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Cohomology and removable subsets

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Abstract. Let X be a (connected and reduced) complex space. A *q-collar* of X is a bounded domain whose boundary is a union of a strongly q-pseudoconvex, a strongly q-pseudoconcave and two flat (i.e. locally zero sets of pluriharmonic functions) hypersurfaces. Finiteness and vanishing cohomology theorems obtained in [19, 20] for semi q-coronae are generalized in this context and lead to results on extension problems and removable sets for sections of coherent sheaves and analytic subsets.

Keywords. Cohomology, pseudoconvexity, analytic extension, removable singularities.

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1 Introduction

Let X be a (connected and reduced) complex space. We recall that X is said to be *strongly q-pseudoconvex* in the sense of Andreotti–Grauert [3] if there exist a compact subset $K \subset X$ and a smooth function $\varphi : X \to \mathbb{R}$, $\varphi \geq 0$, which is strongly q-plurisubharmonic on $X \smallsetminus K$ such that

(a) for every $c > 0$ the subset

$$B_c = \{x \in X : \varphi(x) < c\}$$

is relatively compact in X.

Without loss of generality, we may suppose $\min_X \varphi = 0$. If $K = \varnothing$, X is said to be *q-complete*.

For technical reasons, we also assume that the set of the local minima of φ is discrete (cf. [6]) and that $\min_K \varphi > 0$ whenever $K \neq \varnothing$. In particular, for every $c > 0$ one has $\overline{B}_c = \{\varphi \leq 0\}$.

We remark that, for a space, being 1-complete is equivalent to being Stein.

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Replacing the condition (a) by

(a') for every $0 < a < c$ the subset

$$B_{a,c} = \{x \in X : a < \varphi(x) < c\}$$

is relatively compact in X,

we obtain the notion of *q-corona* (see [3, 4]). A q-corona is said to be *complete* whenever $K = \varnothing$.

The extension problem for analytic objects (basically, sections of coherent sheaves, cohomology classes, analytic subsets) defined on q-coronae was studied by many authors (see e.g. [3, 12, 21, 22, 23]).

In [19, 20] we dealt with the larger class of the *semi q-coronae* which are defined as follows. Consider a strongly q-pseudoconvex space (or, more generally, a q-corona) X, and a smooth function $\varphi : X \to \mathbb{R}$ displaying the q-pseudoconvexity of X. Let $B_{a,c} \subset X$ and let $h : X \to \mathbb{R}$ be a pluriharmonic function such that $K \cap \{h = 0\} = \varnothing$. A connected component of $B_{a,c} \smallsetminus \{h = 0\}$ is, by definition, a *semi q-corona*. If X is a complex manifold, the zero set $\{h = 0\}$ can be replaced by a Levi flat hypersurface.

Finiteness and vanishing cohomology theorems proved there lead to results of this type: depending on q, analytic objects given near the convex part of the boundary of a *semi q-corona* fill in the hole.

In this paper we consider a more general situation. Let X be a strongly q-pseudoconvex space. Then $C = B_{a,c} = B_c \smallsetminus \overline{B}_a$ is a q-corona (with exhaustion function $\psi = \frac{1}{c-\varphi} - \frac{1}{c-a}$).

Let Σ_1, Σ_2 be two Levi flat hypersurfaces in a neighbourhood of $\overline{B}_c$ such that

$$B_c \cap \Sigma_1 \cap \Sigma_2 = \Sigma_1 \cap K = \Sigma_2 \cap K = \varnothing,$$

and $\Sigma_1 \cap B_c \neq \varnothing$, $\Sigma_2 \cap B_c \neq \varnothing$ are nonempty connected subsets. We also assume that $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ where h_1, h_2 are pluriharmonic on $\overline{W}_1, \overline{W}_2$ where $W_1 \Subset B_{c'}$, $W_2 \Subset B_{c'}$, $c' > c$, are neighbourhoods of $\Sigma_1 \cap B_c$, $\Sigma_2 \cap B_c$ respectively. Let Q be the open subset of B_c bounded by $\Sigma_1 \cap \overline{B}_c$, $\Sigma_2 \cap \overline{B}_c$ and a part of $\mathrm{b}B_c$. We assume that Q is connected and that $B_c \smallsetminus \overline{Q}$ has two connected components, B_+ and B_-, and define $C_0 = Q \cap C$, $C_+ = B_+ \cap C$, $C_- = B_- \cap C$. The domain C_0 is called a *q-collar* (see Figure 1). A q-collar is said to be *complete* if $K = \varnothing$. Note that C_+ and C_- are semi q-coronae.

Observe that a q-collar is a difference of two pseudoconvex spaces. Indeed, consider $1/(c - \varphi)$ which is a strongly q-plurisubharmonic exhaustion function for B_c. Let $\varepsilon > 0$ be such that

$$\left\{x \in W_1 : h_1(x) \leq \varepsilon\right\} \cap B_c \Subset W$$

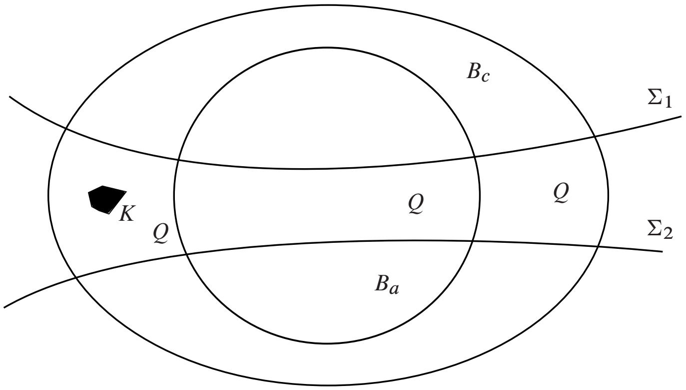

Figure 1. A q-collar $C_0 = Q \cap (B_c \smallsetminus \overline{B}_a)$. In spite of the figure, C_0 is connected.

and let C be the connected component containing $\overline{W} \cap \{h = 0\}$. Define $\widehat{h}$ by $\widehat{h}(x) = h(x)/\varepsilon$ if $x \in C$ and $h(x) \leq \varepsilon$ and $\widehat{h}(x) = 1$ otherwise and define $\psi_1 = -\log(\widehat{h}_1)^2$. Then ψ_1 is plurisubharmonic and positive on $\{0 < h_1 < \varepsilon\}$, constant ($= 0$) on $B_c \smallsetminus \{0 < h_1 < \varepsilon\}$ and $\psi_1 \to +\infty$ when $h_1 \to 0$. Then the function

$$\phi_1 = \begin{cases} \psi_1 \exp(-1/\psi_1^2), & \text{if } \psi_1 > 0, \\ 0, & \text{if } \psi_1 \leq 0, \end{cases}$$

is smooth, plurisubharmonic on $B_c \smallsetminus \{h_1 = 0\}$ and $\phi_1 \to +\infty$ when $h_1 \to 0$. Arguing in the same way, starting from h_2 we construct a function ϕ_2 which is smooth, plurisubharmonic on $B_c \smallsetminus \{h_2 = 0\}$ and $\phi_2 \to +\infty$ when $h_2 \to 0$. It follows that

$$\Phi = \frac{1}{c - \varphi} + \phi_1 + \phi_2$$

is an exhaustion function for Q which is strongly q-plurisubharmonic on $Q \smallsetminus K$.

In order to get the conclusion it is sufficient to apply the same argument starting from B_a.

The results on the cohomology of q-collars, generalizing the ones proved in [19, 20], are established in the first part of the paper (see Section 2). They are applied in Section 3 to study removability. Removability for functions was extensively studied by many authors (see e.g. [24, 17, 14, 9, 16, 18]). We are dealing with removability for sections of coherent sheaves and analytic sets. The main results are contained in Theorems 3.1, 3.3, 3.5, 3.6.

2 Some cohomology

This section is dealing with cohomology of q-collars and some application to extension of sections of coherent sheaves.

2.1 Closed q-collars

Let C_0 be a q-collar in a strongly q-pseudoconvex space X.

Theorem 2.1. *Let* $\mathcal{F} \in \mathrm{Coh}(B_c)$ *and*

$$p(\mathcal{F}) = \inf_{x \in B_c} \operatorname{depth} \mathcal{F}_x .$$

Then, for $q-1 \leq r \leq p(\mathcal{F}) - q - 2$, *the homomorphism*

$$H^r(\overline{Q}, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \longrightarrow H^r(\overline{C}_0, \mathcal{F})$$

(all closures are taken in B_c*), defined by* $(\xi \oplus \eta) \mapsto \xi_{|\overline{C}_0} - \eta_{|\overline{C}_0}$, *has finite codimension.*

If $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ *where* h_1 *and* h_2 *are pluriharmonic functions near* $\Sigma_1 \cap \overline{B}_c$ *and* $\Sigma_2 \cap \overline{B}_c$ *respectively, then*

$$\dim_{\mathbb{C}} H^r(\overline{C}_0, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F}) - q - 2$.

Proof. Consider the Mayer–Vietoris sequence applied to the closed sets $\overline{Q}$ and $\overline{C}$

$$\begin{aligned} \cdots \to H^r(\overline{Q} \cup \overline{C}, \mathcal{F}) \to H^r(\overline{Q}, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \\ \xrightarrow{\delta} H^r(\overline{C}_0, \mathcal{F}) \to H^{r+1}(\overline{Q} \cup \overline{C}, \mathcal{F}) \to \cdots , \end{aligned} \tag{2.1}$$

$\delta(\xi \oplus \eta) = \xi_{|\overline{C}_0} - \eta_{|\overline{C}_0}$. We have

$$\overline{Q} \cup \overline{C} = B_c \smallsetminus U$$

where $U = B_a \smallsetminus (B_a \cap \overline{Q})$. Thus U is q-complete and consequently the groups of compact support cohomology $H^r_c(U, \mathcal{F})$ are zero for $q \leq r \leq p(\mathcal{F}) - q$ (see [3, Proposition 25]).

From the exact sequence of compact support cohomology

$$\cdots \to H^r_c(U, \mathcal{F}) \to H^r(B_c, \mathcal{F}) \to H^r(B_c \smallsetminus U, \mathcal{F}) \to H^{r+1}_c(U, \mathcal{F}) \to \cdots$$

it follows that

$$H^r(B_c, \mathcal{F}) \xrightarrow{\sim} H^r(B_c \smallsetminus U, \mathcal{F}), \tag{2.2}$$

for $q \leq r \leq p(\mathcal{F}) - q - 1$.

Since B_c is q-pseudoconvex,

$$\dim_{\mathbb{C}} H^r(B_c, \mathcal{F}) < \infty$$

for $q \le r$ (see [3, Théorème 16]), and so

$$\dim_{\mathbb{C}} H^r(B_c \smallsetminus U, \mathcal{F}) < \infty$$

for $q \le r \le p(\mathcal{F}) - q - 1$.

From (2.1) we see that

$$\dim_{\mathbb{C}} H^r(B_c \smallsetminus U, \mathcal{F}) = \dim_{\mathbb{C}} H^r(\overline{Q} \cup \overline{C}, \mathcal{F})$$

is greater than or equal to the codimension of the homomorphism δ. This proves that the image of the homomorphism

$$H^r(\overline{Q}, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \longrightarrow H^r(\overline{C}_0, \mathcal{F})$$

(all closures are taken in B_c), defined by $(\xi \oplus \eta) \mapsto \xi_{|\overline{C}_0} - \eta_{|\overline{C}_0}$, has finite codimension provided that $q - 1 \le r \le p(\mathcal{F}) - q - 2$, proving the first assertion of the theorem.

If $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ are as in the second part of the statement, then, since $K \cap (\Sigma_1 \cup \Sigma_2) = \varnothing$, $\overline{Q}$ has a fundamental system of neighborhoods which are q-pseudoconvex spaces, thus, by virtue of [3, Théorème 11] we have

$$\dim_{\mathbb{C}} H^r(\overline{Q}, \mathcal{F}) < \infty$$

for $r \ge q$. On the other hand, $\overline{C}$ is a q-corona, so

$$\dim_{\mathbb{C}} H^r(\overline{C}, \mathcal{F}) < \infty$$

for $q \le r \le p(\mathcal{F}) - q - 1$ in view of [4, Theorem 3].

Summarizing, for $q \le r \le p(\mathcal{F}) - q - 1$ the vector space $H^r(\overline{Q}, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F})$ has finite dimension and for $q - 1 \le r \le p(\mathcal{F}) - q - 2$ its image in $H^r(\overline{C}_0, \mathcal{F})$ has finite codimension. Thus, for $q \le r \le p(\mathcal{F}) - q - 2$, $H^r(\overline{C}_0, \mathcal{F})$ has finite dimension. □

Theorem 2.2. *Assume that $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ where h_1 and h_2 are pluriharmonic functions near $\Sigma_1 \cap \overline{B}_c$ and $\Sigma_2 \cap \overline{B}_c$ respectively, and $\overline{Q} \cap K = \varnothing$. Then*

$$H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_0, \mathcal{F})$$

for $q \le r \le p(\mathcal{F}) - q - 2$ and the homomorphism

$$H^{q-1}(\overline{Q}, \mathcal{F}) \oplus H^{q-1}(\overline{C}, \mathcal{F}) \longrightarrow H^{q-1}(\overline{C}_0, \mathcal{F}) \tag{2.3}$$

is surjective for $p(\mathcal{F}) \ge 2q + 1$.

If $\overline{B}_+$ *is a* 1*-complete space and* $p(\mathcal{F}) \geq 3$*, the homomorphism*

$$H^0(\overline{Q}, \mathcal{F}) \longrightarrow H^0(\overline{C}_0, \mathcal{F})$$

is surjective.

Proof. By hypothesis $\overline{Q}$ has a fundamental system of neighborhoods which are q-complete spaces, so $H^r(\overline{Q}, \mathcal{F}) = \{0\}$ for $q \leq r$ (see [3, Corollaire, p. 250]). From (2.2) it follows that $H^r(\overline{Q} \cup \overline{C}, \mathcal{F}) = \{0\}$ for $q \leq r \leq p(\mathcal{F}) - q - 1$. Thus, from the Mayer–Vietoris sequence (2.1) we derive the isomorphism

$$H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_0, \mathcal{F})$$

for $q \leq r \leq p(\mathcal{F}) - q - 2$ and that the homomorphism (2.3) is surjective if $p(\mathcal{F}) \geq 2q + 1$.

In particular, if $q = 1$ and $p(\mathcal{F}) \geq 3$, the homomorphism

$$H^0(\overline{Q}, \mathcal{F}) \oplus H^0(\overline{C}, \mathcal{F}) \longrightarrow H^0(\overline{C}_0, \mathcal{F})$$

is surjective, i.e. every section $\sigma \in H^0(\overline{C}_0, \mathcal{F})$ is a difference $\sigma_1 - \sigma_2$ of two sections $\sigma_1 \in H^0(\overline{Q}, \mathcal{F})$, $\sigma_2 \in H^0(\overline{C}, \mathcal{F})$. Since B_a is Stein, the cohomology group with compact supports $H^1_c(B_a, \mathcal{F})$ is zero, and so the Mayer–Vietoris compact support cohomology sequence implies that the restriction homomorphism

$$H^0(\overline{B}_c, \mathcal{F}) \longrightarrow H^0(\overline{B}_c \setminus B_a, \mathcal{F}) = H^0(\overline{C}, \mathcal{F})$$

is surjective, hence $\sigma_2 \in H^0(\overline{C}, \mathcal{F})$ is the restriction of $\widetilde{\sigma}_2 \in H^0(B_c, \mathcal{F})$. So σ is the restriction to $\overline{C}_0$ of $(\sigma_1 - \widetilde{\sigma}_{2|\overline{B}_+}) \in H^0(\overline{Q}, \mathcal{F})$, and the restriction homomorphism is surjective. □

Corollary 2.3. *Let* $q = 1$ *and* depth $\mathcal{O}_x \geq 3$ *for every* $x \in B_c$*. Then all holomorphic functions on* $\overline{C}_0$ *extend holomorphically on* $\overline{Q}$*.*

2.2 Open q-collars

Keeping the same notations as above consider an open q-collar C_0. For the sake of simplicity we assume that B_c is q-complete. We also assume that $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$ where h_1 and h_2 are pluriharmonic functions on open neighbourhoods U_1 and U_2 of $\Sigma_2 \cap \overline{B}_c$ and $\Sigma_1 \cap \overline{B}_c$ respectively.

Theorem 2.4. *Let* B_c *be* 1*-complete and* $\mathcal{F}$ *be a coherent sheaf on* B_c *satisfying* depth $\mathcal{F}_x \geq 3$ *for every* $x \in B_c$*. Then the homomorphism*

$$H^0(Q, \mathcal{F}) \longrightarrow H^0(C_0, \mathcal{F})$$

is surjective.

Proof. Let $s \in H^0(C_0, \mathcal{F})$. Fix a couple of positive numbers $\varepsilon = (\varepsilon_1, \varepsilon_2)$ small enough such that Σ_{i,ε_i} defined by $\Sigma_{i,\varepsilon_i} = \{h_i = \varepsilon_i\}$ are connected hypersurfaces, $\Sigma_{i,\varepsilon_i} \cap \overline{B}_c \cap \overline{Q} \neq \varnothing$ and $\Sigma_{i,\varepsilon_i} \cap \overline{B}_c \subset U_i$, for $i = 1, 2$.

Consider the open subset Q_ε of Q bounded by the hypersurfaces $\Sigma_{i,\varepsilon_i} \cap \overline{B}_c$ and by a part of $\mathrm{b}B_c$, and set $C_{0,\varepsilon} = Q_\varepsilon \cap C_0$. In view of Theorem 2.2 there exists a section $\tilde{s}_\varepsilon \in H^0(\overline{Q}_\varepsilon, \mathcal{F})$ which extends $s_{|C_{0,\varepsilon}}$. Now observe that the connected component W of $B_c \setminus \Sigma_1$ containing Σ_2 is Stein. So there exists a strongly pseudoconvex domain $\Omega \Subset W$ such that the domain D_ε bounded by $\Sigma_{2,\varepsilon_2} \cap \overline{B}_c$, $\Sigma_2 \cap \overline{B}_c$ and by a part of $\mathrm{b}B_c$ is relatively compact in Ω. By Theorem 5 of [19] the section $\tilde{s}_\varepsilon$ extends on $\Omega \cap Q$. Thus s extends on Q_ε. In order to conclude the proof we argue as before with respect to the hypersurfaces Σ_{1,ε_1} and Σ_1. □

In particular, we get the extension of holomorphic functions:

Corollary 2.5. *If B_c is a 1-complete space and* depth $\mathcal{F}_x \geq 3$ *for every $x \in B_c$, all holomorphic functions on C_0 uniquely extend on Q.*

Corollary 2.6. *Let X be a Stein space. Let $\Sigma_1 = \{h_1 = 0\} \subset X$ and $\Sigma_2 = \{h_2 = 0\} \subset X$ be the zero set of two pluriharmonic functions, and let S be a bounded real hypersurface of X with boundary $bS \subset \Sigma_1 \cup \Sigma_2$ such that $S \cap \Sigma_1 = bS \cap \Sigma_1 = bA_1$ and $S \cap \Sigma_2 = bS \cap \Sigma_2 = bA_2$ where A_1 is an open set in Σ_1 and A_2 is an open set in Σ_2. Let $D \subset X$ be the relatively compact domain bounded by $S \cup A_1 \cup A_2$ and $\mathcal{F}$ be a coherent sheaf on a neighbourhood of $\overline{D}$ with* depth $\mathcal{F}_x \geq 3$ *for all $\in \overline{D}$. Then every section of $\mathcal{F}$ on S extends to D.*

2.3 Finiteness of cohomology

Results on the cohomology of q-collars obtained in the preceding section concern coherent sheaves defined in larger domains. For the applications that we have in mind it is needed to study cohomology of coherent sheaves which are defined just on collars. This can be done by the same methods used in [20] for semi q-coronae. We briefly sketch the main points of proofs given there focusing on the case $q = 1$. The extension for an arbitrary q demands only technical adjustments. Keeping the same notations as in Section 1 let

$$C_0 = Q \cap (B_c \setminus \overline{B}_a) = Q \cap B_{a,c} = Q \cap \{x \in X : a < \varphi(x) < c\}$$

be an open 1-collar of a Stein space X (see Figure 1, page 1095). Q is the subdomain of B_c bounded by the two Levi flat hypersurfaces $\Sigma_1 = \{h_1 = 0\}$, $\Sigma_2 = \{h_2 = 0\}$. Σ_1 and Σ_2 are defined on a neighbourhood of $\overline{B}_c$ where h_1 and h_2 are pluriharmonic functions near Σ_1 and Σ_2 respectively. Thus Q is a

Stein domain. By Σ_1^0, Σ_2^0 we denote the parts of $\mathrm{b}C_0$ contained in Σ_1 and Σ_2, and by F_1^0, F_2^0 the 1-pseudoconvex and the 1-pseudoconcave part respectively. Since Q is Stein, there exist two families of 1-pseudoconvex hypersurfaces $\{\Sigma_1^\varepsilon\}$, $\{\Sigma_2^\varepsilon\}$, $\varepsilon \searrow 0$, in a neighbourhood of $\overline{Q}$, with the following properties:

(1) Σ_1^ε, Σ_2^ε bound a strip $Q_\varepsilon \subset Q$ and $\Sigma_1^\varepsilon \to \Sigma_1$, $\Sigma_2^\varepsilon \to \Sigma_2$ as $\varepsilon \searrow 0$;

(2) defining $C_0^\varepsilon = Q_\varepsilon \cap B_{a+\varepsilon,c-\varepsilon}$ we obtain an exhaustion $\{C_0^\varepsilon\}$ of the collar C_0.

The bump lemma and the approximation theorem hold for the closed subsets $\overline{C}_0^{\,\varepsilon}$ with the same proof as in [20, Lemma 3.3, 3.9] and this enables us to the following results. Assume that depth $\mathcal{F}_x \geq 3$ for x near to the pseudoconcave part of the boundary of C_0; then

(3) there exists ε_0 sufficiently small such that if $\varepsilon < \varepsilon_0$ the cohomology spaces $H^1(\overline{C}_\varepsilon^{\,+}, \mathcal{F})$ are finite dimensional;

(4) if $\varepsilon < \varepsilon_0$ there exists $\varepsilon_1 < \varepsilon$ such that

$$H^1(C_{\varepsilon'}^+, \mathcal{F}) \simeq H^1(\overline{C}_\varepsilon^{\,+}, \mathcal{F})$$

for every $\varepsilon' \in]\varepsilon_1, \varepsilon[$.

(3) and (4) have an important consequence, namely that for $\mathcal{F}$ Theorem A of Oka–Cartan–Serre holds in the following form (see [20, Corollary 4.2]):

(5) if $\varepsilon, \varepsilon'$ are as in (4), for every compact subset K of $C_{\varepsilon'}^+ \smallsetminus \{\varphi > c - \varepsilon\}$ there exist sections $s_1, \dots, s_k \in H^0(C_{\varepsilon'}^+, \mathcal{F})$ which generate $\mathcal{F}_x$ for every $x \in K$.

As an application we get the following extension theorem for analytic subsets.

Theorem 2.7. *Let X be a Stein space, $C_0 = Q \cap \left(B_c \smallsetminus \overline{B}_a\right) \subset X$ be a complete 1-collar and Y be a closed analytic subset of C_0 such that* depth $\mathcal{O}_{Y,x} \geq 3$ *for x near $\{\varphi = a\}$. Then Y extends to a closed analytic subset on Q.*

Proof. Taking into account (5) the proof runs as in [20, Theorem 4.3 and Corollary 4.4]. □

3 Removable sets

The notion of removable sets was originally given with respect to holomorphic functions and the removability problem was extensively studied (see e.g. [24, 17, 14, 9, 16], and the recent survey [18]). Here we want to study the same problem with respect to larger classes of analytic objects, namely the classes of sections of coherent sheaves, of cohomology classes and of analytic sets.

Let X be a complex space, D be a bounded domain. Let $\mathcal{F}$ be a coherent sheaf on a neighbourhood of $\overline{D}$. A subset L of the boundary bD of D is said to be *removable* for (the sections of) $\mathcal{F}$ or for the cohomology classes with values in $\mathcal{F}$, of a certain degree r, if every section $s \in \Gamma(\mathrm{b}D \smallsetminus L, \mathcal{F})$ or cohomology class $\omega \in H^r(\mathrm{b}D \smallsetminus L, \mathcal{F})$ extends by $\tilde{s} \in \Gamma(\overline{D} \smallsetminus L, \mathcal{F})$ or by $\widetilde{\omega} \in H^r(\overline{D} \smallsetminus L, \mathcal{F})$ respectively.

Similarly, the subset L is said to be *removable* for the (respectively, a given) class of analytic subsets if every analytic subset (of a given class of analytic subsets) defined on a neighbourhood of b$D \smallsetminus L$ extends by an analytic subset of $\overline{D} \smallsetminus L$.

3.1 Coherent sheaves

Given a coherent sheaf $\mathcal{F}$ on a complex space X let us denote by Tor($\mathcal{F}$) the torsion of $\mathcal{F}$; Tor($\mathcal{F}$) is the coherent subsheaf of $\mathcal{F}$ whose stalk at a point $x \in X$ is

$$\mathrm{Tor}(\mathcal{F})_x = \left\{ s_x \in \mathcal{F}_x : \lambda_x s_x = 0 \text{ for some } \lambda \in \mathcal{O}_x,\ \lambda \neq 0 \right\}.$$

It can be proved (see [2]) that the topology of $\mathcal{F}$ is Hausdorff if and only if $\mathcal{F}$ has no torsion, i.e. Tor($\mathcal{F}$) $= \{0\}$. We denote by $T(\mathcal{F})$ the analytic subset supp Tor($\mathcal{F}$).

Given a bounded domain $D \subset X$ let $\mathcal{A}(D)$ be the algebra $C^0(\overline{D}) \cap \mathcal{O}(D)$ and for every compact $L \subset \overline{D}$ let

$$\widehat{L} = \left\{ x \in \overline{D} : |f(z)| \leq \max_L |f|, \forall f \in \mathcal{A}(D) \right\}$$

be the $\mathcal{A}(D)$-envelope of L. We want to prove the following

Theorem 3.1. *Let X be an n-dimensional manifold, D a bounded pseudoconvex domain in X with a connected smooth boundary and L a compact subset of bD such that* b$D \smallsetminus L$ *is a non-empty strongly Levi convex hypersurface. Let $\mathcal{F}$ be a coherent sheaf on X satisfying*

(1) depth $\mathcal{F}_x \geq 3$ *for every $x \in \overline{D}$;*

(2) $\dim_{\mathbb{C}} T(\mathcal{F}) \cap \overline{D} \leq n-2$.

Let U be an open neighborhood of $\overline{D} \smallsetminus (\widehat{L} \cap bD)$ or of $X \smallsetminus (D \cup (\widehat{L} \cap bD))$. Then every section of $\mathcal{F}$ on $U \smallsetminus \overline{D}$ or $U \smallsetminus (X \smallsetminus D)$ uniquely extends to a section on $U \smallsetminus \widehat{L}$ or $D \smallsetminus \widehat{L}$. In particular, if $\widehat{L} = L$, then L is removable for $\mathcal{F}$.

Proof. Before starting the proof, observe that $D \smallsetminus \widehat{L} \subset D$ has no relatively compact (in D) connected components. Indeed, suppose $K \subset D$ is a relatively compact (in D) connected component of $D \smallsetminus \widehat{L}$. Then $bK \subset \widehat{L}$ and by the maximum principle, for every $f \in A(D)$,

$$\max_K |f| \leq \max_{bK} |f| \leq \max_{\widehat{L}} |f| = \max_L |f|,$$

which means $K \subset \widehat{L}$, a contradiction.

The uniqueness is a consequence of the *Kontinuitätsatz* and of hypothesis (1). Indeed let s_1, s_2 be sections of $\mathcal{F}$ on $D \smallsetminus \hat{L}$ such that $s_1 \equiv s_2$ near $\mathrm{b}D \smallsetminus L$. In view of the hypothesis (1), the support of $s_1 - s_2$ is an analytic subset A of $D \smallsetminus \widehat{L}$ with no 0-dimensional irreducible component (see [5, Théorème 3.6 (a), p. 46]). Let A_1 be an irreducible component of A. Since $\mathrm{b}D \smallsetminus L$ is strongly Levi convex, in view of the *Kontinuitätsatz* A_1 cannot touch $\mathrm{b}D \smallsetminus L$ so $\overline{A}_1 \cap \mathrm{b}D \equiv \overline{A}_1 \cap L$. Let $x \in A_1$. Since $x \notin \widehat{L}$, there exists $f \in \mathcal{A}(D)$ such that $\max_L |f| < |f(x)|$. Consider an exhaustion $W_1 \Subset W_2 \Subset \cdots$ by relatively open subsets of A_1, $x \in W_1$. By virtue of the maximum principle, for every k there exists a point $x_k \in \mathrm{b}W_k$ such that $|f(x)| < |f(x_k)|$. Then (passing if necessary to a subsequence) we have $x_k \to y \in \widehat{L}$ as $k \to +\infty$ and consequently $|f(x)| \leq |f(y)| \leq \max_L |f|$, a contradiction.

We now need to show the existence of the extension. In order to prove the extension we consider just the case that U is an open neighborhood of $\overline{D} \smallsetminus L$ or $X \smallsetminus (D \cup L)$ and $\sigma \in \mathcal{F}(U \smallsetminus \overline{D})$, the proof in the other one being similar. In view of the hypothesis (1), given a point $x \in D \smallsetminus \widehat{L}$ there exists $f = f_x \in \mathcal{A}(D)$, $f = u + iv$, $u = u_x$, $v = v_x$ real-valued functions, such that $f(x) = u(x) = 1$ $\max_L |f| < 1$; in particular $\max_L |u| < 1$. Then, if $\varepsilon = \varepsilon_x > 0$ is sufficiently small and $C = C_x = \{u \geq 1 - \varepsilon\}$, we have $C \cap L = \varnothing$. Let $V = V_x$ be an open neighborhood of L such that $C \cap \overline{V} = \varnothing$. Since $\mathrm{b}D \smallsetminus L$ is strongly pseudoconvex, there exists a pseudoconvex domain $D_1 = D_{1,x}$ with a smooth boundary satisfying the following properties:

(i) $D \subset D_1$, $\overline{D}_1 \smallsetminus D \subset U$;

(ii) $\mathrm{b}D_1 \cap \mathrm{b}D \subset V \cap \mathrm{b}D$;

(iii) $\mathrm{b}D_1$ is strongly pseudoconvex at the points of $\mathrm{b}D_1 \smallsetminus \mathrm{b}D_1 \cap \mathrm{b}D$.

Since D_1 is Stein there exists a strongly pseudoconvex $D_2 = D_{2,x} \Subset D_1$ which contains the compact subset $\overline{D} \smallsetminus (V \cap D)$ (hence also $x \in D_2$) and such that $\mathrm{b}(D_2 \cap D) \smallsetminus \mathrm{b}D \Subset V$ (see Figure 2).

The boundary of $D_3 = D_{3,x} = D_2 \cap D$ is piecewise smooth but we may regularize it along $\mathrm{b}D_2 \cap \mathrm{b}D$, thus we may assume that D_3 is a smooth strongly

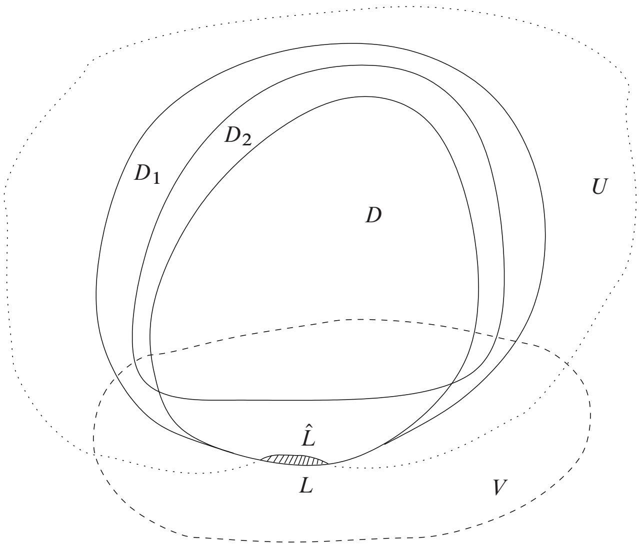

Figure 2. Construction of the three domains D_1, D_2 and $D_3 = D_2 \cap D$.

pseudoconvex domain $D_3 = \{\varrho < 0\}$ where $\varrho = \varrho_x$ is a strongly plurisubharmonic function on a neighbourhood of $\overline{D}_3$ and $d\varrho(z) \neq 0$ along $\mathrm{b}D_3$. By the approximation theorem of Kerzman (see [15]) there exists an open neighbourhood $W = W_x$ of $\overline{D}_3$ such that $\mathcal{O}(W)$ is a dense subalgebra of $\mathcal{A}(D_3)$. It follows that we may assume that

(a) $\sigma \in \mathcal{F}(\mathrm{b}D_3 \cap \{u > 1-\varepsilon\})$ where u is pluriharmonic near $\overline{D}_3$;

(b) $\{u = 1-\varepsilon\}$ is smooth, intersects $\mathrm{b}D_3$ transversally, and

$$\Sigma = \Sigma_x = \{u = 1-\varepsilon\} \cap D_3$$

has a finite number of connected components $\Sigma^1, \dots, \Sigma^k$, each of them Levi flat.

Each Σ^l, $l = 1, \dots, k$, cuts D_3 in different connected components. Since $V \subset \{u < 1-\varepsilon\}$ and $u(x) = 1$, x and V lie in different connected components of $\overline{D} \setminus \Sigma$. Let us call $\Omega = \Omega_x$ a connected component containing x but not V.

Observe that Σ cuts a small neigbourhood of D_3 in a q-collar or a semi q-corona. In view of the extension theorem proved in [19], there exists a unique section $\sigma_x \in \mathcal{F}(\Omega)$ which extends σ. Hence we are able to extend σ to a section defined in x, for any $x \notin \hat{L}$.

In order to finish the proof we have to show that if σ_x, σ_y are two such extensions, defined on Ω_x and Ω_y respectively, then $\sigma_x = \sigma_y$ on $\Omega_x \cap \Omega_y$.

Notice that every connected component of

$$\Sigma_{xy} = \{u_x = 1 - \varepsilon_x\} \cap \{u_y = 1 - \varepsilon_y\}$$

meets the boundary. Indeed, suppose not, then there is a compact component of Σ_{xy}, which implies that the pluriharmonic function u_x has an inner maximum or minimum on the Levi flat hypersurface $\{u_y = 1 - \varepsilon_y\}$, which is a contradiction. Thus $\sigma_x = \sigma_y$ on $\Omega_x \cap \Omega_y$ trivially holds if $\mathcal{F}$ is locally isomorphic to a subsheaf of $\mathcal{O}^N$, in particular if $\mathcal{F}$ is locally free.

In our situation consider the difference $\tau = \sigma_x - \sigma_y$ on $\Omega_x \cap \Omega_y$. Since $\mathcal{F}$ is Hausdorff on $D \smallsetminus T$, $T = T(\mathcal{F})$, we have $\operatorname{supp} \tau \subset T$. Let $x \in \operatorname{supp} \tau$. If $B \subset (\Omega_x \cap \Omega_y)$ is a sufficiently small Stein neighbourhood of x, we have an exact sequence

$$0 \to \mathcal{O}^{p_d} \to \cdots \to \mathcal{O}^{p_1} \xrightarrow{\psi} \mathcal{O}^{p_0} \xrightarrow{\varphi} \mathcal{F} \to 0, \tag{3.1}$$

with $n - d \geq 3$ by hypothesis, and from this we derive the exact sequence

$$H^0(B, \mathcal{O}^{p_1}) \xrightarrow{\psi} H^0(B, \mathcal{O}^{p_0}) \xrightarrow{\varphi} H^0(B, \mathcal{F}) \longrightarrow 0$$

where ψ, φ are defined by matrices (ψ_{ij}), (φ_{rs}) of holomorphic functions on B. Then $\tau = \varphi(s)$, $s = (s_1, \ldots, s_q) \in H^0(B, \mathcal{O}^{p_0})$ and $\varphi(s_y) = 0$ for every $y \in B \smallsetminus T$; consequently

$$s_{|B \smallsetminus T} \in H^0\left(B \smallsetminus T, \operatorname{Ker} \varphi\right) = H^0\left(B \smallsetminus T, \operatorname{Im} \psi\right).$$

Because of (3.1) we have the exact sequence

$$0 \to \mathcal{O}^{p_d} \to \cdots \to \mathcal{O}^{p_2} \to \operatorname{Ker} \psi \to 0$$

proving that

$$\operatorname{depth} \operatorname{Ker} \psi_x \geq 5 \quad \text{if } n \geq 5$$

($\operatorname{Ker} \psi = 0$ if $n = 3$ and $\operatorname{Ker} \psi$ is locally free if $n = 4$). It follows that the first and second local cohomology groups of $\operatorname{Ker} \psi$ with support T vanish,

$$H^1_T(B, \operatorname{Ker} \psi) = H^2_T(B, \operatorname{Ker} \psi) = 0,$$

and consequently, from the local cohomology exact sequence, that

$$H^1(B \smallsetminus T, \operatorname{Ker} \psi) = 0.$$

The exact sequence

$$0 \longrightarrow \operatorname{Ker}\psi \longrightarrow \mathcal{O}^{p_0} \xrightarrow{\psi} \operatorname{Im}\psi \longrightarrow 0$$

now implies that the homomorphism

$$H^0(B \setminus T, \mathcal{O}^{p_1}) \xrightarrow{\operatorname{Ker}\psi} H^0(B \setminus T, \operatorname{Im}\psi)$$

is onto.

It follows that there exist holomorphic functions $g_1, \ldots, g_{p_1}$ on $B \setminus T$ such that

$$s_{1|B\setminus T} = \sum_{j=1}^{p_1} \psi_{1j} g_j, \quad \ldots, \quad s_{q|B\setminus T} = \sum_{j=1}^{p_1} \psi_{qj} g_j.$$

Since $\dim_{\mathbb{C}} T \leq n-2$, the functions $g_1, \ldots, g_{p_1}$ can be holomorphically extended through T by $\widetilde{g}_1, \ldots, \widetilde{g}_{p_1}$. This implies that $s \in H^0(B, \operatorname{Im}\psi)$, so $s = \psi(\widetilde{g})$, $g = (g_1, \ldots, g_{p_1})$, and consequently $\tau = (\varphi \circ \psi)(\widetilde{g}) = 0$.

The proof when U is a neighbourhood of $X \setminus (D \cup L)$ is similar starting by a pseudoconvex domain D_1 with a smooth boundary satisfying the following properties:

(i) $D_1 \subset D$, $D \setminus \overline{D}_1 \subset U$;

(ii) $\mathrm{b}D_1 \cap \mathrm{b}D \subset V \cap \mathrm{b}D$;

(iii) $\mathrm{b}D_1$ is strongly pseudoconvex at the points of $\mathrm{b}D_1 \setminus \mathrm{b}D_1 \cap \mathrm{b}D$. □

Remark 3.2. In view of a theorem by Alexander and Stout [1], the connectedness of $D \setminus \widehat{L}$ is certainly satisfied if $\widehat{L} \cap bD = L$. Indeed, the connected components A_i of $D \setminus \hat{L}$ and B_i of $bD \setminus (\hat{L} \cap bD)$ are in a 1-1 correspondence given by

$$A_i \leftrightarrow B_i \iff bA_i \cap bD = B_i.$$

Since $L = \hat{L} \cap bD$, and $bD \setminus L$ is connected, also $D \setminus \hat{L}$ is connected.

If L is a Stein compact we have the following

Theorem 3.3. *Let X be a locally irreducible Stein space, D be a bounded domain in X with a connected smooth boundary $\mathrm{b}D \subset X_{\mathrm{reg}}$ and $L \subset \mathrm{b}D$ be a Stein compact. Let $\mathcal{F}$ be a coherent sheaf on X satisfying*

(1) depth $\mathcal{F}_x \geq 3$ *for every $x \in X$;*

(2) $\dim_{\mathbb{C}} T(\mathcal{F}) \leq n-2$.

Then every section of $\mathcal{F}$ on $bD \setminus L$ uniquely extends to a section on $\overline{D} \setminus L$.

Proof. Let

$$p(\mathcal{F}) = \inf_{x\in X} \text{ depth } \mathcal{F}_x$$

and $\{U_\alpha\}$ be a fundamental system of Stein neighbourhoods of L. Then for the compact support cohomology groups we have

$$H_c^j(U_\alpha, \mathcal{F}) = 0$$

for $j \leq p(\mathcal{F}) - 1$ and every α. Moreover, if $H_L^j(X, \mathcal{F})$ denotes the j^{th} local cohomology group with support in L, we have the isomorphism

$$H_L^j(X, \mathcal{F}) = \varinjlim_{U_\alpha} H_c^j(U_\alpha, \mathcal{F})$$

(see [5, Corollaire 2.16]) hence

$$H_L^j(X, \mathcal{F}) = \{0\}$$

for $j \leq p(\mathcal{F}) - 1$.

From the local cohomology exact sequence

$$\cdots \to H^j(X, \mathcal{F}) \to H^j(X \smallsetminus L, \mathcal{F}) \to H_L^{j+1}(X, \mathcal{F}) \to \cdots ,$$

in view of the fact that X is a Stein space, we then obtain

$$H^j(X \smallsetminus L, \mathcal{F}) = \{0\}$$

for $1 \leq j \leq p(\mathcal{F}) - 2$. In particular, since $p(\mathcal{F}) \geq 3$ we have

$$H^1(X \smallsetminus L, \mathcal{F}) = \{0\}.$$

Let $s \in H^0(bD \smallsetminus L, \mathcal{F})$. Applying the Mayer–Vietoris sequence to the following closed partition of $X \smallsetminus L$,

$$X \smallsetminus L = (\overline{D} \smallsetminus L) \cup [X \smallsetminus (D \cup L)],$$

we get the exact sequence

$$H^0(\overline{D} \smallsetminus L, \mathcal{F}) \oplus H^0(X \smallsetminus (D \cup L), \mathcal{F}) \to H^0(bD \smallsetminus L, \mathcal{F}) \to H^1(X \smallsetminus L, \mathcal{F}).$$

Since $H^1(X \smallsetminus L, \mathcal{F}) = \{0\}$ the first homomorphism is onto, so the section s is a difference $s = s_1 - s_2$ of two sections

$$s_1 \in H^0(\overline{D} \smallsetminus L, \mathcal{F}), \quad s_2 \in H^0(X \smallsetminus (D \cup L), \mathcal{F}).$$

Hence, in order to end our proof, we have to extend the section s_2. Consider an open Stein neighbourhood U of L. Since, by hypothesis, $p(\mathcal{F}) \geq 3$ we have $H^1_c(U, \mathcal{F}) = \{0\}$ and consequently, again from the cohomology exact sequence

$$H^0(X, \mathcal{F}) \to H^0(X \smallsetminus U, \mathcal{F}) \to H^1_c(U, \mathcal{F}) \to \cdots ,$$

we deduce that the homomorphism

$$H^0(X, \mathcal{F}) \to H^0(X \smallsetminus \overline{U}, \mathcal{F})$$

is onto. In particular, there exists a global section $\tilde{s}_2$ which extends $s_{2|X \smallsetminus U}$.

If we choose a smaller Stein neighbourhood $V \supset L$, we get a second extension $\tilde{s}_{2,V}$ of $s_{2|X \smallsetminus U}$ which agrees with s_2 on the bigger set $X \smallsetminus V$. The difference $\tilde{s}_2 - \tilde{s}_{2,V}$ is a section on X with (compact) support in U (since out of U they both agree with s_2). Hence its support S is a discrete set of points. Since $p(\mathcal{F}) \geq 3$, $S = \varnothing$, which means that $\tilde{s}_2$ is actually an extension of s_2. Thus, $\tilde{s} = s_1 - \tilde{s}_2$ is a section of $\mathcal{F}$ on $\overline{D} \smallsetminus L$ which extends s. This concludes the proof. □

Corollary 3.4. *Let X be a locally irreducible Stein space, D be a bounded domain in X with a connected smooth boundary* $\mathrm{b}D \subset X_{\mathrm{reg}}$ *and* $L \subset \mathrm{b}D$ *be a Stein compact. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that*

(1) depth $\mathcal{F}_x \geq 3$ *for every* $x \in X$*;*

(2) $\dim_{\mathbb{C}} T(\mathcal{F}) \leq n-2$.

Let U be an open neighbourhood of $\overline{D} \smallsetminus L$. Then every section of $\mathcal{F}$ on $U \smallsetminus \overline{D}$ uniquely extends to a section on $U \smallsetminus L$.

Proof. Define a smooth domain D' such that $U \supset \overline{D}' \supset \overline{D}$ and $\mathrm{b}D' \cap \overline{D} = L$. Any section s of $\mathcal{F}$ on $U \smallsetminus \overline{D}$ gives a section of $\mathcal{F}$ on $\mathrm{b}D' \smallsetminus L$. Hence, applying Theorem 3.3 to D' and L we get the desired extension. □

Theorem 3.3 can be slightly improved if X is a Stein manifold. Indeed, in that case, under the same hypothesis for D, we are allowed to assume that $\mathcal{F}$ is defined only in a neighbourhood of $\overline{D}$. The proof uses the fact that every domain W of X has the envelope of holomorphy $\widehat{W}$ (see [13, 7]). We recall that $\widehat{W}$ is the set of all continuous characters $\chi : \mathcal{O}(W) \to \mathbb{C}$ (or, equivalently, the set of all closed maximal ideals of $\mathcal{O}(X)$) equipped with the weak topology. The complex structure on $\widehat{W}$ is such that

(i) the map $\mathrm{j} : W \to \widehat{W}$ associating to a point $x \in W$ the point evaluation $f \mapsto \delta_x(f) = f(x)$, $f \in \mathcal{O}(W)$, is a biholomorphism $W \simeq \mathrm{j}(W)$ such that $\mathrm{j}^* : \mathcal{O}(\widehat{W}) \to \mathcal{O}(W)$ is an isomorphism of Fréchet algebras;

(ii) if $f \in \mathcal{O}(W)$, the function $\widehat{f} : \mathcal{S}(X) \to \mathbb{C}$ defined by $\widehat{f}(\chi) = \chi(f)$ is a holomorphic extension of f;

(iii) the restriction map $\mathcal{O}(X) \to \mathcal{O}(W)$ gives a holomorphic map $\widehat{p} : \widehat{W} \to X$ making $\widehat{W}$ a domain over X.

Theorem 3.5. *Let D be a bounded domain of a Stein manifold X with a connected smooth boundary and $L \subset \mathrm{b}D$ be a Stein compact. Let $\widehat{p} : \widehat{D} \to X$ be the envelope of holomorphy of D and $\mathcal{F}$ be a coherent sheaf on a neighbourhood W of $\overline{\widehat{p}(\widehat{D})}$ satisfying*

(1) depth $\mathcal{F}_x \geq 3$ *for every $x \in W$;*

(2) $\dim_{\mathbb{C}} T(\mathcal{F}) \leq n - 2$.

Then every section of $\mathcal{F}$ on $bD \smallsetminus L$ uniquely extends to $\overline{D} \smallsetminus L$.

Proof. Let $\widehat{W}$ be the envelope of holomorphy of W, $\widehat{p} : \widehat{W} \to X$ be the canonical projection and $j : W \to \widehat{W}$ be the canonical open embedding of W into $\widehat{W}$. $j^* : \mathcal{O}(\widehat{W}) \to \mathcal{O}(W)$ is an isomorphism. In particular $\widehat{p}^*\mathcal{F}$ is a coherent sheaf on $\widehat{W}$ with the same depth as $\mathcal{F}$, which extends $j_*\mathcal{F}$. At this point we argue as in the proof of Theorem 3.3. □

3.2 Analytic sets

As for analytic sets, results of removability are obtained arguing as in the proof of Theorem 3.1 taking into account Theorem 2.7. Precisely

Theorem 3.6. *Let X be an n-dimensional manifold, D be a bounded pseudoconvex domain in X with a connected smooth boundary and L be a compact subset of bD. Assume that* $\mathrm{b}D \smallsetminus (\widehat{L} \cap bD)$ *is a non-empty strongly Levi convex hypersurface.*

Let U be an open neighborhood of $\overline{D} \smallsetminus (\widehat{L} \cap bD)$ and Y be a closed, analytic subset of $U \smallsetminus \overline{D}$ such that depth $\mathcal{O}_{Y,x} \geq 3$ *for every $x \in U \smallsetminus \overline{D}$. Then Y extends to an analytic subset $\widetilde{Y}$ of $(D \smallsetminus \widehat{L}) \cup U$.*

3.3 Obstructions to extension

The extension theorems proved in the above sections state that, under appropriate conditions, analytic objects like *CR*-functions, sections of coherent sheaves, analytic subsets defined on $bD \smallsetminus L$ ($bD \smallsetminus L$ being connected) extend – uniquely – to $D \smallsetminus \widehat{L}$ where $\widehat{L}$ is the envelope of L with respect to the algebra $\mathcal{A}(D)$ of holomorphic functions continuous up to the boundary.

Natural problems arise about minimality. In order to state the problem in all generality, given a compact subset L of bD we fix a class C of analytic objects and we consider the family L_{C} of all compact subsets $\widetilde{L}$ of $\overline{D}$, partially ordered by inclusion, satisfying the following properties:

(i) $\widetilde{L} \cap bD = L$;

(ii) every analytic object of C defined on $bD \smallsetminus L$ extends – uniquely – to $D \smallsetminus \widetilde{L}$.

Suppose that $\mathsf{L}_{\mathsf{C}} \neq \varnothing$; then there exists in L_{C} some minimal element L^0_{C}. One natural problem arises: is L^0_{C} unique? In general, due to polidromy phenomena, the answer could be negative. A second observation is that, at least in the cases already considered, if we have unicity then for the minimal compact L^0_{C}, we have the inclusions

$$L \subset L^0_{\mathsf{C}} \subset \widehat{L}.$$

Trivial examples show that the two extremal cases may actually occur. Moreover L^0_{C} heavily depends upon the class C. For instance, let $D = \mathbb{B}^n \subset \mathbb{C}^n$, $L = b\mathbb{B}^n \cap \{z_{n-2} = \cdots = z_n = 0\} = \mathbb{S}^1 \times \{0\}^{n-1}$, $n \geq 5$, and C_1 be the class of holomorphic functions, and C_2 be the class of analytic sets of codimension 3. Then the minimal compacts are

$$L^0_{\mathsf{C}_1} = L \subsetneq \widehat{L} = L^0_{\mathsf{C}_2},$$

as shown by the fact that the analytic set

$$\bigcup_{k \in \mathbb{Z}} \left\{ z_{n-2} = z_{n-1} = 0, z_n = \frac{1}{k} \right\}$$

does not extend through $\widehat{L}$.

4 The unbounded case

Some of the previous results extend to unbounded domains. The following is of particular interest.

Theorem 4.1. *Let X be a complex space and D be a strongly pseudoconvex unbounded domain with a connected boundary. Assume that there exists a sequence $\{p_k\}$ of pluriharmonic functions near $\overline{D}$ such that*

(1) $D_k = \{x \in D : p_k(x) > 0\} \subsetneq D_{k+1} = \{x \in D : p_{k+1}(x) > 0\}$;

(2) $D_k \Subset X$ *and* $D = \bigcup_{k \geq 1} D_k$.

Let $\mathcal{F}$ be a coherent sheaf on a neighbourhood U of $\overline{D}$ such that

(3) depth $\mathcal{F}_x \geq 3$ *for every* $x \in U$*;*

(4) $\dim_{\mathbb{C}} T(\mathcal{F}) \leq n-2$.

Then every section of $\mathcal{F}$ on $U \smallsetminus \overline{D}$ uniquely extends to a section on U.

Proof. Fix a section σ of $\mathcal{F}$ on $U \smallsetminus \overline{D}$. Consider the domain D_k. Since D is strongly pseudoconvex, using the bump lemma we find a Stein neighbourhood $V_k \subset U$ of $\overline{D}_k$. We may assume that the function p_k is defined on V_k, so $\mathrm{b}D_k \cap \mathrm{b}D$ is a Stein compact L_k, and we are in position to apply Theorem 3.3 and obtain a unique section $\hat{\sigma}_k$ of $\mathcal{F}$ on $V_k \smallsetminus L_k$ extending σ. Repeating this argument for every k, thanks to the uniqueness of extension we get the conclusion. □

Remark 4.2. If $X = \mathbb{C}^n$, conditions (1) and (2) are implied by the following one:

($\star$) if $\overline{D}^\infty$ denotes the closure of $D \subset \mathbb{C}^n \subset \mathbb{CP}^n$ in $\mathbb{CP}^n$, then there exists an algebraic hypersurface V such that $V \cap \overline{D}^\infty = \varnothing$.

Under this condition the extension of analytic sets (with discrete singularities) of dimension at least two holds, see [10].

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