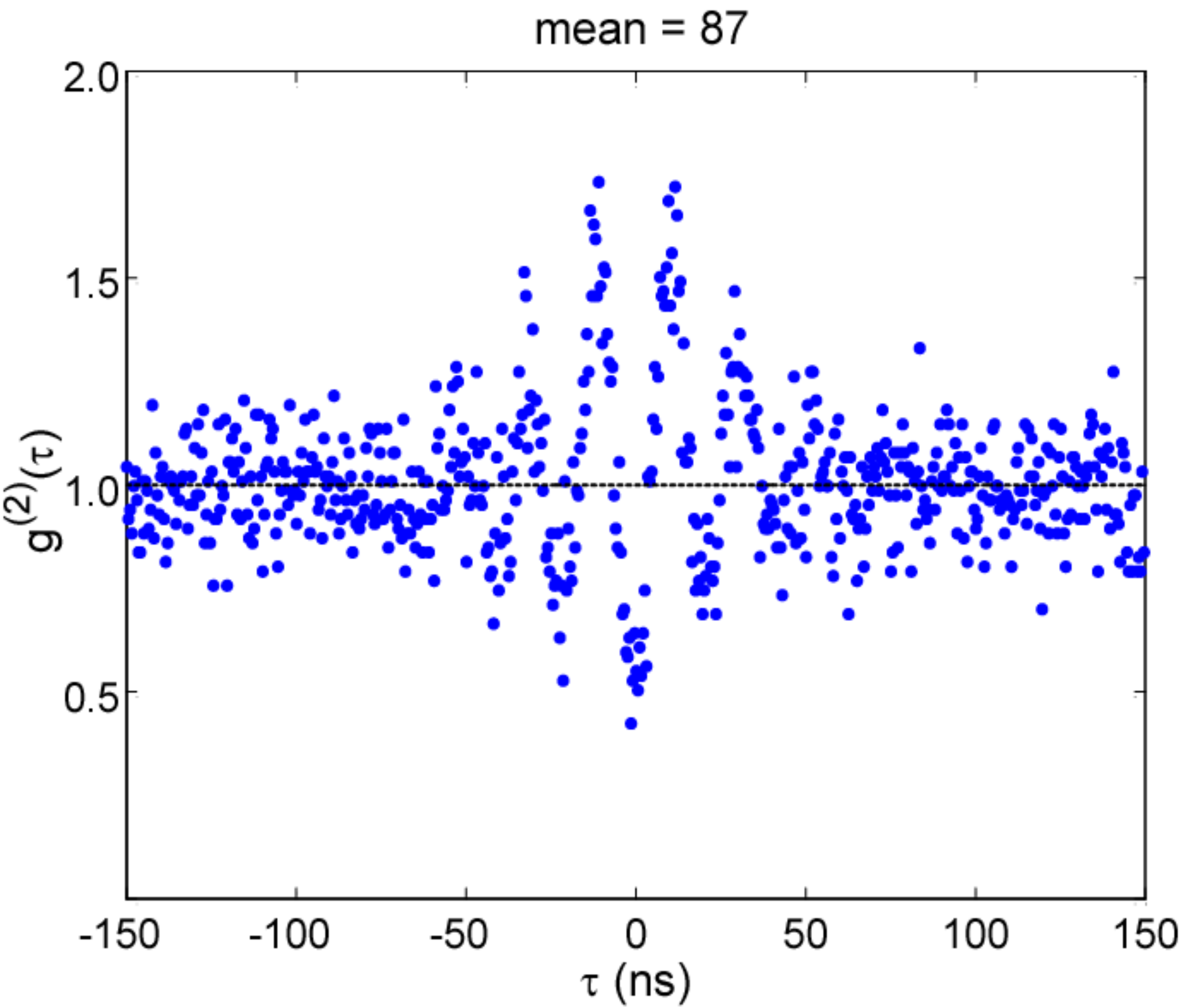


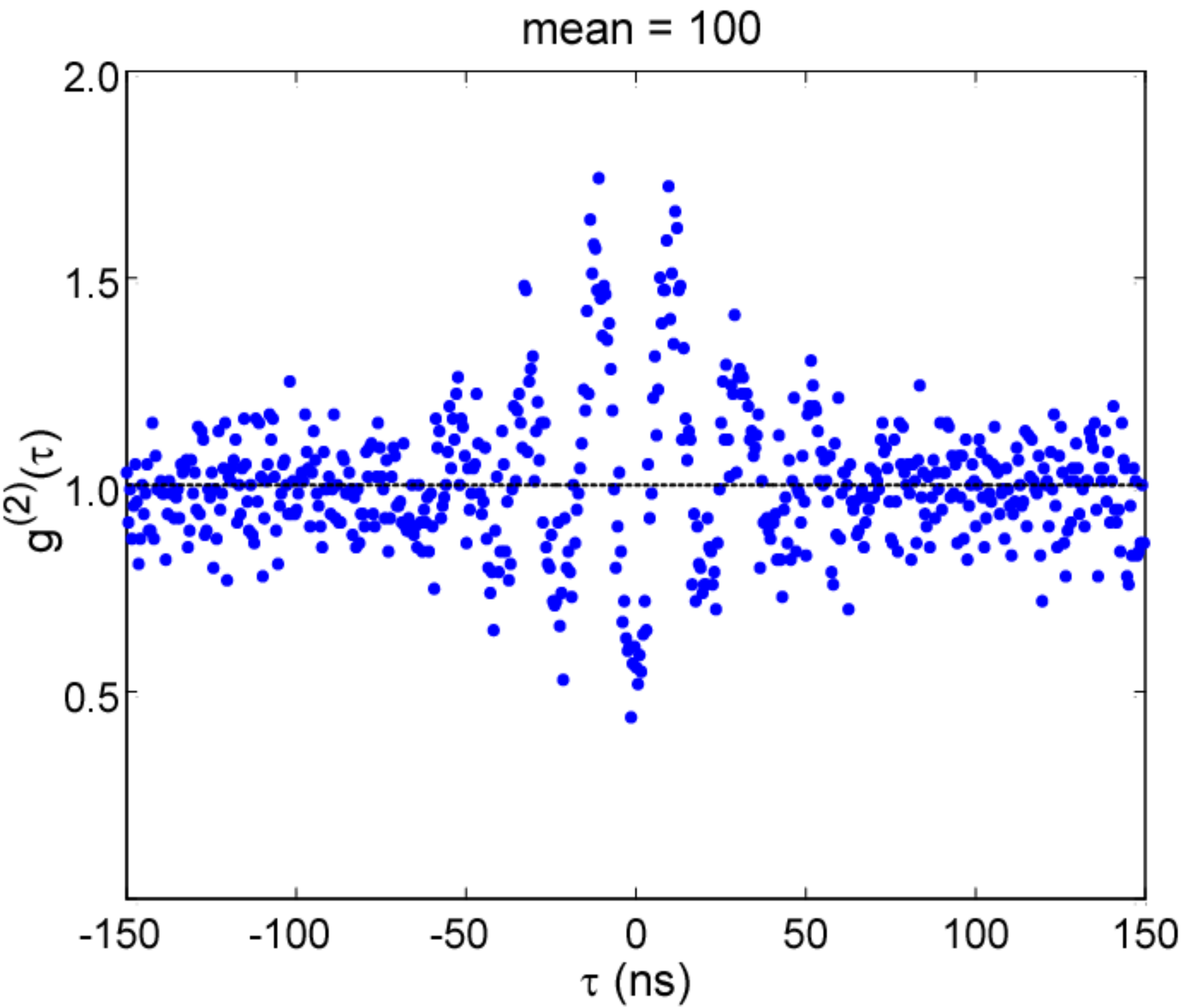
mean = 50
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)


mean = 62
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

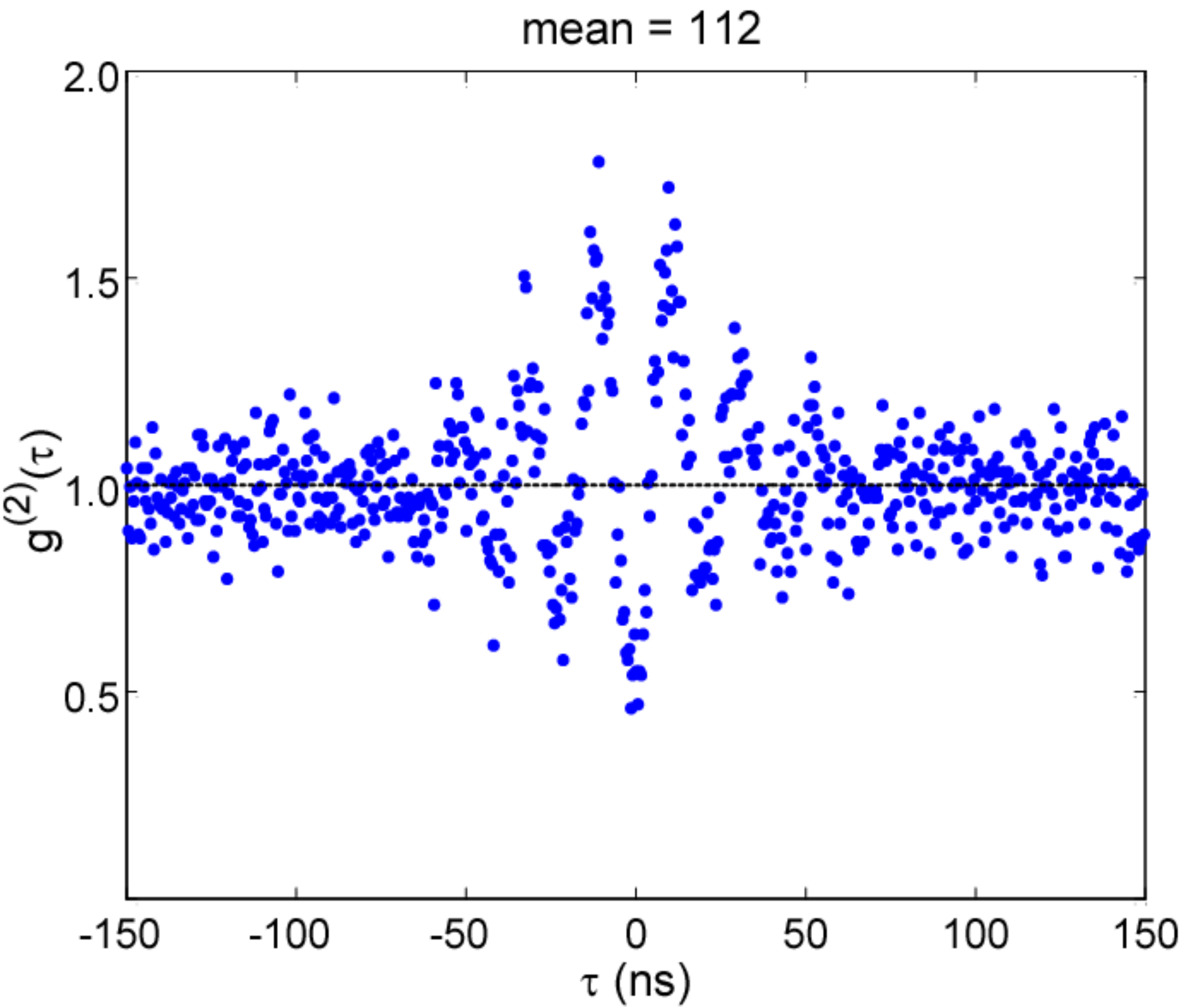


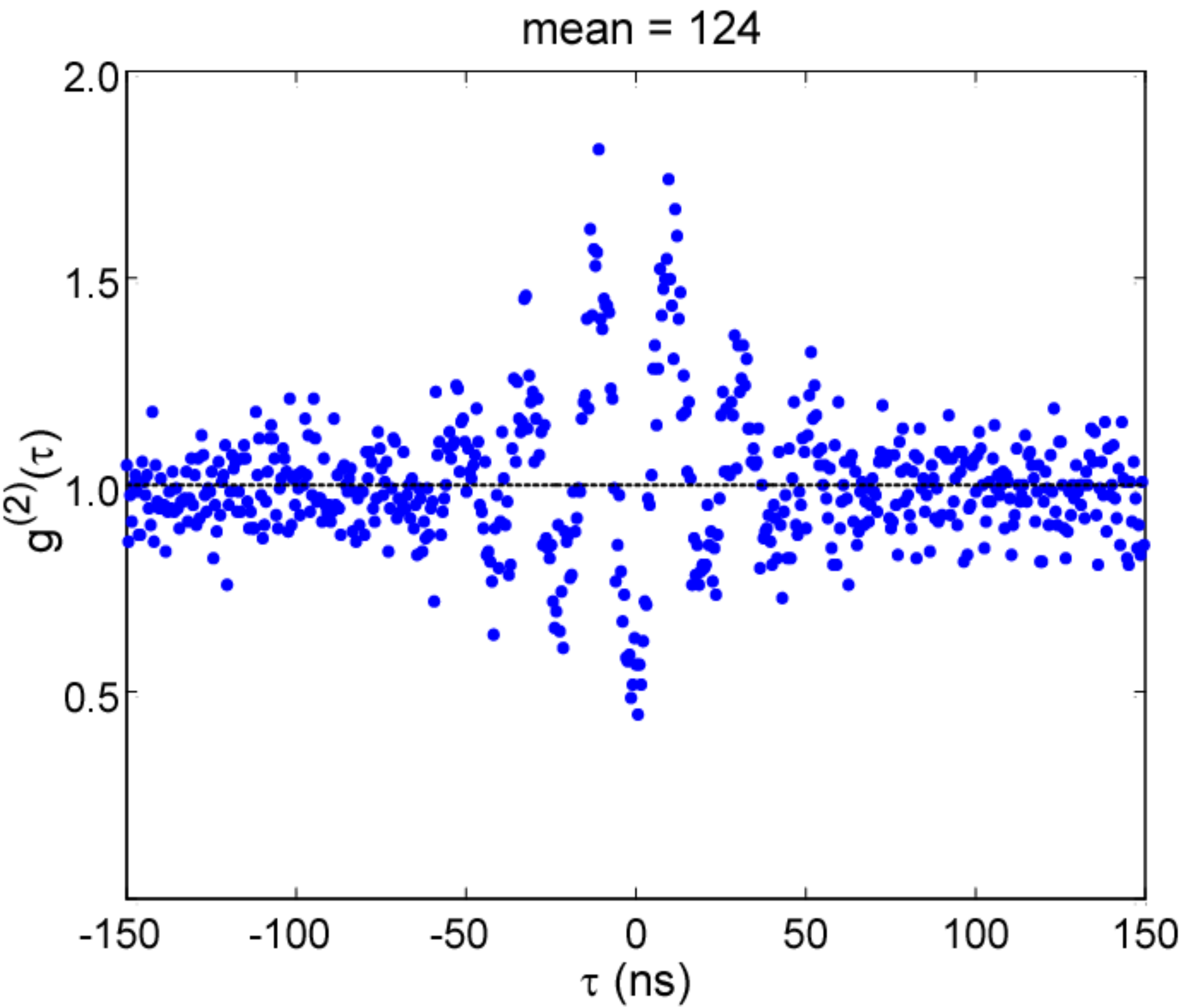
mean = 75
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

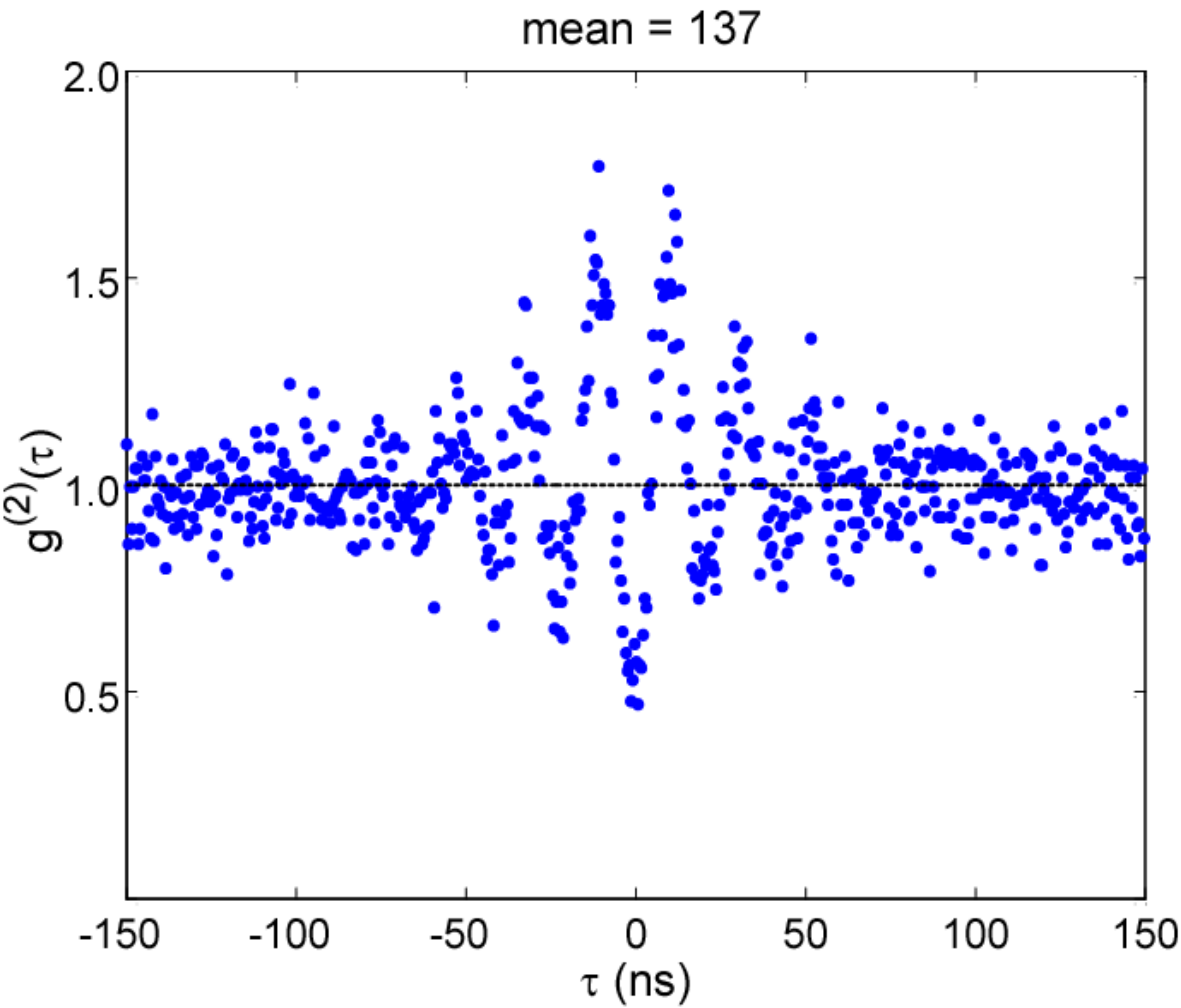

mean = 87
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

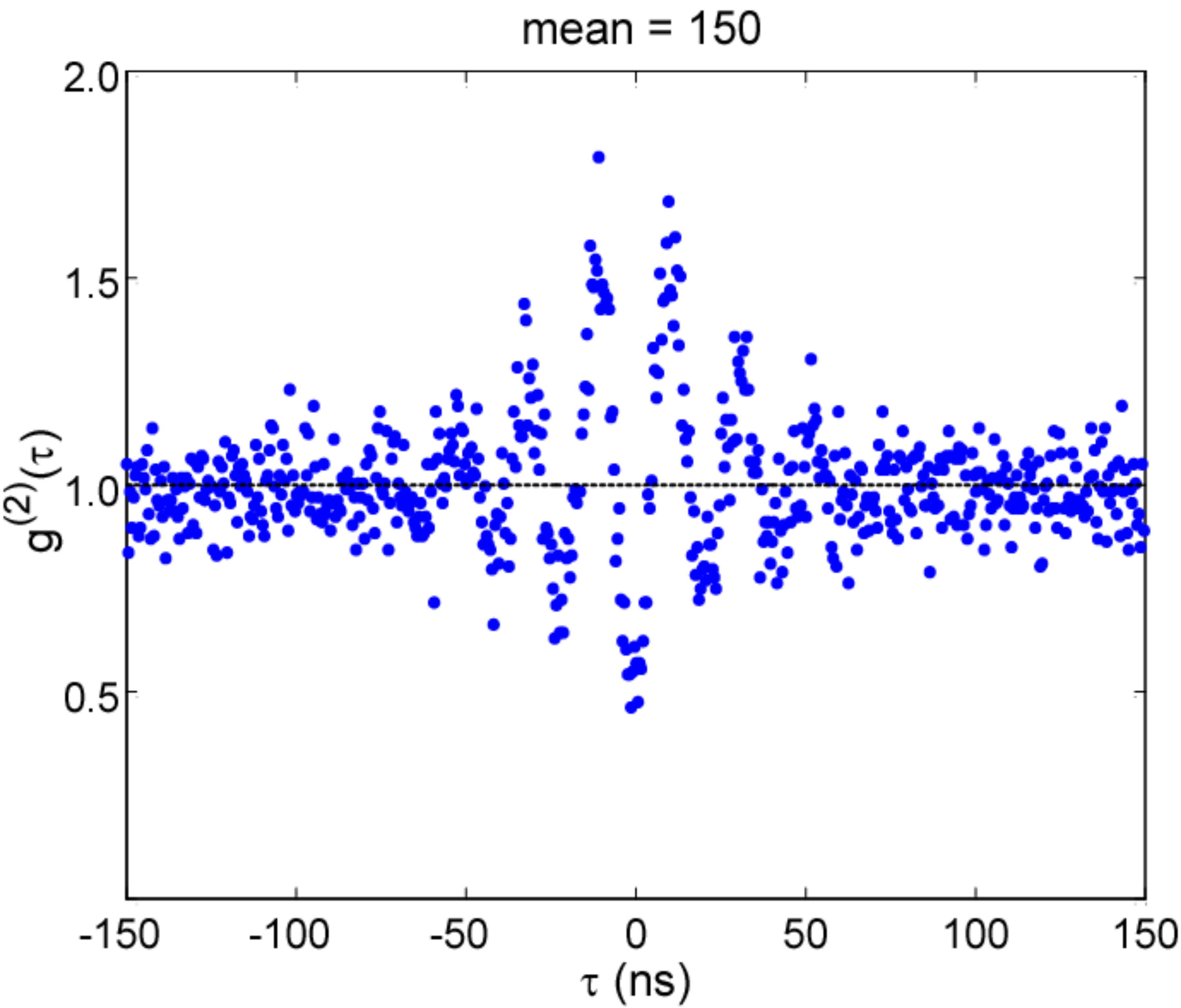


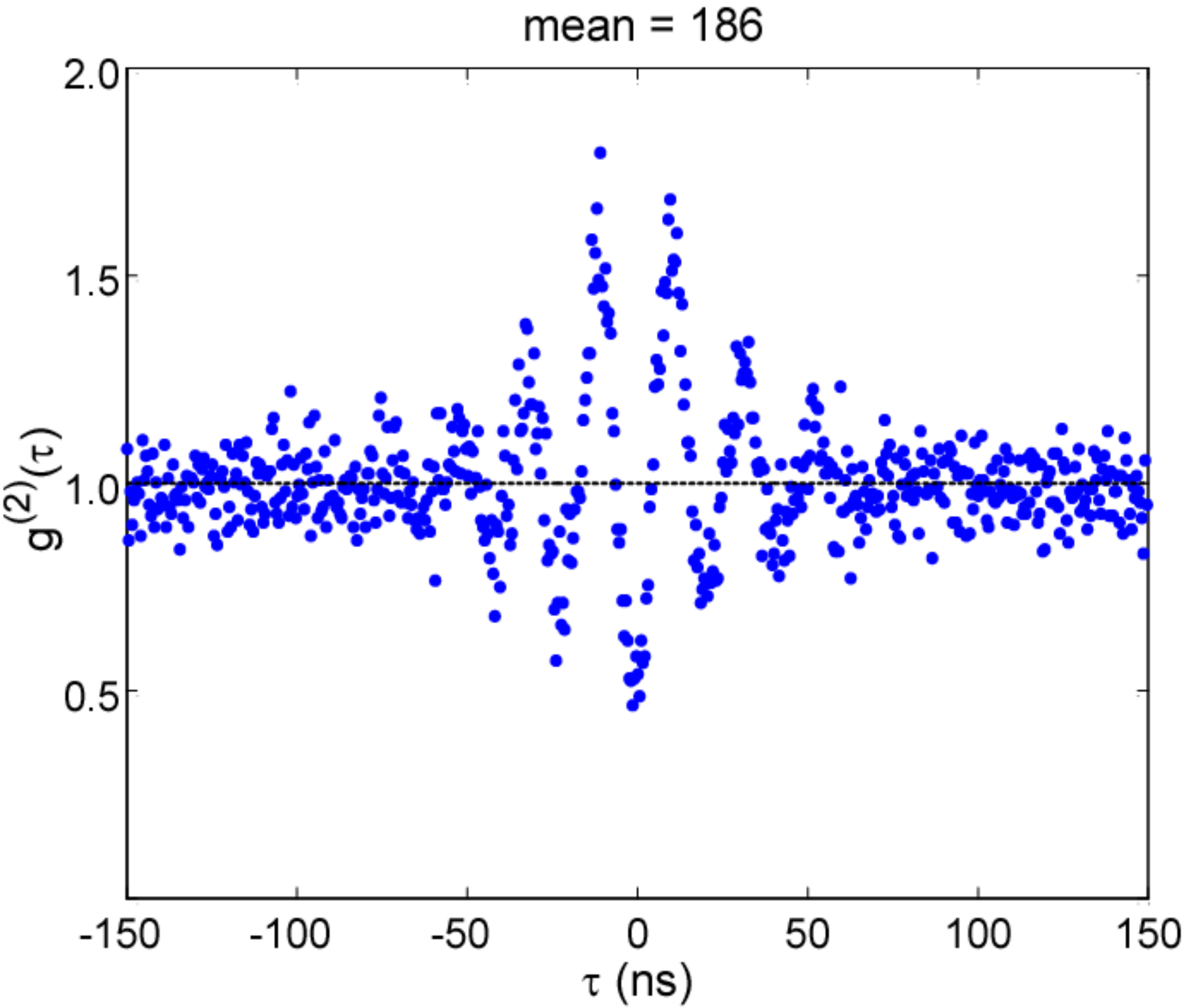
mean = 100
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

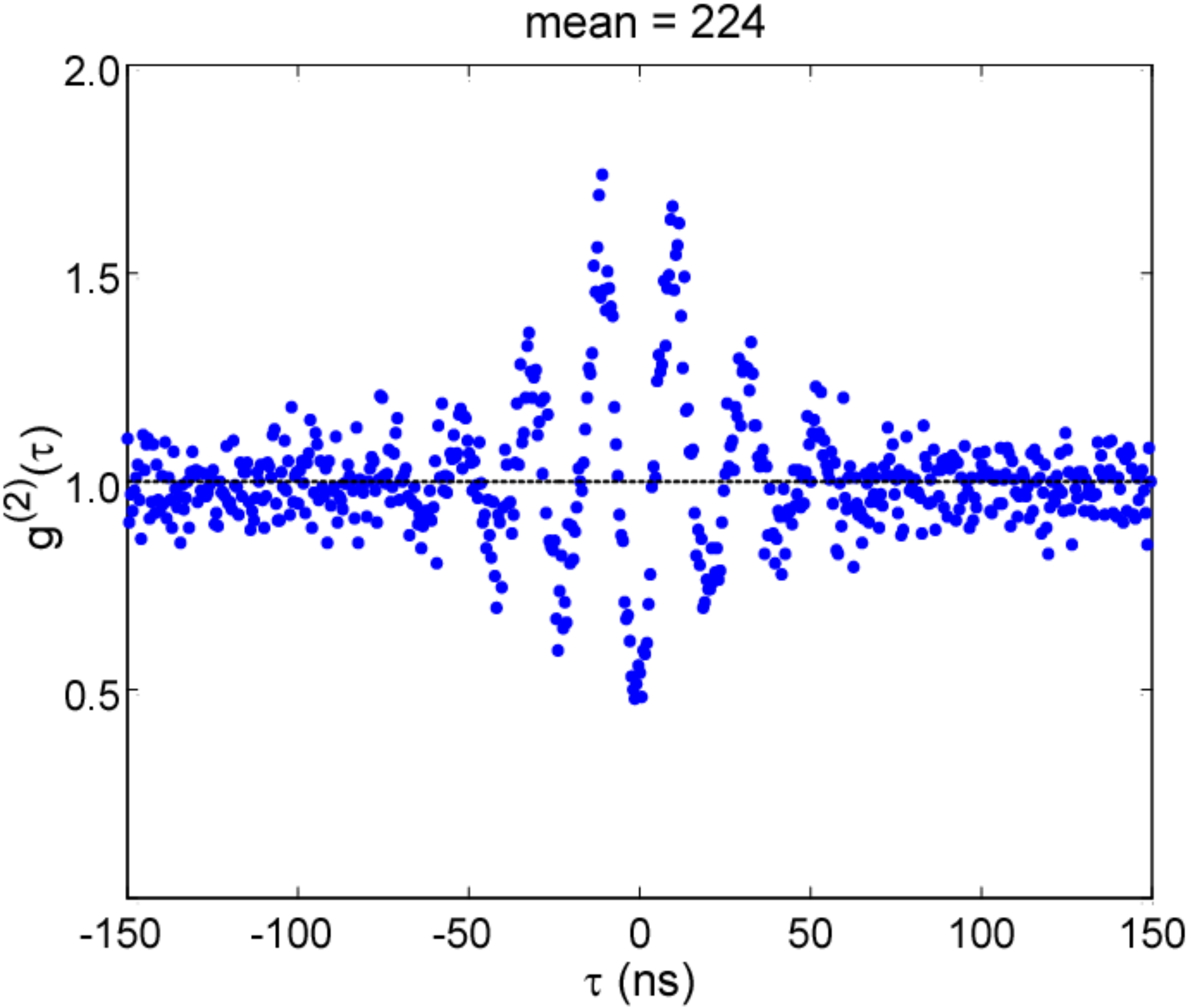

mean = 112
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)


mean = 124
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

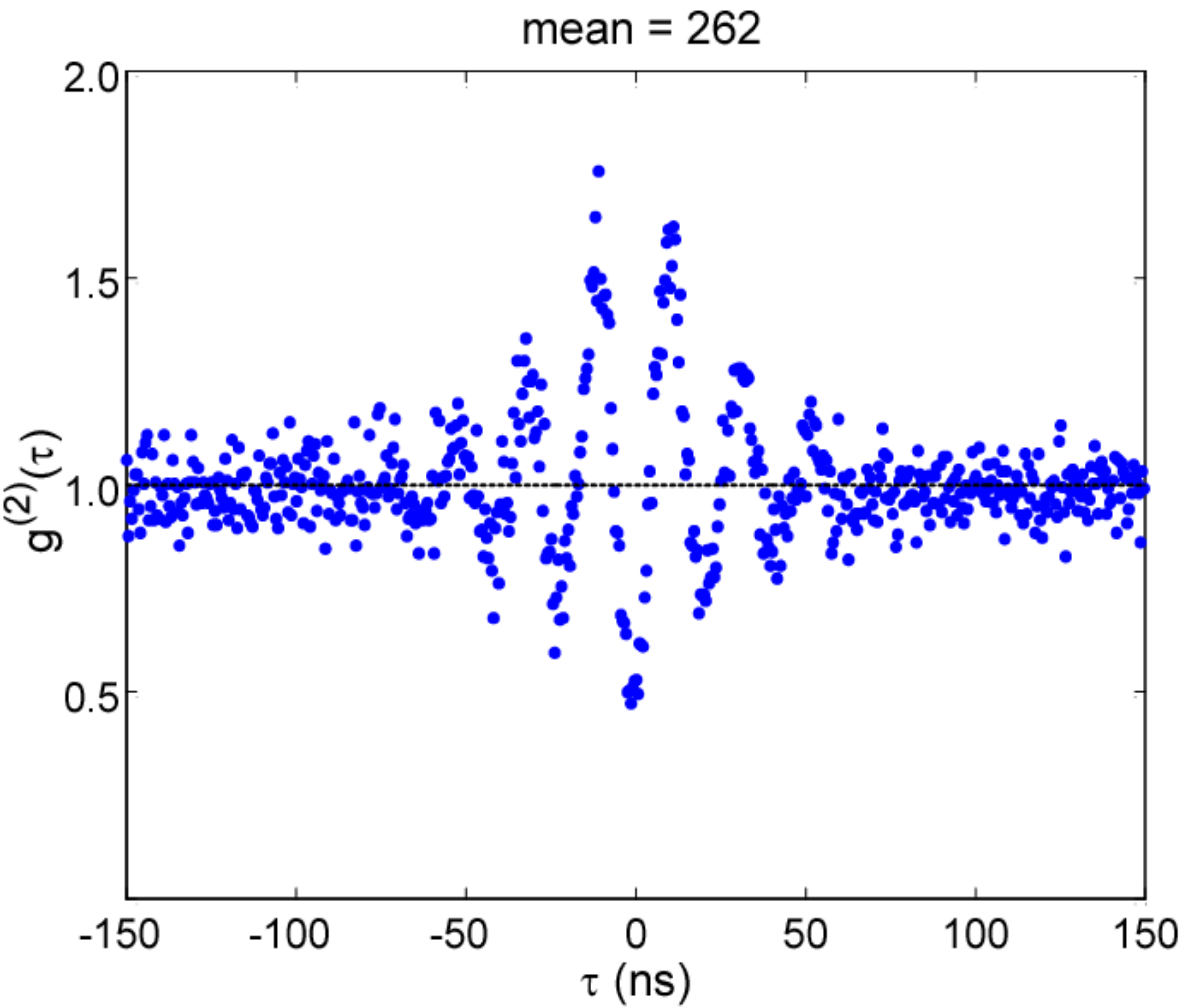

mean = 137
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

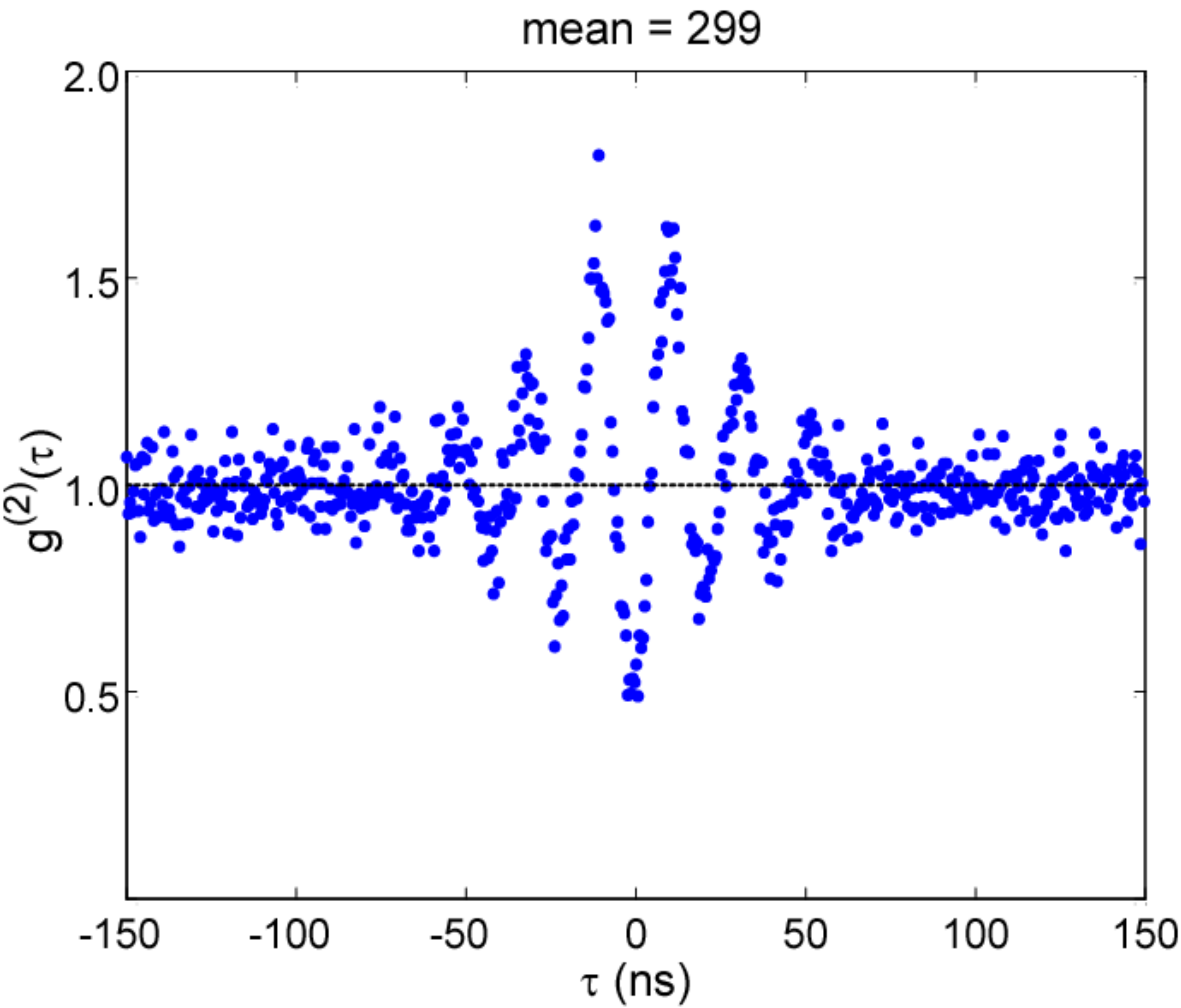

mean = 150
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

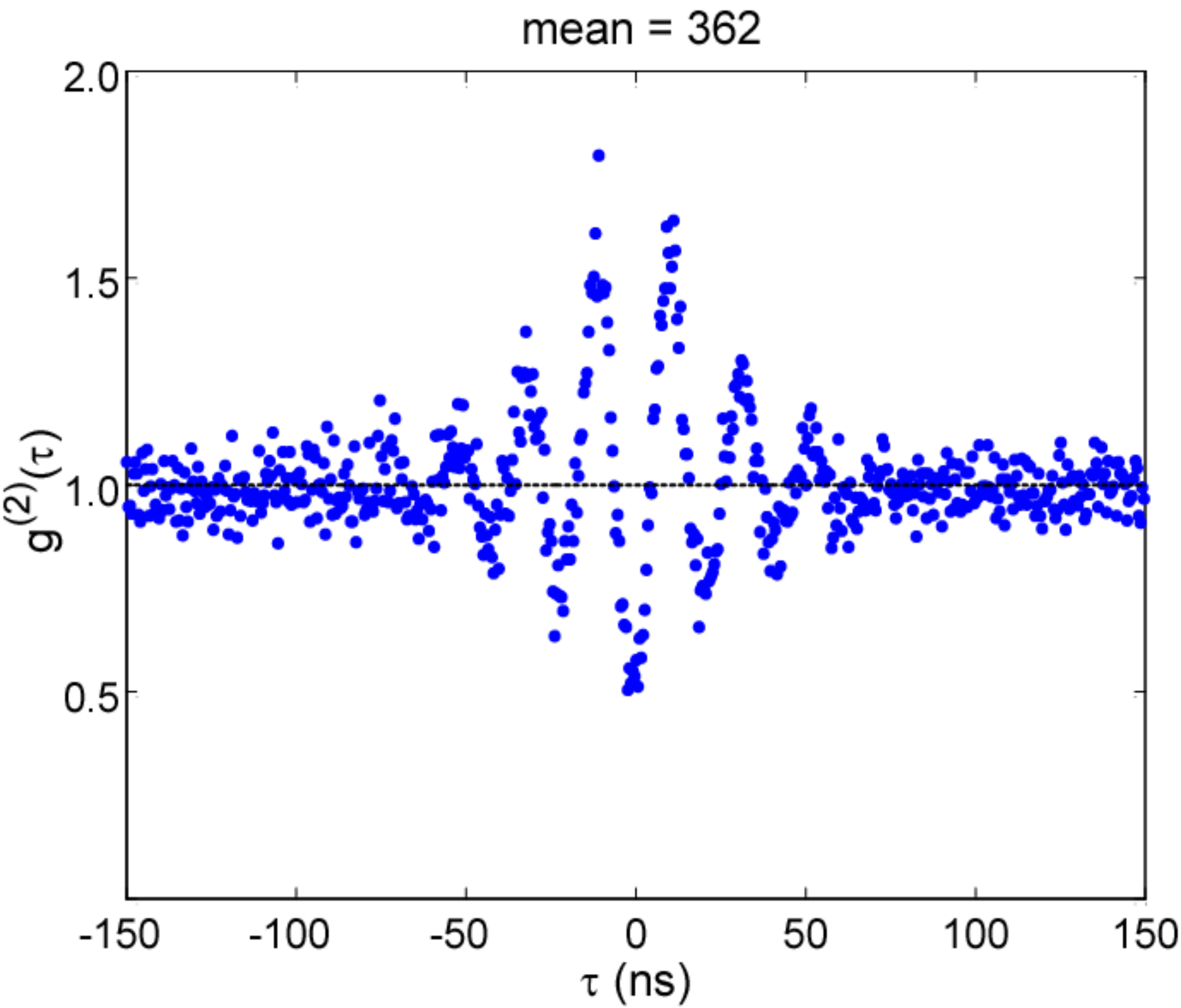

mean = 186
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

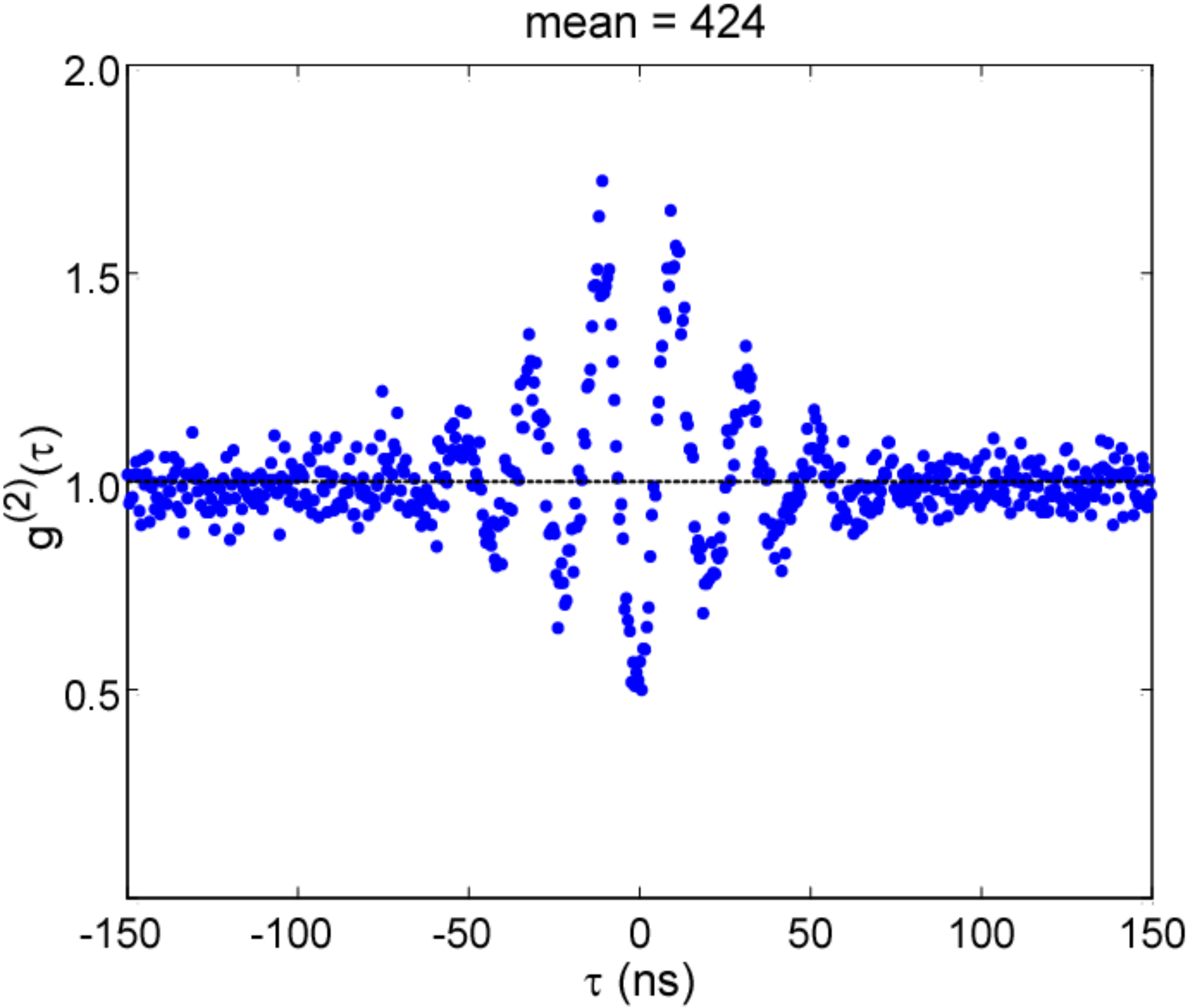


mean = 224
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

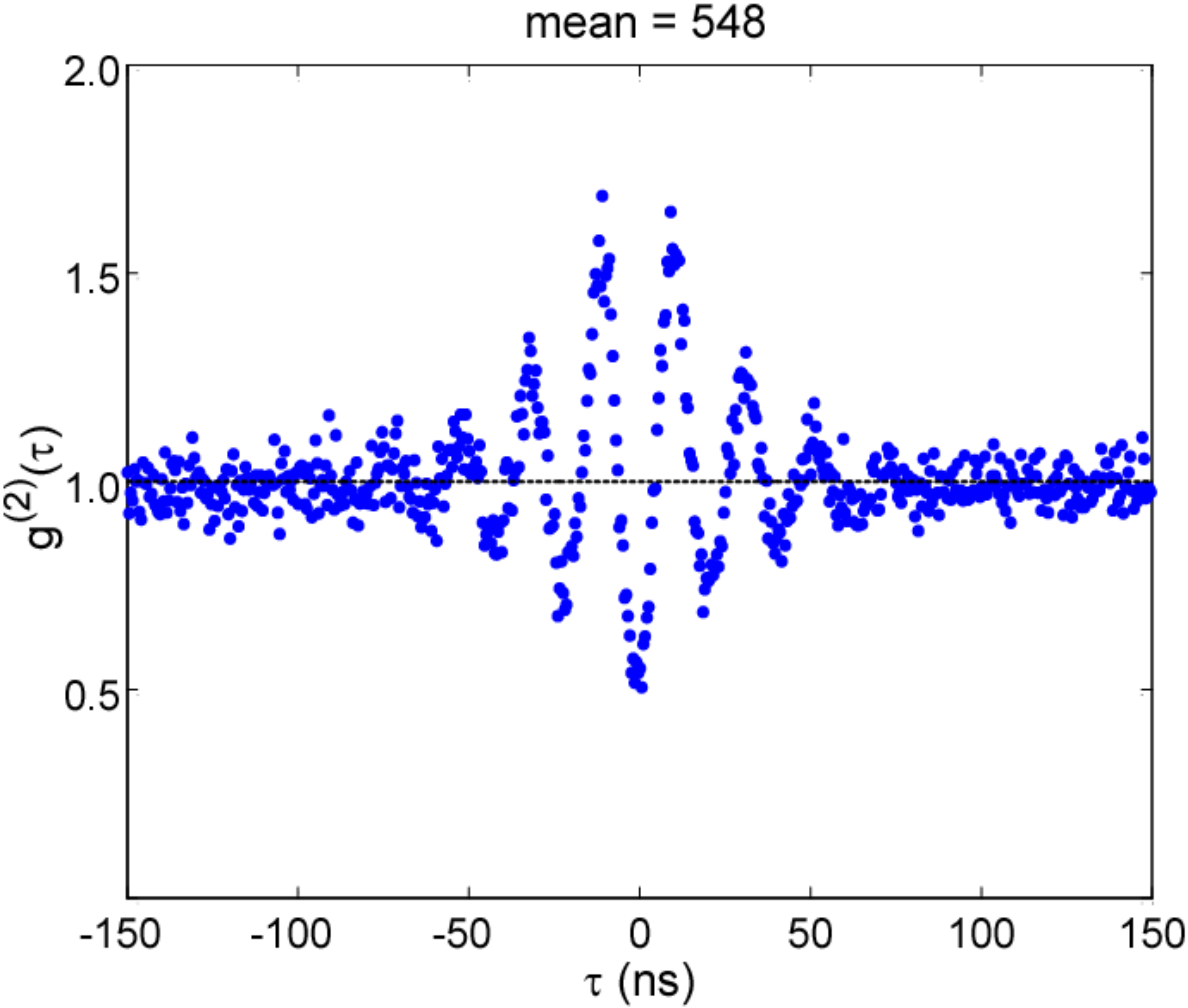

mean = 262
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

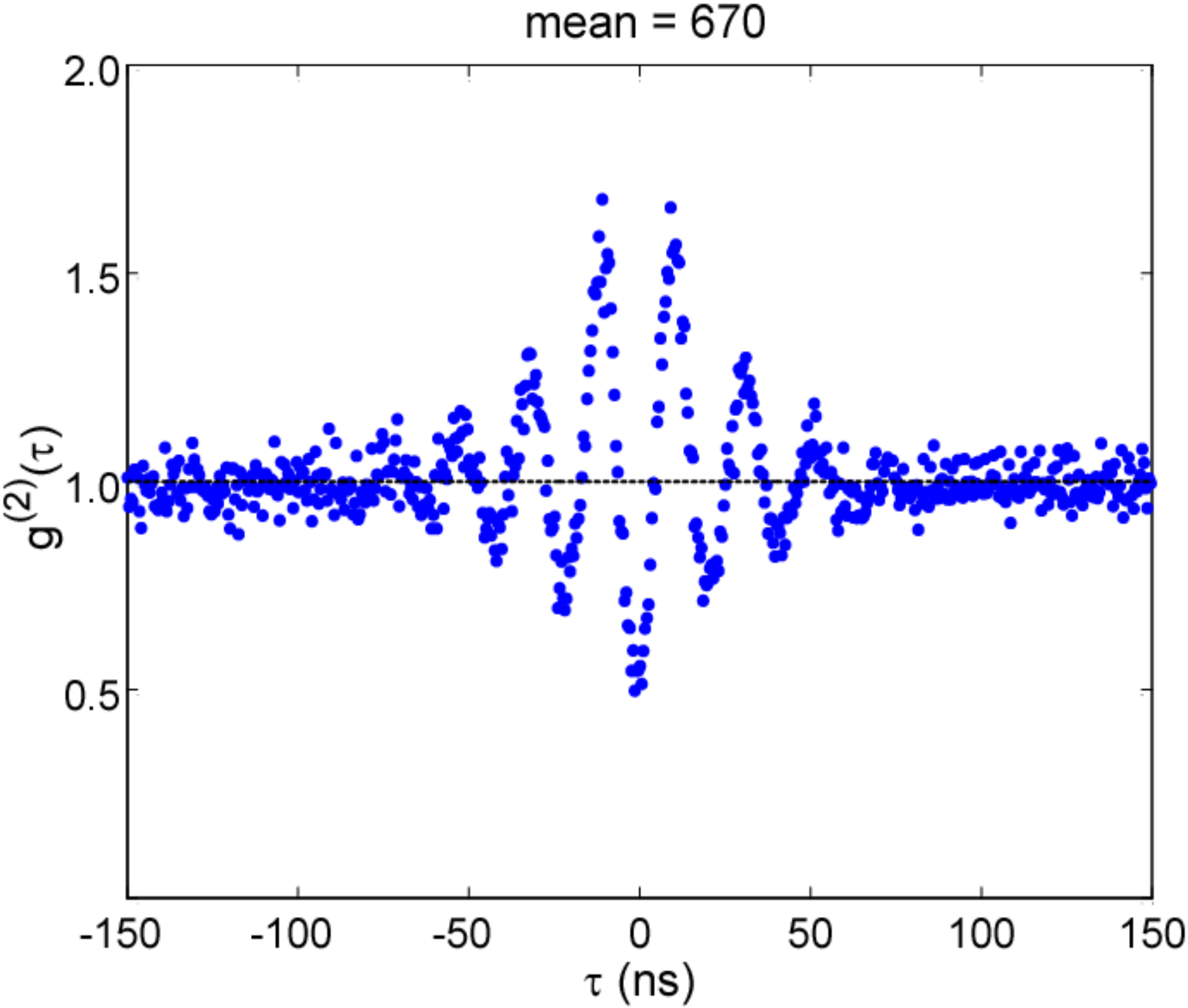

mean = 299
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

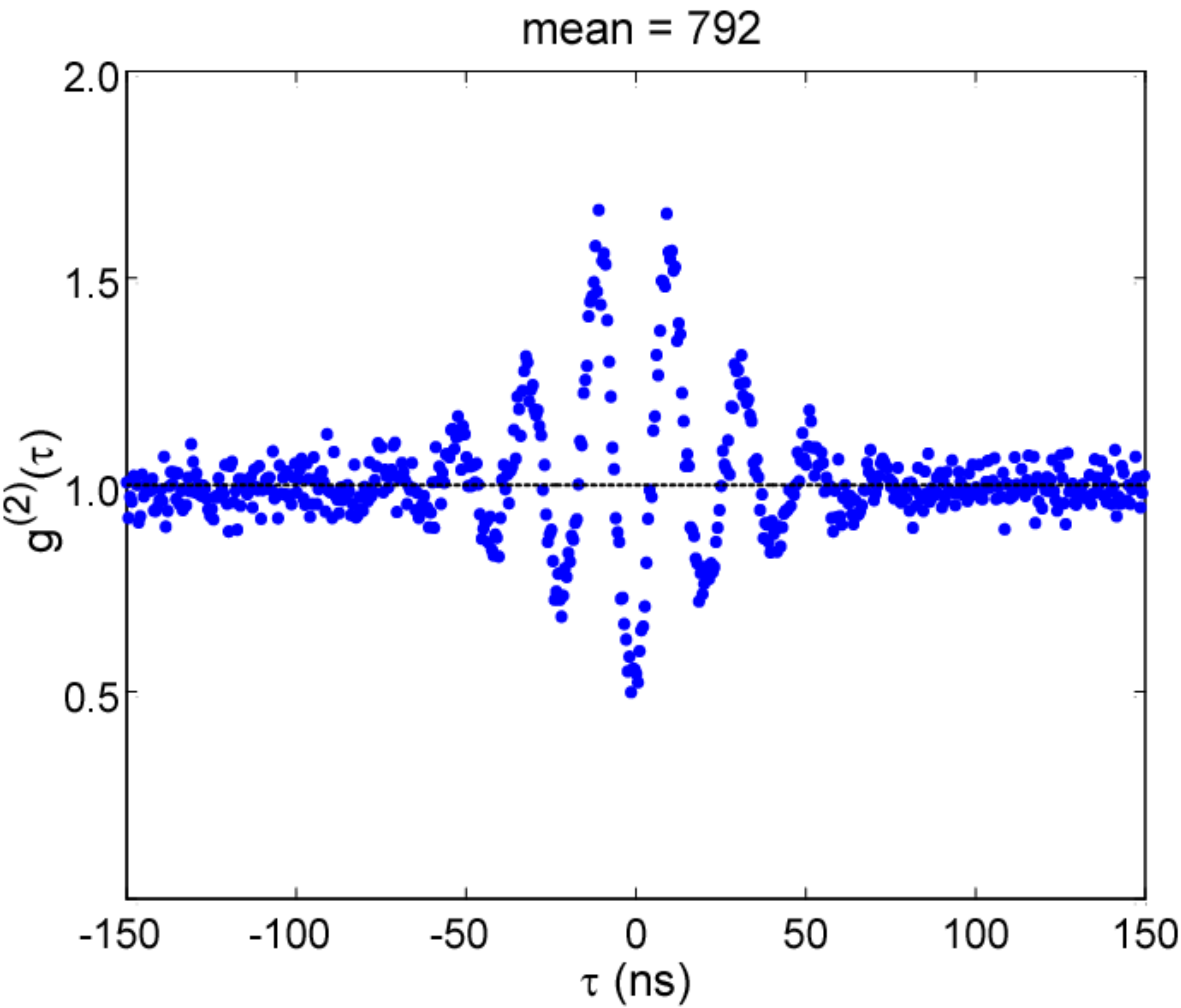

mean = 362
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)



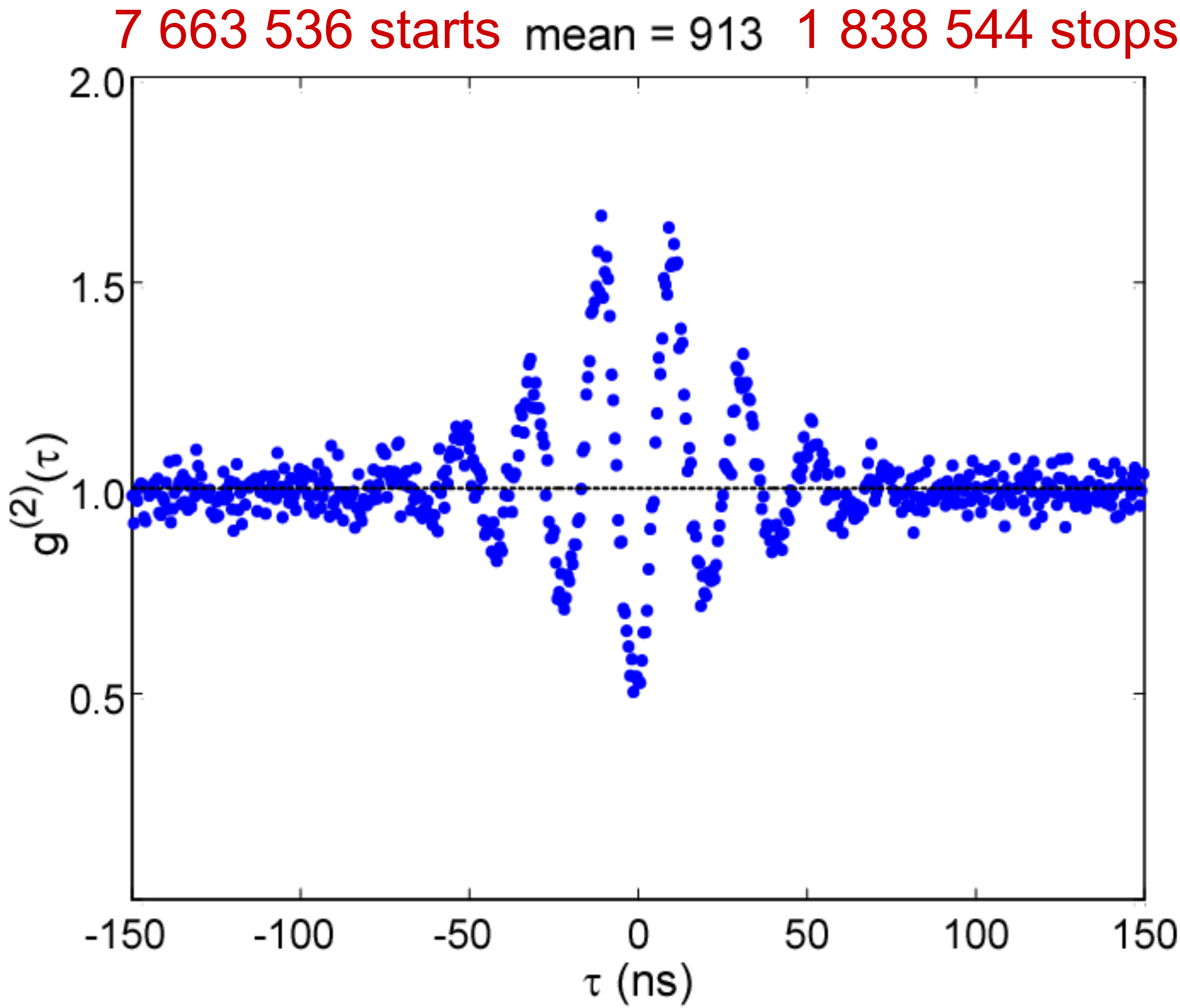
mean = 424
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)

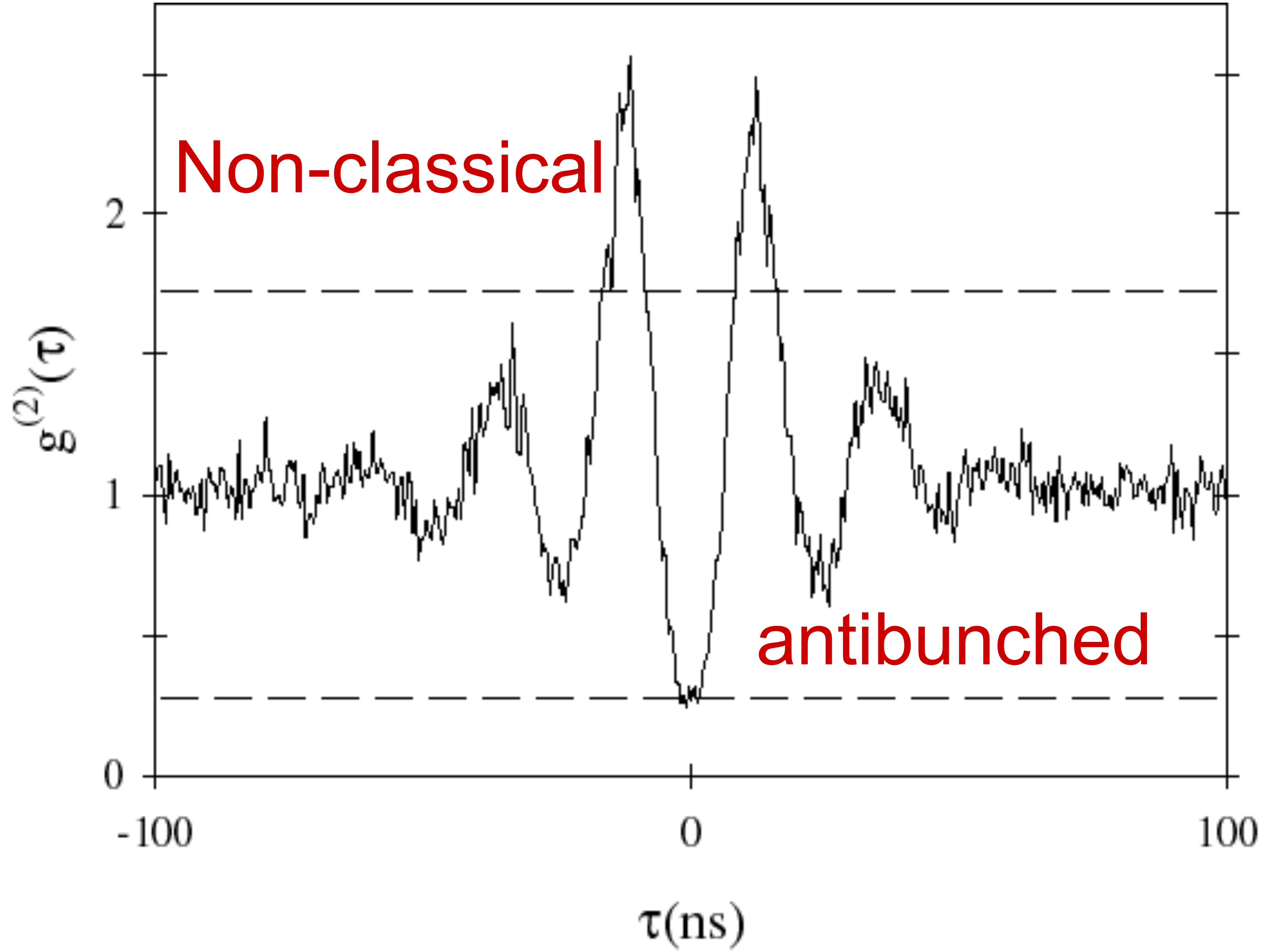

mean = 548
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)


mean = 670
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)



mean = 792
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)


7 663 536 starts
mean = 913
1 838 544 stops
2.0
1.5
1.0
0.5
$g^{(2)}(\tau)$
-150
-100
-50
0
50
100
150
τ (ns)



Classically $g^{(2)}(0) > g^{(2)}(\tau)$ and also $|g^{(2)}(0)-1| > |g^{(2)}(\tau)-1|$

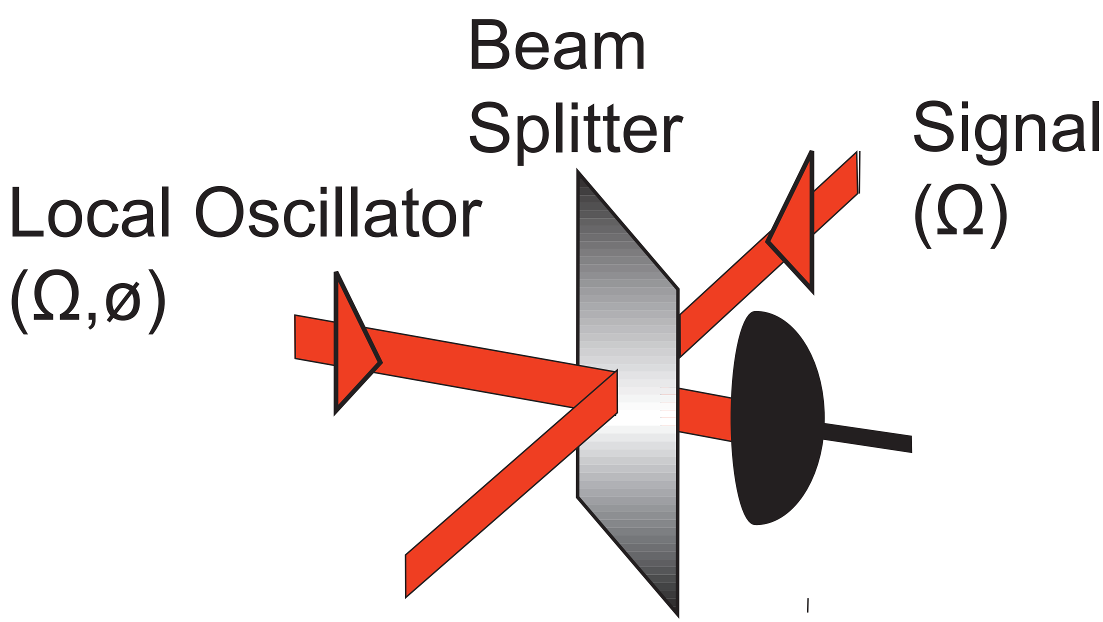
Reference that includes pulsed sources:
Zheyu Jeff Ou
"*Quantum Optics for Experimentalists*"
World Scientific, Singapore 2017

Reference that includes pulsed sources:
Zheyu Jeff Ou
"*Quantum Optics for Experimentalists*"
World Scientific, Singapore 2017

- The main object of interest in quantum optics is the optical FIELD. That is what is quantized.

- Can we measure the FIELD of a state with an average of ONE PHOTON in it?

Homodyne detection



Photocurrent ~ |L.O. cos(ø)+S|^2

Interfere two fields: A local oscillator (LO) and a signal (S). The resulting photocurrent has a term proportional to the amplitude of S and also depends on the cosine of the phase difference ϕ between LO and S.

$$|LO\cos(\phi) + S|^2 = |LO|^2 + 2\,LO\,S\cos(\phi) + |S|^2$$

Review of shot noise :

Shot noise happens whenever the transport of energy is through a finite number of discreet particles. For example, electric charge *e* (Schottky 1918). If the number of particles is small and it follows a Poisson distribution (random independent events), it can be the dominant noise.

- The mean of a Poisson distribution is n
- The variance of a Poisson distribution n
- The signal to noise ratio $n^{1/2}$
- A Poisson distribution with n large approximates a Gaussian.
- The current spectral density (i) of noise is: $(2e|i|)^{1/2}$ with units of [A/Hz$^{1/2}$].
- The power of the noise depends on the detection bandwith and the Resistance R: $P(\nu)=R2e|i|\Delta\nu$.

Review of Coherent States $|\alpha>$

The coherent state $|\alpha>$ is the eigenstate of the annihilation operator:

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$$

Its amplitude (complex): α
Its mean squared: $\alpha^*\alpha=|\alpha|^2$
Its uncertainty: 1/2

They are states with the minimum uncertainty allowed by quantum mechanics. Equal on both quadratures

Relation with the harmonic oscillaros:
Quadratures of the electromagnetic field

$$E_{\mathrm{R}} = \left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} \cos(\theta) X \qquad \text{and} \qquad E_{\mathrm{I}} = \left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} \sin(\theta) X$$

$$H = \hbar\omega\left(P^2 + X^2\right), \qquad \text{with} \qquad [X, P] \equiv XP - PX = \frac{i}{2} I$$

$$\left(X - \langle X\rangle\right)|\alpha\rangle = -i\left(P - \langle P\rangle\right)|\alpha\rangle$$

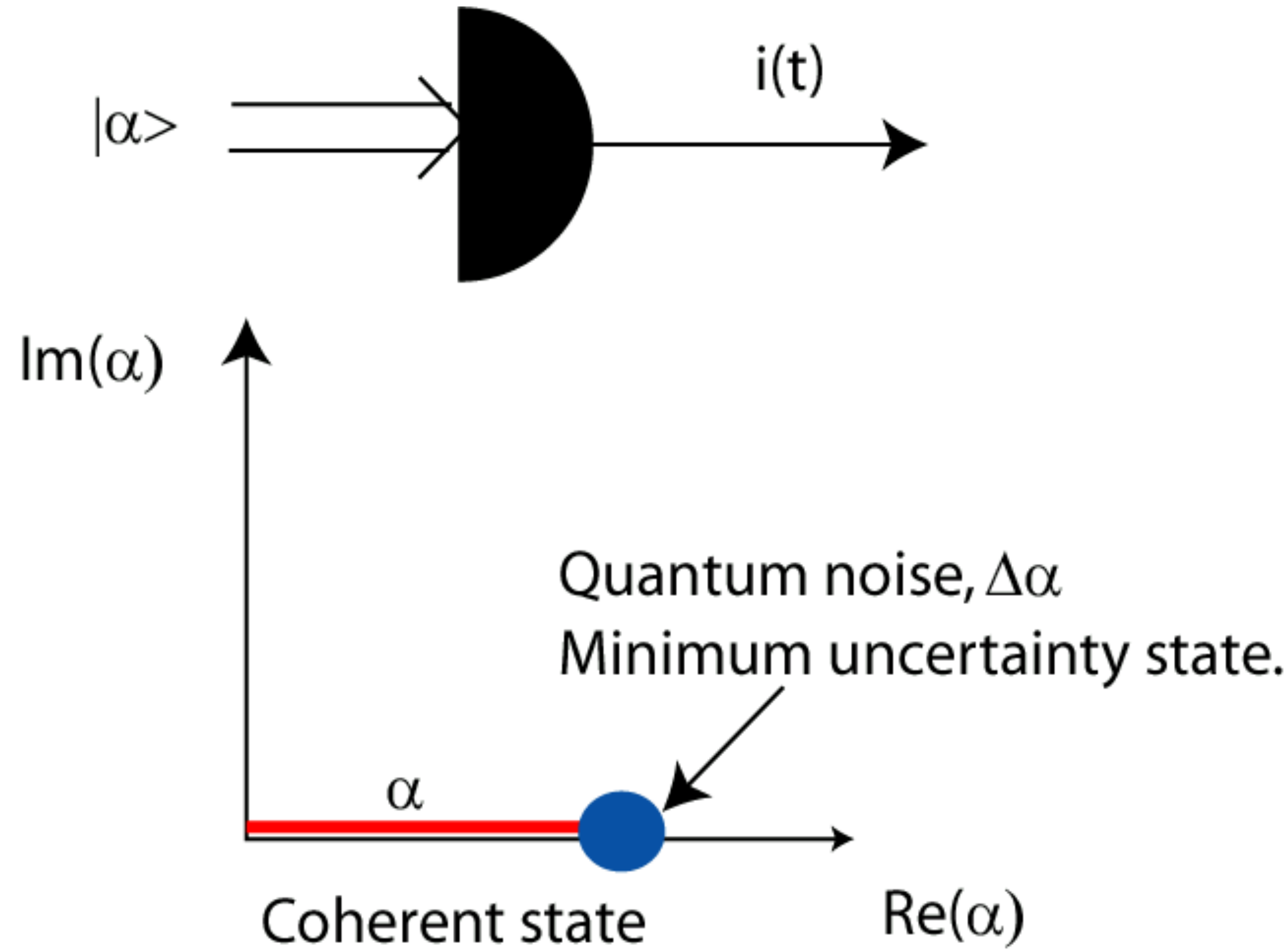
$$\left(X + iP\right)|\alpha\rangle = \langle X + iP\rangle\,|\alpha\rangle$$

States of minimum uncertainty :

$$\langle\alpha|\left(X - \langle X\rangle\right)^2 + \left(P - \langle P\rangle\right)^2|\alpha\rangle = 1/2$$

Relation with Fock states (Poisson)

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}$$



Perfect detector $i(t) = |\alpha + \Delta\alpha|^2$

$$i(t) = |\alpha|^2 + 2\alpha\,\Delta\alpha + |\Delta\alpha|^2 \quad ; \quad \langle\alpha^*\alpha\rangle = n$$

DC ~ n   Shot noise ~ $n^{1/2}$ neglect.

Correlation functions tell us something about the fluctuations.

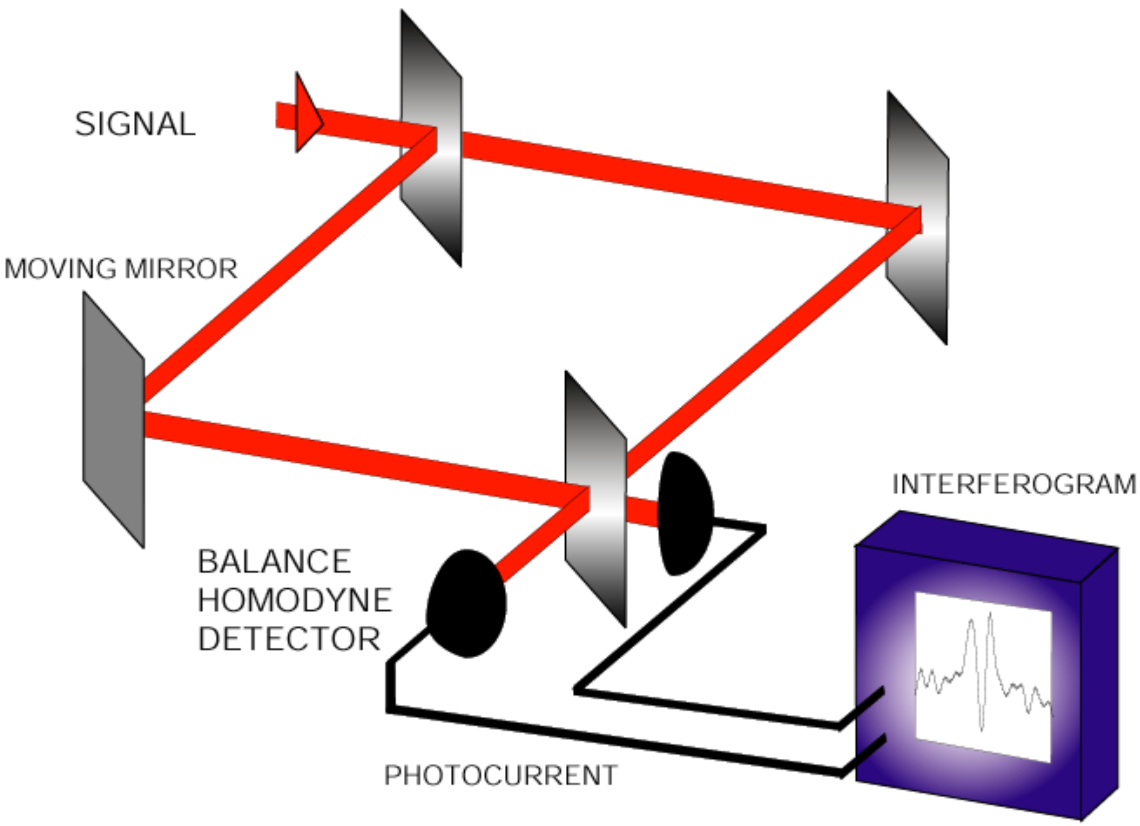
Correlations have classical bounds.

They are conditional measurements.

Can we use them to measure the field associated with a FLUCTUATION of one photon?

Mach Zehnder Interferometer Wave-Wave Correlation

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle I(t)\rangle}$$

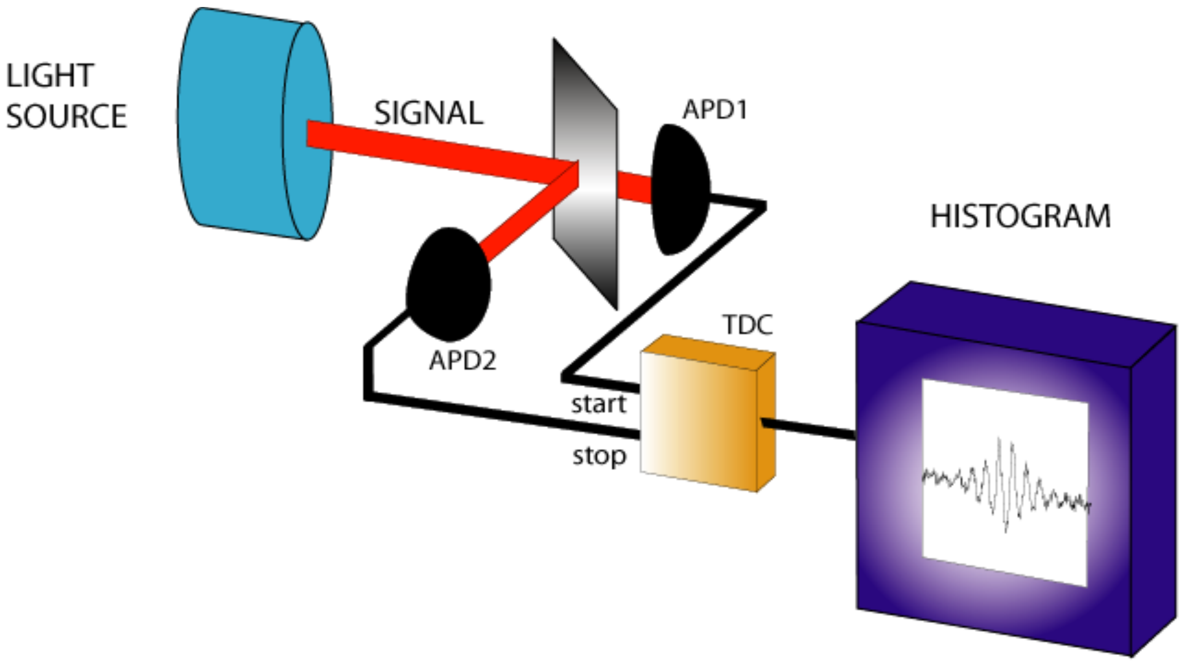


Spectrum of the signal:

$$F(\omega) = \frac{1}{2\pi}\int exp(i\omega\tau)g^{(1)}(\tau)d\tau$$

Basis of Fourier Transform Spectroscopy

Hanbury Brown and Twiss Intensity-Intensity Correlations



$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2}$$

Cauchy-Schwarz

$$2I(t)I(t+\tau) \leq I^2(t) + I^2(t+\tau)$$

The correlation is largest at equal time

$$g^{(2)}(0) \geq 1$$

$$g^{(2)}(0) \geq g^{(2)}(\tau)$$

Intensity correlation function measurements:

$$g^{(2)}(\tau) = \frac{\left\langle : \hat{I}(t)\hat{I}(t+\tau) : \right\rangle}{\left\langle \hat{I}(t) \right\rangle^2}$$

Gives the probability of detecting a photon at time $t + \tau$ given that one was detected at time t. This is a conditional measurement:

$$g^{(2)}(\tau) = \frac{\left\langle \hat{I}(\tau) \right\rangle_c}{\left\langle \hat{I} \right\rangle}$$

A strategy starts to appear:

Correlation function; Conditional measurement.

Detect a photon: Prepare a conditional quantum mechanical state in our system.

The system has to have at least two photons.

Do we have enough signal to noise ratio?

$$|LO|^2 \qquad + \qquad 2\,LO\,S\cos(\phi)$$

SHOT NOISE SIGNAL

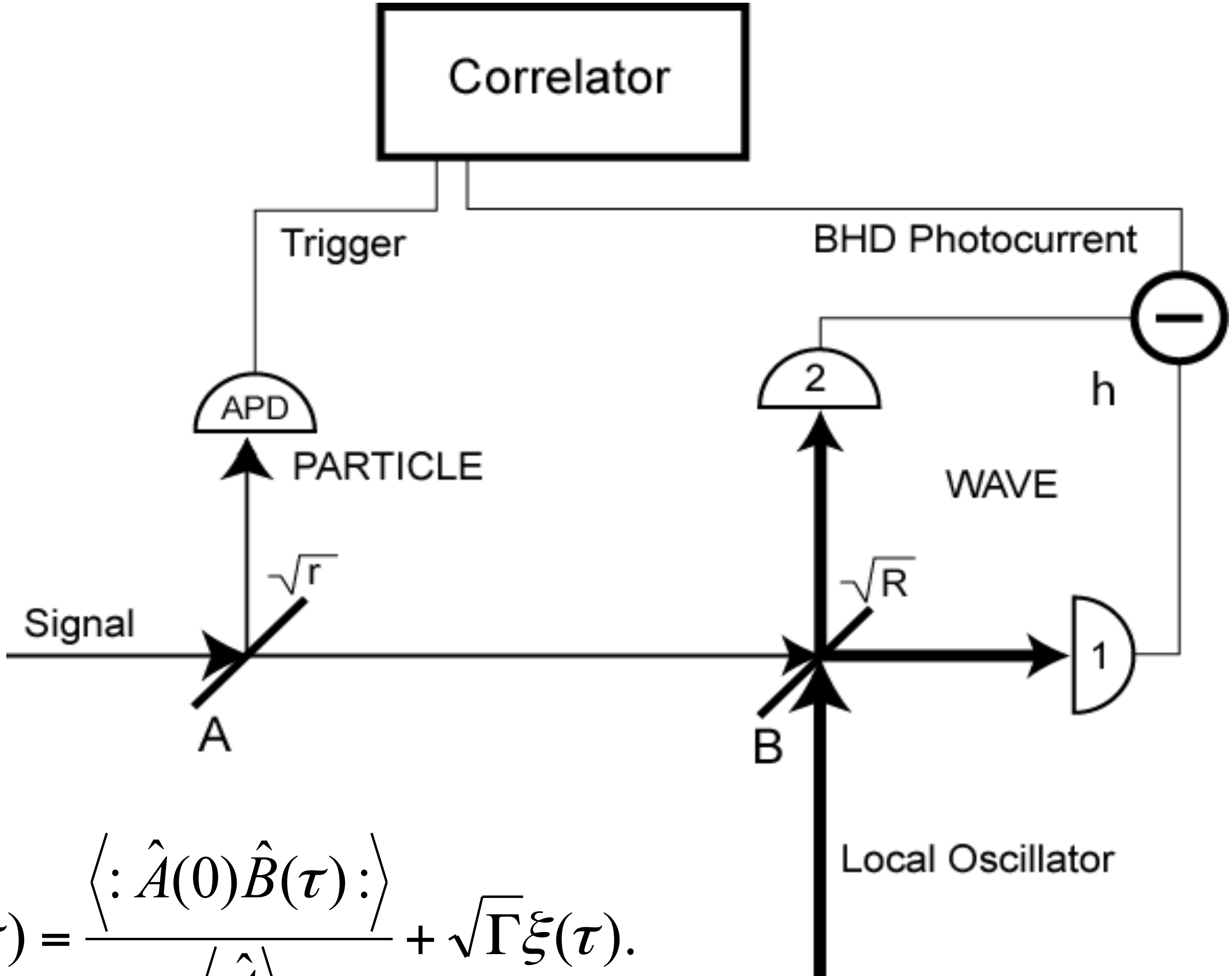
How to correlate fields and intensities?

Detection of the field: Homodyne.

Conditional Measurement: Only measure when we know there is a photon.

Source: Cavity QED

The Intensity-Field correlator.



$$\mathrm{H}(\tau) = \frac{\left\langle : \hat{A}(0)\hat{B}(\tau) : \right\rangle}{\left\langle \hat{A} \right\rangle} + \sqrt{\Gamma}\xi(\tau).$$

Condition on a Click

Measure the correlation function of the Intensity and the Field:

$<I(t)\, E(t+\tau)>$

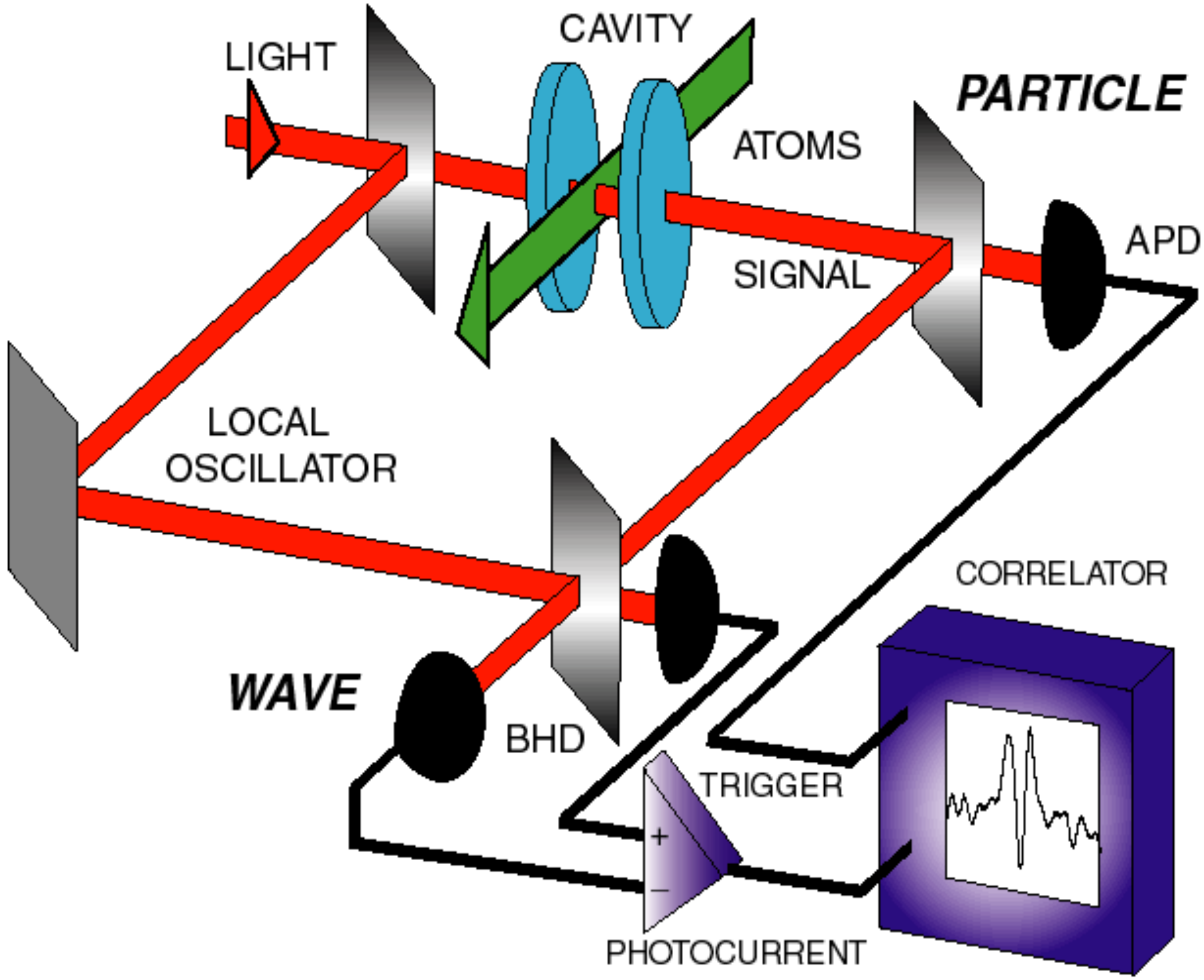
Normalized form:

$h_\theta(\tau) = <E(\tau)>_c / <E>$

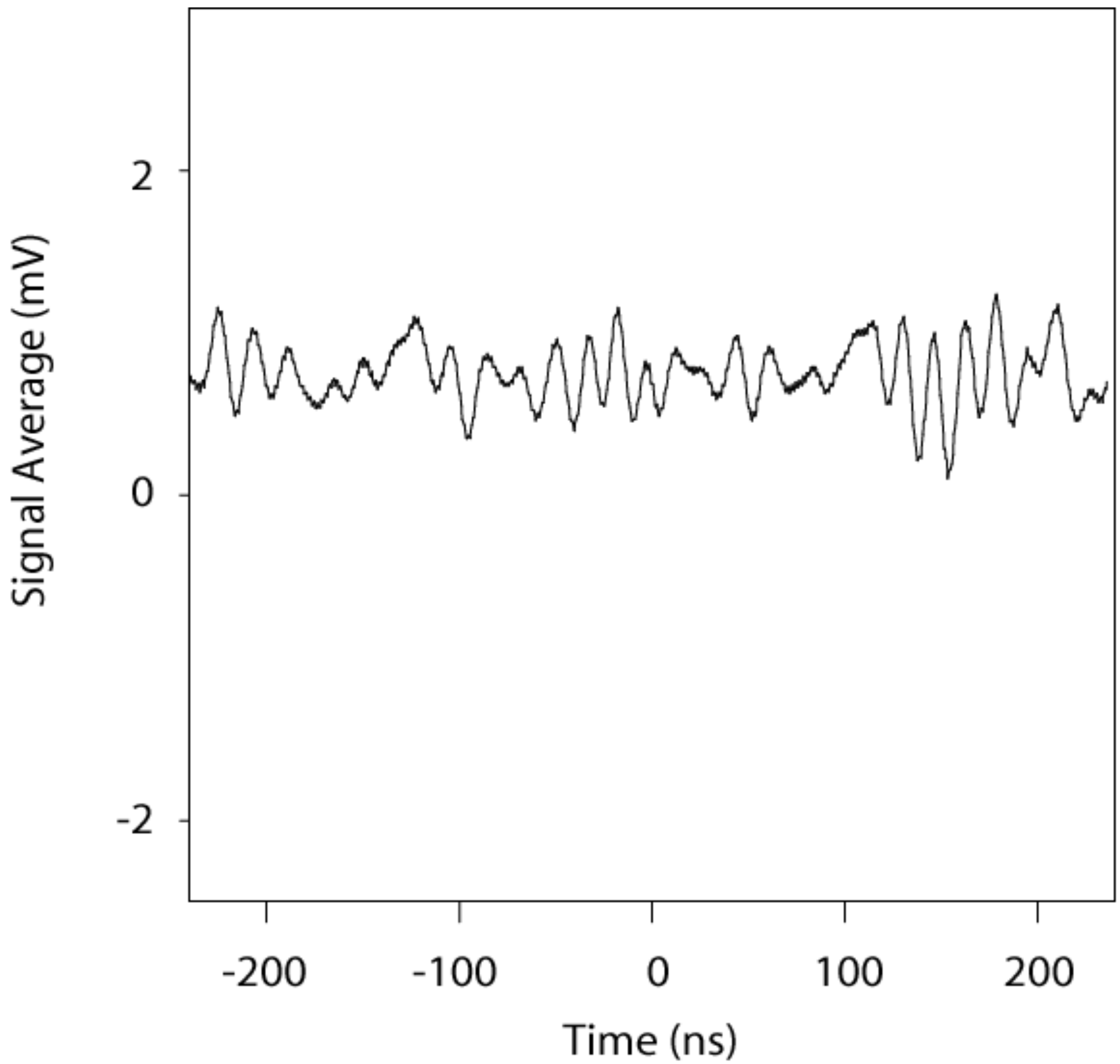
From Cauchy Schwartz inequalities:

$$0 \le \bar{h}_0(0) - 1 \le 2$$

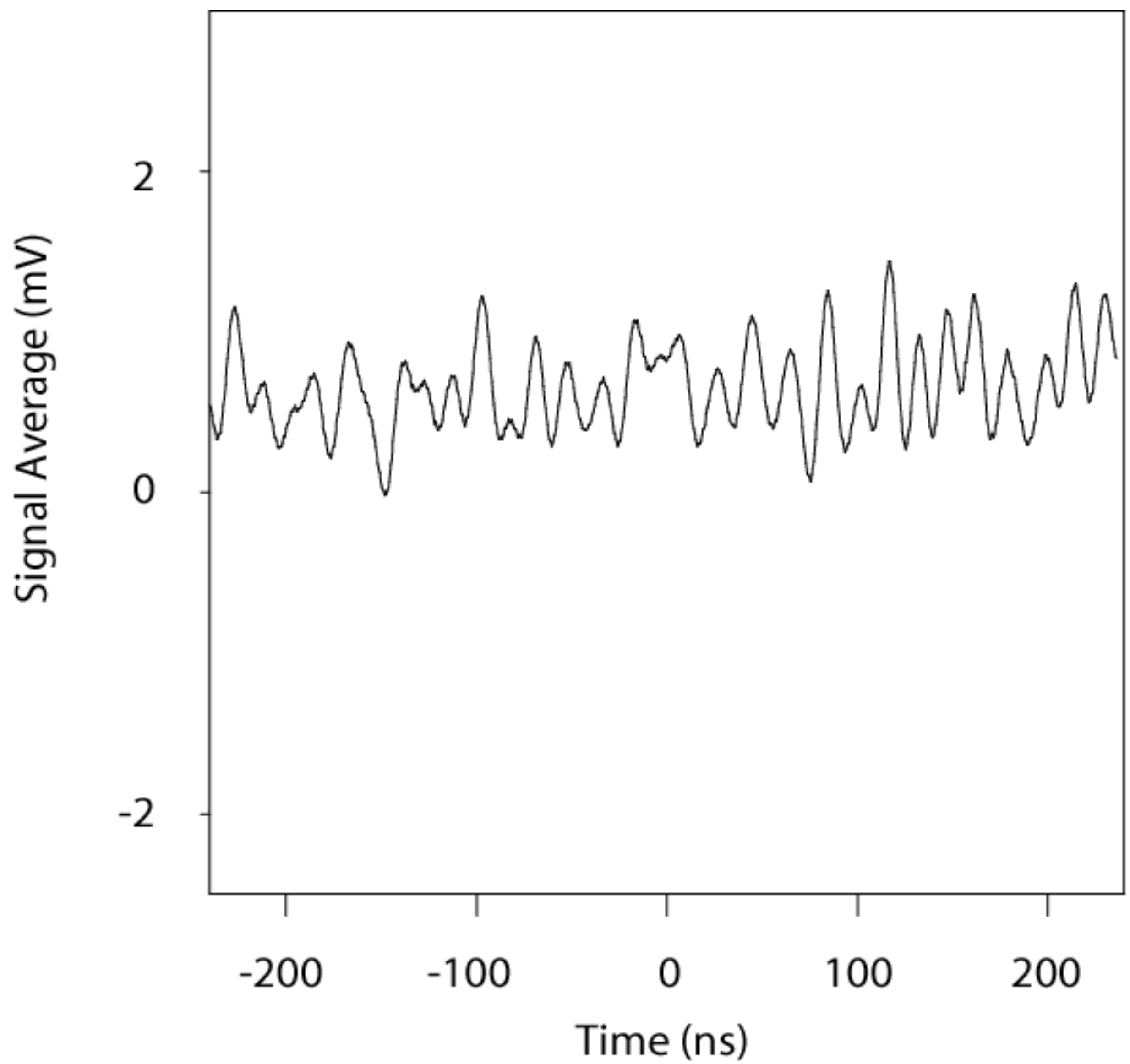
$$\left|\bar{h}_0(\tau) - 1\right| \le \left|\bar{h}_0(0) - 1\right|$$

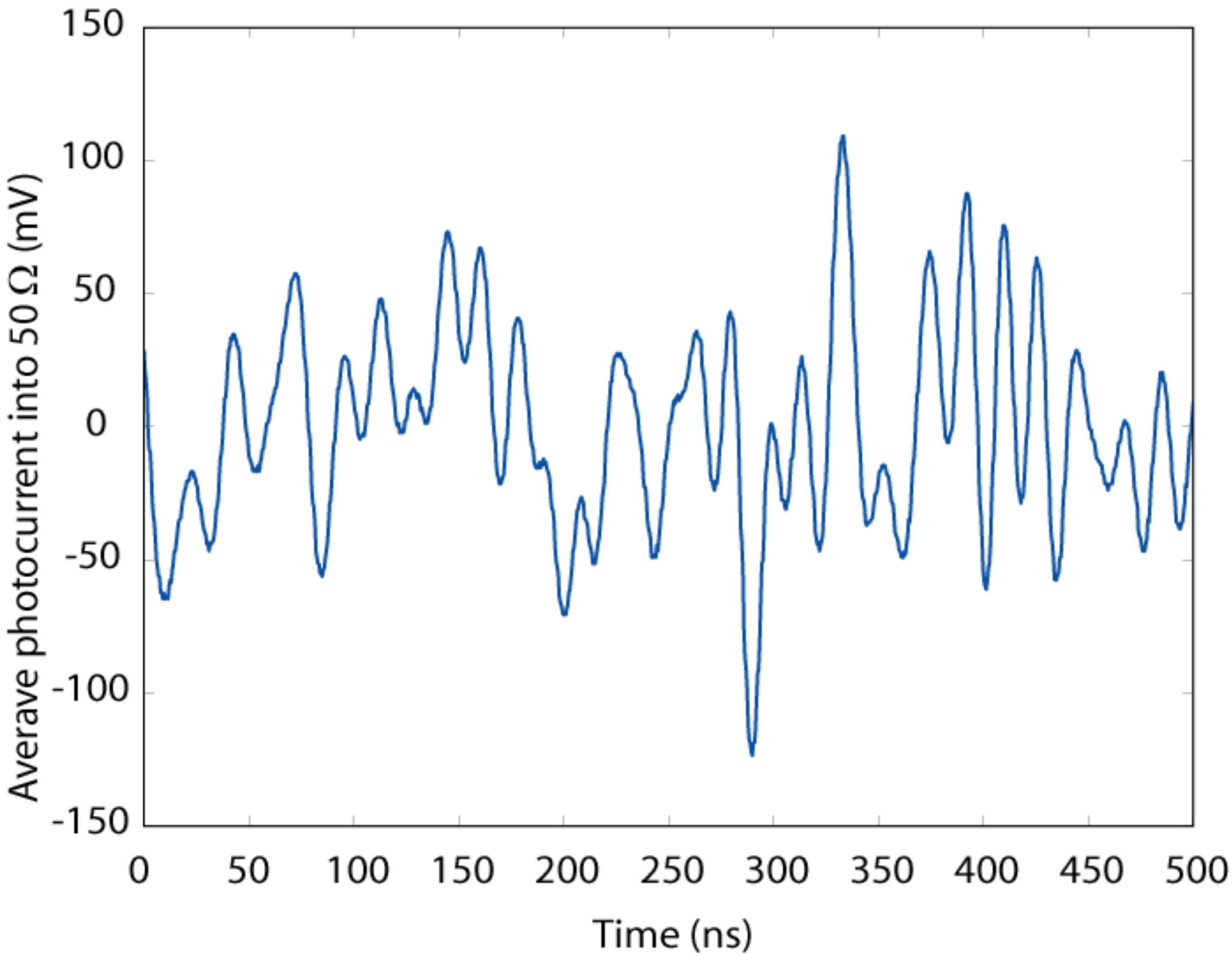

CAVITY
LIGHT
PARTICLE
ATOMS
APD
SIGNAL
LOCAL
OSCILLATOR
CORRELATOR
WAVE
BHD
TRIGGER
+
−
PHOTOCURRENT

Photocurrent average with random conditioning

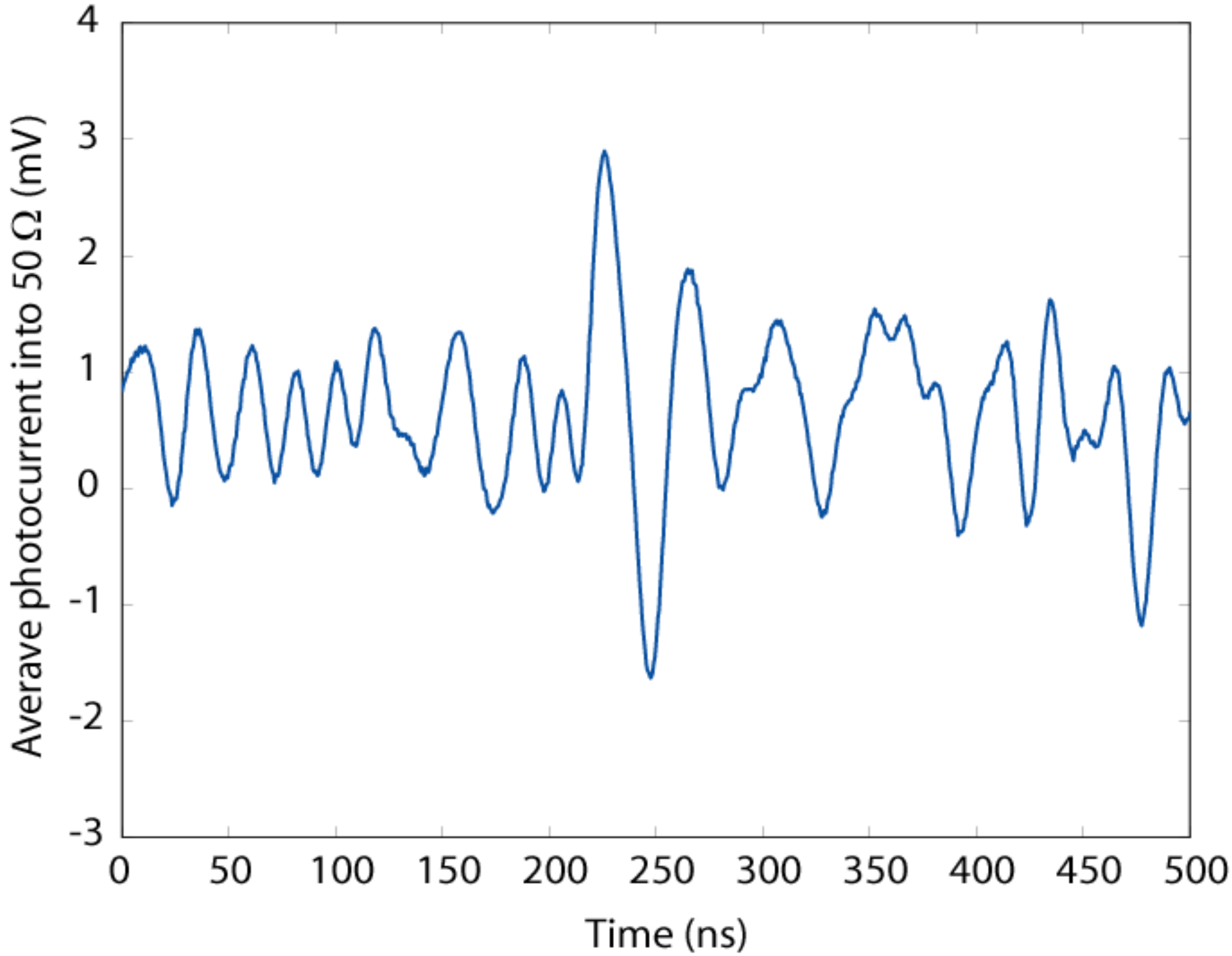


Conditional photocurrent with no atoms in the cavity.

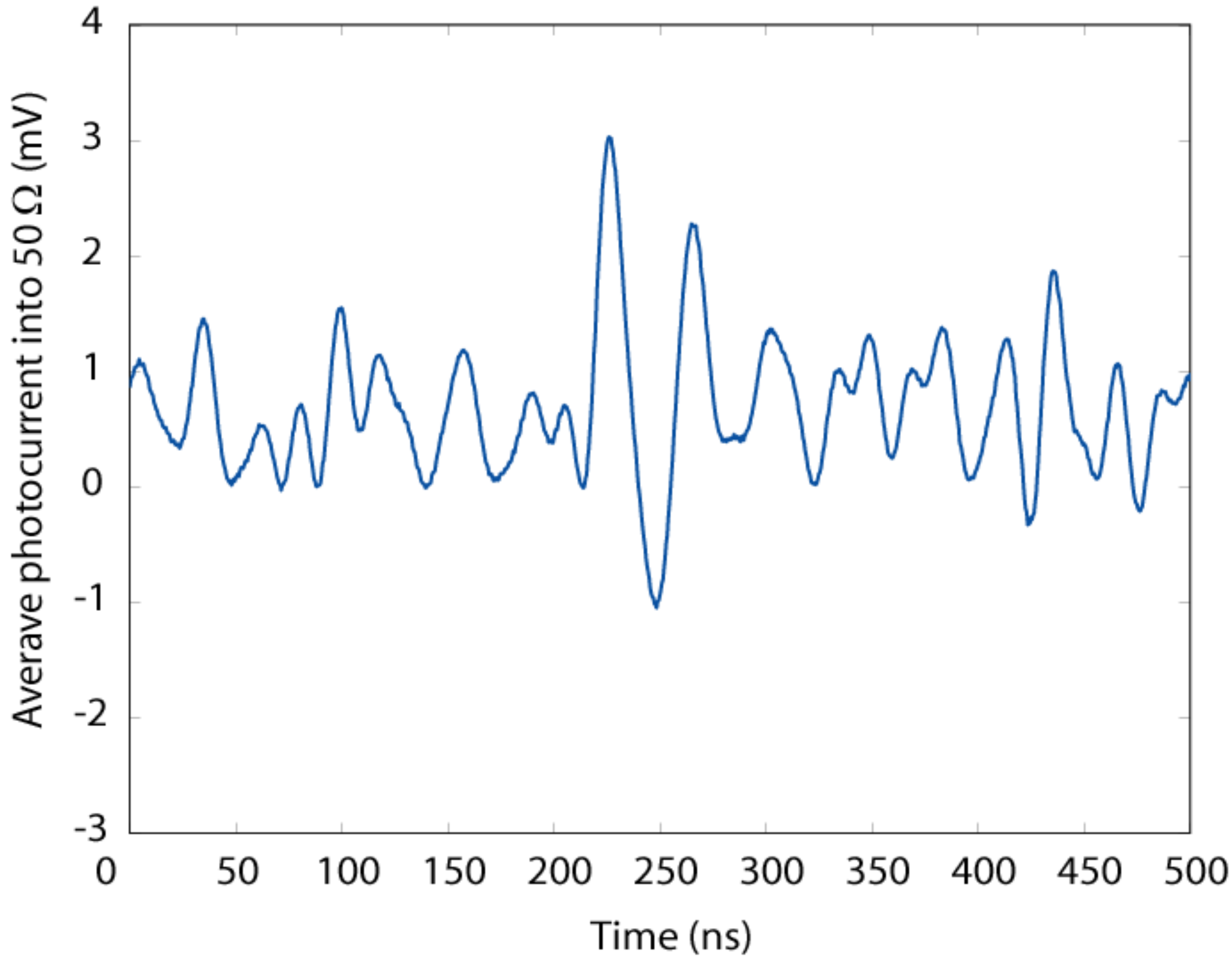




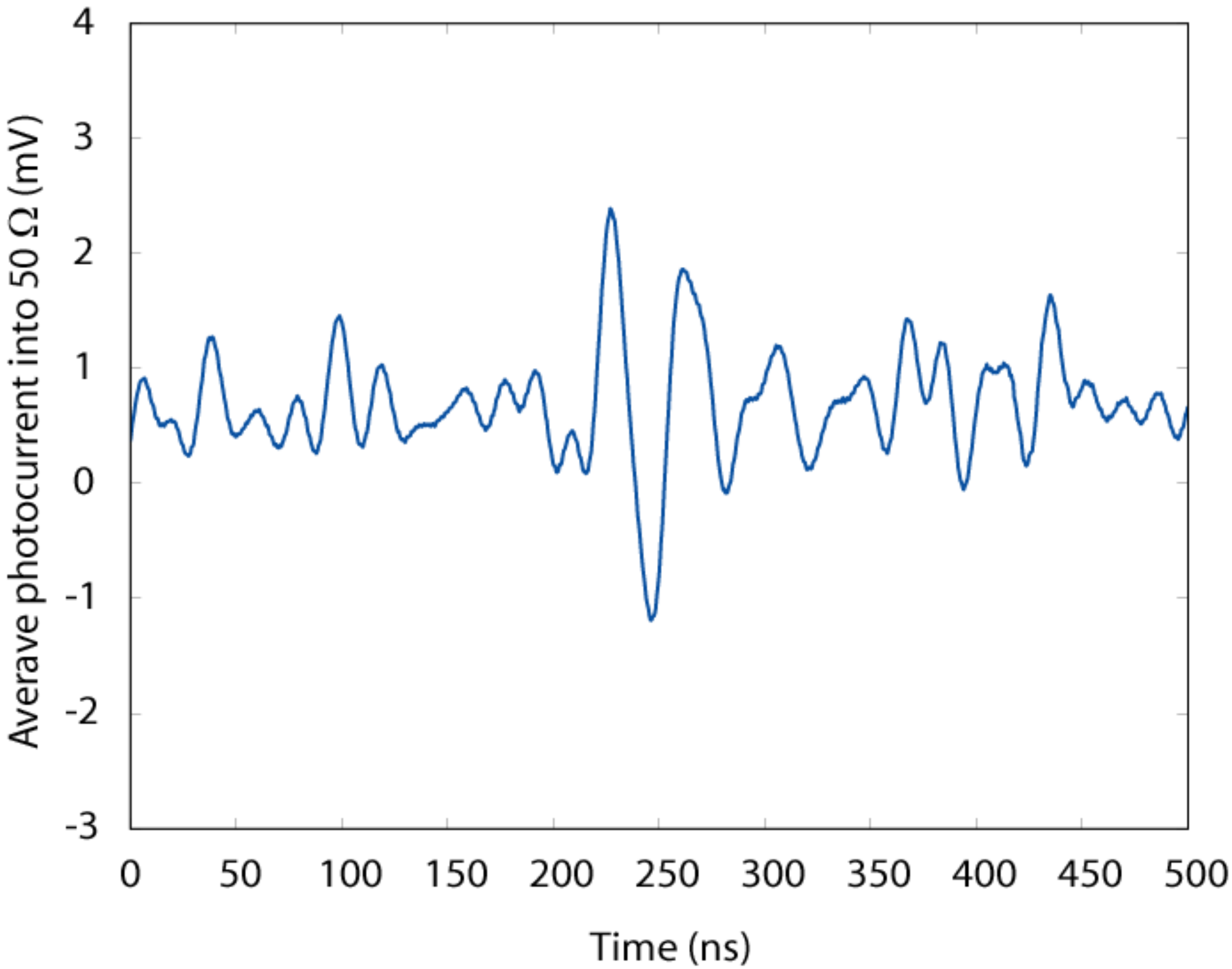
After 1 average



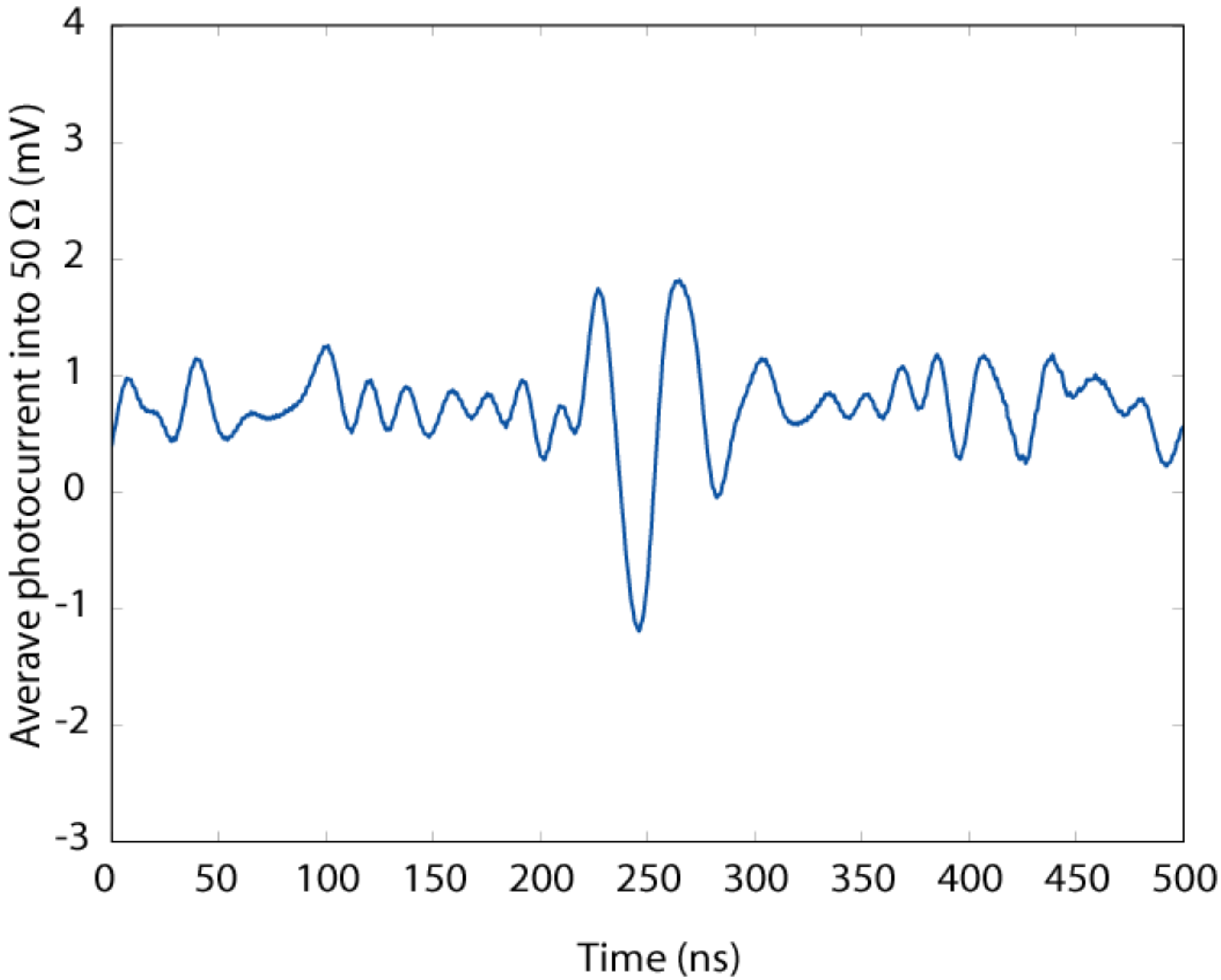
After 6,000 averages



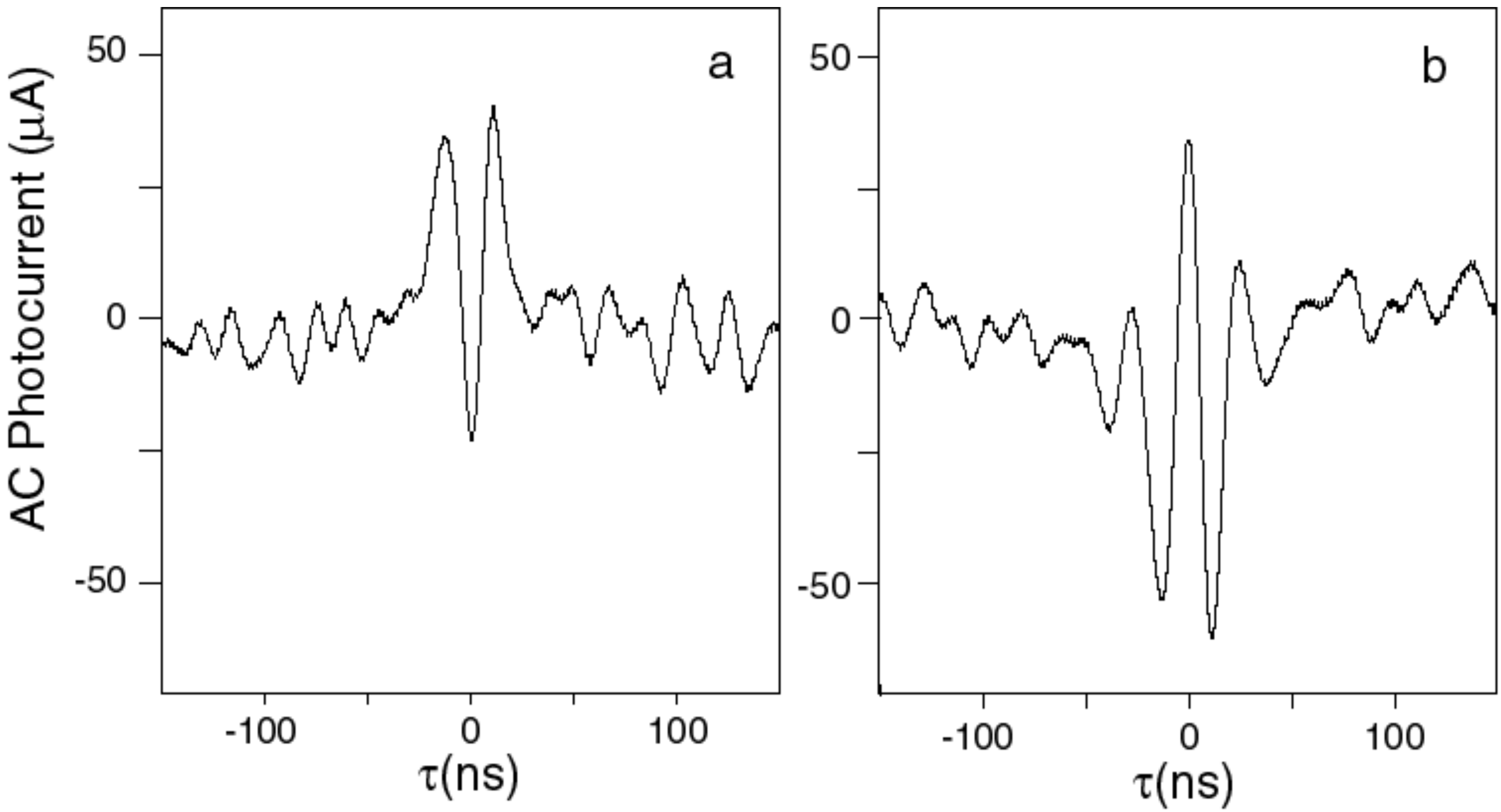
After 10,000 averages



After 30,000 averages

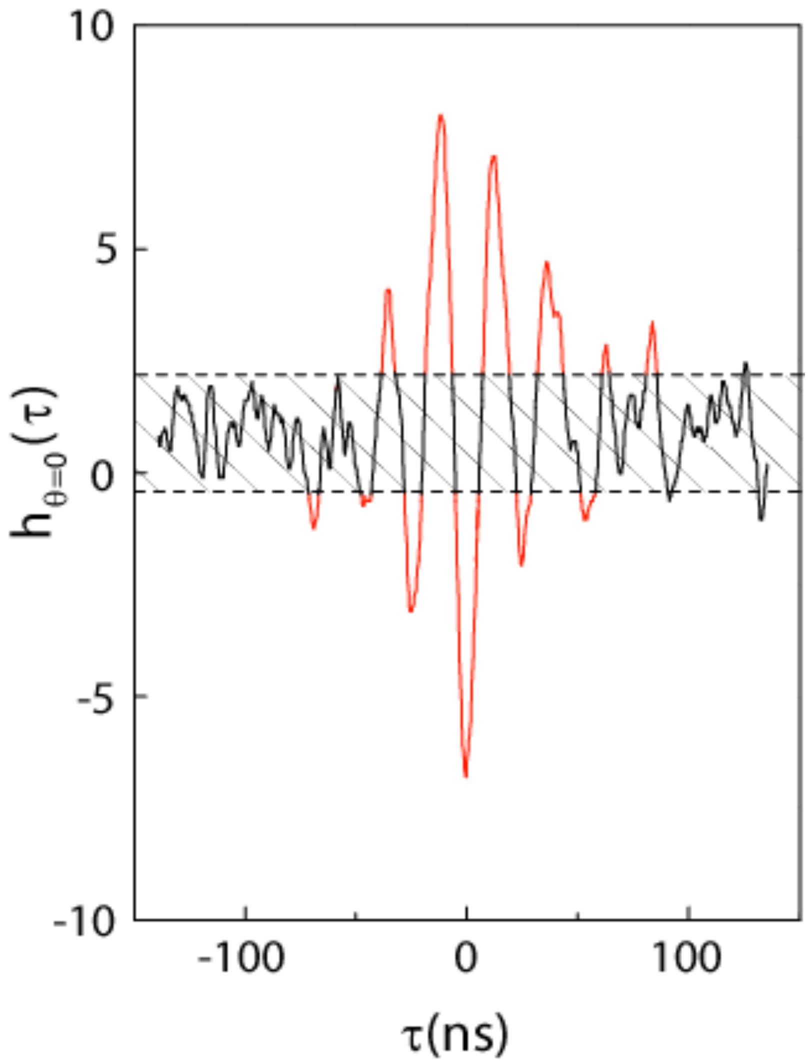


After 65,000 averages



Flip the phase of the Mach-Zehnder by 146°

Monte Carlo simulations for weak excitation:



Atomic beam N=11

This is the conditional evolution of the field of a fraction of a photon [B(t)] from the correlation function.

$$h_\theta(\tau) = \langle E(\tau)\rangle_c / \langle E\rangle$$

The conditional field prepared by the click is:

$A(t)|0\rangle + B(t)|1\rangle$ with $A(t) \approx 1$ and $B(t) << 1$

We measure the field of a fraction of a photon!

Fluctuations are very important.

Conditional dynamics in cavity QED at low intensity:

$$\left|\Psi_{ss}\right\rangle = \left|0,g\right\rangle + \lambda\left|1,g\right\rangle - \frac{2g}{\gamma}\lambda\left|0,e\right\rangle + \frac{\lambda^2 pq}{\sqrt{2}}\left|2,g\right\rangle - \frac{2g\lambda^2 q}{\gamma}\left|1,e\right\rangle$$

$$\lambda = \langle \hat{a} \rangle, \quad p = p(g,\kappa,\gamma) \text{ and } q = q(g,\kappa,\gamma)$$

A photodetection conditions the state into the following non-steady state from which the system evolves.

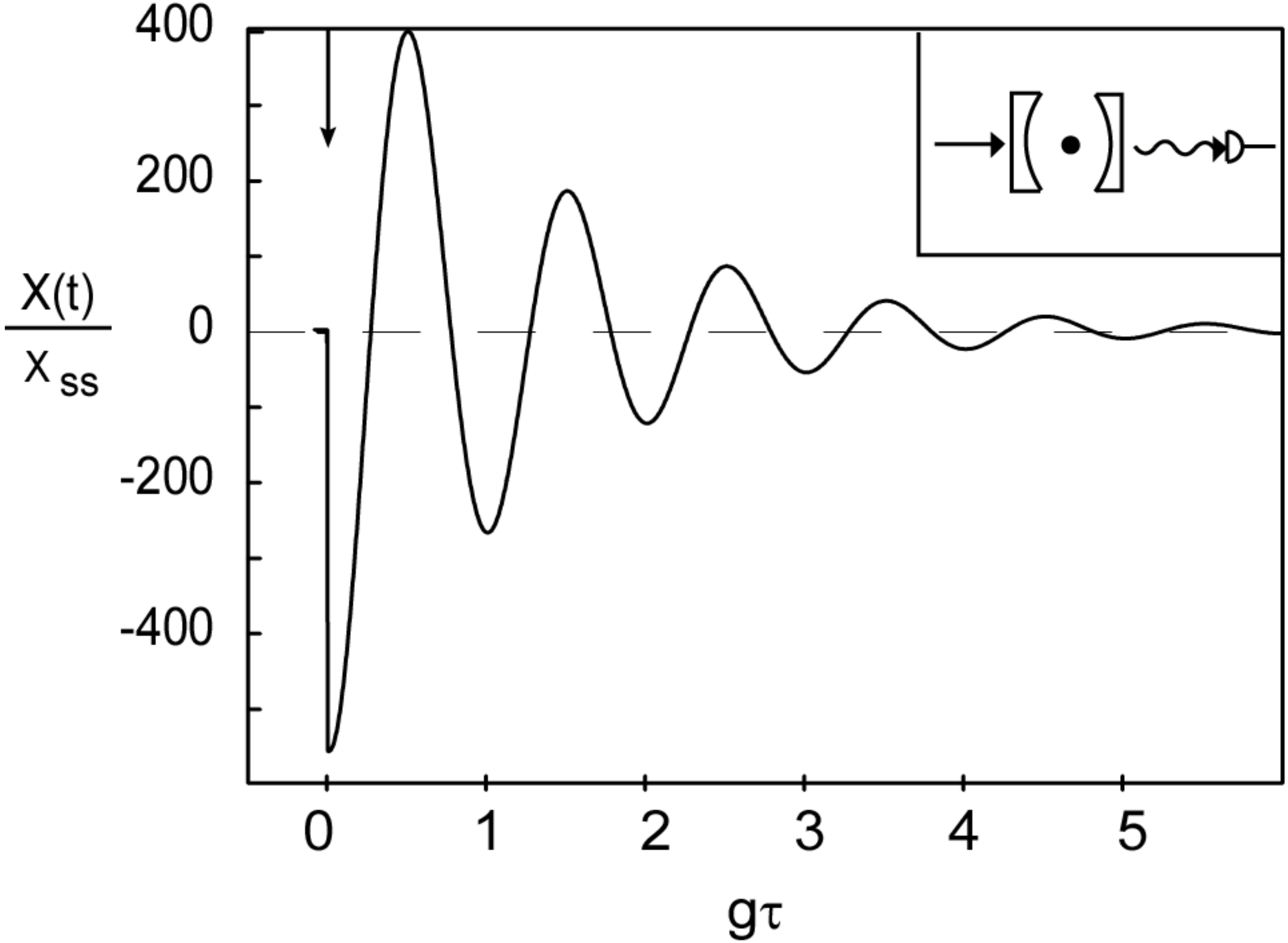
$$\hat{a}\left|\Psi_{ss}\right\rangle \Rightarrow \left|\Psi_c(\tau)\right\rangle = \left|0,g\right\rangle + \lambda pq\left|1,g\right\rangle - \frac{2g\lambda q}{\gamma}\left|0,e\right\rangle$$

$$\left|\Psi_c(\tau)\right\rangle = \left|0,g\right\rangle + \lambda\left[f_1(\tau)\left|1,g\right\rangle + f_2(\tau)\left|0,e\right\rangle\right] + O\left(\lambda^2\right)$$

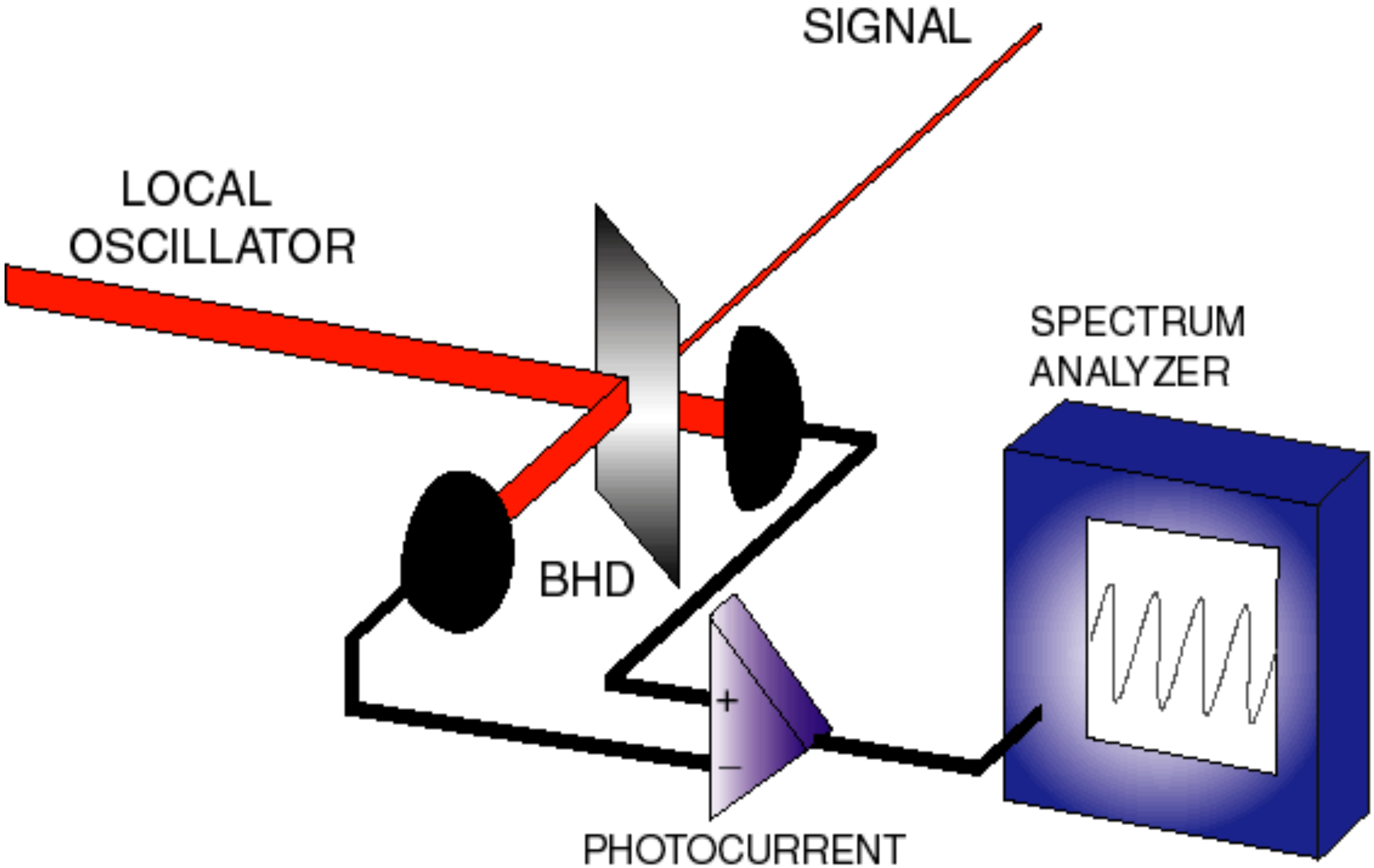
Field

Atomic Polarization

Regression of the field to steady state after the detection of a photon.



Detection of the Squeezing spectrum with a balanced homodyne detector (BHD).

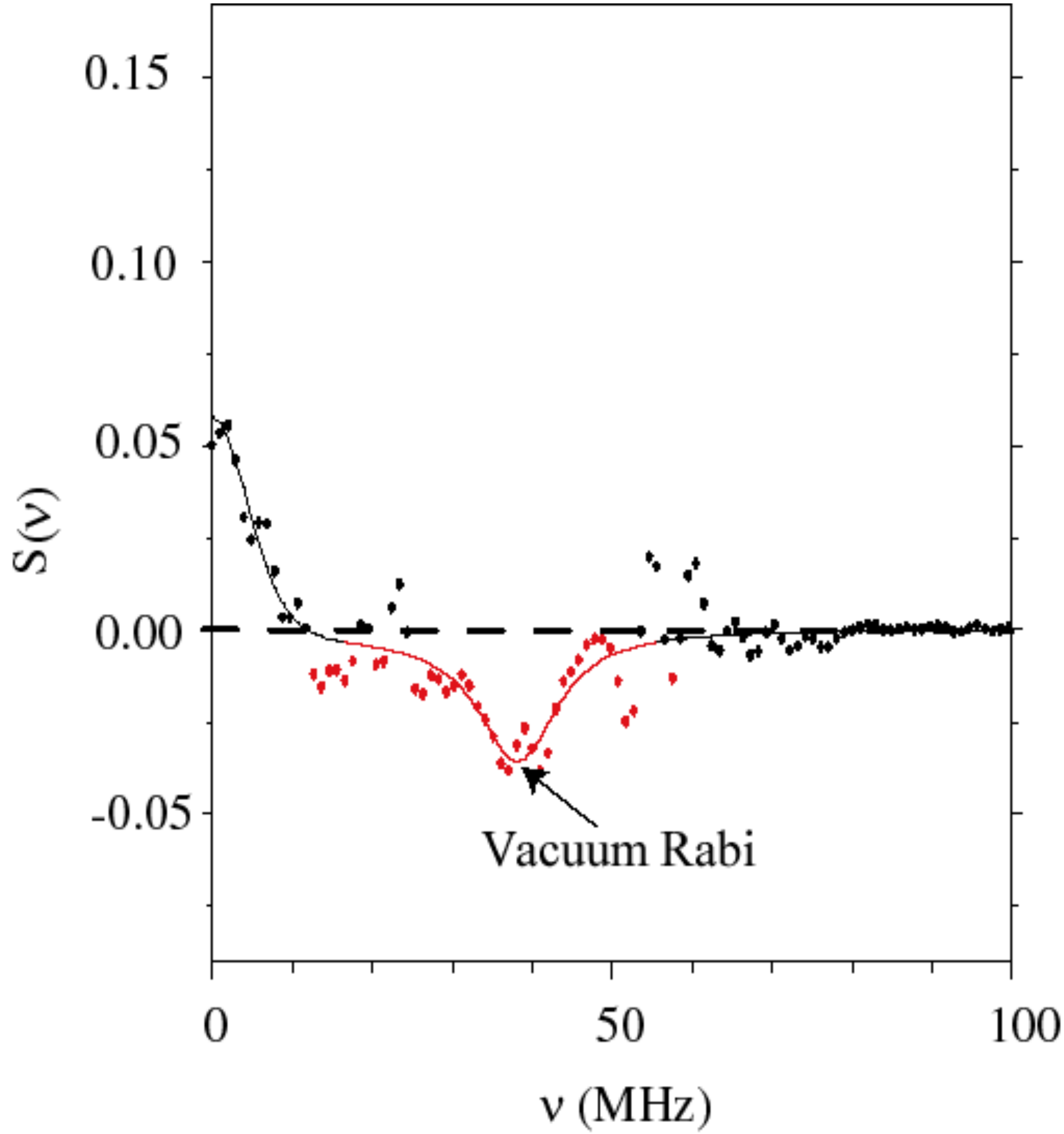


The fluctuations of the electromagnetic field are measured by the spectrum of squeezing. Look at the noise spectrum of the photocurrent.

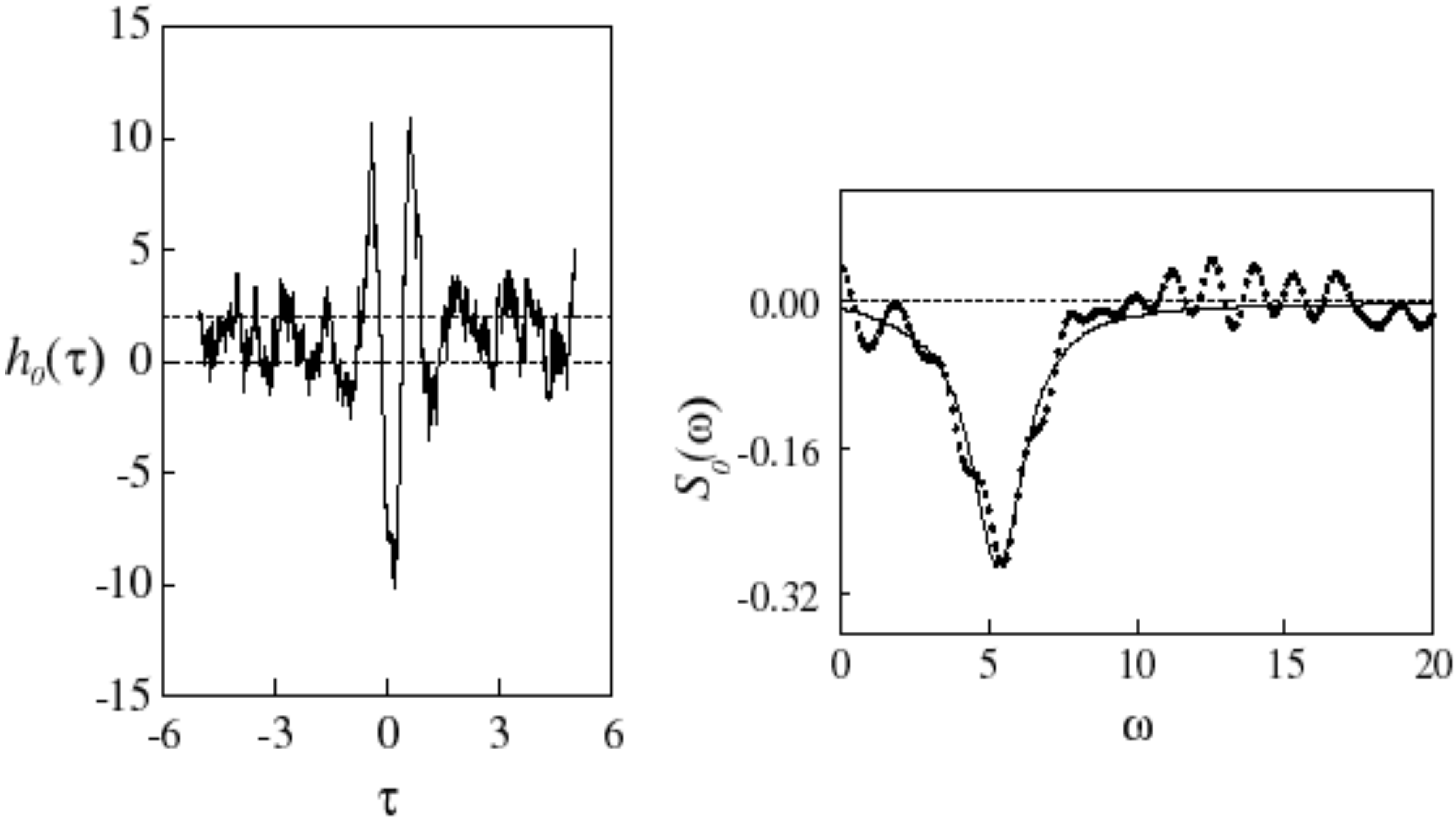
$$S(\nu,0^\circ) = 4F\int_0^\infty \cos(2\pi\nu\tau)[\bar{h}_0(\tau)-1]d\tau,$$

F is the photon flux into the correlator.

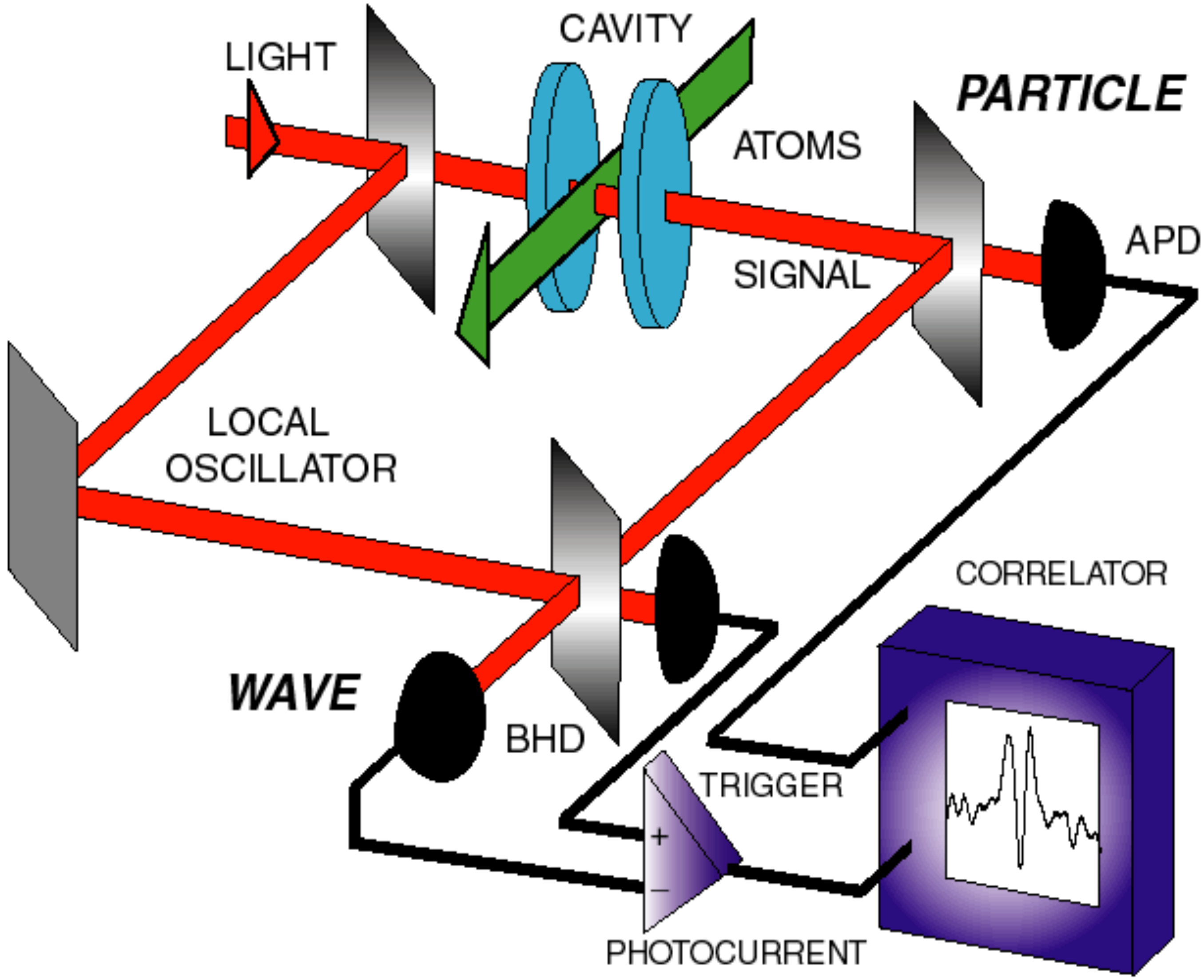
Spectrum of Squeezing from the Fourier Transform of $h_0(t)$

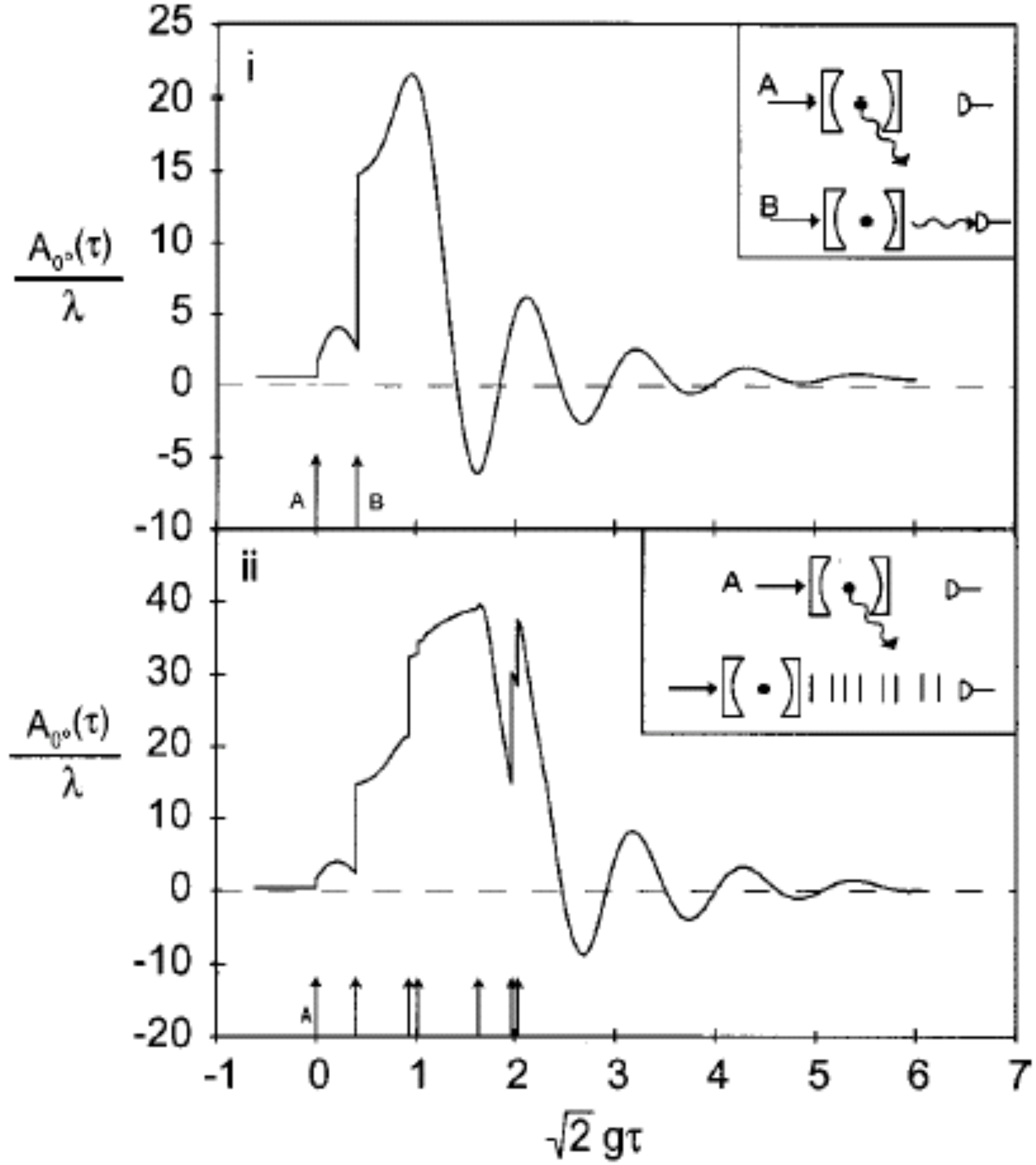


Monte Carlo simulations of the wave-particle correlation and the spectrum of squeezing in the low intensity limit for an atomic beam.

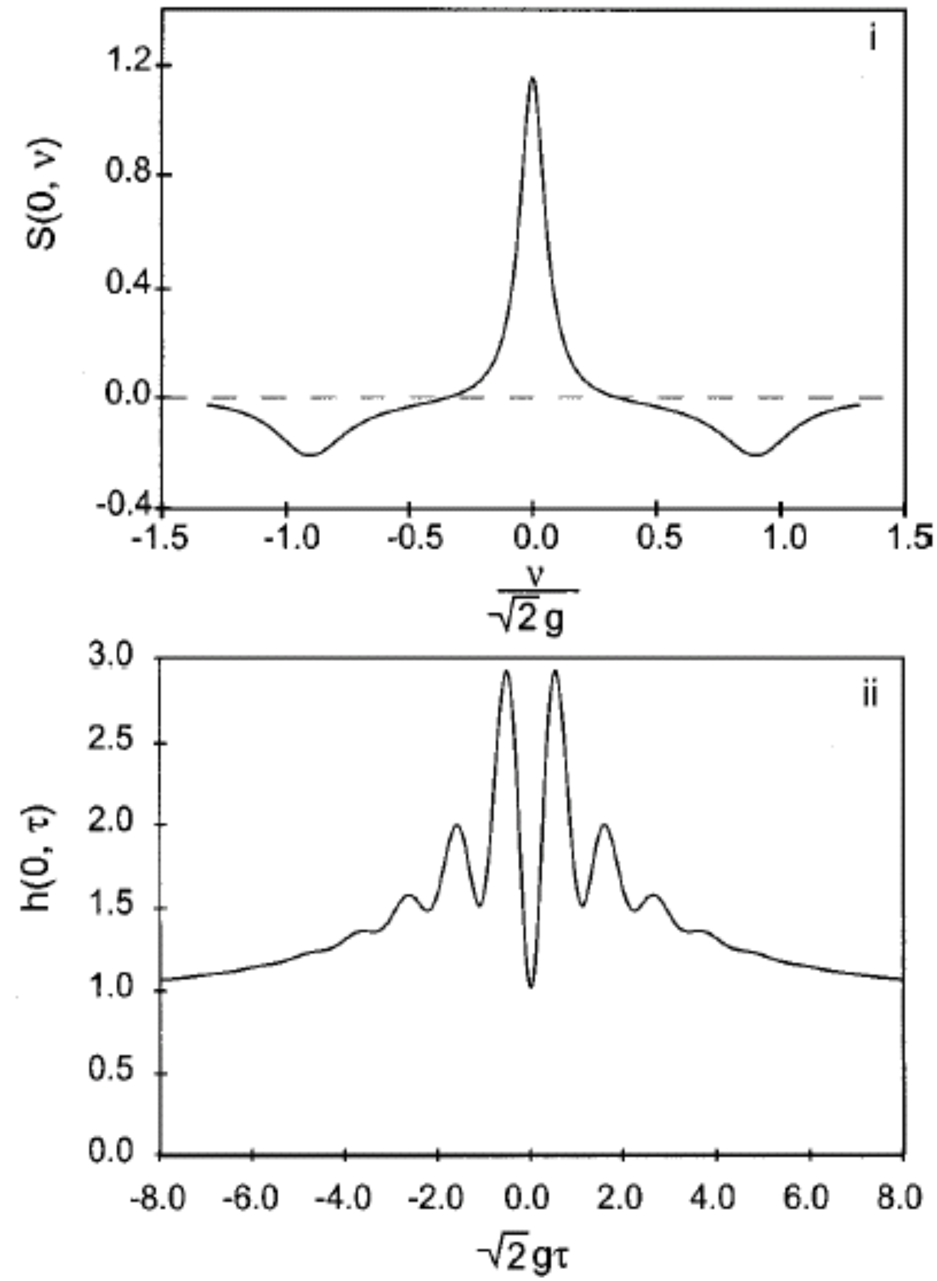


It has upper and a lower classical bounds


LIGHT
CAVITY
ATOMS
PARTICLE
APD
SIGNAL
LOCAL
OSCILLATOR
CORRELATOR
WAVE
BHD
TRIGGER
+
–
PHOTOCURRENT

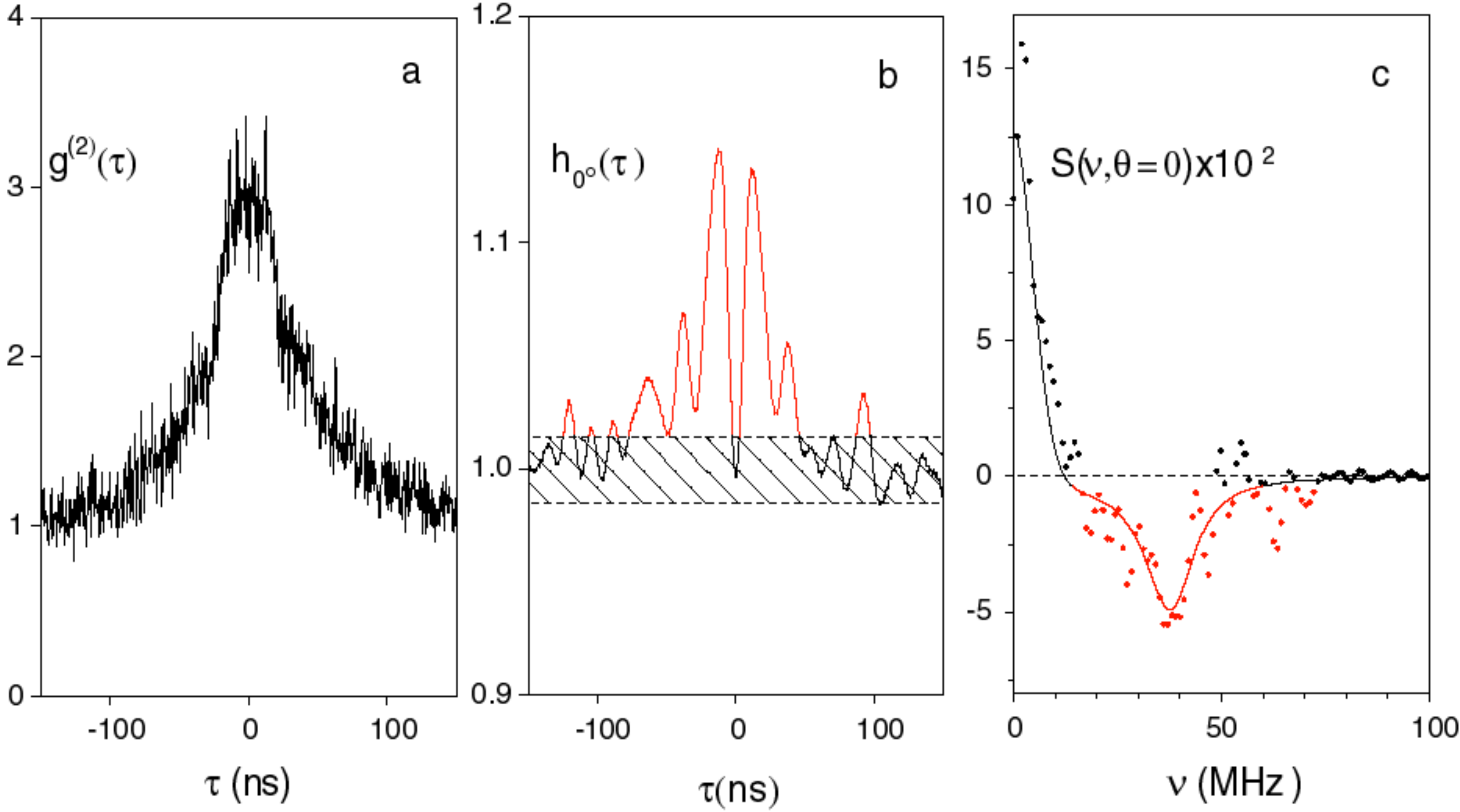


Single quantum trajectories simulation of cavity QED system with spontaneous emission.



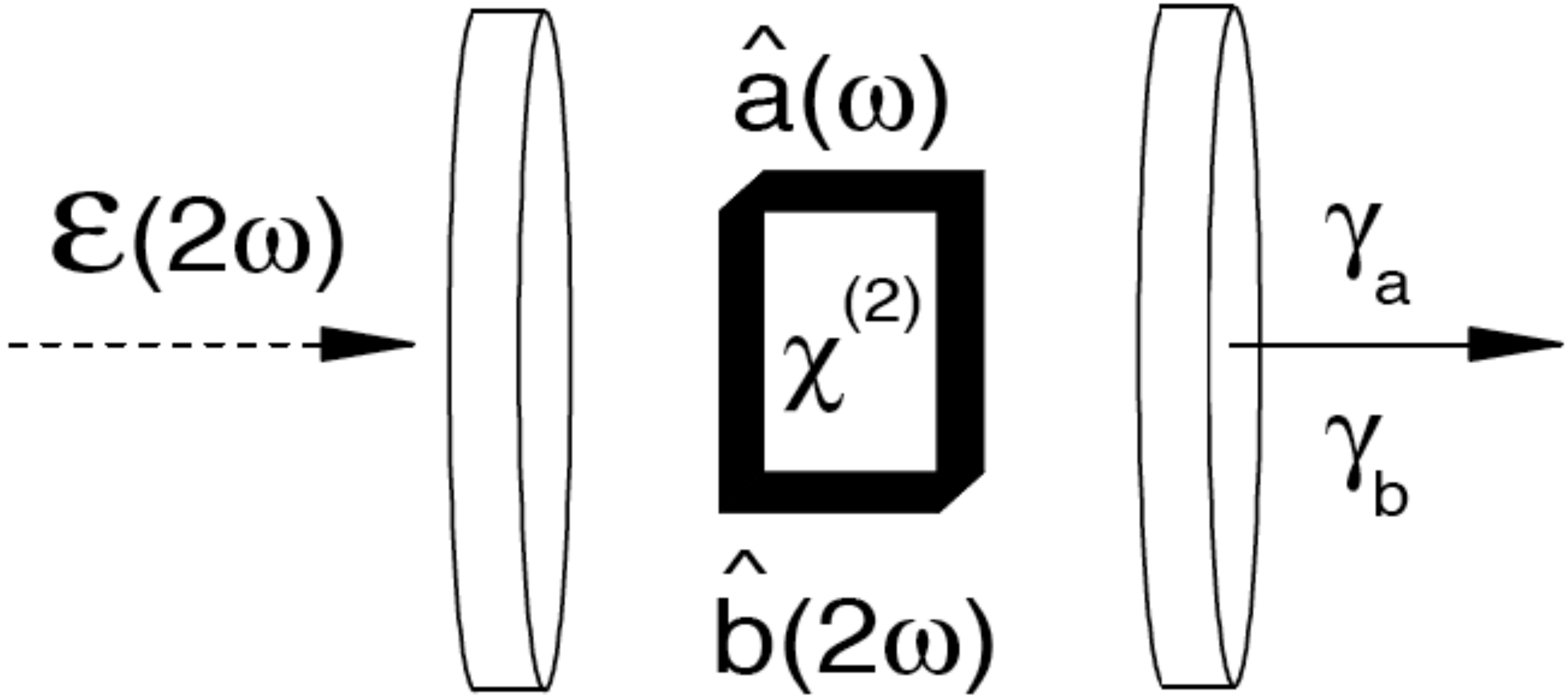
(i) Spectrum of squeezing obtained from the averaged (ii) h(0,t) correlation function that shows the effects of spontaneous emission.

Classical $g^{(2)}$ Non-classical h Squeezing



N=13; $1.2n_0$

Optical Parametric Oscillator



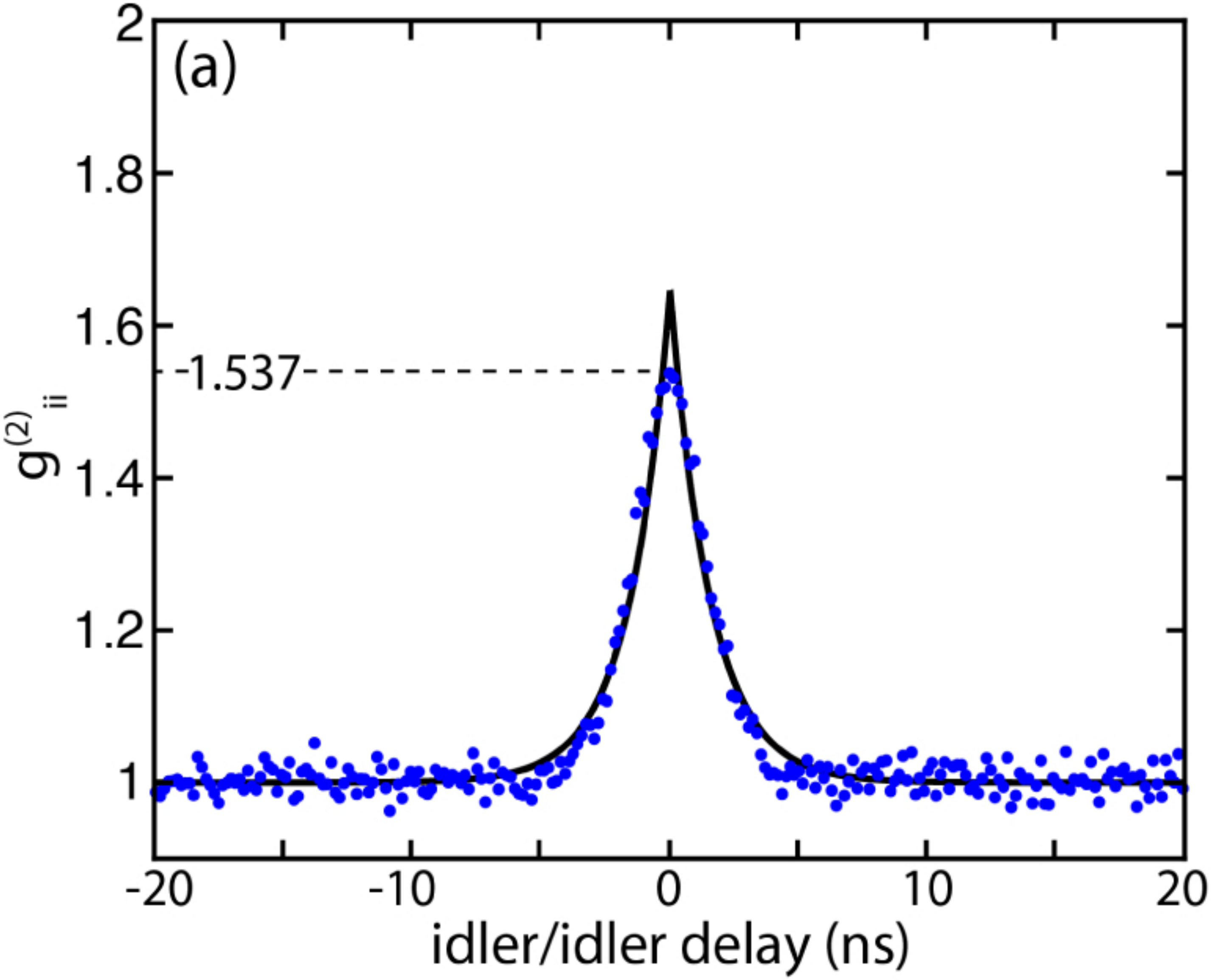
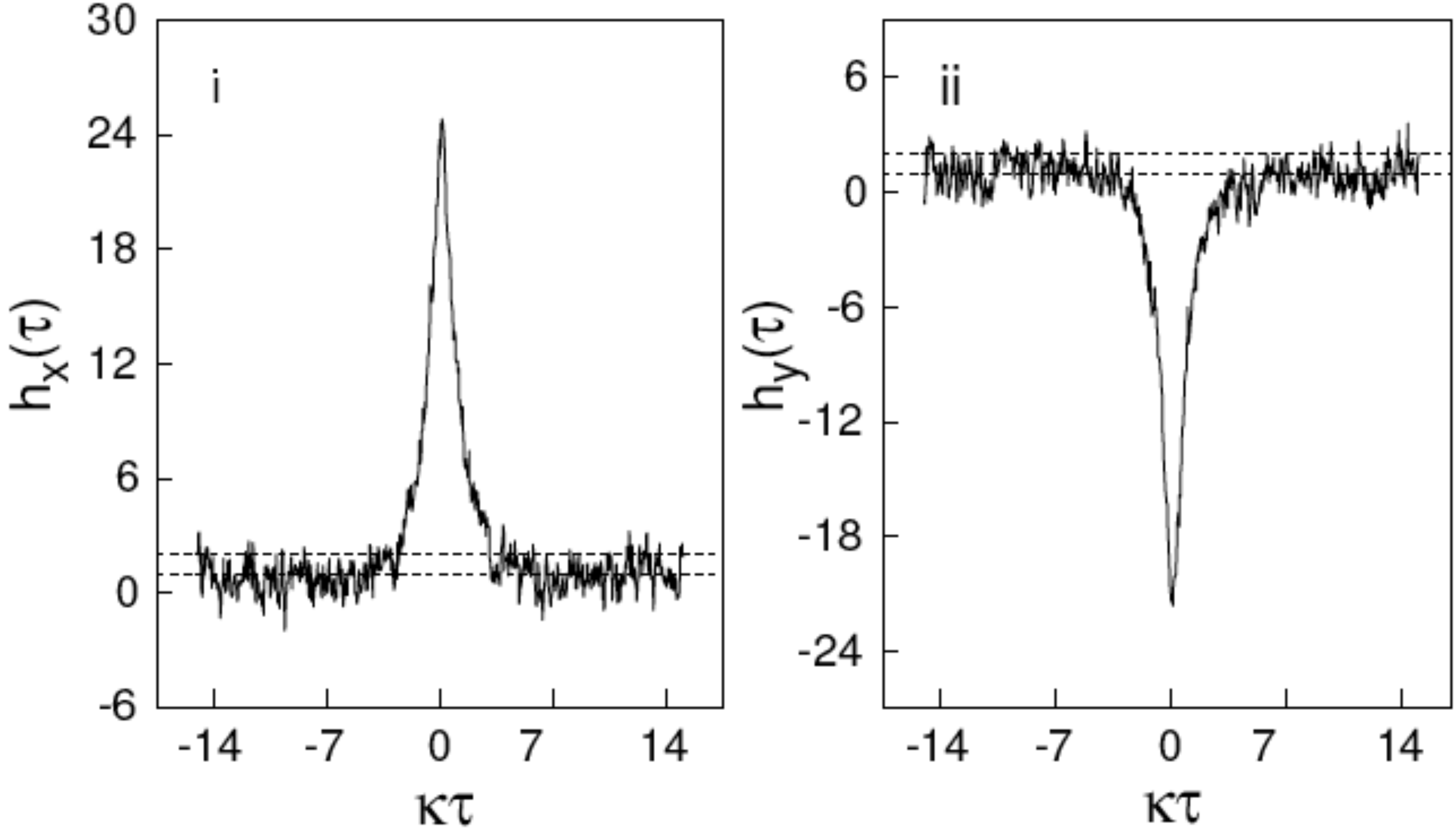


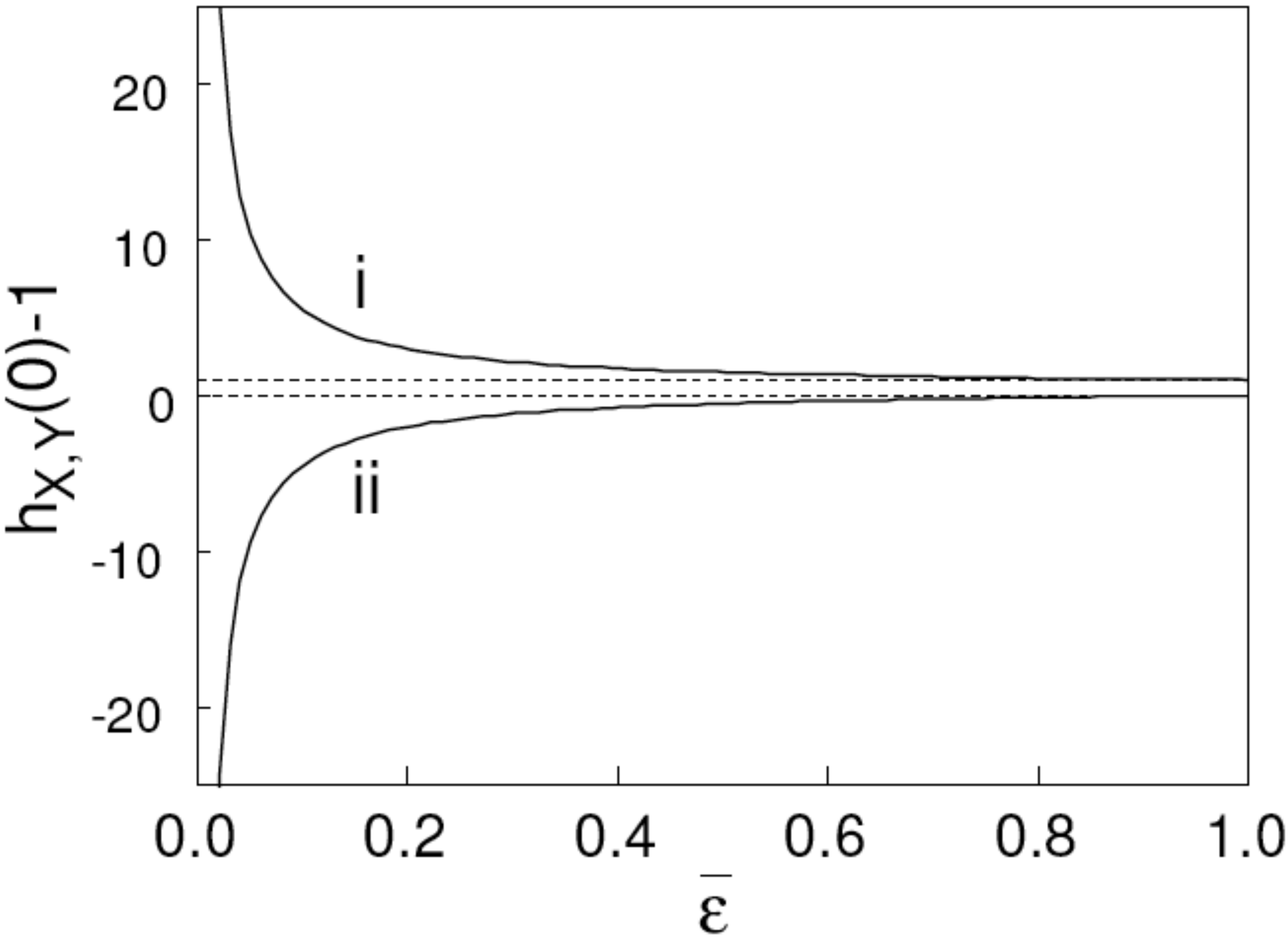
Fig. 4

Citation
Christian Reimer, Lucia Caspani, Matteo Clerici, Marcello Ferrera, Michael Kues, Marco Peccianti, Alessia Pasquazi, Luca Razzari, Brent E. Little, Sai T. Chu, David J. Moss, Roberto Morandotti, "Integrated frequency comb source of heralded single photons," Opt. Express **22**, 6535-6546 (2014);
https://www.osapublishing.org/oe/abstract.cfm?uri=oe-22-6-6535

Calculation of $h_\theta(\tau)$ in an OPO with the classical bounds



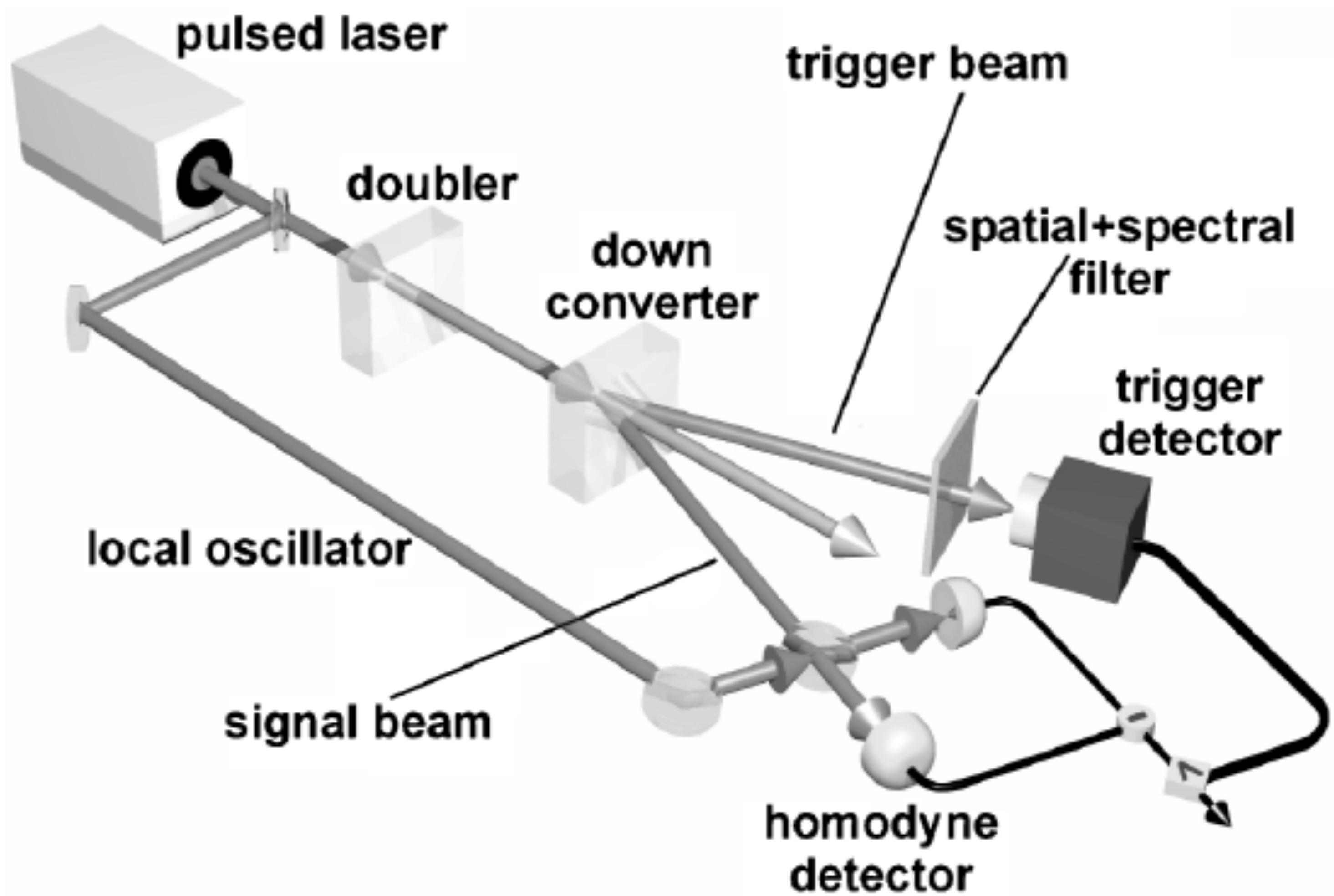
Maximum of $h_\theta(\tau)$ in an OPO below threshold



Quantum State Reconstruction of the Single-Photon Fock State

A. I. Lvovsky,* H. Hansen, T. Aichele, O. Benson, J. Mlynek,† and S. Schiller‡

Phys. Rev. Lett. 87, 050402 (2001)



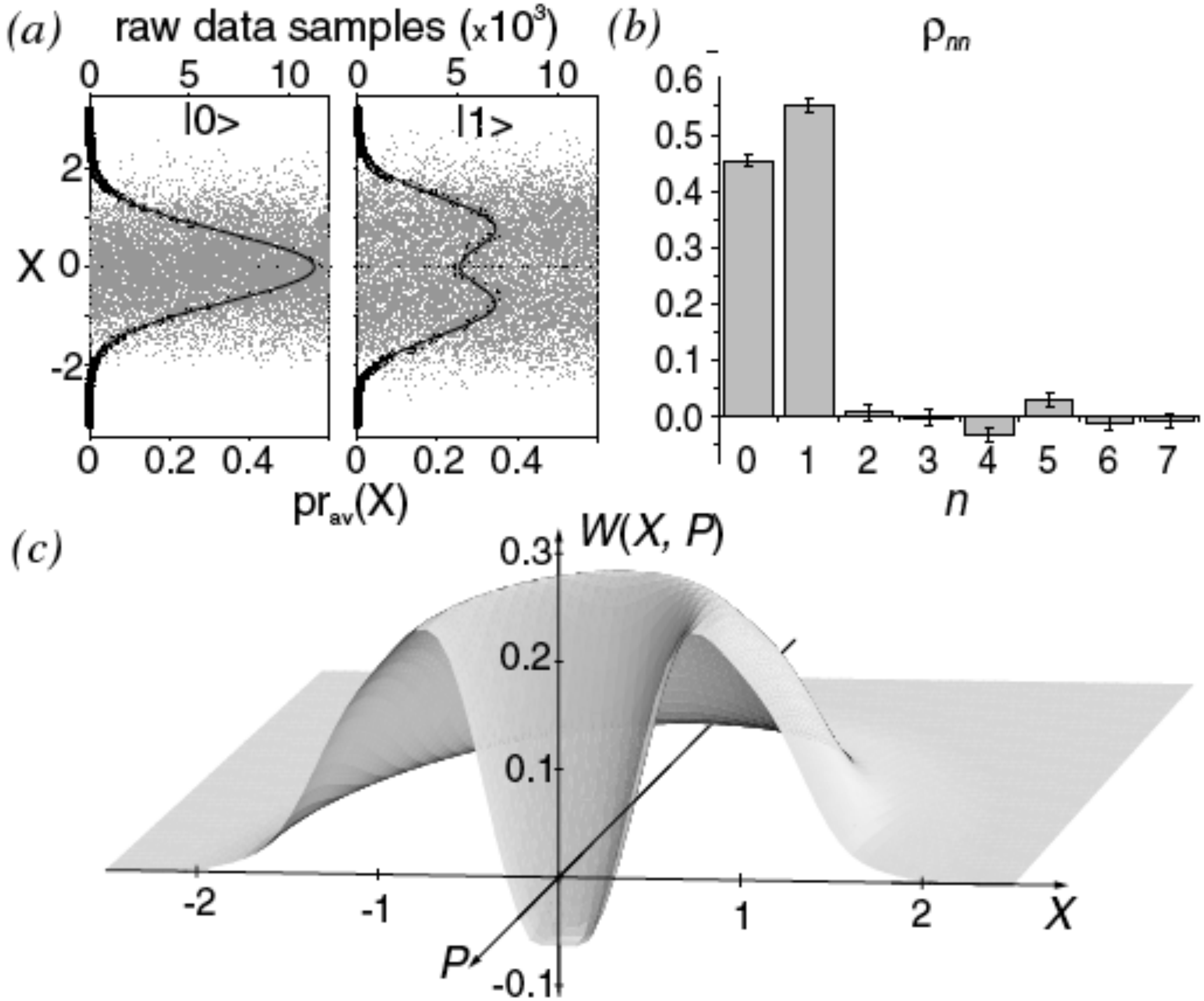


FIG. 4. Experimental results: (a) raw quantum noise data for the vacuum (left) and Fock (right) states along with their histograms corresponding to the phase-randomized marginal distributions; (b) diagonal elements of the density matrix of the state measured; (c) reconstructed WF which is negative near the origin point. The measurement efficiency is 55%.

Conclusions:

- The wave-particle correlation $h_\theta(\tau)$ measures the conditional dynamics of the electromagnetic field. The Spectrum of Squeezing $S(\Omega)$ and $h_\theta(\tau)$ are Fourier Transforms of each other.
- Many applications in many other problems of quantum optics and of optics in general: microscopy, degaussification, weak measurements, quantum feedback.
- Possibility of a tomographic reconstruction of the dynamical evolution of the electromagnetic field state.

Review Article:
H. J. Carmichael, G. T. Foster, L. A. Orozco, J. E. Reiner, and P. R. Rice, “Intensity-Field Correlations of Non-Classical Light”. Progress in Optics, Vol. 46, 355-403, Edited by E. Wolf Elsevier, Amsterdam 2004.

Merci