# Discrete Mathematics in Computer Science

## D4. Inference

Malte Helmert, Gabriele Röger

University of Basel

December 11, 2023

# D4.1 Inference Rules and Calculi

# Inference: Motivation

- up to now: proof of logical consequence with semantic arguments
- no general algorithm
- solution: produce formulas that are logical consequences of given formulas with syntactic inference rules
- advantage: mechanical method that can easily be implemented as an algorithm

# Inference Rules

- Inference rules have the form

$$\frac{\varphi_1, \dots, \varphi_k}{\psi}.$$

- Meaning: "Every model of $\varphi_1, \dots, \varphi_k$ is a model of $\psi$."
- An axiom is an inference rule with $k = 0$.
- A set of inference rules is called a calculus or proof system.

German: Inferenzregel, Axiom, (der) Kalkül, Beweissystem

# Some Inference Rules for Propositional Logic

Modus ponens $$\frac{\varphi,\ (\varphi \rightarrow \psi)}{\psi}$$

Modus tollens $$\frac{\neg\psi,\ (\varphi \rightarrow \psi)}{\neg\varphi}$$

$\wedge$-elimination $$\frac{(\varphi \wedge \psi)}{\varphi} \qquad \frac{(\varphi \wedge \psi)}{\psi}$$

$\wedge$-introduction $$\frac{\varphi,\ \psi}{(\varphi \wedge \psi)}$$

$\vee$-introduction $$\frac{\varphi}{(\varphi \vee \psi)}$$

$\leftrightarrow$-elimination $$\frac{(\varphi \leftrightarrow \psi)}{(\varphi \rightarrow \psi)} \qquad \frac{(\varphi \leftrightarrow \psi)}{(\psi \rightarrow \varphi)}$$

# Derivation

## Definition (Derivation)

A derivation or proof of a formula $\varphi$ from a knowledge base KB is a sequence of formulas $\psi_1, \ldots, \psi_k$ with

- $\psi_k = \varphi$ and
- for all $i \in \{1, \ldots, k\}$:
  - $\psi_i \in$ KB, or
  - $\psi_i$ is the result of the application of an inference rule to elements from $\{\psi_1, \ldots, \psi_{i-1}\}$.

German: Ableitung, Beweis

# Derivation: Example

## Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \land R) \rightarrow S)\}$

Task: Find derivation of $(S \land R)$ from KB.

1. $P$ (KB)
2. $(P \rightarrow Q)$ (KB)
3. $Q$ (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
5. $R$ (1, 4, Modus ponens)
6. $(Q \land R)$ (3, 5, $\land$-introduction)
7. $((Q \land R) \rightarrow S)$ (KB)
8. $S$ (6, 7, Modus ponens)
9. $(S \land R)$ (8, 5, $\land$-introduction)

# Correctness and Completeness

**Definition (Correctness and Completeness of a Calculus)**

We write $\mathrm{KB} \vdash_C \varphi$ if there is a derivation of $\varphi$ from KB in calculus $C$.
(If calculus $C$ is clear from context, also only $\mathrm{KB} \vdash \varphi$.)

A calculus $C$ is correct if for all KB and $\varphi$
$\mathrm{KB} \vdash_C \varphi$ implies $\mathrm{KB} \models \varphi$.

A calculus $C$ is complete if for all KB and $\varphi$
$\mathrm{KB} \models \varphi$ implies $\mathrm{KB} \vdash_C \varphi$.

Consider calculus $C$, consisting of the derivation rules seen earlier.
Question: Is $C$ correct?
Question: Is $C$ complete?

German: korrekt, vollständig

# D4.2 Summary

# Summary (Consequence and Inference)

- knowledge base: set of formulas describing given information; satisfiable, valid etc. used like for individual formulas
- logical consequence $\mathrm{KB} \models \varphi$ means that $\varphi$ is true whenever (= in all models where) KB is true
- A logical consequence $\mathrm{KB} \models \varphi$ allows to conclude that KB implies $\varphi$ based on the semantics.
- A correct calculus supports such conclusions on the basis of purely syntactical derivations $\mathrm{KB} \vdash \varphi$.

# Further Topics

There are many aspects of propositional logic
that we do not cover in this course.

- resolution: a commonly used proof system for formulas in CNF
- other proof systems, for example tableaux proofs
- algorithms for model construction, such as the Davis-Putnam-Logemann-Loveland (DPLL) algorithm.

$\rightsquigarrow$ Foundations of AI course