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Hermann Weyl and his Phenomenological Research within Infinitesimal Geometry

Author:
Flavio Baracco

Supervisors:
Marcello D'Agostino

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Contents

Introduction

The present work focuses on the mathematical and philosophical works of Hermann Weyl (1885-1955). Weyl was a leading mathematician at the beginning of the twentieth century and his major contributions have concerned several fields of research, both within pure mathematics and theoretical physics. Many of them were pioneering works at that time and, most of all, they were carried out in the light of his peculiar philosophical view. As few mathematicians of his time, Weyl was able to manage both scientific and philosophical issues with an impressive competence. For this reason he represented a very peculiar figure among scientists and mathematicians of his time. This dissertation aims to clarify these works both from a philosophical and a mathematical perspective. Specifically, I will focus on those works developed through the years 1917-1927.

The first chapter aims to shed some light on the philosophical reasons that underlie Weyl's foundational studies during this period. I will explore these works especially with respect his attempt to establish a connection between a descriptive analysis of phenomena and their exact determination. I will focus both on his mathematical formulation of Euclidean space and on his analysis of phenomenal continuum pointing out the main features of these studies. Weyl's investigations on the relations between what is intuitively given and the mathematical concepts through which we seek to construct the given in geometry and physics do not seem to be carried out by chance. These investigations indeed could be better understood within the phenomenological framework of Husserl's philosophy. Husserl's distinction between descriptive and exact concepts delineates the difference between a descriptive analysis of a field of inquiry and its exact determination. Clarifying how they are related is not an easy task. Nevertheless, Husserl points out that a connection might be possible if we were able to establish a connection by means of some idealizing procedure intuitively ascertained. Within this phenomenological framework we should interpret Weyl's investigations on the relation between phenomenal knowledge and theoretical construction.

In the second chapter I will focus on Weyl's mathematical account of the continuum within the framework of his pure infinitesimal geometry developed mainly in *Raum-Zeit-Materie*. It deserves a special attention. Weyl indeed seems to make use of infinitesimal quantities and this fact appears to be rather odd at that time. The literature on this issue is rather poor. For this reason I've tried to clarify Weyl's use of infinitesimal quantities considering also Weyl's historical context. I will show that Weyl's approach has not to be understood in the light of modern differential geometry. It has instead to be understood as a sort of algebraic reasoning with infinitesimal quantities. This approach was not so unusual at that time. Many mathematicians, well-known to Weyl, were dealing with kind of mathematics although many of these studies were works in progress. In agreement with that, Weyl's analysis of the continuum has to be understood as a work in progress as well. In the following Weyl's studies in combinatorial topology are proposed. I will then suggest that both these approaches should be understood within the phenomenological framework outlined in the first chapter. The latter, however, attempts to establish a more faithful connection between a descriptive analysis of the continuum and its exact determination and for this reason it can be regarded as an improvement with respect to the former from a phenomenological point of view.

Finally, in the third chapter I will attempt a phenomenological clarification of Weyl's view. In the first and second chapter Weyl's studies are clarified showing how they are related with the phenomenological framework of Husserl's philosophy. Despite this, the theoretical proposal revealed by them is not so easy to understand. That issue seems to be shared by many other contemporary studies. The relevant literature on this author dealing with a phenomenological interpretation seems often to be hardly understandable. I'm going to outline the main problems involved in this field of research and how they are related with the peculiarity of Husserl's framework. I will then suggest a way to improve these studies. Specifically, I will attempt a phenomenological clarification of Weyl's writings. To this aim, I will argue for an approach that makes use of Husserl's writings as a sort of "analytic tools" so that a sort of phenomenologically-informed reconstruction of Weyl's thought can be achieved. I will finally consider Weyl's notion of surface as a case study to show a concrete example of this kind of reconstruction.

1 Weyl's Phenomenological Framework

Our examination of the continuum problem contributes to critical epistemology's investigation into the relation between what is immediately (intuitively) given and the formal (mathematical) concepts through which we seek to construct the given in geometry and physics.

H. Weyl[1]

Several scholars support the view that Weyl's researches have undergone many changes along his life. Among them there is no common agreement, but many authors set an early phase connected with Weyl's adherence to intuitionism and a later phase usually referred as Weyl's symbolic constructivism. This chapter aims to show that previous interpretations are reasonable but they miss the phenomenological framework according to which Weyl's studies should be understood. I will consider a selection of Weyl's body of works in the period 1917-1927 focusing on Weyl's attempt to establish a connection between a descriptive analysis of phenomena and their exact determination. I will show that some underlying Husserlian issues seem to have been overlooked in literature. Husserl's distinction between descriptive and exact concepts delineates the difference between a descriptive analysis of a field of inquiry and its exact determination. Clarifying how they are related is not an easy task. Nevertheless, Husserl points out that a connection might be possible if we were able to establish a connection by means of some idealizing procedure intuitively ascertained. Within this phenomenological framework a more uniform interpretation of Weyl's researches can be put forward, at least in period 1917-1927.

1.1 Weyl's Foundational Studies as Changeable View

Hermann Weyl (1881-1955) was a leading mathematician at the beginning of the twentieth century. His major contributions have concerned several fields of research, both in pure mathematics and theoretical physics, and, most importantly, his pioneering work was carried out in the light of his unique philosophical view. As only mathematicians of his time, Weyl dealt with both scientific and philosophical issues with great skill,

[1]Weyl (1994, p. 2).

becoming a very unique figure among scientists and mathematicians of his time.[2] Although Weyl was a well-known mathematician, philosophers of mathematics have started getting interested in his work only recently, and, even though several authors have tried to uncover the philosophical framework that underlies Weyl's studies. Many of them did not identify a coherent perspective in his philosophical view, arguing that his foundational research changed over the years.

Both Sieroka (2009) and Mancosu (2010a) recognize at least two main tendencies in Weyl's work between 1917 and 1927. A first phase mainly characterized by his criticisms against set theory and classical analysis, and by his rejection of Hilbert's formalism and adherence to the intuitionistic-oriented account of Husserl and then Brouwer.[3] A second phase instead is characterized by his tendency toward a sort of constructivism and his reconciliation with Hilbert's formalism.[4] A similar interpretation is also supported by Da Silva (1997), Bell (2004) and Folina (2008). They all identify a changeable perspective along the years 1917-1927, from an intuitionistic-oriented approach towards a constructivist account. All these studies, of course, shed light on many further details.[5] Anyway our concern here is just to show that many scholars set an early phase connected with Weyl's adherence to intuitionism and a later phase usually regarded as Weyl's symbolic constructivism.[6]

Not everyone, however, follows this interpretation. Scholz, for instance, gives a more uniform interpretation of Weyl's researches. In Scholz (2000) he suggests a constructive reading of Weyl's works since the publication of *Das Kontinuum*. He argues that Weyl was strongly influenced by the constructive philosophy of Fichte.[7]

A similar constructive interpretation is also defended by Tieszen (2000) although

[2]For a general introduction to Weyl's scientific and philosophical works see, for instance, Scholz (2001). See also Bell and Korté (2016).

[3]Actually, they further make a distinction between the intuitionistic approach related to Husserl and the one related to Brouwer. Mancosu, for instance, claims: "In the years following the publication of *The Continuum*, Weyl familiarized himself with the works of Brouwer. [...] This lead him to abandon his previous approach to foundational matters and join the intuitionistic camp" (Mancosu 2010a, p. 271).

[4]Their reconstructions anyway do not coincide. Sieroka, for instance, stress on the influence of Fichte in Weyl's elaboration of a sort of "formalistic-constructivist approach". He says: "[...] after his brief alliance with Brouwer in 1921 [...] Weyl came to believe in the superiority of Fichtean constructivism over the passive Husserlian viewing of essences, which seems not to do justice to the creativity of work done in mathematics and theoretical physics" (Sieroka 2009, p. 93).

[5]In Da Silva (1997), for instance, Weyl's predictivism is clarified by reference to Husserl's theory of meaning proposed in *Logische Untersuchungen*. In the abstract Da Silva states: "In this paper I discuss the version of predicative analysis put forward by Hermann Weyl in *Das Kontinuum*. [...] More specifically, I analyze Weyl's philosophical ideas in connexion with the work of Husserl, in particular *Logische Untersuchungen* and *Ideen I*. I believe that this interpretation of Weyl can clarify the views on mathematical existence and mathematical intuition which are implicit in *Das Kontinuum*" (Da Silva 1997, p. 277).

[6]Specific aspects concerning Weyl's intuitionistic phase are further analysed in Bell (2000), van Atten et al. (2002) and Adams and Luo (2010). For further details about Weyl's constructivism see, for instance, Bell (2004).

[7]In Scholz (2000) he says: "[...] we have to look into the philosophical background of the concept of continuum and space, he dealt with in that time. This background was, at least as far as the concept of continuum, space and construction of concepts is concerned, stronger rooted in Fichte than in Husserl who usually is claimed as the main reference philosopher for Weyl" (Scholz 2000, p. 200). See also Scholz (1994, 2005).

he does not underestimate the role of Husserl's philosophy. He proposes a constructive reading of Husserl and he suggests that the philosophical framework of Weyl's mathematical constructivism should be understood in the light of the transcendental idealism which find its roots in Kant, Fichte and Husserl.[8]

The following sections aim to support a more uniform interpretation of Weyl's researches in the period 1917-1927.[9] Specifically, I will focus on three main works regarding this period, *Das Kontinuum* (1918), *Raum-Zeit-Materie* (1921) and *Philosophy of Mathematics and Natural Science* (1949).[10] In some aspects my interpretation will be close to Tieszen's reading although it will focus on some important Husserlian issues that I think were overlooked.[11]

1.2 The Mathematical Form of the Euclidean Space

Weyl's researches on the nature of intuitive space represent an important body of works. The space of intuition pertains to our experience of spatiality and it has not be confused with any conceptualization of it. We "have to differentiate carefully between phenomenal knowledge or insight" and "theoretical construction" (Weyl 1949, p. 61). The first is expressed by statement such as "this leaf (given to me in a present act of perception) has this green color (given to me in a present act of perception)" (Weyl 1949, p. 61). The second instead is characterized by rational principles and it allows us to "'jump over its own shadow', to leave behind the stuff of the given, to represent the *transcendent*" (Weyl 1949, p. 66). Mathematics and physics allows us to gain this sort of theoretical construction. Weyl's mathematical formulation of *affine geometry* is an attempt in that direction. He aims to develop a mathematical account of our intuitive space that is not "demanding the reduction of all truth to the intuitively given" (Weyl 1949, p. 65).

As for any intuitively given field of inquiry we should be able first to identify which *basic categories of objects* (*Grundkategorien*) and *primitive relations* among those objects

[8]Tieszen further clarifies that transcendental idealism gives us the general framework for a proper understanding of Weyl's view, but it does not mean that Weyl follows this philosophical framework in all details. He observes: "Weyl's constructivism is motivated by transcendental idealism and the view that intuition is the central source of knowledge. Of course this does not mean that his views on foundations, especially in their technical details, are precisely those of Fichte or Husserl" (Tieszen 2000, p. 278).

[9]In Weyl's obituary appeared in the *Biographical Memoirs of Fellows of the Royal Society* in 1957, the period 1917-1927 is described as the period when Weyl "was at the height of his powers" (Newman 1957, p. 306). It was a rich and stimulating period for Weyl's mathematical and philosophical production and several works were published at that time. For this reason the decade 1917-1927 represents an important period to focus on.

[10]*Philosophy of Mathematics and Natural Science* (1949) represents the revised English version of the first German edition published in 1927 (Weyl 1927). The text was translated by O. Helmer with the help of J. Weyl, but it was reviewed by Hermann Weyl himself. It is basically unchanged with respect the first edition, except for six essays that Weyl added at the end. For this reason I will refer to this English edition, unless it would be appropriate to quote directly from the German edition.

[11]A constructive interpretation is also defended by Gauthier (2006) although he does not stress on any sort of idealistic framework. He just observes: "[...] beyond the general philosophical attitude which one could readily characterize as Kantian (or neo-Kantian or Husserlian), there remains the constant concern for the constructivist foundations of science" (Gauthier 2006, p. 268).

(*ursprünglichen Relationen*) pertain to it.[12] To each primitive relation is associated a *primitive judgment scheme* (*ursprünglichen Urteilsschema*) which "yields a meaningful proposition" only when each blank of the relation is filled by an object of its own category (Weyl 1994, p. 41). In the first part of *Das Kontinuum* Weyl deals with this subject matter and he gives some examples. The proposition "this leaf is green", whose judgment scheme is "G(x): x is green" is meaningful (*sinnvoll*) since the blank x is affiliated with the category "visible thing" and it is filled by the object "leaf", that is a visible thing (Weyl 1994, p. 5).[13] Weyl then aims to avoid any mathematical account that makes use of judgment schemes that yield meaningless propositions. He remarks that "anyone who forgets that a proposition with such a structure can be meaningless is in danger of becoming trapped in absurdity" (Weyl 1994, p. 6).[14] For this reason Weyl considers only "well-structured" primitive judgment schemes. From them we can derive further judgment schemes by applying some principles of logical construction, making any further appeal to intuition. Weyl refers to them as *complex judgment schemes* and to the associated relations as *derived relations*. What sort of new judgment schemes "will unfold before our intuition in the development of the life of the mind can certainly not be anticipated *a priori*" (Weyl 1994, p. 113).[15] Despite that, these principles "can be set down once and for all (just like the elementary forms of logical inference)" (Weyl 1994, p. 113).[16] Among them Weyl identifies those judgments that express a state of affairs concerning the given field of inquiry. They are called *pertinent judgements* and they allow us to acquire a "complete knowledge of the objects of the basic categories as far as they are connected by the basic relations" (Weyl 1949, p. 7). Therefore a meaningful mathematical analysis of a intuitively given field of inquiry starts with the recognition of its basic categories and primitive relations. Upon them a mathematical theory can be then logically developed, making any further appeal to intuition.

Following this path Weyl develops affine geometry recognizing two "fundamental categories of objects", i.e. *spatial-point category* and *translation category* (Weyl 1952, p. 18). Weyl simply refers to them also as the category of *points* and of *vectors*. Few primitive relations are found among these objects, i.e. the *axioms*:

[12]We are considering properties among relations as a special case.

[13]A proposition instead turns out to be meaningless (*sinnlos*) when this condition is not satisfied. For instance, the judgment scheme "H(x): x is honest" does not yield a meaningful proposition if x is filled by the object "leaf". Weyl's theory of meaning is well described in Tieszen (2000).

[14]Actually Weyl seems to believe that a proper intuitive analysis of the given field of inquiry will prevent us to be trapped in such absurdities. He says: "Perhaps meaningless propositions can appear only in thought about language, never in thought about things" (Weyl 1994, p. 5).

[15]This quotation is taken from an article published in 1919 (Weyl 1919). It has been translated in English and added as an appendix in Weyl (1994).

[16]Weyl's principles of logical construction belong to a more comprehensive "logical critique of language" (Weyl 1949, p. 7). Specifically, he speaks in terms of *pure grammar* by referring to Husserl's *Logische Untersuchungen*. See Weyl (1994, 113, note 2). He makes reference to Husserl's philosophy of logic also in the preface of *Das Kontinuum* claiming: "Concerning the epistemological side of logic, I agree with the conceptions which underlie Husserl's *Logische Untersuchungen*. The reader should also consult the deepened presentation in Husserl's *Ideen* which places the logical within the framework of a comprehensive philosophy" (Weyl 1994, p. 2).

- two vectors $\vec{a}$ and $\vec{b}$ uniquely determine the vector $\vec{a} + \vec{b}$;
- a real number λ and a vector $\vec{a}$ uniquely determine the vector $\lambda\vec{a}$;
- a point A and a point B determine a vector $\vec{a} = \vec{AB}$;
- the first two operations satisfy the common laws concerning addition and multiplication (commutative law, associative law, ...);
- if A is any point and $\vec{a}$ any vector, there is one and only one point B such that $\vec{a} = \vec{AB}$;
- if $\vec{a} = \vec{AB}$ and $\vec{b} = \vec{BC}$, then $\vec{a} + \vec{b} = \vec{AC}$.

Weyl claims that the notion of *multiplication* together with its laws can be derived from the ones of *addition* in the case of rational numbers. However, he extends them for any real number "in accordance with the principle of continuity". He further observes that in this case they have to be formulated as separate axioms "because they cannot be derived in the general form from the axioms of addition by logical reasoning alone" (Weyl 1952, p. 17). As we will see later this remark is related with Weyl's efforts to properly understand the nature of continuum. Weyl's studies on the continuum indeed are often characterized by a continuous tension between a temporary solution and a call for a better solution.

Weyl then observes that all concepts that may be defined, by logical reasoning alone, from the basic notions of vector and point and their primitive relations "belong to affine geometry" (Weyl 1952, p. 18). It is possible, for instance, to define the concept of *straight line* and of *plane*:

- given a point O and a vector $\vec{e}$, the end-points of all vectors $\vec{OP}$ which have the form $\lambda\vec{e}$ constitute a straight line;
- given a point O, a vector $\vec{e}_1$, and a vector $\vec{e}_2$ which is not of the form $\lambda\vec{e}_1$, then the end-points of all vectors $\vec{OP}$ that have the form $\lambda_1\vec{e}_1 + \lambda_2\vec{e}_2$ constitute a plane.

It is then possible to derive the totality of all possible formations concerning that field of inquiry out of a few basic notions and relations. Moreover, all theorems that can be logically deduced within this framework constitute "the doctrine of affine geometry" (*Lehrgebäude der affinen Geometrie*) (Weyl 1952, p. 18). In this sense geometry turns out to be a "*theory of space*" (Weyl 1949, p. 18).[17]

Weyl further introduces the notion of *n-dimensional linear vector-manifold* (*n-dimensionale lineare Vektor-Mannigfaltigkeit*), that is constituted by all vectors of the form $\lambda_1\vec{e}_1 + \cdots + \lambda_n\vec{e}_n$ (where $\vec{e}_1, \ldots, \vec{e}_n$ are n linearly independent vectors, i.e. vectors whose linear

[17] A similar remark can be found with respect to Weyl's *pure infinitesimal geometry*. The more general framework of pure infinitesimal geometry improves our analysis of space to such an extent that Weyl refers to it as "the climax of a wonderful sequence of logically-connected ideas, and in which the result of these ideas has found its ultimate shape, is a true *geometry*, a doctrine of *space itself*" (Weyl 1952, p. 102).

combination vanishes only when all the coefficients vanish).[18] We have affine geometry in the case $n = 3$. He then formulates the last axiom pertaining to affine geometry, i.e. the *dimensional axiom*. This axiom states that in affine geometry (3-dimensional linear manifold) there are 3 linearly independent vectors, but every 4 are linearly dependent on one another.[19]

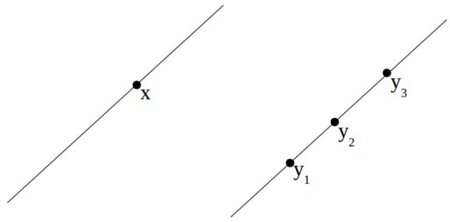
This mathematical conceptualization however is not unique. Any field of inquiry allows us to identify certain categories of objects or primitive relations and not others. Nevertheless, their choice can be "arbitrary to a considerable extent" (Weyl 1949, p. 20). They are not uniquely determined by the field of inquiry. The difference between "essentially originary and essentially derived notions lies beyond the competence of the mathematician" (Weyl 1949, p. 20). The classical concept of space that pertains to *Euclidean geometry*, for instance, represents another possible conceptualization of the space of intuition. Specifically, Euclidean geometry is able to account for his homogeneity rather well. In this case we are concerned with three categories of objects: *spatial-point*, *line*, and *plane*. They are not defined "but assumed to be intuitively given" (Weyl 1949, p. 3). Moreover, few primitive relations are associated with these categories:

- the relation of *incidence* (a point lies on a line, a line lies in a plane, a point lies in a plane);
- the relation of *betweenness* (a point z lies between the point x and y);
- and the relation of *congruence* (congruence of line segments and of angles).

Weyl observes then that the category of points "reflects the intuitive homogeneity of space" (Weyl 1949, p. 8). Any judgment scheme "P(x)" whose blank x is affiliated with this category and which is derived from the primitive judgement schemes without any reference to individual spatial-points, lines, or planes, "is always true either of *each* or of *no*" point (Weyl 1994, p. 16). For instance, the property "P(x): there exists a line such that the point x lies on it" is always true for any point. Differently, the property "P(x): there exist three points y_1, y_2, y_3 lying on a line (y_2 being between y_1 and y_3) such that x is between y_1 and y_2 and it is also between y_2 and y_3" is always false.

[18]For a detailed account of the notion of manifold (*Mannigfaltigkeit*) from the middle of the 19th century to the middle of the 20th century, see Scholz (1999). The historical development of this concept is complex and it is not always easy to recognize which meaning each author ascribes to it. For a better understanding of Weyl's notion of manifold, however, we can observe what he says with respect to the notion of *surface* in *Die Idee der Riemannschen Fläche*: "[...] the concept 'two-dimensional manifold' or 'surface' will not be associated with points in three-dimensional space; rather it will be a much more general abstract idea. If any set of objects (which will play the role of points) is given and a continuous coherence between them, similar to that in the plane, is defined, then we shall speak of a two-dimensional manifold" (Weyl 2009, p. 16). Therefore, Weyl's notion of two-dimensional manifold can be interpreted as the "abstract form" of a surface, and more generally the notion of manifold can be broadly understood as the "abstract form" of a given field of inquiry.

[19]He further adds that a point O and 3 linearly independent vectors constitute a *coordinate system*. This system allows us to refer to a point by its *coordinates* $\lambda_1, \lambda_2, \lambda_3$ by means of the relation $\vec{OP} = \lambda_1\vec{e}_1 + \lambda_2\vec{e}_2 + \lambda_3\vec{e}_3$.



For this reason Weyl refers to this category as a *homogeneous category*. In this sense, therefore, this mathematical conceptualization enables us to account for the intuitive homogeneity of space.

Although Weyl recognizes the possibility of different conceptualizations of the space of intuition, it does not mean that is completely a matter of choice. Weyl seems to believe that in some cases we should prefer a conceptualization to another. The axiomatic construction of affine geometry, for instance, seems to be a better conceptualization of the space of intuition. It forms "a system that, also in logical respect, is of a much more transparent and homogeneous structure than the purely geometrical axioms of Euclid or Hilbert" (Weyl 1949, p. 69).[20] This theoretical construction reveals "a wonderful harmony between the given on one hand and reason on the other" (Weyl 1949, p. 69). The derived concepts of straight line and plane, moreover, "correspond to those which suggest themselves most naturally from the logical standpoint" (Weyl 1949, p. 69). For these reasons Weyl claims for a better mathematical conceptualization of what is intuitively given in the case of affine geometry.

To conclude, let's point out the main features that seem to characterize these researches. Firstly, they imply a distinction between two kinds of knowledge. The first deals with our sense perception and Weyl refers to it as a phenomenal knowledge. The second instead seems to pertain to a domain of mathematical concepts. Weyl refers to it as a sort of theoretical construction. They are two different kinds of knowledge but Weyl seems to believe in the possibility of establishing a connection between them. Upon few basic notions and relations that are intuitively grasped in some way he attempts to formulate a mathematical conceptualization of the space of intuition. Our mathematical knowledge of real world should rely on these intuitive insights although it is not limited to this source of knowledge. The mathematical knowledge aims to represent the "transcendent" and a mathematical theory logically developed upon these intuitive insights can achieve this sort of transcendence. This, however, is not an easy task. Weyl seems to suggest that different mathematical conceptualizations are possible. There is not a unique approach although it is not completely a matter of choice. In some cases we should prefer a conceptualization to another. This poses the problem of finding which mathematical conceptualization best suits what is intuitively given.

[20]Weyl ascribes a primary role to an algebraic treatment of geometry since "the numbers are to a far greater measure than the objects and relations of space a free product of the mind and therefore transparent to the mind" (Weyl 1949, p. 22). Since the logical structure of the axiomatic construction of affine geometry reflects the operational field of linear algebra, this approach is preferred over Euclid's or Hilbert's one. As we will see, anyway, there are other philosophical reasons underlying this preference.

1.3 The Continuum

In *Raum-Zeit-Materie* Weyl remarks that his axiomatic formulation of affine geometry is still far from being satisfactory. It lacks a proper understanding of the continuity. As we have observed above Weyl does not deduce the notion of multiplication and its laws from the ones of addition in the case of real numbers because the continuum "is so difficult to fix precisely, from the logical structure of geometry" (Weyl 1952, p. 17).[21] For this reason Weyl deals with the nature of continuum in several works aiming at a better understanding of this issue.

In *Philosophy of Mathematics and Natural Science*, a long historically-informed philosophical treatise, he explores the main attempts that have been suggested along the history of mathematics. Although the problem of continuum has a long-standing tradition, it still remains one of the most troublesome issue in mathematics. The most "essential character of the continuum" regards its being capable of infinite division (Weyl 1949, p. 41). This feature, like any other feature intuitively given in a continuum, has been the source of many mathematical paradoxes. Understanding the nature of continuum is of crucial importance to such an extent that Weyl regards mathematics itself as "the science of infinite" (Weyl 1949, p. 66). His efforts to understand the nature of continuum, however, do not represent uniquely an attempt to solve a mathematical issue. These studies indeed are strongly motivated by his desire of a mathematical formulation of physical notions of space and time.[22] In the treatise *Das Kontinuum* Weyl deals with this issue. Specifically, he aims to understand to what extent our mathematical theories of space and time reflect the experience we have of them. Since we experience them as two continuous entities, our mathematical theories should reflect their continuous nature. Understanding the nature of continuum then turns out to be especially important for a proper understanding of the real world. It contributes "to critical epistemology's investigation into the relations between what is immediately (intuitively) given and the formal (mathematical) concepts through which we seek to construct the given in geometry and physics" (Weyl 1994, p. 2).

We shall now focus on the mathematical formulation of these continua developed in *Das Kontinuum*. Weyl makes clear that the object of his investigation concerns

[21]In *Das Kontinuuum* Weyl remarks: "[. . .] the *continuity* given to us immediately by intuition (in the flow of time and in motion) has yet to be grasped mathematically as a totality of discrete "stages" in accordance with that part of its content which can be conceptualized in an "exact" way. More or less arbitrarily axiomatized systems (be they ever so "elegant" and "fruitful") cannot further help us here" (Weyl 1994, p. 24).

[22]The beginning of the twentieth-century is marked by deep changes in several fields of scientific knowledge and Weyl plays a key role within this context. In particular the arising theory of general relativity together with the development of non-euclidean geometries transform the way mathematicians and physicists were used to look at the external world. The usual notions of space and time are called into question. With this respect Weyl pays close attention on their philosophical and foundational aspects. Our understanding of the external world is indeed essentially based on these two notions. For instance, the fundamental physical notion of motion results from them. Motion plays a fundamental role in every branch of physics, particularly in the context of general relativity. Many changes in the way physicists were used to look at real world revolves around this notion. Thus it is not surprising that Weyl begins his philosophical introduction of *Raum-Zeit-Materie* making reference to this notion, where the three fundamental physical notions of space, time and matter "enter into intimate relationship" (Weyl 1952, p. 1).

the *phenomenal continuum*. By temporal continuum he means the constant form of our experiences of consciousness by virtue of which they appear to us to flow by successively. He further explains that by experience he does not mean "real psychical or even physical processes" which occur in an individual, "belong to a real world and, perhaps, correspond to the direct experiences" (Weyl 1994, p. 88). He mean what we experience, exactly as we experience it. The phenomenal time has then to be understood as a pure experience. It refers to the direct perception we have of it and it has not be confused with the time of physics or any other notion of time derived from a certain view of the world.[23] Weyl aims to develop a mathematical theory of this phenomenal continuum. Then, like before we should identify which kind of basic categories and primitive relations pertain to this field of inquiry. That doesn't seem to be an easy task anyway. Weyl need to postulate the possibility that a "now" is intuitively given in order to have "some hope of connecting phenomenal time with the world of mathematical concepts" (Weyl 1994, p. 88). Under this assumption we are able to dissolve the phenomenal time into isolated time-points, rigidly punctual "now". By the identification of such a sequence of time-points we can then grasp this species of time in an exact way. These time-points belong to a basic category, i.e. *time-point category*. Moreover, the following primitive relations can be associated with these points:

- the binary relation $E_{arlier}(A, B)$: A is earlier than B;
- the quaternary relation $E_{qual}(A, B, A', B')$: A is earlier than B, A' is earlier than B', and AB is equal to $A'B'$.[24]

We might then develop a mathematical theory of time upon them, but Weyl observes that we should be able first to deal with some issues. We would like to differentiate conceptually all time-points in the given continuum, but we are not able to do it by means of these relations. Phenomenal time is homogeneous. As in the previous case of the homogeneity of the space of intuition, we can show that any judgment scheme (whose blank x is affiliated with time-point category and it is derived, without any reference to individual time-points, from those primitive judgement schemes associated to the above primitive relations) is always true either of each or of no point. Therefore, a single time-point "can only be given by being specified individually", i.e. by a direct intuition (Weyl 1952, p. 8). There is no inherent property that we can ascribe to a specific time-point in order to differentiate it from all the others.

[23]Weyl follows implicitly Husserl's approach. Husserl supports the need for a preliminary act in the analysis of experience. He refers to it as *epoché* and it is conceived as the suspension of any judgment about the natural world, setting aside all objective thesis and focusing on the phenomenon as it presents itself. See Husserl (1913a).

[24]This equality refers to the equality of experiential content of the two *time spans* AB and $A'B'$, into which falls every time-points that is later than $A(A')$, but earlier than $B(B')$. Weyl actually remarks that such an equality might be very controversial. He prefers to dwell on it anyway. As we will see the previous postulate is an even bigger issue.

The situation would change if we were able to establish an *isomorphism* between the domain of time-points and the domain of real numbers (as they are constructed in *Das Kontinuum*).[25] Each time-point will then be associated with a definite real number and viceversa. Specifically, we first need to fix two time-points O and E by a direct intuition such that $E_{arlier}(O, E)$ holds. Then we can "fix conceptually further time-points P by referring them to the unit-distance OE" (i.e. the time span OE taken as unit) (Weyl 1922, 8). This is done by establish a connection between a time-point P and the relation $R_t(P, O, E)$ that can be expressed in the form $OP = t * OE$. Our mathematical theory of time will have the same structure of real numbers if this relation, logically derived from the above primitive relations, reflects Weyl's construction of a real number. If it is the case then an isomorphism between the two domains could be established and we could associate a real number t to each time-point P. Moreover, we could speak in terms of *co-ordinate system* centred at O (with OE as unit length) and t would represent the *abscissa* with respect to this co-ordinate system. Weyl speaks also in terms of *transfer principle* (*Übertragungsprinzips*). By means of a isomorphic mapping between two domains "is possible to transfer any insights gained in one field to the isomorphic field" (Weyl 1949, p. 26).[26] For this reason, the idea of isomorphism turns out to be "of fundamental importance for epistemology" (Weyl 1949, p. 25). Weyl further remarks that the notion of isomorphism "induce us to conceive of an axiom system as a *logical mold* (*Leerform*) of possible sciences" (Weyl 1949, p. 25). A concrete interpretation is given "when designata have been exhibited for the names of the basic concepts, on the basis of which the axioms become true propositions" (Weyl 1949, p. 25). "Pure mathematics, in the modern view, amounts to a general hypothetico-deductive theory of relations; it develops the theory of logical 'mold' without binding itself to one or the other among the possible concrete interpretations" (Weyl 1949, p. 27). In agreement with Husserl, Weyl observes that the notion of *formalization* reflects a point of view "without which an understanding of mathematical methods is out of the question" and suggests the reader to "compare Husserl, *Logische Untersuchungen*, I, Section 67-72" (Weyl 1949, p. 27).

Finally observe that such a conceptualization of phenomenal time relies on the individual exhibition of the time-point O. Only by such an intuitive act we are able to differentiate time-points within temporal continuum. Weyl ascribes this fact to "the unavoidable residue of the eradication of the ego" in that theoretical construction of the world whose existence can be given only "as the intentional content of the

[25] Weyl's construction of real numbers in *Das Kontinuum* is logically developed upon the basic category of natural numbers and the primitive relation "S(n',n): n' is the successor of n". He develops this construction in details and many other notions are introduced, such as the notions of *set* and *function*. For further details, see Mancosu (2010a). For an axiomatic interpretation, see Feferman (1988).

[26] Similar considerations are also supported in the case of a mathematical theory of space. With respect to space Weyl states: "[. . .] for example, Descartes' construction of coordinates maps the space isomorphically into the operational domain of linear algebra" (Weyl 1949, p. 25). Weyl further claims that a mathematical theory of time or space cannot be pursued as an independent axiomatic science but it should rely on this transfer principle. We should transfer any result pertaining to analysis into the domain of time-points by means of "a transfer principle based on the introduction of a coordinate system" (Weyl 1994, p. 96).

processes of consciousness of a pure, sense-giving ego" (Weyl 1994, p. 94).[27]

Therefore, if that is the case then we should be able to confirm it by a direct inspection of phenomenal time. That is, our intuition should confirm us "whether this correspondence between time-points and real numbers holds or not" (Weyl 1994, p. 90). However, that is not the case. Our "intuition of time provides no answer" (Weyl 1994, p. 90). The main reason relies of the fact that such an interrogation is meaningless. Our mathematical theory of time indeed fails to satisfy a fundamental criterium for any theoretical construction, i.e. the time-point category "lacks the required support in intuition" (Weyl 1994, p. 90). Any judgment scheme involving this category cannot be filled by time-points given in an individual intuition. What is given in consciousness presents itself "not simply as a being" but "as an enduring and changing being-now" (Weyl 1994, p. 91). This being-now is "in its essence, something which, with its temporal position, slips away" (Weyl 1994, p. 92).[28] For this reason, a mathematical theory of time that dissolves the phenomenal time into time-points turns out to be inadequate. It is due to the continuous nature of phenomenal time. A time-point "exists only as a "point of transition" [...] always only an *approximate*, never an *exact* determination is possible" (Weyl 1994, p. 92).[29]

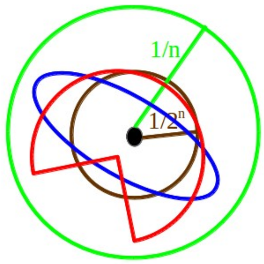
Similar considerations are also put forward in the case of spatial continuum. In *Das Kontinuum* Weyl deals with the phenomenal continuum of spatial extension and following his previous work *Die Idee der Riemannschen Fläche* he attempts to conceptualize the continuous connectedness of the points on a two-dimensional surface. In the case of temporal continuum we needed to postulate that a "now" was intuitively given. Now we need to assume that an exact "here" can be fixed. Nevertheless, a fixed spatial-point "cannot be exhibited in any way", again an exact determination is never possible (Weyl 1994, p. 92). This continuum does not consist of isolated individual points. Moreover, Weyl observes that even accepting this postulate other issues arise. Under this assumption we can regards a spatial surface as a "two-dimensional manifold" of surface-points (Weyl 2009, p. 16). Its continuous connectedness can be then grasped by

[27]Weyl inherits this conception of real world from Husserl. He explicitly makes reference to Husserl's *Ideen* when he claims: "[...] the real world, and every one of its constituents with their accompanying characteristics, are, and can only be given as, intentional objects of acts of consciousness" (Weyl 1952, p. 4).

[28]He further claims that it might be possible to place "points" in this intuitive continuum if we, in reflection, extricate ourselves from this stream and place ourselves in front of the constant now, treated as an object, which spans a changing experiential content. Then the phenomenal time becomes a flow, in which we can place points. Nevertheless, we cannot speaks of rigidly punctual "now". To every point there "corresponds a definite experiential whole; and if consciousness stands at a certain point, then it has the corresponding experiential whole; only this *is*" (Weyl 1994, p. 91). Moreover, even if we hold fast to the individual points in their isolation from one another, further problems arise. This conceptualization faces us the following situation. In each experience belonging to a time-point, we have also more or less clear memories of the experiences we had in the past time-points, and in turn, each of these experiences contains more or less clear memories of the experiences we had in all earlier moments. Then the flow would consist of infinitely many mutually related systems of infinitely many memories, one packed inside another, but "clearly, we experience none of this; and besides, such a system of point-like moments of experience fitted endlessly into one another in the form of a completely apprehended unity is absurd" (Weyl 1994, p. 91).

[29]Weyl recommends reading Husserl's phenomenological description of time (Husserl 1913a, §81,§82) for further details. He makes also reference to Bergson's philosophy (Bergson 1907).

means of the notion of *neighbourhood*. Given two surface-points P and Q, and a relation N that satisfies certain conditions, we say that Q lies in the n-neighbourhood of P if the relation $N(P, Q; n)$ holds. This relation aims to represent the structural properties involved in the common notion of neighbourhood $|x - x_0| < n$ so that all ideas of continuity in a two-dimensional surface can be developed within this abstract scheme, free from any residue of intuitive knowledge. Although this approach offers many advantages, the reduction of continuous connectedness to the concept of neighbourhood isn't satisfactory. When a relation $N(P; Q; n)$ establishes the nth neighbourhood of P, "much more occurs than is given by the continuous connectedness itself" (Weyl 1994, p. 106). In the case of the plane, for instance, "we could choose the interior of the circle of radius $1/n$ about a point as the nth neighbourhood of that point, but the circle of radius $1/2^n$ would serve just as well" (Weyl 1994, p. 107). Furthermore, we could employ several other shapes in place of the circular ones (elliptical, square, ...).



No clear-cut answer "is yet at hand to the question of how we shall establish the link between the given and the mathematical in a perspicuous manner" (Weyl 1994, p. 107). Dissolving the phenomenal continuum into isolated spatial-points then turns out to be deeply unsatisfactory.[30]

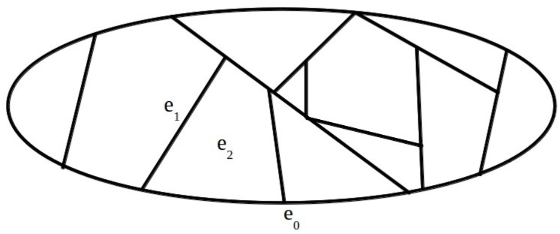
Nevertheless, these studies on the nature of space and time are not pointless. On the contrary, they are of great importance for our understanding of the real world. Those abstract schemata involved in our mathematical theories "must underlie the exact science of domains of objects in which continua play a role" (Weyl 1994, p. 108). Weyl indeed believes that a sort of "Logos" dwells into reality and we can try to reveal it as much as possible. Our mathematical theories are not a matter of choice just as "our inability to connect up the continuous with the schema of the whole numbers is not just a matter of personal preference" (Weyl 1994, 93, note 11). In this sense he claims that his construction of analysis "contains a *theory of the continuum* which must establish its own reasonableness (beyond its mere logical consistency) in the same way as a physical theory" (Weyl 1994, p. 93).

In the following years Weyl revises his mathematical approach to the continuum. Our previous approaches rely on the assumption that is possible to exhibit a time-point or spatial-point in an individual intuition. This assumption, however, violates the

[30] As we have already observed, Weyl's studies are often characterized by a continuous tension between a temporary solution and a call for a better solution. For this reason these considerations are not in conflict with previous mathematical conceptualizations of space. In this occasion Weyl is showing us the underlying problems concerning a finer analysis of a mathematical theory of time or space.

essence of continuum, which by its very nature cannot be shattered into a multitude of individual elements. It is not the relation of the element to the set, but that of the part to the whole, that should underlie the analysis of the continuum.[31] The continuum "falls under the notion of the 'extensive whole', which Husserl characterizes as that 'which permits a dismemberment of such a kind that the pieces are by their nature of the same lowest species as is determined by the undivided whole"' (Weyl 1949, p. 52).[32] An attempt to improve these studies is first proposed in *Über die neue Grundlagenkrise der Mathematik* published in 1921. In this paper Weyl emphasises on "the inner groundlessness of the foundations" upon which rests the current mathematics (Weyl 1921a, p. 86). Following Brouwer's ideas, he then attempts to offer a different approach to the concept of the real number and that of the continuum. At that time Weyl was deeply impressed with the works of Brouwer and his foundational viewpoint to the extent that he refers to Brouwer as "the revolution" (Weyl 1921a, p. 99).[33] In the last pages of this polemical paper he stresses on the need for a different mathematical treatment of the continuum of a two-dimensional manifold.

He first formulates the schema S concerning the *topological structure* of the manifold. It consists of finitely many *corners* e_0 (elements of level 0), *edges* e_1 (elements of level 1) and *surface pieces* e_2 (elements of level 2).



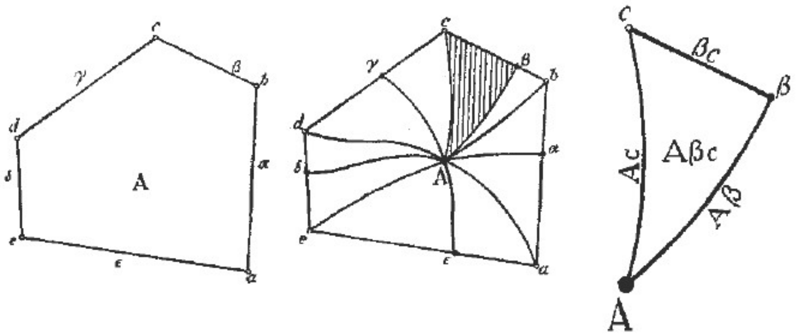
Few basic properties can be established. Each surface is limited by certain edges and each edges by certain corners. These properties represent the content of the schema S, i.e. the *topological framework* of the manifold. This schema "has to satisfy certain requirements, which can easily stated" (Weyl 1921a, p. 115). Weyl then outlines a *process of division* by dividing each edge into two edges by means of one of their points. Analogously, each surface piece is divided into triangles by means of lines from a center, arbitrarily chosen within it, to the corners associated to the surface piece.[34]

[31]"[. . .] sie dadurch gegen das Wesen des Kontinuums verstößt, als welches seiner Natur nach gar nicht in eine Menge einzelner Elemente zerschlagen werden kann. Nicht das Verhältnis von Element zur Menge, sondern dasjenige des Teiles zum Ganzen sollte der Analyse des Kontinuums zugrunde gelegt werden" (Weyl 1988, p. 5).

[32]Weyl is referring to Husserl's *Logische Untersuchungen*. See Husserl (1970a, vol II, 29).

[33]Observe that he wasn't always in agreement with Brouwer, but instead he carried out his own foundational account. For a comparison between them, see van Atten et al. (2002).

[34]The picture and the following remarks are taken from another paper published in 1940 with the title *The Mathematical Way of Thinking* (Weyl 1940). In this occasion this account is better explained.



The picture shows an example focusing on one surface piece. In this case the surface piece is a pentagon and it is shown the first step of the process of division from S to S'. We can easily identify each element arising from the process of division. For instance, the edge β is divided by means of an arbitrary point on it so that two new edges emerge, i.e. βc and βb. Moreover, the arbitrary point within the surface piece A leads to the division into triangles. Then we obtain the new surface piece $A\beta c$. All other elements can be identified in a similar way. As the process of division goes on all elements can be properly named. Weyl then observe that we can carry out a general pattern. Given the initial schema S, any symbol $e_2e_1e_0$ represents a surface piece e_2' of the subdivided scheme S'. Through iteration of this symbolic process we obtain a sequence of derived schemes $S, S', S'', S''', \ldots$ so that what "we have done is nothing else than devise a systematic cataloguing of the parts created by consecutive subdivisions" (Weyl 2012, p. 76). A point in this continuum is caught by a *sequence* $e\ e'\ e'' \ldots$ which starts with a surface piece e of S and in which the surface piece $e^{(n)}$ of the scheme $S^{(n)}$ is followed by one of the surface piece $e^{(n+1)}$ of $S^{(n+1)}$ into which $e^{(n)}$ breaks up by our subdivision. From the surface pieces of the initial topological framework, i.e. the schema S, we then reach the points of the manifold. It is achieved by a process of division which is iterated infinitely many times. This mathematical conceptualization is able to account for the essential character of the continuum which relies on the relation between part and whole, where "*every part of it can be further divided without limitation*" (Weyl 1921a, p. 115). A point in a manifold must be seen as a *limiting idea* (*Grenzidee*). The concept of a point is indeed "the idea of the *limit* of a division extending *in infinitum*" (Weyl 1921a, p. 115). For this reason Weyl is convinced that everyone "feel how truly the new analysis conforms to the intuitive character of the continuum" (Weyl 1921a, p. 117).[35]

To conclude, Weyl's researches on the nature of the continuum seem to be characterized by similar features to the previous studies. They underlie a distinction between two kinds of knowledge, one related to sense perception and another to the domain of mathematical concepts. Again, what is intuitively given seems to be the starting point. Our mathematical understanding of the continuum should rely on some intuitive

[35]Weyl sketches how we can develop a full mathematical analysis on such a manifold. However, he was aware that several issues should have been addressed and his research works on *combinatorial topology* aims to further develop this approach. He published two important contributions in that direction in 1923 and 1924. See Weyl (1923a, 1924). See also Scholz (2000).

insights and a theoretical construction should be developed upon those basic notions and relations that are intuitively given. Moreover, also in this case our mathematical conceptualizations are not a matter of choice. Weyl seems to suggest that a sort of "Logos" dwells into reality and these studies allow us to grasp those abstract schemata that underlie what is immediately given. Nevertheless, the analysis of the continuum turns out to be much more complicated. Several problems arise and they all are related to the mistaken assumption that it is possible to exhibit a time-point or spatial-point in an individual intuition. For this reason Weyl aims at improving his analysis of the continuum in later works. His research works in topology deal with this issue. They regard a point in a continuum as a limiting idea, i.e. the idea of the limit of a division extending *in infinitum*. According to Weyl this approach is a more faithful analysis of the continuum. Weyl's research works seems then to be characterized by a constantly looking for the mathematical conceptualization that best suits what is intuitively given.

1.4 A Phenomenological Framework

Weyl's studies can be better understood within the philosophical framework of Husserl's phenomenology. Edmund Husserl (1859-1938) came to Göttingen as *extraordinarius* professor of philosophy in 1901. In 1904 Weyl moved to Göttingen to study mathematics and physics and in 1908, under Hilbert's supervision, he received his doctorate. In the years 1904-1913 Husserl and Weyl overlapped at this university. Historical records show that they know each other, but Weyl's truly interest in phenomenology was kindled by his future wife Helene Joseph (1893-1948). She specifically came to Göttingen to be a student of Husserl in 1911 and since then her philosophical thinking was deeply influenced by phenomenology. In the years that followed the period in Göttingen the Weyls became friends with Husserl and his family to the extent that Gerhard Husserl, Edmund's youngest son, escaped from Germany during the Nazism and stayed for some time with the Weyls in Princeton.[36] Weyl send a copy of *Das Kontinuum* and *Raum-Zeit-Materie* to Husserl and he received from him a copy of the second edition of the sixth logical investigation of *Logische Untersuchungen*. Four letters from Husserl to Weyl have been preserved. They clearly provide documentation of the close affiliation of Weyl with phenomenology in the years 1917-1927.[37]

Previous studies bear a strong imprint of Husserl's influence. Several Husserlian issues underlie Weyl's investigations. Weyl's distinction between phenomenal and conceptual knowledge and his theory of meaning, for instance, strongly suggest a Husserlian influence on these studies. Moreover, Weyl makes explicit reference to

[36]Weyl (1948, p. 381). For further details concerning the personal contacts between Weyl and Husserl, see Ryckman (2005, §5).

[37]The correspondence is published in Schuhmann (1996) and Van Dalen (1984). Few excerpts are translated and discussed in Ryckman (2005, §5). See also Tonietti (1988). For a French translation that also includes a noteworthy letter from Weyl to Husserl, see Lobo (2009).

Husserl's writings several times. We have shown some of them, but many others examples can be found.[38] As observed in the introduction several works in the recent literature have shown Husserl's influence on Weyl's scientific investigations. In what follows I will focus on some Husserlian issues that seem to have been overlooked in literature. In the light of these considerations a more uniform interpretation of Weyl's researches in the period 1917-1927 will be put forward.

In *Ideen I* Husserl stresses on the distinction between *descriptive sciences* and *exact sciences*. Both are eidetic sciences but they are essentially different. Geometry is a good example of exact science. It is an *axiomatic science* that operates with *exact concepts*, which express *ideal essences*. Starting with few basic concepts and by means of few primitive axioms, it derives all ideally possible spatial forms. All these "derived essences" are not usually intuited. That is, geometry does not grasp each essence directly but they are reached by mediate reasoning. For this reason, Husserl refers to these sciences also as *explicative sciences*. Moreover, geometry "can be completely certain of dominating actually by its method all the possibilities and of determining them exactly" (Husserl 1982, p. 163). Husserl refers to this "fundamental logical property" in terms of *definite manifold* (Husserl 1982, p. 163). A field of inquiry is articulated as a definite manifold if, out of a few basic concepts and a given set of axioms, it is possible to derive the totality of all possible formations concerning that field. A descriptive science is instead purely descriptive and it operates with *inexact concepts*, which express *morphological essences*. It investigates its field of inquiry by means of a direct seeing of essences. In this sense we can refer to phenomenology as a descriptive science. Its phenomenological descriptions are based on a direct seeing of essences.[39] Nonetheless, such a difference between descriptive and exact sciences does not exclude the fact that they might be coexist as two correlate investigations in the same field of inquiry. A field of inquiry, for instance, might be articulated as a definite manifold. However, that is not a matter of choice. This fact "must be demonstrable in immediate intuition" (Husserl 1982, p. 165). One of the necessary conditions has to be "the exactness in '*concept-formation*', which is by no means a matter of free choice and logical technique" (Husserl 1982, p. 165). The exactness of the basic concepts has to be grounded on the descriptive analysis of the field of inquiry itself so that *their meaning is completely clarified* within this phenomenal domain. There must exist some idealizing procedure, intuitively ascertained, that substitute morphological essences with ideal essences. Husserl further observes that these ideal essences, grasped by such an idealization, have to be regarded as a kind of "limits", that is *limiting ideas* (*Grenzideen*) in the sense of Kant. In this way it might be possible

[38]For instance, he also refers to Husserl's analysis of abstraction in his paper *Der Circulus Vitiosus in der Heutigen Begründung der Analysis* as a proper clarification of this notion in spite of a mistaken empiricist theory of abstraction. See Weyl (1994, p. 111).

[39]Husserl observes, however, that not being an exact science does not make phenomenology an inadequate science. Our prejudices on the well-known exact sciences, as geometry, shouldn't lead us to fail to recognize that "transcendental phenomenology, as a descriptive science of essence, belongs however to a *fundamental class of eidetic sciences totally different* from the one to which the mathematical sciences belong" (Husserl 1982, p. 169).

to deal with this field as a definite manifold.[40] An important case is represented by the relationship between intuitive space and geometry. The former is extensively described by the various Husserl's eidetic investigations on our spatial experience.[41] These phenomenological descriptions constitutes a *descriptive material eidetic science of space*. Geometry instead is an eidetic science dealing with all possible *spatial forms* by means of exact concepts, that is an *exact material eidetic science of space*. Clarifying all the connections between these two sciences is not an easy task.[42] In *Ideen I* Husserl himself admits that further investigations are needed for "a clarification of the so-little understood relationship between 'descriptive' and 'explanatory' science" (Husserl 1982, p. 165). This field of phenomenological researches belong to the more general issue concerning the complex relationship between phenomenology and ontology. Anyway, shedding light on Husserl's complex view on this issue is not our purpose.[43] It shall be sufficient to point out that a connection between a descriptive analysis of a field of inquiry and its exact determination has to be established by means of some idealizing procedure intuitively ascertained. A connection of that sort, moreover, is important if we wish to develop an exact determination of that very field of inquiry or we can say of that *regional ontology*. Within this phenomenological framework we should interpret Weyl's investigations.

Weyl's researches on the nature of intuitive space aims to uncover the structure of space that underlie the domains of objects immediately given in our experience of space. Whereas "in examining a real object we have to rely continually on our sense perception in order to bring to light ever new features, capable of *description in concepts of vague extent only*", the structure of space "can be exhaustively characterized with the help of a few *exact concepts* and in a few statements, the *axioms*, in such a manner that all geometrical concepts can be defined in terms of those basic concepts and every true geometrical statement follows as a logical consequence from the axioms" (Weyl 1949, 3, *my emphasis*). Once intuition has "furnished us with the necessary basis" we shall "enter into *the region of deductive mathematics*" (Weyl 1952, 16, *my emphasis*). In this sense geometry turns out to be a "*theory of space*" (Weyl 1949, 18, *my emphasis*). Moreover, "the scientific theory in question is said to be *definite* (*definit*) according to Husserl" (Weyl 1949, p. 18).[44] Weyl's preference for the axiomatic construction of

[40]Husserl observes: "In the eidetic province of reduced phenomena (either as a whole or in some partial province) [. . .] the pressing question of whether, besides the descriptive procedure, one might not follow - as a counterpart to descriptive phenomenology - an idealizing procedure which substitutes pure and strict ideals for intuited data and might even serve as the fundamental means for a mathesis of mental processes" (Husserl 1982, p. 169).

[41]See, for instance, Husserl (1997).

[42]In a manuscript dated 1910 Husserl asks himself if this sort of idealization is unique and necessary or it may be arbitrary to a considerable extent. In the latter case, several exact material eidetic sciences of space might be equally well-founded. Some of these remarks are discussed in Sinigaglia (2011).

[43]For further details, see Husserl (1982, §72-§75) and Husserl (1980, §13-§17). In later years Husserl revises his analysis of idealization within his historical reflection on the origins of philosophical and scientific thought. Important remarks concerning the origin of geometry can be found in his *Krisis*. See Husserl (1970b, p. 353).

[44]Husserl's notion of *definiteness* (*Definitheit*) has been the subject of debate, especially in relation to the modern notion of *completeness*. A number of different interpretations of this notion has been proposed in the literature. See, for instance, Ortix Hill (1995), Majer (1997), Da Silva (2000) and Centrone (2010).

affine geometry with respect to Euclid's or Hilbert's one can be also better understood within these framework. The former theoretical construction takes account of the idealizing procedure involved in the constitution of the ideal essences of line and plane, whereas the latter does not.[45] The meaning of the exact concepts that express these ideal essences are then better clarified within the phenomenal domain of intuitive space. Affine geometry therefore reveals "a wonderful harmony between the given on one hand and reason on the other" because reflects more accurately the descriptive analysis of this field of inquiry (Weyl 1949, p. 69).

Similarly, Weyl's mathematical conceptualizations of the continuum find their roots in this phenomenological framework. In *Das Kontinuum* Weyl attempts to establish a connection between something given in the "*morphological description* of what presents itself in intuition" and "something constructed in a logical conceptual way" (Weyl 1994, 49, *my emphasis*). Nevertheless, any idealizing procedure can be intuitively ascertained with respect to the constitution of the category of point. His research works in topology improve this approach developing a theoretical construction that takes account of the idealizing procedure involved in the constitution of the ideal essence of point. This ideal essence is then expressed by an exact concept whose meaning can be clarified within the phenomenal domain of intuitive continuum. Weyl further observes that an improvement of this approach should take account that the process of division itself shouldn't be regarded as given in a exact way. In reality one must imagine that the divisions are given only vaguely, with a limited accuracy. For an exact division contradicts the essence of the continuum. But as the division progresses, the accuracy will increase indefinitely.[46] Topological studies allow us to address these problems exactly "even though the continua to which they are addressed may not be given exactly but only vaguely, as is always the case in reality" (Weyl 1949, p. 90). They represent an intermediate level of analysis. A rational analysis of continua indeed "proceeds in three steps: (1) *morphology*, which operates with *vaguely circumscribed types of forms*; (2) *topology*, which, *guided by conspicuous singularities* or even in free construction, places into the manifold a vaguely localized but combinatorially exactly

For a detailed account of the various notions of completeness which occurred in connection with the development of the axiomatic method in the late nineteenth and early twentieth century mathematics, see Awodey and Reck (2002). For a better understanding of Weyl's notion of definiteness, however, we should observe that he distinguishes it from the notion of *completeness* (*Vollständigkeit*). He claims that in a complete system of axioms for every pertinent general proposition a the question 'does a or $\neg a$ hold?' is decidable by logical inference on the basis of the axioms, but in that case "mathematics would thereby be trivialized" (Weyl 1949, p. 24). Intuition and "the life of the scientific mind pose the problem, and these cannot be solved by mechanical rules like computing exercise" (Weyl 1949, p. 24). Cf. Centrone (2010, §3.6.2) for a comparison between Husserl's notion of *Definitheit* and Hilbert's notion of *Vollständigkeit*.

[45] Affine geometry does not take account of any idealizing procedure regarding the basic categories of objects, i.e. the category of points and of vectors. For this reason it establishes only a partial connection between a descriptive analysis of a field of inquiry and its exact determination. Anyway, as we have observed, Weyl's studies should be always understood as constantly improving.

[46] "In der Wirklichkeit mußman sich vorstellen, daßdie Teilung auf der 0^{ten} Stufe Σ_0 [on S] nur vage, mit einer beschränkten Genauigkeit gegeben ist; denn eine exakte Teilung widerspricht dem Wesen des Kontinuums. Aber bei fortschreitender Teilung soll sich auch die Genauigkeit, mit der die anfänglichen Ecken und Seiten und die auf den vorhergehenden Stufen neu eingeführten festgelegt sind, unbegrenzt steigern" (Weyl 1988, p. 8).

determined skeleton; and (3) *geometry* proper, whose *ideal structures* could only be carried with exactness into a real continuum after this has been spun over with a subdivision net of a fineness increasing *ad infinitum*" (Weyl 1949, 91, *my emphasis*). Husserl's influence on this sort of analysis becomes clear from Weyl's reference to O. Becker (1889-1964), which wrote a *Habilitationsschrift* in 1922 on the phenomenological foundations of geometry and relativity theory under Husserl's direction. For a "more careful phenomenological analysis of the *contrast between vagueness and exactness and of the limit concept*, the reader may be referred to the work by O. Becker", his *Beiträge zur phänommenologischen Begründung der Geometrie und ihrer physikalischen Anwendungen*[47] (Weyl 1949, 91, *my emphasis*). Becker indeed further develops this sort of analysis improving especially these foundational aspects involved in the connection between a descriptive analysis of a field of inquiry and its exact determination.[48]

1.5 Conclusion

Weyl's researches then turn out to be an attempt to establish a connection between a descriptive analysis of phenomena and their exact determination within the phenomenological framework we have outlined. He attempts to manage this connection in several ways. For this reason we shouldn't interpret the variety of Weyl's studies as a changeable view along the years 1917-1927. They instead should be understood as different attempts to attain a theoretical construction that is as much phenomenologically grounded as possible. In this sense we can also speak in terms of Weyl's *phenomenological constructivism*.[49] To be clear, I am not claiming that Weyl's studies can be properly defined as phenomenological researches. Weyl himself admits that he touches only "lightly on the philosophical implications" since he is not "in a position to give such answers to the epistemological questions involved" as his conscience would allow him to uphold (Weyl 1952, p. 2). In *Das Kontinuum*, for instance, he remarks that his researches on the continuum are "only a slightly illuminating surrogate for a genuine philosophy of the continuum" since his task "is mathematical rather than epistemological" (Weyl 1994, p. 97). In another occasion he further admits: "[...] it strikes me as very difficult to give a precise analysis of the relevant mental acts" (Weyl 1995, p. 454). For this reason my interpretation is suggesting that Weyl's researches are develop taking account of this phenomenological framework but they should be enlightened by deeper phenomenological analysis.

[47] Becker (1923).

[48] In a letter to Weyl dated April 9, 1922, Husserl wrote: "Dr Becker also found it necessary in the first part of his work to enter into the general fundamental questions concerning the theorization of vague experiential data, with its vague continuity, and to sketch a constitutive theory of the continuum" (Mancosu Paolo 2010b, p. 282). For further details, one can consult the correspondence between Weyl and Becker. They are discussed in Mancosu Paolo (2010a,b). See also Lobo (2009).

[49] In Tieszen (2000) Tieszen speaks in term of Weyl's *mathematical constructivism*. His reading of Weyl's writings is indeed close to mine. We both recognize a sort of constructivism that is strongly influenced by Husserl's phenomenology. However, my interpretation has revealed some important Husserlian issues underlying Weyl's studies that Tieszen seems to overlook. They instead are relevant to a proper understanding of Weyl's research studies.

These considerations could be extended beyond the few examples we have shown. Weyl's development of *infinitesimal geometry* especially should be understood within this framework. In *Raum-Zeit-Materie* Weyl faces the arising theory of general relativity and he aims to develop the theoretical construction of real world whose meaning is phenomenologically clarified within the domain of our experience.[50] To conclude, these are the philosophical reasons that underlie Weyl's famous remark in *Raum-Zeit-Materie*:

> The investigations about space that have been conducted in chapter II appear to me to offer a good example of the essential analysis (*Wesenanalyse*) striven for by phenomenological philosophy (Husserl), an example that is typical for such cases where a non-immanent essence is dealt with.[51]

[50]In a letter to Weyl dated April 12, 1923, Becker remarks that Weyl's works on general relativity has first "made possible a complete phenomenological foundation for geometry (in the sense of 'world geometry')". He further adds that "the same idealistic conception" underlies both Weyl's theory of continuum and his infinitesimal geometry. See Mancosu Paolo (2010a, p. 309).

[51]Weyl (1921b, 133, *my translation*).

2 Weyl's Research within Infinitesimal Geometry

[...] we now pass on to the systematic development of pure infinitesimal geometry [...] which, in my opinion, is the climax of a wonderful sequence of logically-connected ideas, and in which the result of these ideas has found its ultimate shape, is a true geometry, a doctrine of space itself.

H. Weyl[1]

Pure infinitesimal geometry plays a crucial role in Weyl's mathematical understanding of real world. He started to develop it around 1917 with the goal of achieving a mathematical and philosophical foundations of relativity theory. It has certainly represented a pioneering work in the scientific landscape of that time, both from a mathematical and physical point of view. It represents in fact a new systematic mathematical approach, which will give birth to the modern differential geometry in later years. Moreover, Weyl's pure infinitesimal geometry aims to revisit the fundamental physical concepts of space, time and matter in the light of the arising new world view opened by Einstein's relativity theory. In this chapter I will focus my attention on Weyl's mathematical conceptualization of the continuum in *Raum-Zeit-Materie*. This mathematical approach deals with a fundamental notion within Weyl's pure infinitesimal geometry, i.e. the notion of infinitesimal quantity. I will clarify its meaning within this framework. Then I will show that these studies have to be understood as works in progress and they can be connected with Weyl's investigations in combinatorial topology. They both follow the phenomenological guidelines illustrated in the previous chapter. The works on combinatorial topology, however, are more faithful from a phenomenological perspective. In this sense they have to be regarded as foundational works with respect to the mathematical conceptualization of the continuum in *Raum-Zeit-Materie*.

[1]Weyl (1952, p. 102).

2.1 A New Physical Understanding of the Real World

In 1854 Bernhard Riemann inaugurates a new approach to geometry with his inaugural lecture *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*. The lecture is often unclear, the main notions are briefly illustrated and there are not calculations. However, it contains some groundbreaking ideas that will transform the way mathematicians and physicists explore the external world. Although these ideas did not spread out at the beginning, they will spread out after his death in 1866. From now on several mathematicians and physicists began to develop what later would became the modern differential geometry. This new mathematical approach was based on the physical principle that we gain knowledge of the real world from the behaviour of its infinitesimal parts. Starting from local properties then we are able to understand the global properties that characterize the real world as a whole. It had so many impacts on the development of mathematics between the late 19th century and the beginning of 20th century. One of most merits is surely that it enabled the formulation of Einstein's general theory of relativity. In 1916 Einstein's publication of *Die Grundlage der allgemeinen Relativitätstheorie* gave birth to a new revolutionary theory. From then on several scholars began to deal with the many physical and mathematical problems connected with Einstein's novel world view. Hermann Weyl played a key role within this context. In fact his *pure infinitesimal geometry* was developed for this very purpose. Weyl addresses this issue to his summer course at ETH of Zurich in 1917. The first 1918 edition of *Raum-Zeit-Materie* is based on these lectures. There will be anyway five editions in the following years. Whereas the second edition is a reprint of the first, in the third edition Weyl reviews completely the mathematical approach. Our analysis will be based on the fourth edition, published in 1921.

2.2 Weyl's Infinitesimal Quantities

The notion of manifold plays a crucial role within the new geometrical description of real world suggested by the theory of general relativity. Weyl introduces the systematic development of his pure infinitesimal geometry clarifying his notion of space:

> We shall make the sole assumption about space that it is an n-dimensional continuum [manifold].[2]

Space has then to be understood as an *n-dimensional continuous manifold* whose elements are spatial points.[3] The main feature is that each of its points can be identified by giving n coordinates $x_1, \dots, x_n$ which are continuous functions on the manifold. These

[2]Weyl (1952, p. 102).

[3]Weyl deals with the general case of n dimensions for two main reasons: "To recognize the perfect mathematical harmony underlying the laws of space, we must discard the particular dimensional number $n = 3$. [...] It seems to me to be one of the chief objects of mathematical instruction to develop the faculty of perceiving this simplicity and harmony, which we cannot fail to observe in the theoretical physics of the present day. It gives us deep satisfaction in our quest for knowledge. [...] but also because we shall later require four-dimensional geometry for concrete physical problems such as are introduced by the theory of relativity, in which Time becomes added to Space in a four-dimensional geometry" (Weyl

functions anyway are not defined globally on the whole manifold, but only locally. That is, for any arbitrary point P of the manifold it is possible to represent each point in a certain domain surrounding P by the n coordinates $x_1, \ldots, x_n$. If $\bar{x}_1, \ldots, \bar{x}_n$ is another coordinate system for the same point P then the following relations hold

$$x_i = f_i(\bar{x}_1, \ldots, \bar{x}_n) \quad \forall i$$

in which the f_i's are continuous functions and they have continuous derivatives

$$a^i_k = \frac{\partial f_i}{\partial \bar{x}_k}$$

in such a way that the determinant of the matrix a^i_k is non-vanishing.

The most important aspect that characterizes the space refers to his being a *continuous* manifold. Space indeed is characterized by his continuous extension.[4] Therefore, any mathematical account of space has to take into account this fact. Moreover, since the continuum is capable of infinite division, then the issue of continuity turns out to be strictly related with a proper mathematical understanding of the infinite. Weyl remarks this connection in *Philosophy of Mathematics and Natural Science*:

> In a different form than in the sequence of integers *we encounter the infinite in the continuum*, which is capable of infinite division. Cases of special importance are the continua of time and of space.[5]

In order to deal properly with the continuum we need to clarify its essential character:

> The essential character of the continuum is clearly described in this fragment due to Anaxagoras: "Among the small there is no smallest, but always something smaller. For what is cannot cease to be no matter how far it is being subdivided".[6]

Therefore this kind of "infinite in the continuum" represents one of the most difficult problems involved in our attempts to develop a mathematical theory of the continuum. Weyl suggests that the notion of infinitesimally small were proposed in order to deal with this issue:

> Three attempts have been made in the history of thought to conceive of the continuum as Being in itself. According to the first and most radical the continuum consists of countable discrete elements, atoms. [...] *The second*

1952, p. 23). Therefore, there are very good epistemological reasons to go beyond the particular case of $n = 3$. The general case indeed seems to have an epistemic value. It allows us to better recognize the "mathematical harmony underlying the laws of space". Moreover, Einstein's relativity requires 4-dimensional geometry since time and space need to be handled together. For these reasons Weyl develops his pure infinitesimal geometry in the most general case.

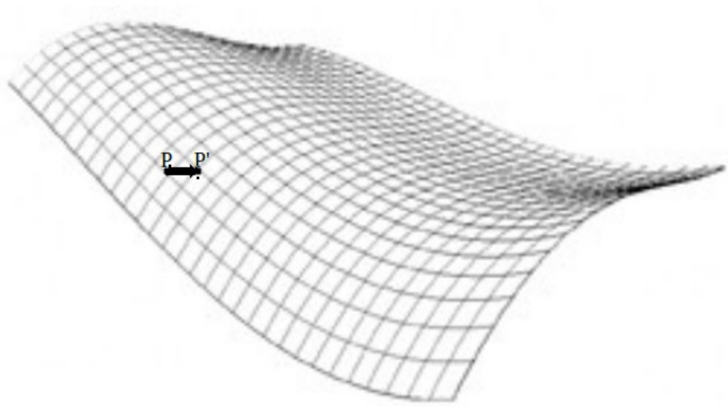
[4]In general relativity we have to consider a 4-dimensional continuous manifold, where we handle together the continuous extension of space and time.

[5]Weyl (1949, 38, *my emphasis*).

[6]Weyl (1949, p. 41).

> *attempt is that of the infinitely small*. [...] The third attempt to 'save' the continuum in the Platonic sense may be seen in the modern set-theoretic foundations of analysis.[7]

Mathematical investigations dealing with the notion of infinitesimally small therefore have been an attempt to understand the nature of continuum. Specifically, they conceive the continuum as a whole made up of infinitely many infinitesimal parts. For this reason, according to Weyl any mathematical account involving the notion of infinitesimally small has to be considered an attempt to deal with the problematic nature of the continuum. Specifically, this is the case of Weyl's studies on infinitesimal geometry in *Raum-Zeit-Materie*. The notion of *infinitesimally small* is indeed of fundamental importance within this framework. Given a point of the manifold P we can speak of infinitesimal neighborhood consisting of all points P' infinitely near to P. The notion of *infinitesimal displacement* then arises within this context.[8] Given a point of the manifold P' infinitely near to the point P we can speak of an infinitesimal displacement PP' that transfers the point P to the point P'.[9]



In *Raum-Zeit-Materie* Weyl introduces this notion with the following words:

> The relative co-ordinates dx_i of a point $P' = (x_i + dx_i)$ infinitely near to the point $P = (x_i)$ are the components of a *line element* (Linienelementes) at P or of an *infinitesimal displacement* (infinitesimalen Verschiebung) PP' of P. The transformation to another co-ordinate system is effected for these components by formula (18), in which a_{ik} denote the values of the respective derivatives at the point P.[10]

where formula (18) refers to:[11]

[7]Weyl (1949, 42-45, *my emphasis*).

[8]The picture represents a special case, i.e. a 2-dimensional manifold.

[9]Weyl refers to it also as a *line element at P*. In this respect it is worth noting that this notion had been already introduced in the past. Gauss, Riemann and the first mathematicians dealing with the arising differential geometry already used this notion.

[10]Weyl (1952, p. 103). The English translation shows a mistake. The index i in the first appearance of "dx" is not shown.

[11]Observe that this formula is invertible since the real numbers a^i_k have a non-vanishing determinant.

$$dx_i = \sum_{k=1}^{n} a^i_k d\bar{x}_k$$

A fundamental question then arises: how we can properly understand all that? It seems to refer to a sort of displacement that is infinitesimally small. With respects to his components he also seems to refer to *infinitesimal quantities*. But speaking of infinitesimal quantities seems to be odd at that time. It is usually believed indeed that the concept of infinitesimal was banished from mathematics at the beginning of nineteenth century.[12] Since antiquity the notion of infinitesimally small had been troublesome. Several attempts along the history were provided in order to give a proper mathematical formulation of this notion. A peculiar role were played by the infinitesimal quantities involved in the calculus elaborated by Leibniz and Newton. The calculus was widely used by mathematicians and physicists and it deeply contributed in the development of scientific knowledge. A good number of historians and philosophers of mathematics usually embrace the following idea. In the second half of nineteenth century, the use of infinitesimal quantities became subject of many criticisms and it was reformulated using the $(\epsilon - \delta)$-definition of limit. Cauchy, Bolzano, Weierstrass, Cantor, Dedekind were the main figures involved in what is usually called the rigourisation of calculus. Not only calculus, but all branches of mathematics were revisited. In the late 19th and the early 20th century there was a huge emphasis on foundational issues in mathematics. Within this context, infinitesimals have been shown to be incoherent and for this reason they disappeared from mathematics. This commonplace view is well depicted by the words of Bertrand Russell (1872-1970). In his paper *Recent work on the Principles of Mathematics*, published in 1901, he remarks:

> The banishment of the infinitesimal has all sorts of odd consequences, to which one has to become gradually accustomed.[13]

After the works of Cantor, Dedekind and Weierstrass any kind of infinitesimal quantity that had been of interest to many mathematicians turned out to be a ill-posed notion. Such a view is supported in both editions of his famous *The Principles of Mathematics*. With respect to infinitesimal line segment, for instance, Russell states:

> We found that the differential and integral calculus has no need of the infinitesimal, and that, though some forms of infinitesimal are admissible, the most usual form, that of infinitesimal segments in a compact series, is not implied by either compactness or continuity, and is in fact self-contradictory. [...] It remains to apply to space, time, and motion, the three chief results of this discussion, which are (1) the impossibility of infinitesimal segments, (2) the definition of continuity, and (3) the definition and the consistent doctrine of the infinite.[14]

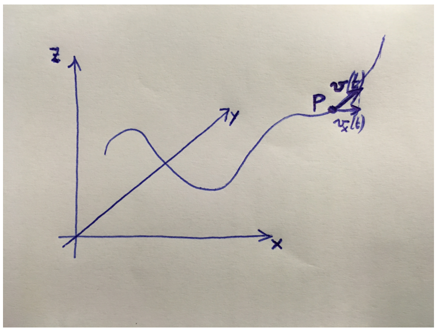
[12] Ehrlich (2006, p. 2).
[13] Russell (1901, p. 371).
[14] Russell (1937, p. 368).

This commonplace view is also well depicted by the following remark:

> nonstandard analysis [...] created by Abraham Robinson in the early 1960s, used techniques of mathematical logic and model theory to introduce a rigorous theory of both [non-Cantorian] infinite and infinitesimal numbers. This, in turn, required a reevaluation of the long-standing opposition, historically, among mathematicians to infinitesimals in particular.[15]

Robinson's *nonstandard analysis* indeed is usually regarded as a reassessment of Leibniz's infinitesimals. Using the methods of mathematical logic Robinson allows us to reformulate this concept on a solid basis. These quantities are incorporated into the real number system without violating any of the usual rules of arithmetic.[16]

For these reasons Weyl's commitment to any sort of infinitesimal quantity seems to be odd. In secondary literature scholars usually approach this issues in two different ways. Some authors tend to keep Weyl's terminology of infinitesimal displacements or infinitely near points. They aim to be historically correct and for this reason they describe Weyl's view in the way he conceives it. Nevertheless, the way they discuss Weyl's works seems to underlying the following hypothesis. Although Weyl is engaged with this kind of notions, he is not really referring to infinitesimal quantities. His approach has instead to be understood as what is usually called an *abuse of notation*. Mathematicians, and more often physicists, sometimes tend to use a mathematical notation in a way that is not properly correct but that simplifies the presentation without introducing errors. Moreover, it often suggests the intuition underlying the mathematical exposition.[17] A typical example is given by the usage of Leibniz's notation by physicists in mechanics. Let $x(t)$ be the x-coordinate of the position of a moving point P in the space. It is a function of time and its derivative with respect to time gives us the x-component of the velocity $v(t)$ of the moving point, i.e. $v_x(t)$.



Using Leibniz's notation we shall indicate the velocity as

$$v_x = \frac{dx}{dt}$$

[15] Dauben (1992, pp. 113-114).

[16] Robinson (1966).

[17] Be aware that the notion of intuition in this case refers to a vague idea that our mathematical exposition aims to grasp. It has not to be confused with Husserl's notion of intuition in any sense.

The notation dx/dt refers to a derivative, but it can be thought as a fraction with dx being divided by dt. For this reason physicists are used to manipulate such a notation as if it were a fraction. For instance, the distance crossed by the moving point along the x-direction over a certain amount of time can be estimated as[18]

$$(1) \qquad \frac{dx}{dt} = v_x$$

$$(2) \qquad dx = v_x dt$$

$$(3) \qquad x(t) = \int_0^x dx = \int_0^t v_x dt$$

The two quantities dx and dt are regarded as being infinitesimal quantities that can be multiplied or divided by each other or by real numbers. It allows us to obtain $dx = v_x dt$ (the value of v_x at a given time t is multiplied by dt). This notation can be understood as an equality between two infinitesimal quantities that can be integrated. That is, we can perform an infinite sum of these quantities obtaining the finite value of $x(t)$. This conception properly represents the intuition underlying the physical notion of instantaneous velocity. It simplifies the exposition and it turns out to be very useful. For this reason it is widely used by physicist and engineers. Nevertheless, it is not completely correct from a mathematical point of view.[19] The notation $\frac{dx}{dt}$ refers in fact to a derivative. A proper mathematical exposition should be the following[20]

$$(1') \qquad \frac{dx}{dt} = v_x$$

$$(2') \qquad \int_0^t \frac{dx}{dt} dt = \int_0^t v_x dt$$

$$(3') \qquad x(t) = \int_0^t v_x dt = V_x(t) - V_x(0)$$

with V_x be the function whose derivative is v_x. In this case dx and dt are not regarded as quantities that can be multiplied or divided by each other. In the second line dt denotes just the fact we are integrating over t. It is just a part of the notation of the integral over t, that is $\int dt$. By itself it does not mean anything.

In this sense then Weyl's terminology seems to be understood by these authors. They usually speaks about infinitesimal displacements, but they are aware that any infinitesimal quantity is properly involved. Although it is not always clear how we can clarify such an abuse of notation, it seems what they have in mind.[21]

[18]We are assuming that the moving point starts to move at $x(0) = 0$.

[19]We are referring to the present state of affairs, of course.

[20]We are assuming again that the moving point starts to move at $x(0) = 0$.

[21]In Ryckman (2005), for instance, we read: "[...] Weyl simply assumed that in the tangent space covering each manifold point P, there is an affine linear space of vectors centered on P in that line elements dx radiating from P are *infinitely small vectors*" (Ryckman 2005, 150, *my emphasis*). Ryckman

Other authors instead have tried to clarify Weyl's terminology referring to modern differential geometry.[22] That is, they have attempted to reconstruct Weyl's mathematical works in the light of the modern approach. They refer to dx_i as an element of a basis in the cotangent space. Specifically, let M be a smooth n-manifold and $T^*_P M$ its cotangent space at P, then $(dx_1, \ldots, dx_n)$ represents a basis for this space. Each dx_i can be seen as a function

$$\begin{array}{rcccc} dx_i & : & T_P M & \longrightarrow & \mathbf{R} \\ & & v & \longmapsto & dx_i(v) \end{array}$$

where $T_P M$ is the tangent space at P, whose elements are the tangent vectors v. Therefore an infinitesimal displacement should be interpreted as a *function* that operates on tangent vectors and gives back a real number. As we can see any infinitesimal quantity is involved.

Approaching history of mathematics in such a way is not always well-accepted by historians of mathematics. There are several problems involved in such works and it is not always clear to what extent we can refer to them as historical studies. Nevertheless there are good reasons to approach in such a way. For instance, it might shed light on new aspects that only a modern approach can reveal us. At the present stage anyway we do not need to face this issue. We are only interested in pointing out that if such an approach is legitimate then it has at least to account for author's implicit intentions. I believe that it is not so obvious in this case. Weyl's philosophically-informed works aim to formulate not only a new geometry but also a new approach to mathematics itself. In that period of great changes within mathematics, indeed, foundational issues were on the agenda of many mathematicians. And Weyl was a leader mathematician in this regard, too. Generally speaking, moreover, he often seems to be quite critical of any formal approach that embraces a set theoretic framework, as it might be considered modern mathematics in several respects. For these reasons, it seems that a modern interpretation can be misleading in this case. Without a serious scrutiny examination of his implicit intentions, we should be very careful to reconstruct Weyl's account in the light of the modern differential geometry. It carries with it several assumptions concerning the nature of mathematics that might not be shared by Weyl. We need then to consider in greater details how the notion of infinitesimal quantity is used within Weyl's framework. It will shed light on its meaning clarifying to what extent it can be understood in the light of modern differential geometry.

A first attempt to clarify Weyl's notion of infinitesimal displacement seems to go through the notion of curve. In *Philosophy of Mathematics and Natural Science* he says

speaks in terms of infinitely small vectors although his reference to the notion of *tangent space* seems to underlie a modern conception of differential geometry. Shortly after he states: "This furnishes the Christoffel symbols with a geometric interpretation, by relating them to the parallel displacement of a vector along a path connecting a point P to another point P' in the infinitesimal region (tangent space T_P) around P ($T_{P'}M = T_P M$)" (Ryckman 2005, 150, *my emphasis*). Weyl's notion of *infinitesimal region* is clearly interpreted as the modern notion of tangent space T_P. Since the modern approach of differential geometry does not involve any infinitesimal quantity, Ryckman seems to deal with Weyl's notion of infinitesimal displacement dx as if it was an abuse of notation.

[22]See, for instance, Scholz (2001, p. II.4).

that:

> [...] the components $(dx_1, \ldots, dx_n)$ of all infinitesimal vectors $\overrightarrow{PP^*}$ issuing from P transform according to linear formulas
>
> $$(1) \qquad dy_i = \sum_{k=1}^{n} a_{ik} dx_k, \qquad dx_k = \sum_{k=1}^{n} b_{ki} dy_i$$
>
> the coefficients a_{ik}, b_{ki} of which depend on the point P but not on P^*. (Infinitesimal quantities may be avoided by introducing an imaginary time τ and letting a point move in the manifold according to an arbitrary law $x_k = x_k(\tau)$. Suppose the point passes P at the moment $\tau = 0$; its velocity at that moment will be a vector at P with the x-components $u_k = (dx_k/d\tau)_{\tau=0}$. The y-components v_i of the same velocity are related to the x-components by the equations (1),
>
> $$v_i = \sum_{k=1}^{n} a_{ik} u_k, \qquad u_k = \sum_{k=1}^{n} a_{ik} u_k$$
>
> which hold for all possible velocities in P).[23]

Then it seems that the use of infinitesimal displacements can be understood as an abuse of notation. They actually refers to the components of the velocity of a moving point in a manifold. This however does not seem a fair interpretation. An abuse of notation indeed refers to a notation that is meaningless taken in itself. (2), (3) have to be understood as they (2′), (3′). They are not two different but equivalent mathematical approaches to the same issue. If it were the case then Weyl shouldn't declare "infinitesimal quantities may be avoided". Such a claim indeed seems to require that speaking of infinitesimal quantities is meaningful in some sense and you can avoid them by another approach. Had Weyl regarded the use of infinitesimal quantities just an abuse of notation then he would have claimed something like "infinitesimal quantities refer to", "infinitesimal quantities actually mean" or the like. Of course it might be the case that he is just an inaccurate way of expressing himself. There are however several other occasions when Weyl seems to follow such an interpretation. For instance, dealing with geodesic lines he remarks:

> If a point which is in motion carries a vector (which is arbitrarily variable) with it, we get for every value of the time parameter s not only a point
>
> $$P = (s) : x = x_i(s)$$

[23]Weyl (1949, p. 85). The notation is slightly changed, but the connection with the previous case seems to be clear enough without further remarks.

> of the manifold, but also a vector at this point with components $v^i = v^i(s)$ dependent on s. The vector remains stationary at the moment s if
>
> $$\frac{dv^i}{ds} + \Gamma^i_{\alpha\beta} v^\alpha \frac{dx^\beta}{ds} = 0$$
>
> (This will *relieve the minds of those who disapprove of operations with differentials; they have here been converted into differential co-efficients*).[24]

Again Weyl seems to imply that operations with differentials does not coincide with operations with components of the velocity of a moving point in a manifold. Therefore it is not an abuse of notation. Another instance is represented when he deals with tensor fields. He operates with infinitesimal quantities and he finally remarks:

> *If one is inclined to distrust these perhaps too venturesome operations with infinitesimal quantities the differentials may be replaced by differential co-efficients*. [...] Let a point (st) of our manifold be assigned to every pair of values of two parameters s, t (in a certain region encircling $s = 0, t = 0$). Let the functions $x_i = x_i(st)$, which represents this "two-dimensional motion" (extending over a surface) in any co-ordinate system x_i, have continuous first and second differential co-efficients. For every point (st) there are two velocity vectors with components dx_i/ds and dx_i/dt.[25]

The problems analysed by means of the notion of infinitesimal quantity is then reformulated in term of differential co-efficients. He further adds the following remark:

> The connection between *this view and that which uses infinitesimal* consists in the fact that the latter [with differential co-efficient] is applied in rigorous form to the infinitesimal parallelograms into which the surface $x_i = x_i(st)$ is divided by the co-ordinate lines $s = const.$ and $t = const.$.[26]

Weyl is clearly accepting as meaningful two different mathematical approaches, one which uses infinitesimals and another that does not. Therefore, we are not dealing with a meaningless notation. This last remark however points out also another aspect. Weyl speaks in terms of "rigorous form" with respect to the view that does not make use of infinitesimals. In another passage he claims:

> The foregoing argument, based on infinitesimals, become rigorous as soon as we interpret d and δ in terms of the differentiations $\frac{d}{ds}$ and $\frac{d}{dt}$, as was done earlier.[27]

[24]Weyl (1952, 114, *my emphasis*). A proper clarification of the notion of geodesic line is not required here to understand our point.

[25]Weyl (1952, 107, *my emphasis*). A proper clarification of Weyl's notion of tensor field is not required here to understand our point.

[26]Weyl (1952, 108, *my emphasis*).

[27]Weyl (1952, p. 120).

Although Weyl is accepting an approach based on infinitesimal as meaningful, he seems to regard it as a view that is less satisfactory than the other approach without infinitesimals.

We can therefore conclude pointing out the key points. Weyl's use of infinitesimal displacements is not an abuse of notation, but it has to be meaningfully understood as a different mathematical approach.[28] This, however, does not mean that such a view is not in need of clarification. Weyl indeed seems to accept it while thinking that further investigations are needed.[29]

In order then to clarify which sort of mathematical approach underlies Weyl's use of infinitesimal quantities, let's focus on the remark that follows the introduction of the notion of infinitesimal displacement:

> The infinitesimal displacements play the same part in the development of Tensor Calculus as do displacements in Chapter I.[30]

For this reason we need first to clarify the notion of displacement within the framework of tensor calculus in first chapter of *Raum-Zeit-Materie*. Weyl introduces the tensor calculus within the context of Euclidean geometry although it extends beyond the limits of this geometry. This calculus plays indeed a crucial role in his infinitesimal geometry as we will see.

We now aim to show how Weyl develops his tensor calculus within the context of Euclidean geometry. We need first to take a look at his approach to affine and metrical geometry.[31] Affine geometry is erected deductively on an axiomatic basis that rules the relations between two fundamental categories of objects, i.e. *point* and *vector*.[32] The axioms are the followings:

- two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ uniquely determine the vector $\boldsymbol{a} + \boldsymbol{b}$
- a real number λ and a vector $\boldsymbol{a}$ uniquely determine the vector $\lambda \boldsymbol{a}$
- the first two axioms satisfy the usual laws concerning addition and multiplication (commutative law, associative law, ...)

[28]If the two approaches can be regarded as equivalent in some sense it is not our concern here. We are only pointing out that the view with infinitesimal has to be understood by itself.

[29]Several years later he seems to support the same view. In a lecture delivered at the National Academy of Sciences in Washington on March 1955 he remarks: "The use of infinitesimal quantities like the differentials $d\epsilon$, $d\eta$ is horrifying to modern mathematicians. But it is easy to replace them by *differential quotients* to which nobody objects." Here it seems that Weyl does not agree completely with modern mathematicians regarding the use of infinitesimal quantities as horrifying. He says "is horrifying *to modern mathematicians*". Nevertheless, he is inclined to follow them in order to be more rigorous. Again he seems to accept the use of infinitesimal quantities although further investigations are needed. See Weyl (2012, p. 206).

[30]Weyl (1952, p. 103).

[31]We do not aim to give an exhaustive exposition of his foundational approach to Euclidean geometry. At the present stage we just need to clarify the main concepts in order to properly understand the development of his tensor calculus.

[32]The term "vector" denotes a *displacement* in the space: "we shall use the term vector to denote a translation or a displacement $\boldsymbol{a}$ in the space" (Weyl 1952, p. 16). This denotation will change after the introduction of the notion of tensor.

- a point A and a point B uniquely determine a vector $\boldsymbol{a} = \vec{AB}$ (B is called the end-point of the vector $\boldsymbol{a}$ whose starting-point is at A)
- a point A and a vector $\boldsymbol{a}$ uniquely determine a point B such that $\vec{AB} = \boldsymbol{a}$

All concepts that may be defined, by logical reasoning alone, from the fundamental categories of vector and point, and the above basic axioms belong to affine geometry. It is possible to define, for instance, the notions *straight line* and *plane*:

- given a point O and a vector $\boldsymbol{e}$, the end-points of all vectors $\vec{OP}$ which have the form $\lambda \boldsymbol{e}$ constitute a straight line
- given a point O, a vector $\boldsymbol{e}_1$, and a vector $\boldsymbol{e}_2$ which is not of the form $\lambda \boldsymbol{e}_1$, then the end-points of all vectors $\vec{OP}$ that have the form $\lambda_1 \boldsymbol{e}_1 + \lambda_2 \boldsymbol{e}_2$ constitute a plane

Moreover, a point O and n linear independent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ (i.e. their linear combination only vanishes when all the coefficients vanish) constitute a *coordinate system*, denoted as $O|\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$.[33] Every vector $\boldsymbol{x}$ can be presented in one and only one way in the form

$$\boldsymbol{x} = \epsilon_1 \boldsymbol{e}_1 + \epsilon_2 \boldsymbol{e}_2 + \cdots + \epsilon_n \boldsymbol{e}_n$$

where the real numbers ϵ_i are called the *components* of the vector $\boldsymbol{x}$. If P is the end-point of the vector $\vec{OP}$, then ϵ_i are called the *coordinates* of P.

Weyl then introduces another fundamental notion. A *linear transformation* is a transformation

$$T : \epsilon^i \longrightarrow \bar{\epsilon}^i$$

$$\epsilon^i = \sum_k a^i_k \bar{\epsilon}^k$$

that describes the relation between certain variables ϵ^i and other variables $\bar{\epsilon}^i$ (where a^i_k is a matrix of real numbers having a non-vanishing determinant). This relation is clearly linear. Weyl speaks in terms of linear transformation in the case of variables as being real numbers (e.g. ϵ^i and $\bar{\epsilon}^i$) or in the case of variables as being vectors. For instance, let consider a change of coordinate system in affine geometry. Weyl refers to the transformation of the fundamental vectors $\boldsymbol{e}_i$

$$\bar{\boldsymbol{e}}_i = \sum_k a^k_i \boldsymbol{e}_k$$

[33] Weyl refers to the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ also as *fundamental vectors* in a given coordinate system.

as a linear transformation between the vectors $\boldsymbol{e}_i$ and the transformed vectors $\bar{\boldsymbol{e}}_i$.[34] Weyl then introduces the notions of *contra-gredient* and *co-gredient* transformation. Two linear transformations[35]

$$T_1 : \epsilon^i \rightarrow \bar{\epsilon}^i$$

$$\epsilon^i = \sum_k a^i_k \bar{\epsilon}^k$$

$$T_2 : \eta^i \rightarrow \bar{\eta}^i$$

$$\eta_i = \sum_k b^k_i \bar{\eta}_k$$

are said to be contra-gredient to one another if, given $M_{T_1} = (a^i_k)$ and $M_{T_2} = (b^k_i)$, we have

$$M_{T_1} = ((M_{T_2})^T)^{-1}$$

(i.e. one matrix is the transpose of the inverse of the other matrix).[36] Two linear transformations are said to be co-gredient to one another if instead we have

$$M_{T_1} = M_{T_2}$$

(i.e. they transform according to the same matrix). The variables of the first transformation are said to be transformed *contra-grediently* (*co-grediently*) with respect to the ones of the second transformation.

Let now consider a coordinate system $O|\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. If we pass into another coordinate system $\bar{O}|\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \ldots, \bar{\boldsymbol{e}}_n$ whereby

$$\bar{\boldsymbol{e}}_i = \sum_k a^k_i \boldsymbol{e}_k$$

[34]I believe this approach might be better understood if we shed light on Weyl's approach to algebra and his view concerning the notion of *quantity* (Größe). Weyl in fact speaks in terms of quantities referring to both geometrical and numerical entities. He aims to deal with them from an algebraic point of view. He seems to want to subsume them under a universal concept of number. Referring to the issue of measuring, for instance, he declares: "it is unnecessary to introduce special fractions for each domain of quantities (Größengebiet). Since their laws are independent of the nature of these domains of quantities, it is more expedient to define them in purely arithmetical terms" (Weyl 1949, 31, *the translation is slightly changed*). And in a footnote, he continues: "this is in line with the oldest mathematical tradition, that of Sumerians. [...] the Greeks abandon the algebraic road [...] the post-classical Occident, partly stimulated by the algebraic achievements of the Arabs, reversed this development. There was little justification, however, for the modern viewpoint *subsuming all quantities (Größen) under a universal concept of number*, before Dedekind gave Eudoxus's analysis of the irrational its constructive twist" (Weyl 1949, 31, *my emphasis*). Despite of these remarks, we do not need any further clarifications on this issue at the moment. Just keep in mind that geometry and algebra are often interconnected within Weyl's framework.

[35]With the assumption that $|a^i_k| \neq 0$ and $|b^i_k| \neq 0$. We will always assume such a condition unless otherwise specified.

[36]This condition is equivalent to the following one: two linear transformations are said to contra-gredient to one another if $\sum_i \eta_i \epsilon^i = \sum_i \bar{\eta}_i \bar{\epsilon}^i$ holds.

then the components of a vector $\boldsymbol{x}$ ($\boldsymbol{x} = \epsilon^1\boldsymbol{e}_1 + \epsilon^2\boldsymbol{e}_2 + \cdots + \epsilon^n\boldsymbol{e}_n$) undergo the transformation

$$\epsilon^i = \sum_k a_k^i \bar{\epsilon}^k.$$

This means that the components ϵ^i are transformed contra-grediently with respect to the fundamental vectors $\boldsymbol{e}_i$.[37] In metrical space, moreover, Weyl observes that we may characterize a vector $\boldsymbol{x}$ by the values of its scalar product with the fundamental vectors, i.e.[38]

$$\epsilon_i = (\boldsymbol{x} \cdot \boldsymbol{e}_i).$$

If we pass into another coordinate system these components undergo the transformation

$$\bar{\epsilon}_i = \sum_k a_i^k \epsilon_k.$$

This means that the components ϵ_i are transformed co-gredient with respect to the fundamental vectors $\boldsymbol{e}_i$. In the first case Weyl speaks of *contra-variant components* of the vector $\boldsymbol{x}$. In the second case instead he speaks of *co-variant components* of the vector $\boldsymbol{x}$.[39]

Another fundamental notion is the concept of *linear form*. A linear form is a function L mapping

$$\boldsymbol{x} \longrightarrow L(\boldsymbol{x}) \in \mathbf{R}$$

and satisfying

$$L(\boldsymbol{a} + \boldsymbol{b}) = L(\boldsymbol{a}) + L(\boldsymbol{b}) \quad L(\lambda\boldsymbol{a}) = \lambda L(\boldsymbol{a})$$

for any vector $\boldsymbol{a}, \boldsymbol{b}$ and any real number λ. It can also be expressed in the form

$$L(\boldsymbol{x}) = a_1\epsilon_1 + \cdots + a_n\epsilon_n = \sum_i a_i\epsilon_i$$

where $\epsilon_1, \ldots, \epsilon_n$ are the components of the vector $\boldsymbol{x}$ in a given coordinate system $O|\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. The real numbers $a_1, \ldots, a_n$ are called the *coefficients* of the linear form. The real numbers $\epsilon_1, \ldots, \epsilon_n$ are called the *variables* of the linear form. Weyl,

[37]As we have observed this is equivalent to the condition $\sum_i \epsilon^i\boldsymbol{e}_i = \sum_i \bar{\epsilon}^i\bar{\boldsymbol{e}}_i$. The transformation of the components is usually derived by this very condition. Such a condition has a clear physical meaning. Weyl wish indeed to regard a vector (i.e. a displacement in the space) as invariant of the observer.

[38]Weyl refers to metrical geometric to the one that endows the affine space with metrical properties. He speaks in terms of *metrical groundform*, a symmetrical bilinear form. It is nothing more than the scalar product between vectors.

[39]Observe that co-variant components of a vector $\boldsymbol{x}$ are allowed only in metrical space. In affine space any scalar product is defined.

moreover, generalizes this concept speaking of *n-linear form*. A n-linear form can be expressed in the form

$$A(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} \epsilon_{i_1}, \ldots, \epsilon_{i_n}$$

We shall now introduce Weyl's notion of *tensor*. Given a coordinate system $O|\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ let us consider the trilinear form

$$A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sum_{i,k,l} a^{l}_{i,k} \epsilon^i \eta^k \omega_l$$

where ϵ^i, η^k are the contra-variant components of the vectors $\boldsymbol{x}, \boldsymbol{y}$ and ω_l are the co-variant components of the vector $\boldsymbol{z}$. It is called a *doubly co-variant, singly contra-variant tensor of the third degree* if its expression in the coordinate system $\bar{O}|\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \ldots, \bar{\boldsymbol{e}}_n$ is given by[40]

$$A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}) = \sum_{i,k,l} \bar{a}^{l}_{i,k} \bar{\epsilon}^i \bar{\eta}^k \bar{\omega}_l$$

that is

$$\sum_{i,k,l} a^{l}_{i,k} \epsilon^i \eta^k \omega_l = \sum_{i,k,l} \bar{a}^{l}_{i,k} \bar{\epsilon}^i \bar{\eta}^k \bar{\omega}_l$$

The variables ϵ^i, η^k are transformed contra-grediently to the fundamental vectors $\boldsymbol{e}_i$ and the variables ω_l co-grediently to the same. Moreover, the new coefficients $\bar{a}^{l}_{i,k}$ are estimated in terms of $a^{l}_{i,k}$ and a^{k}_{i}. The coefficients $a^{l}_{i,k}$ are called *components of the tensor* in the coordinate system $O|\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Similarly, the coefficients $\bar{a}^{l}_{i,k}$ are called components of the tensor in the coordinate system $\bar{O}|\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \ldots, \bar{\boldsymbol{e}}_n$. Furthermore, they are called *co-variant in the indices i,k* (which are associated with the variables to be transformed contra-grediently) and *contra-variant in the index l* (which are associated with the variables to be transformed co-grediently). Moreover, observe that a tensor is fully known if its component in a coordinate system are given, but it is independent of the coordinate system. That is, it is a function that remains invariant by any coordinate change. It indeed ascribes the same real number to the same vectors in any coordinate system. The change of coordinate system changes only its algebraic formulation.[41] Finally, observe that it is clearly a particular example of tensor. The general case easily follows and Weyl speaks of *r-fold co-variant, s-fold contra-variant tensor of (r+s)th degree*.

A vector can be identified with a contra-variant tensor of the first degree. Given a coordinate system $O|\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$, let consider the components a^i of a vector $\boldsymbol{x}$. The linear form

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[40]The coordinate system obtained by the transformation $\bar{e}_i = \sum_k a^k_i \boldsymbol{e}_k$.

[41]Such a notion of invariance is of fundamental importance within a mathematical investigation of physical reality. Geometrical laws and physical laws indeed has to be understood as invariant of the observer. Weyl's tensor calculus gives a mathematical formulation of this fact.

$$A(\boldsymbol{y}) = a^1\epsilon_1 + \cdots + a^n\epsilon_n$$

having the variables ϵ^i (components of the vector $\boldsymbol{y}$) can be uniquely assigned to the vector $\boldsymbol{x}$. It is a contra-variant tensor of the first degree. For this reason Weyl identifies any vector with its associated tensor:

> From now on we shall no longer use the term "vector" as being synonymous with "displacement" but to signify a "tensor of the first order", so that we shall say, *displacements are contra-variant vectors*. [...] The present use of the word vector agrees with its usual significance which includes not only displacements but also every quantity which, after the choice of an appropriate unit, can be represented uniquely by a displacement.[42]

In the light of these considerations Weyl's notion of infinitesimal displacement can be better understood. First, let observe that there is a fundamental difference between the notion of displacement in affine or metrical geometry and that of infinitesimal displacement within infinitesimal geometry:

> It must, however, be noticed that, here, *a displacement is essentially bound to a point*, and that there is no meaning in saying that the infinitesimal displacements of two different points are the equal or unequal. [...] we cannot talk of a vector or tensor simply, but must talk of a *vector* or *tensor* as being *at a point* P.[43]

Let PP', QQ' be two infinitesimal displacements given by

$$PP' = dx_1\boldsymbol{e}_1 + \cdots + dx_n\boldsymbol{e}_n,$$

$$QQ' = dy_1\boldsymbol{e}_1 + \cdots + dy_n\boldsymbol{e}_n.$$

If we would identify them by their components, i.e.

$$dx_i = dy_i \quad \forall i$$

that doesn't need to be true in another coordinate system. That is, after the transformations[44]

$$dx_i = \sum_{k=1}^{n} a_k^i d\bar{x}_k, \quad dy_i = \sum_{k=1}^{n} a_k^i d\bar{y}_k,$$

[42]Weyl (1952, p. 37).

[43]Weyl (1952, p. 103).

[44]We follow Weyl in using this notation, but observe that there is an abuse of notation. The index i is attached as suffix to dx_i but it actually behave as it were at the top in the summation. I believe Weyl is adopting this abuse since it turns out to be easier to read the sum $\frac{\partial f_i}{\partial \bar{x}_k} d\bar{x}_k$.

it doesn't necessarily follow that $d\bar{x}_i = d\bar{y}_i$ $(\forall i)$. The coefficients a^i_k $(a^i_k = \partial f_i/\partial \bar{x}_k)$ in fact are not constant but they vary depending on the point P. However, the concept of equality between infinitesimal displacements should be understood as invariant of the observer. Therefore the notion of infinitesimal displacement has to be bound to a point. The same holds true for the notion of tensor. Within the context of infinitesimal geometry Weyl then speaks of a *tensor at a point P*. Let be $O|\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ a coordinate system associated to a certain neighbourhood of P. The trilinear form

$$A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sum_{i,k,l} a^l_{i,k} \epsilon^i \eta^k \omega_l$$

is called a *doubly contra-variant, singly co-variant tensor at P of the third degree* if its expression in the coordinate system $\bar{O}|\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \ldots, \bar{\boldsymbol{e}}_n$ is given by[45]

$$A(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}) = \sum_{i,k,l} \bar{a}^l_{i,k} \bar{\epsilon}^i \bar{\eta}^k \bar{\omega}_l$$

that is

$$\sum_{i,k,l} a^l_{i,k} \epsilon^i \eta^k \omega_l = \sum_{i,k,l} \bar{a}^l_{i,k} \bar{\epsilon}^i \bar{\eta}^k \bar{\omega}_l.$$

The variables ϵ^i, η^k are transformed co-grediently to the differential dx_i and the variables ω_l contra-grediently to the same. That is, the following relations hold

$$\epsilon^i = \sum_k a^i_k \bar{\epsilon}^k,$$

$$\eta^i = \sum_k a^i_k \bar{\eta}^k,$$

$$\bar{\omega}_i = \sum_k a^k_i \omega_k.$$

The coefficients $a^l_{i,k}$ are called *components of the tensor at P* in the coordinate system $O|\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ under consideration. Furthermore, they are called *contra-variant in the indices i,k* (which are associated with the variables to be transformed co-grediently) and *co-variant in the index l* (which are associated with the variables to be transformed contra-grediently). Finally, observe that this is clearly a particular example of tensor. The general case easily follows and Weyl speaks of *r-fold contra-variant, s-fold co-variant tensor at P of (r+s)th degree*.

Similarly, the coefficients $\bar{a}^l_{i,k}$ are called components of the tensor at P in the coordinate system $\bar{O}|\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \ldots, \bar{\boldsymbol{e}}_n$. Moreover, the new coefficients $\bar{a}^l_{i,k}$ are calculated in terms of $a^l_{i,k}$ and a^k_i. Moreover, observe that a tensor at P is fully known if its component in a coordinate system are given, but it is independent of the coordinate system. That is, it is a function that remains invariant by any coordinate change. It

[45]The coordinate system obtained by the transformation $\bar{e}_i = \sum_k a^k_i e_k$.

indeed ascribes the same real number to the same vectors in any coordinate system. The change of coordinate system changes only its algebraic expression.

Weyl then identifies an infinitesimal displacement at P with a contra-variant tensor at P of the first degree. Given a coordinate system $O|\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$, let consider the components dx_i of an infinitesimal displacement PP'. The linear form[46]

$$A(\boldsymbol{y}) = dx_1\epsilon^1 + \cdots + dx_n\epsilon^n$$

(where the variables ϵ^i are the components of the vector $\boldsymbol{y}$) can be uniquely assigned to the infinitesimal displacement PP'. It is a contra-variant tensor of the first degree. For this reason Weyl identifies any infinitesimal displacement with its associated tensor:

> Here, again, we shall call tensors of the first order *vectors*. [...] Infinitesimal quantities of this type [contra-variant tensor at P of the first degree] are the line elements in P.[47]

Therefore infinitesimal displacements PP' of P (i.e. line elements at P) are infinitesimal quantities associated with those kind of tensors. The connection is established by means of the components of the infinitesimal displacement that define the linear form.

A fundamental remark is worth mentioning here. Both an infinitesimal displacement and its components are infinitesimal quantities. Nevertheless they are two different entities. The first is a geometric entity, whereas the latter are numerical entities. The connection between infinitesimal displacements and tensors supports our previous remarks that geometry and algebra are strongly interconnected within Weyl's framework.[48] Weyl indeed develops his investigations within infinitesimal geometry using mainly tensor calculus.[49] He usually speaks in terms of components of infinitesimal displacements and not in terms of infinitesimal displacements themselves. For this reason, we can recognize a sort of *algebraic reasoning with infinitesimal quantities*.[50] The following two quotations show such an approach:

[46]We follow again Weyl's notation but observe that it can be misleading. The index i attached to dx_i has to be understood as a proper suffix in this summation. It refers to the sum between the coefficients dx_i and the variables ϵ^i of the linear form. The previous abuse of notation (see footnote 44) instead has to be properly understood referring to the co-gredient relation between the transformation of the variables ϵ^i and the one of the differential dx_i.

[47]Weyl (1952, p. 104). This passage has be to understood in connection with the previous claim within affine (or metrical) geometry. "Again" refers to the fact that also within infinitesimal geometry the term "vector" will usually signify a tensor and not some sort of displacement.

[48]See footnote 34.

[49]The use of tensor calculus becomes possible within infinitesimal geometry only locally, i.e. within a certain neighborhood of P. As Weyl states: "One here uses the exceedingly fruitful mathematical device of making a problem "linear" by reverting to infinitesimal small quantities. The whole of *Tensor Algebra*, by whose operations only tensors *at the same point* are associated, *can now be taken over from Chapter I*" (Weyl 1952, p. 104).

[50]The same view is shared by Laugwitz that declares: "*Raum-Zeit-Materie* [...] is a perfect example of infinitesimal mathematics in action" (Laugwitz 1986, 241, *my translation*).

> We perform the following construction. From the point $P = P_{00}$ we draw two line elements with components dx_i and δx_i, which lead to the two infinitely near points P_{10} and P_{01}. [...] let the components dx_i have increased by δdx_i, so that
>
> $$\delta dx_i = x_i(P_{11} - x_i(P_{01}) - x_i(P_{10} - x_i(P_{00})$$
>
> [...] we get, in particular [...] $(\frac{\partial f_i}{\partial x_k} - \frac{\partial f_k}{\partial x_i})dx_i \delta x_k$.[51]

> Let the vector $\boldsymbol{x} = \boldsymbol{x}(P_{00})$ with components ϵ^i be given at the point P_{00}. The vector $\boldsymbol{x}(P_{10})$ that is derived from $\boldsymbol{x}(P_{00})$ by parallel displacement along the line element dx is attached to the end point P_{10} of the same line element. If the components of $\boldsymbol{x}(P_{10})$ are $\epsilon^i + d\epsilon^i$ then $d\epsilon^\alpha = -d\gamma^\alpha_\beta \epsilon^\beta = -\Gamma^\alpha_{\beta i}\epsilon^\beta dx_i$.[52]

Weyl's approach then seems to be a proper example of "infinitesimal mathematical reasoning in action" and it has to be understood as a proper mathematical account sui generis.[53]

A question then arises: how is that possible? As we have remarked above, Weyl's commitment to any sort of infinitesimal quantity seems to be odd at that time. It might be the case that Weyl's approach is unorthodox with respect to his time. There are good reasons to follow this path. Weyl has a novel approach both to mathematical and physical investigations. Nevertheless it does not explain why he never emphasises too much this issue. He adopts such an approach, but he does not seem to be much worried about any strong justification of it. Although Weyl's insights may be very concise, and sometimes unclear, he always attempts to make explicit his underlying reasons. This issue regarding the use of infinitesimals instead seems to be accepted without too much concerns. Therefore, there are two possible cases. Or our interpretation is misleading or this situation can be understood within the historical context of that time. Indeed, if such an approach was not so weird at that time then Weyl's lack of emphasis on this issue would become more comprehensible. A mathematician indeed lays out the results of his research by addressing to his contemporary. He manages which issues need a closer attention and which do not according to the mathematical audience he is addressing his works. Therefore, if Weyl's mathematical audience was not so hostile to such an use of infinitesimals then our reconstruction turns out to be feasible. In the following section we are going to show that this was the case. Weyl's audience, at least part of it, was not so hostile to such an approach. In fact, several mathematicians were dealing with infinitesimal quantities in their work, especially within the context of the arising modern algebra. Although a proper account of this kind of mathematics was still lacking, many studies were aiming at a

[51] Weyl (1952, p. 107).

[52] Weyl (1952, p. 119). I just aim to show Weyl's reasoning with infinitesimals. A full mathematical understanding of these quotations is not my goal.

[53] As we have shown above, however, Weyl embraces this approach but he seems to believe that further investigation are needed.

proper justification of it. For these reasons, Weyl's approach was not unusual within this context.

2.3 Setting the Historical Context

A remarkable paper in 2006 by Philip Ehrlich has brought out that the commonplace view about infinitesimals within mathematics was misleading. In his *The Rise of non-Archimedean Mathematics and the Roots of a Misconception I: The Emergence of non-Archimedean Systems of Magnitudes*, we can read the following remark:

> Having accepted along with Russell that infinitesimals had indeed been shown to be incoherent, and that (with the possible exception of constructivist alternatives) the nature of the infinite and the continuum had been essentially laid bear by Cantor and Dedekind, following the development of nonstandard analysis in 1961, a good number of historians and philosophers of mathematics (as well as a number of mathematicians and logicians) readily embraced the now commonplace view that is typified the following remarks:
>
> > In the nineteenth century infinitesimals were driven out of mathematics once and for all, or so it seemed. [P. Davis and R. Hersh 1972, p. 78]
>
> [...] What is not so well know in these communities, however, is that *whereas most late nineteenth- and pre-Robinsonian twentieth-century mathematicians banished infinitesimal from the calculus, they by no means banished them from mathematics*. [...] between the early 1870s and the appearance of Abraham Robinson's work on nonstandard analysis in 1961 there emerged a large, diverse, technically deep and philosophically pregnant body of consistent non-Archimedean mathematics of the (non-Cantorian) infinitesimally large and the infinitesimally small.[54]

Mathematical investigations on infinitesimal quantities were not so atypical at that time. Indeed, if we look at mathematical works that were not within the domain of calculus we discover that infinitesimals were not banished at all. In the second half of the 19th century calculus had become the target of many criticisms. Any appeal to infinitesimal quantities began to be considered with mistrust and more generally it started to be questioned to what extent our understanding of calculus should rely on our geometrical intuition. The arithmetization of analysis was an attempt to deal with these issues. Thanks to the works of Weierstrass, Cantor and Dedekind and their theories of real numbers calculus became independent of any notion of infinitesimal quantity. Especially, Weierstrass's fundamental $(\epsilon - \delta)-$definition of limit based on a set theoretic framework represented a landmark in the history of calculus. Cantor and Dedekind helped to develop an arithmetical account of continuity based on a

[54] Ehrlich (2006, p. 2).

rigorous definition of real number. From then on mathematicians started to embrace this new approach on calculus and infinitesimal quantities were considered imprecise and unreliable notions. Nevertheless it wasn't the case with respect other branches of mathematics. First of all, not all mathematicians agreed with the arising new conceptions of real number. For instance, in 1867 Hankel expressed his disappointment in response to Weierstrass's distinction between a formal notion of real number and the notion of geometrical magnitude:

> Every attempt to treat the irrational numbers formally and without the concept of magnitude must lead to the most abstruse and troublesome artificialities, which, even if they can be carried through with complete rigor, as we have every right to doubt, do not have a higher scientific value.[55]

Few years later du Bois-Reymond and Otto Stolz developed non-Archimedean magnitude systems that Robinson described as:

> a modest but rigorous theory of non-Archimedean systems.[56]

These number systems acquired large audience with the publication of Stolz's textbook *Vorlesungen über Allgemeine Arithmetik*.[57] Systems of magnitudes that are non-Archimedean were presented here as alternative systems to those Archimedean systems of magnitudes that are employed within calculus. Although they were just presented as alternative systems and they have little to do with calculus, these studies laid the groundwork for the late 19th century pioneering investigations of *non-Archimedean Größensysteme* that have led to the 20*th century theory of ordered algebraic systems*. Among them Stolz (1883, 1884, 1885, 1891), Schur (1899), Bettazzi (1890), **veronese1889:grandezze**; **veronose1891:fondamenti**; Veronese (1894) and Levi-Civita (1898, 1892-93). In particular the pioneering work of Veronese *Fondamenti di Geometria* published in 1891 attempts to develop a non-Archimedean ordered field of line segments. His proposal is often obscure and badly formulated. Nevertheless, many insights will became central concepts in the modern theory of ordered algebraic systems. In the following years Veronese had to support his account against several criticisms by authors such as Cantor, Peano, Vivanti, Killing, klein and Schoenflies. The publication in 1892-93 of Levi-Civita's paper *Sugli Infiniti ed Infinitesimi Attuali quali Elementi Analitici* helped Veronese's body of works to be better accepted. Anyway, it was only after the publication of Hilbert's *Grundlagen der Geometrie* in 1899, and later in 1907 of Hahn's *Über die Nichtarchimedischen Größensysteme*, that non-Archimedean theories of magnitudes began to be regarded as a legitimate field of research. Hilbert's revolutionary publication is well-know for its innovative contributions in the foundational investigations of mathematics at the beginning of 20th century. However, it is

[55]Hankel (1867, p. 46).
[56]Robinson (1967, p. 39).
[57]Stolz (1885).

much less know for its contribution to non-Archimedean geometry. In 1902 Poincaré reviewed Hilbert's work and in this occasion he affirms:

> But the most original conception of Professor Hilbert is that of non-Archimedean geometry, in which all the axioms remain true except that of Archimedes. For this it was necessary, in the first place, to construct a *system of non-Archimedean numbers* [...] Our ordinary numbers come in as particular cases among these *non-Archimedean numbers*.[58]

We have then pointed out that mathematical studies dealing with infinitesimal quantities were still alive at that time.[59] In the following I will focus on those mathematicians that had an influence on Weyl. Among them we can find Poincaré, Hilbert, Levi-Civita and others.

Hilbert had definitely a strong influence on Weyl. In 1904 Weyl moved to Göttingen to study mathematics and physics and in 1908, under Hilbert's supervision, he received his doctorate. At that time Hilbert, Minkowski and Klein were leading mathematicians at this university. Felix Klein in particular promoted the cultural development of this university in many ways. He introduced weekly discussion meetings and several other activities. Under his editorship *Mathematische Annalen* became one of the most attractive mathematical journals in the world. The Göttingen's reputation as one of the best research facilities throughout the world in Europe at the beginning of the 20th century is surely due also to Klein. Weyl followed several lectures and recalling this exciting period, in 1944 Weyl writes:

> Hilbert and Minkowski were the real heroes of the great and brilliant period, unforgettable to those who lived through it, which mathematics experienced during the first decade of this century in Göttingen. Klein ruled over it like a distant god, *divus Felix*, from above the clouds; the peak of his mathematical productivity lay behind him.[60]

In *Philosophy of Mathematics and Natural Science* Weyl points out that though the notion of infinitesimal quantity is vague, a non-Archimedean theory of quantities is logically sustainable. With this respect he quotes Hilbert:

> As a matter of fact, it is not impossible to build up a consistent "non-Archimedean" theory of quantities in which the axiom of Eudoxus (usually named after Archimedes) does not hold [and in a footnote] Compare, for instance, Hilbert, *Grundlagen der Geometrie*, Chapter II, 12.[61]

[58]Poincaré (1902, pp. 10,11).

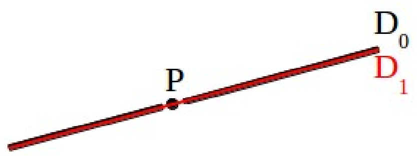
[59]A detailed account of these studies between the late 19th century and the beginning of the 20th century can be found in the above mentioned paper of Ehrlich (2006) and also in Ehrlich (1994a).

[60]Weyl (2012, p. 85).

[61]Weyl (1949, p. 45). In this occasion Weyl quotes also Leibniz and Wallis's notion of *anguli contactus*, i.e. the angle between a circle and its tangent. Weyl seems to think of it as a good approach to infinitesimal quantities.

As Poincaré points out, Hilbert's *Grundlagen der Geometrie* represents an outstanding work in several respects. In no small part the proof of consistency of a non-Archimedean geometry represents one the most significant achievements of that time. In the system of non-Archimedean numbers that Hilbert proposes to model a non-Archimedean geometry are defined relations of equality and inequality as well operations of addition and multiplication. The standard laws are satisfied except for the axiom of Archimedes. Poincaré imagines a space of three dimensions in which the co-ordinates of a point would be measured by non-Archimedean numbers while the usual analytic expressions would hold. In this space we face a particular situation. Let D_0 be an ordinary straight line and D_1 be the corresponding non-Archimedean straight line.[62] Let further P be an ordinary point on D_0 that divides it in two half-rays S and S' but it does not belong to any of them. Then Poincaré concludes:

> Then there will be on D_1 an infinity of new points as well between P and S as between P and S'. [...] In short, our ordinary space is only a part of the non-Archimedean space.[63]



Poincaré then seems to suggest that we may use infinitesimal quantities to identify infinitesimal neighborhood around a point in the space.[64] Regardless whether it was the case or not, from the publication of the *Grundlagen* the logical possibility of a non-Archimedean theories of quantities began to be accepted among mathematicians. At least, it was the case among mathematicians working in Göttingen where Hilbert was teaching.

Another important mathematician was surely Tullio Levi-Civita. In 1901 he and Gregorio Ricci-Curbastro published *Méthodes de calcul différentiel absolu et leurs applications* on the invitation of Felix Klein.[65] This paper develops the arising theory of tensor calculus and it would constitute the main reference upon which Weyl will develop his tensor calculus in *Raum-Zeit-Materie*. Weyl explicitly recalls it in a note:

> The systematic form which we have here given to the tensor calculus is derived essentially from Ricci and Levi-Civita: Méthodes de calcul différentiel absolu et leurs applications, Math. Ann., Bd. 54 (1901).[66]

[62]Each point on D_0 is represented by the usual co-ordinates whereas each point on D_1 is represented by the non-Archimedean co-ordinates. The two straight lines anyway are the same geometrical line.

[63]Poincaré (1902, p. 11). The picture shows the situation, but of course it can't be completely faithful.

[64]Observe that Weyl was a serious reader of Poincaré as we will see later. Poincaré's *analysis situs*, for instance, represents a fundamental starting point for Weyl's research in combinatorial topology.

[65]Ricci and Levi-Civita (1901).

[66]Weyl (1952, p. 53).

Levi-Civita's notion of infinitesimal parallel displacement is welcomed by Weyl as a fundamental step in a better mathematical understanding of the real world. In the preface of the completely revised third edition of *Raum-Zeit-Materie* he declares:

> I have now undertaken a thorough revision which affects Chapters II and IV above all. The discovery by Levi-Civita, in 1917, of the conception of infinitesimal parallel displacements suggested a renewed examination of the mathematical foundation of Riemann's geometry.[67]

The importance of infinitesimal parallel displacement within Weyl's pure infinitesimal geometry is not our concerns here.[68] Levi-Civita's foundational studies instead have to be taken into consideration. In his paper *Sugli Infiniti ed Infinitesimi Attuali quali Elementi Analitici* he considers the pioneering work of his teacher Giuseppe Veronese and he attempts to reformulate Veronese's insights from an analytic point of view:

> Prof. Veronese, in his masterwork *Fondamenti di Geometria a più Dimensioni e a più Specie di Unità Rettilinee Esposti in Forma Elementare* (Padova, Tip. del Seminario, 1891) [...] being led by those studies to discuss and reform the principles of all geometry, he brought [...] new and fruitful views. Among these, for our purpose, we will just mention the abstract possibility of infinite and limited infinitesimal segments and the consequent admissibility of new segments, even infinite or infinitesimal, which behave with respect to the former like those ones behave with respect to the finite ones (infinitives and infinitesimals of several orders). Hence, in order to represent these entities, he ascribes to them some numbers and establishes their fundamental operations; these numbers, however, can not take advantage of the analytical tools [...], so it is not entirely inappropriate to present this very subject from an absolutely analytical point of view.[69]

As we have observed above this paper helped Veronese's investigations to be better accepted. A special attention should be given to Veronese's notion of *infinitesimal field* within a non-Archimedean Riemannian space. Let consider a point P and two line segments PP' and PQ such that the first is infinitesimal to the second.[70] Let G the

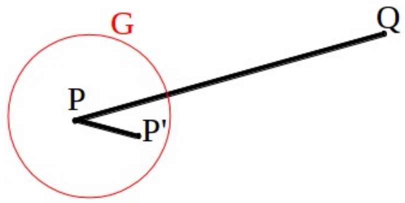
[67]Weyl (1952, p. x).

[68]This notion will play a crucial role in establishing an affine relationship within the manifold. From a physical point of view it allows us to establish a field of gravitation.

[69]"Il chiar.mo Prof. Veronese, nella sua opera magistrale *Fondamenti di Geometria a più Dimensioni e a più Specie di Unità Rettilinee Esposti in Forma Elementare* (Padova, Tip. del Seminario, 1891) [...] essendo condotto da quegli studii a discutere e a riformare i principii di tutta la geometria, apportò [...] vedute nuove e feconde. Tra queste, per lo scopo nostro, ci limiteremo ad accennare la possibilità astratta di segmenti infiniti ed infinitesimi limitati e la conseguente ammissibilità di nuovi segmenti, pur essi infiniti od infinitesimi, che, di fronte ai primi, si comportino come quelli di fronte ai finiti (infiniti ed infinitesimi dei vari ordini). Quindi, per rappresentare questi enti così introdotti, egli coordina loro dei numeri e ne stabilisce le operazioni fondamentali; codesti numeri tuttavia non possono adoperarsi con vantaggio quali strumenti analitici [...] pertanto non del tutto inopportuno di presentare questo stesso soggetto da un punto di vista assolutamente analitico" (Levi-Civita 1892-93, 1, *my translation*). Levi-Civita will further develop these studies in Levi-Civita (1898).

[70]That is nPP' is smaller than PQ for any positive integers n.

region of the space which contains all and only those line segments PP'' that are finite relative to PP'.[71] Then G is an infinitesimal field surrounding P.



In his later work *Sulla Geometria non-Archimedea* Veronese recalls these studies:

> In this geometry [non-Archimedean Riemannian geometry], *in an infinitesimal field surrounding a point, if we consider only segments which are finite relative to one another or which mutually satisfy the Archimedean axiom, then Euclidean geometry holds*. This theorem has since been proved by Levi-Civita for the non-Archimedean geometry of Euclid and Bolyai-Lobatchevsky.[72]

Therefore both Veronese and Levi-Civita seem to accept as logically sound a mathematical account of infinitesimal neighborhood involving infinitesimal quantities. In this way they aim to establish a local Euclidean geometry within a non-Euclidean geometry. Then the use of infinitesimal quantities within this context seems to be a viable option. The arising Riemann's approach to geometry indeed seems to require a new completely approach to the subject matter, as Ricci observes:

> The applications of Differential Calculus in geometry have recently developed to such an extent that they constitute a vast scientific domain by themselves [...] it seems to be the case that Infinitesimal Geometry, constituted in organic unity, proceeds in its own way.[73]

The historical context that we have outlined above shows that Weyl's use of infinitesimal quantities was not so odd. Indeed, mathematical accounts involving infinitesimal quantities were still far from being banished from mathematics. Several mathematicians, well-known to Weyl, were dealing with this kind of mathematics. Although the use of infinitesimal quantities was not without problems, many research studies were undertaken towards a resolution of those problems. Therefore, Weyl's remarks on infinitesimal quantities can be better understood within this context. As we have seen Weyl's approach has to be regarded as a work in need of improvement and that is in

[71] That is, for each such PP'' there are positive integers m and n such that nPP'' is longer than PP' and mPP' is longer than PP''.

[72] Veronese (1994, p. 172). The proof that Veronese is referring is contained in the above-mentioned paper of Levi-Civita.

[73] "Le applicazioni geometriche del Calcolo Differenziale hanno recentemente assunto un tale sviluppo da costituire da sole un vastissimo dominio scientifico [...] sembra mettere in maggior evidenza la opportunità che la *Geometria Infinitesimale* costituita in unità organica proceda oramai per vie sue proprie" (Ricci 1898, 1,2, *my translation*).

agreement with all other works at that time.[74] In the following section I will focus on this very issue. I will show that previous phenomenological investigations provide the guidelines according to which Weyl aims to improve these studies.

2.4 A Phenomenological Foundation

Previous phenomenological considerations provide the main guidelines according to which Weyl attempts to improve these studies. Specifically, Weyl's investigations on the relations between descriptive analysis of phenomena and its exact determination have to be connected with Weyl's attempt to provide a consistent infinitesimal geometry based on an adequate approach to the continuum. In order to point out this connection we need first to clarify in which sense Weyl's studies have to be considered as works in need of improvement. While the mathematical approach involved in the first two editions of *Raum-Zeit-Materie* was not regarded as completely satisfactory by the author, the third edition instead appears to him good enough. This edition was strongly influenced by the previous works of Levi-Civita and Hessenberg.[75] In the preface of the third edition Weyl declares:

> The development of pure infinitesimal geometry in Chapter II, in which every step follows quite naturally, clearly, and necessarily, from the preceding one, is, I believe, the final result of this investigation as far as the essentials are concerned.[76]

In his paper *Reine Infinitesimalgeometrie*, published shortly before the third edition of *Raum-Zeit-Materie*, he speaks of three levels of analysis:

> The construction of the geometry of proximity [Nahegeometrie] goes through three levels. At the first level lies the *continuum* in the sense of Analysis situs, free of any metric determination; in physical terms, *the empty world*. At the second level lies the *affine connection* continuum [...] in physics, the affine connection is presented as a *field of gravitation* [...] At

[74]To be clear, at that time foundational issues within mathematics constituted a field of leading researches and a balance between current mathematical investigations and foundational issues was not an exception. For instance, several mathematicians would have agreed that a study within calculus at the beginning of the twentieth-century was lacking a satisfactory understanding of the concept of number. Nevertheless such a study was considered good enough within the research field of calculus. Similar considerations can actually be done for any research in any time. At that time anyway there was not neither the "right answer" nor the "common answer". A set theoretic approach was not universally accepted and several issues concerning the nature of mathematical objects were still opened and vividly debated.

[75]Levi-Civita (1917) and Hessenberg (1917).

[76]Weyl (1952, p. x).

> the third level, finally, lies the *metric* continuum - physically: the "*ether*", whose states are expressed by material and electrical phenomena.[77]

There are many aspects regarding this quotation that should be considered.[78] Nevertheless we are going to focus only on those aspects that are connected with the issue of continuity. On this very point we will shed light on the connections we are searching. Specifically, the issue of continuity is addressed in the first level. This stage concerns the "empty world", i.e. a 4-dimensional continuum deprived of any further structure. On this level then the mathematical analysis of continuum, which has to be considered *in the sense of Analysis situs*, plays a fundamental role. Weyl seems to be generally satisfied with his account of infinitesimal geometry, as we have pointed out. Nevertheless he expresses repeatedly the need for further investigations about this level of analysis. In *Reine Infinitesimalgeometrie* he writes:

> Because of the difficulties to grasp the intuitive nature [anschauliche Wesen] of the continuous connection by means of a purely logical construction, a fully satisfactory analysis of the concept of *n-dimensional manifold* is not possible today.[79]

And again in *Raum-Zeit-Materie*:

> We now pass on to the systematic development of pure infinitesimal geometry, which will be traced through three stages; from the *continuum*, which eludes closer definition, by way of *affinely connected manifolds*, to *metrical space*.[80]

In both cases Weyl clearly remarks that a mathematical analysis of the continuous connectedness is still far from satisfactory. Despite of these remarks he fully carries out his infinitesimal geometry. For these reasons we should understand these remarks only as an indication of those aspects that should be improved with further investigations. This interpretation is also supported by various other remarks contained in *Raum-Zeit-Materie*. For instance, in the first chapter Weyl deals with the mathematical formulation of Euclidean space. Formulating the axioms that rule affine geometry he states that it might be possible to deduce the axioms of multiplication from the axioms of addition as long as we refer to rational numbers. In the case of real numbers instead this wouldn't be easy. For this reason Weyl formulates them as separate axioms. This approach prevent us to deal with the challenging issue of continuity:

[77] "Der Aufbau der Nahegeometrie vollzieht sich sachgemäß in drei Stufen. Auf der ersten Stufe steht das aller Maßbestimmung bare *Kontinuum* im Sinne der Analysis situs - physikalisch gesprochen, *die leere Welt*; auf der zweiten das *affin zusammenhängende* Kontinuum [...] in der Physik erscheint der affine Zusammenhang als *Gravitationsfeld* [...] auf der dritten endlich das *metrische* Kontinuum - physikalisch: *der "Äther"*, dessen Zustände sich in den Erscheinungen der Materie und Elektrizität kundgeben" (Weyl 1918b, 385, *my translation*).

[78] For a French translation of this paper that also includes other important Weyl's writings and several historical remarks, see Chorlay (2015).

[79] "Infolge der Schwierigkeit, das anschauliche Wesen des stetigen Zusammenhangs durch eine rein logische Konstruktion zu erfassen, ist eine voll befriedigende Analyse des Begriffs der *n-dimensionalen Mannigfaltigkeit* heute nicht möglich" (Weyl 1918b, 386, *my translation*).

[80] Weyl (1952, p. 102).

> By refraining from reducing multiplication to addition we are enabled through these axioms to banish continuity, which is so difficult to fix precisely, from the logical structure of geometry.[81]

Therefore this axiomatic approach allows us to proceed even if we do not have a satisfactory understanding of continuity. Deferring this issue to future researches is pointed out also in the previous passage:

> We fix an exact "here", a point in space, as the first element of continuous spatial extension, which, like time, is infinitely divisible. Space is not a one-dimensional continuum like time. The principle by which it is continuously extended cannot be reduced to the simple relation of "earlier" or "later". We shall refrain from inquiring what relations enable us to grasp this continuity conceptually.[82]

The analysis of continuous spatial extension is then regarded as more complex than the one-dimensional continuum of time. Shortly before Weyl suggests a mathematical analysis of temporal continuum. Although he is not completely satisfied with this sort of analysis, he outlines anyway the main relations between idealized time-points.[83] Grasping the main relations concerning the spatial continuum instead seems to be harder. For this reason he refrains from such kind of inquiries. Nevertheless he continues to develop his account of affine geometry. Again we see that Weyl's critical remarks on this issue have to be understood only as an indication for further investigations.

We have then shown that the issue of continuity has to be considered one of the main aspects that should be improved. According to Weyl indeed a proper mathematical analysis of continuum is still lacking. Moreover, as we have seen above, Weyl's studies involving the notion of infinitesimally small have to be considered as an attempt to deal with the problematic nature of the continuum. Since the infinitesimally small involves infinitesimal quantities, that seems to be in agreement with Weyl's previous remarks. That is, the use of infinitesimal quantities has to be considered as a mathematical approach that should be improved in some way.[84]

Now I will shed light on the guidelines that Weyl seems to follow in order to improve these works. As we have seen in the previous chapter Weyl's investigations on the continuum finds their roots within a phenomenological framework. They attempt to establish a connection as best as they can between descriptive analysis of phenomena and its exact determination. Specifically, these studies have to interpreted as attempts to substitute morphological essences with ideal essences, by means of some idealizing procedure intuitively ascertained, which allow us to grasp those abstract schemata that underlie what is immediately given. This process allow us to

[81]Weyl (1952, p. 17).

[82]Weyl (1952, p. 11).

[83]In *Das Kontinuum* he deals extensively with this issue, as we have seen in the previous chapter.

[84]To be clear, I am not arguing that the issue of continuity and that of the use of infinitesimal quantities are equivalent. I'm just pointing out that they seems to be strongly connected.

grasp only some features of the complexity of the phenomenal continuum. A sort of tricky "Logos" rules the nature of phenomenal continuum:

> There is more at work here [phenomenal continuum] than heavy-handed schematizing or cognitive economizing devised for fulfilling our practical tasks and objectives. Here we discover genuine reasons which lays bare the "Logos" dwelling the reality.[85]

Nevertheless the abstract schemata provided by our mathematical conceptualization aim to represent as best as they can the exact science of domains of objects involved in the phenomenal continuum:

> Those abstract schemata supplied us by mathematics must underlie the exact science of domains of objects in which continua play a role.[86]

The mathematical analysis of the continuum in *Raum-Zeit-Materie* seems to follow that pattern. As we have seen in the previous chapter Weyl aims to establish a connection between the real numbers and the phenomenal continuum given the assumption that the phenomenal continuum can be described as an infinite collection of *isolated points*.[87] The structural properties of real numbers then reflects those intuitively given relations revealed by the descriptive analysis of the continuum. Specifically, Weyl refers to the real numbers with respect to temporal continuum and to real number triads with respect to the spatial continuum. This is the case also in *Raum-Zeit-Materie* where the space is regarded as an n-dimensional continuum manifold. The connection between this continuum and real numbers is given by the notion of *local co-ordinates*:

> It [spatial continuum] may accordingly be referred to n-coordinates $x_1, x_2, \ldots, x_n$ of which each has a definite numerical value at each point of the manifold; different value-systems of the co-ordinates correspond to different points.[88]

These co-ordinates are continuous function defined only locally on the manifold. That is, if P is an arbitrary point of the manifold then there exists a certain domain surrounding the point P that must be representable singly and reversible by the value-system of n-coordinates. Investigations on local aspects generally characterizes Weyl's approach in *Raum-Zeit-Materie*.[89] One of main reasons concerns the epistemological value of a

[85] Weyl (1994, p. 93).

[86] Weyl (1994, p. 108).

[87] Weyl will express his disappointment with respect to this assumption in several occasions. Recall anyway that such a disappointment is not an issue within Weyl's view. As we have shown his phenomenological research aim to establish the best possible mathematical conceptualization but it is always a matter of degree of approximation. We will come back on it later.

[88] Weyl (1952, p. 103). Recall that Weyl is approaching the subject matter leaving the value of n undetermined developing his infinitesimal geometry in the most general case. See footnote 3.

[89] A detailed analysis on this issue is elaborated in Chorlay (2007, 2015).

geometry that gains knowledge of the real world from the behaviour of its infinitesimal parts. As we have observed any mathematical analysis in the infinitesimally small reflects our understanding of the nature of the continuum. Actually, Riemann's geometry had provided the main essential innovations with this respect:

> Riemann's geometry is Euclidean geometry formulated to meet the requirements of continuity.[90]

In this way then we have established an isomorphism between spatial continuum and the domain of real numbers. That allows us to deal with the relations pertaining to the domain of this continuum by means of the relations pertaining to the real numbers.[91]

Weyl refers to real numbers or any other numerical entities with the notion of *quantity* (Größe). On this very notion we should shed some light. In fact, it is very crucial within Weyl's framework. For instance, Weyl speaks in terms of *physical quantities* that vary continuously in space and are expressed as a function of the coordinates.[92] Moreover Weyl refers to the coordinates as *quantities ascribed to a certain point*:

> The characteristic of an n-dimensional manifold is that each of the elements composing it [...] may be specified by the giving of n quantities, the "coordinates", which are continuous functions within the manifold.[93]

Therefore we could say that our understanding of the real world is determined by the relations between all these quantities. The algebraic structure of these quantities then plays a fundamental role in revealing the abstract schemata underlying the real word.[94] The development of the whole tensor calculus follows this very approach. In fact, the notion of tensor is developed with the aim of representing a geometrical or physical quantity from a purely algebraic point of view:

> A quantity in geometry and physics will be called a tensor if it defines uniquely a linear algebraic form depending on the co-ordinate system in the manner described above; and conversely the tensor is fully characterized if this form is given.[95]

Weyl's use of infinitesimal quantities seems to be better understood within this framework. Once we have established an isomorphism between the spatial continuum

[90]Weyl (1952, p. 91). Riemann inaugurated a new approach able to afford a better understanding of real world with his inaugural lecture *Über die Hypothesen, welche der Geometrie zu Grunde liegen*. See Riemann (2013). For an English translation, see Riemann (2016).

[91]See previous chapter for further details.

[92]Weyl (1931, p. 49).

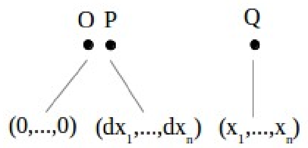
[93]Weyl (1952, p. 84).

[94]Within the contest of Euclidean geometry Weyl clearly expresses this view: "the whole of affine geometry merely teaches us that space is *a region of three dimensions in linear quantities* (ein dreidimensionales lineares Größengebiet)" (Weyl 1952, p. 25). He uses the German term "Größengebiet" that we may translate also as field of quantities. Therefore Euclidean space is a field of quantities which are linearly related.

[95]Weyl (1952, p. 37). Recall that Weyl seems to subsume all type of quantities under a universal concept of number. See footnote 34.

and this domain of quantities then the following problem arises. Given three points O, P, Q of the manifold, let's consider the case where P is infinitely near to the point O and Q is at a finite distance from the point O. Then we can ask ourselves which quantities can properly represent this state of affair.

If we conceptualize P by means of n infinitesimal quantities and Q by n finite quantities, then this state of affair within the phenomenal continuum seems to be well represented by the corresponding state of affair within the domain of real numbers.[96]



Weyl did not express such a view, but all we have said so far seems to support this interpretation or at least a similar one.[97] Of course, if it is the case, then we should be able to develop a coherent non-Archimedean field of quantities. As we have seen in the previous section that was the goal of many studies at that time. Weyl himself seemed to be inclined to this idea. He clearly asserts:

> As a matter of fact, it is not impossible to build up a consistent 'non-Archimedean' theory of quantities in which the axioms of Eudoxus (usually named after Archimedes) does not hold.[98]

The main problems of dealing with infinitesimal quantities arise when we use them within the context of calculus, but this is not the case. Our aim here is just to identify a domain of quantities that is a proper mathematical conceptualization of the spatial continuum. The existence of certain physical domains to which we can ascribe a peculiar field of quantities that cannot be identified with the usual number system is clearly supported by Weyl. In a lecture given in 1931 he remarks:

> We revert to the Greek viewpoint that every subject has an associated intrinsic number realm that must be derived from within it. We experience this reversal not only in geometry but also in the new quantum physics. According to quantum physics, the physical magnitudes associated with a particular physical setup [...] admit of an addition and a non-commutative

[96] I have fixed the coordinate system center at O for the sake of simplicity. The general case follows immediately.

[97] At least as far as I know. There might exist other unpublished material where Weyl shed light on this issue.

[98] Weyl (1949, p. 45).

> multiplication, and thus give rise to *a system of algebraic magnitudes intrinsic to it that cannot be viewed as a sector of the system of real numbers*.[99]

This approach might reveal some problems when it has to deal with infinitesimal geometry fully developed also with respect to global aspects. Nevertheless, as we have observed *Raum-Zeit-Materie* is mainly focused on local aspects and these problems might have been the subject matter for future development.[100] Anyway, even if it is the case our previous considerations have shown that Weyl himself regards his studies as works in need of improvement. Then it does not seem to be a serious challenge for our interpretation.

Despite of these considerations Weyl's later works do not seem to follow this non-Archimedean path involving infinitesimal quantities. We do not find any relevant work that goes in such a direction although a better systematization of this approach seems to be in agreement with Weyl's view.[101] There might be several reasons but the most significant reason find its root in his phenomenological attitude. Indeed, there are few relevant issues that lead Weyl to embrace a different path. They concern the first steps involved in a mathematical conceptualization of the spatial continuum. That is, the process of substitution of morphological essences with exact essences by means of some idealizing procedure intuitively ascertained. As we have seen the previous mathematical conceptualization underlies an important assumption. It is possible to describe the spatial continuum as an infinite collection of isolated points. This very assumption is strongly criticized by Weyl in several occasions. For instance, in *Riemanns geometrische Ideen, ihre Auswirkung und ihre Verknüpfung mit der Gruppentheorie* referring to such an assumption he declares:

> It thereby violates the essence of continuum, which *by its very nature can not be shattered into a multitude of individual elements*. It is not the relation of the element to the set, but that of the part to the whole, that should underlie the analysis of the continuum.[102]

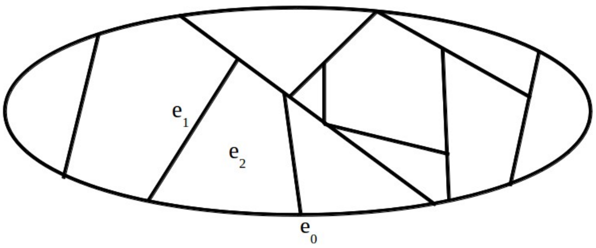
[99]Weyl (2012, p. 38).

[100]A closer analysis of *Die Idee der Riemannschen Fläche* might shed light on this issue. In this book Weyl attempts to handle both local and global properties on a Riemann surface. See Weyl (1913).

[101]An attempt toward such a direction might be realized within the framework of modern nonstandard analysis. In a recent work van Atten explores some approaches to nonstandard analysis and he makes a few suggestions on how Weyl's infinitesimal geometry could be reconstructed within the framework of nonstandard analysis. See Van Atten (2018). Another path might be to reconstruct Weyl's research within the framework of synthetic differential geometry. This kind of geometry proposes a formalization of differential geometry in the language of topos theory. Synthetic differential geometry, apart from being intrinsically of mathematical interest, provides a rigorous understanding of the vague notion of infinitesimal displacement. For this reason, it might be useful to clarify Weyl's infinitesimal approach. See, for instance, Lavendhomme (1996) and Kock (2006). See also Ehrlich (2018). I remain neutral to what extent nonstandard analysis or synthetic differential geometry may be adequate to improve Weyl's studies. I just observe that they seem to be at least a better attempt to reconstruct Weyl's mathematical works with respect to the usual differential geometry.

[102]"Sie dadurch gegen das Wesen des Kontinuums verstößt, als welches seiner Natur nach gar nicht in eine Menge einzelner Elemente zerschlagen werden kann. Nicht das Verhältnis von Element zur Menge, sondern dasjenige des Teiles zum Ganzen sollte der Analyse des Kontinuums zugrunde gelegt werden" (Weyl 1988, 5, *my emphasis*, *my translation*).

For this reason any improvement of the previous works should take into account this fact. It has to identify those idealizing procedures that can be intuitively ascertained more adequately. Weyl's research works on *combinatorial topology* follow this very path.[103] In 1921 Weyl published an important paper on foundational issues in mathematics.[104] In *Über die neue Grundlagenkrise der Mathematik* he emphasises on the lack of foundation upon which rests the current mathematics. Following Brouwer's ideas, he then attempts to offer some different approaches to the concepts of real number and the continuum. In the last pages of this polemical paper he stresses on the need of a different mathematical treatment of the continuum within the context of a two-dimensional continuous manifold. He first of all formulates the schema S of its *topological structure*. It consists of finitely many *corners* e_0 (elements of level 0), *edges* e_1 (elements of level 1) and *surface pieces* e_2 (elements of level 2).



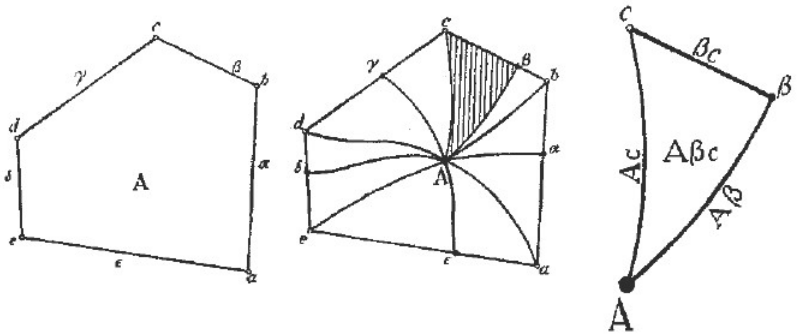
Few basic properties can be established. Each surface is limited by certain edges and each edges by certain corners. These details represent the content of the schema S, named also the *topological framework* of the manifold. It has clearly to satisfy certain conditions.[105] Weyl then outlines a *process of division* by dividing each edge into two edges by means of one of their points. Analogously, each surface piece is divided into triangles by means of lines from a center, arbitrarily chosen within it, to the corners associated to the surface piece. The picture below shows an example focusing on one surface piece. In this case the surface piece is a pentagon and it is shown the first step of the process of division from S to S'.[106]

[103]Recall Weyl's remarks concerning the goal of a rational analysis of continua: "the rational analysis of continua proceeds in three steps: (1) morphology [...] (2) topology [...] (3) geometry [...] The three steps described reveal the sensual-categorial ambivalence of geometry [...] For a more careful phenomenological analysis of the contrast between vagueness and exactness and of the limit concept, the reader may be referred to the work by O. Becker" (Weyl 1949, p. 90). See previous chapter for further details.

[104]Weyl (1921c). For an English translation, see Weyl (1921a).

[105]Weyl did not go into details in this occasion. He simply remarks: "it has to satisfy certain requirements, which can easily stated" (Weyl 1921a, p. 115).

[106]The picture below and the following remarks are taken from another paper published in 1940 with the title *The Mathematical Way of Thinking*. In this occasion his approach is better explained. See Weyl (1940).



We can easily identify each element arising from the process of division. For instance, the edge β is divided by means of an arbitrary point on it so that two new edges emerge, i.e. βc and βb. Moreover, the arbitrary point within the surface piece A leads to the division into triangles. Then we obtain the new surface piece $A\beta c$. All other elements can be identified in a similar way. As the process of division goes on all elements can be properly named. Weyl then observe that we can carry out a general pattern. Given the initial schema S, any symbol $e_2 e_1 e_0$ represents a surface piece e_2' of the subdivided scheme S'. Through iteration of this symbolic process we obtain a sequence of derived schemes $S, S', S'', S''', \ldots$ so that:

> What we have done is nothing else than devise a systematic cataloguing of the parts created by consecutive subdivisions. A *point* of our continuum is caught by a sequence
>
> $$e\ e'\ e'' \ldots$$
>
> which starts with a 2-cell e of S_0 [surface piece of S] and in which the 2-cell $e^{(n)}$ of the scheme $S^{(n)}$ is followed by one of the 2-cells $e^{(n+1)}$ of $S^{(n+1)}$ into which $e^{(n)}$ breaks up by our subdivision.[107]

From the surface pieces of the initial topological framework, i.e. the schema S, we then reach the points of the manifold. It is achieved by a process of division which is iterated infinitely many times. This mathematical conceptualization is able to account for the essential character of the continuum which relies on the relation between part and whole:

> It is part of the nature of the continuum that *every part of it can be further divided without limitation*. The concept of a point must be seen as an idea of a limit [Grenzidee], "point" is the idea of the limit of a division extending *in infinitum*.[108]

A point then is seen as *an idea of a limit* [Grenzidee]. It does not exist within the spatial continuum, we do not have any direct intuition of it. We can anyway intuit the divisions that might involve the parts of the whole. Only a direct intuition of the

[107] Weyl (2012, p. 76).
[108] Weyl (1921a, p. 115).

process of division is possible and a point can be grasped within the continuum only as a "limit". In the light of previous consideration we can interpreted this remark within a phenomenological framework. As we have seen an ideal essence is an ideal "limit". Therefore, this mathematical conceptualization refers to an idealizing procedure that is intuitively ascertained more adequately. It does not make reference to points as they were immediately given. On the contrary it takes into account only what is immediately given, that is the process of division. By means of it then we are able to refer to the points as the limits of this process. Their position is given with limited but arbitrarily great accuracy. In this very sense Husserl speaks of ideal essences. To be precise, the schema and the divisions themselves shouldn't be regarded as given in a exact way. A proper phenomenological description should regard this process of division according to its essential inexact nature. Weyl is indeed well aware of that:

> In reality one must imagine that the division on the 0^{th} level Σ_0 [on S] is given only vaguely, with a limited accuracy; for an exact division contradicts the essence of the continuum. But as the division progresses, the accuracy with which the initial vertices and sides and those newly introduced at the previous stages are set will increase indefinitely.[109]

This conceptualization then should be improved although it represents an important step forward in understanding the relations between a descriptive analysis of the continuum and its exact determination. Again this approach is in agreement with Weyl's practice. His investigations aim indeed to establish the best possible mathematical conceptualization but it is always a matter of degree of approximation:

> The application of the arithmetic continuity forms [de facto the process of division] to vividly presented continua can of course not be understood mathematically in a general and perfect way.[110]

As we already remarked, Weyl's studies are often characterized by a continuous tension between a temporary solution and a call for a better solution. Weyl's approach is truly Husserlian even in this sense. Phenomenology indeed should be understood not as fixed doctrine but as an continuous investigation aiming at a better and better clarification. Anyway, despite of these critical remarks, Weyl proudly declares:

> Everyone will [. . .] feel how truly the new analysis conforms to the intuitive character of the continuum.[111]

[109] "In der Wirklichkeit mußman sich vorstellen, daßdie Teilung auf der 0^{ten} Stufe Σ_0 nur vage, mit einer beschränkten Genauigkeit gegeben ist; denn eine exakte Teilung widerspricht dem Wesen des Kontinuums. Aber bei fortschreitender Teilung soll sich auch die Genauigkeit, mit der die anfänglichen Ecken und Seiten und die auf den vorhergehenden Stufen neu eingeführten festgelegt sind, unbegrenzt steigern" (Weyl 1988, 8, *my translation*). He also remarks that the above simplified description will be enough for the present purposes although it should be slightly altered to do justice to the inseparability of parts in a continuum.

[110] "Die Anwendung der arithmetischen Kontinuitätsformen auf anschaulich vorgelegte Kontinua läßt sich natürlich mathematisch gar nicht allgemein und einwandfrei erfassen" (Weyl 1988, 10, *my translation*).

[111] Weyl (1921a, p. 117).

Weyl further claims that the main mathematical concepts such as limit, convergence and continuity can be easily defined within this context. This conceptualization then allows us to establish a mapping between the spatial continuum and this symbolic construction. We might develop this construction making reference to the real numbers, but it would be an act of violence:

> The introduction of numbers as coordinates by reference to the particular division scheme of the open one-dimensional continuum is an act of violence whose only practical vindication is the special calculatory manageability of the ordinary number continuum with its four basic operations.[112]

Each process of division indeed identifies its own arithmetical scheme and the one associated with the real numbers is just a particular case:

> From this point of view, the continuum of real numbers is only a single, not particularly excellent case. The corresponding division scheme consists of a mutually infinite alternating sequence of elements e_0 and e_1,
>
> $$\ldots, e_0, e_1, e_0, e_1, e_0, e_1, \ldots,$$
>
> in which each e_1 is bounded by the next preceding and following e_0. The ever-repeated process of normal division correspond to the dual-fractional development.[113]

We have then dispensed with any reference to the real numbers. Replacing the points (regarded as "limits") by their symbols, we have turned the intuitively given manifold of the spatial continuum into a symbolic construction. From now on we can make reference only to this abstract construction and its properties. Any aspect involved in our understanding of the real world can be properly represented within this framework:

> A certain such 4-dimensional scheme can be used for the localization of events, of all possible here-nows; physical quantities which vary in space and time are functions of a variable point ranging over the corresponding symbolically constructed 4-dimensional topological space. [...] The causal structure, of which we talked before, will have to be constructed within the medium of this 4-dimensional world, i.e. out of the symbolic material constituting our topological space.[114]

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[112]Weyl (1949, p. 90).

[113]"Von diesem Standpunkt aus ist das Kontinuum der reellen Zahlen nur ein einzelner, nicht besonders ausgezeichneter Fall. Das ihm korrespondierende Teilungsschema besteht aus einer beiderseitig unendlichen alternierenden Folge von Elementen 0. und 1. Stufe (e_0 und e_1),

$$\ldots, e_0, e_1, e_0, e_1, e_0, e_1, \ldots,$$

in welcher jedes e_1 durch das nächstvorhergehende und nächstfolgende e_0 begrenzt wird. Der immer wiederholte Prozeßder Normalteilung entspricht der Dualbruchentwicklung" (Weyl 1988, 8, *my translation*).

[114]Weyl (2012, p. 77). In this occasion Weyl is referring to the real world conceived as a four-dimensional manifold. The same remarks, of course, applies to the case of any n-dimensional manifold.

Despite of these achievements there are several issues that should be faced. Firstly, we need to establish a connection between a given continuum and its symbolic scheme up to isomorphisms. We should indeed recognize when two different symbolic scheme are not intrinsically different. As the notion of affine figure in affine geometry or the notion of congruent figure in plane geometry, the notion of *isomorphism* has to play a fundamental role in combinatorial topology. That, however, is not an easy task and in 1940 it still represents a serious challenge:

> The problem of establishing the criterion of isomorphism for two finite schemes in finite combinatorial form is one of the outstanding unsolved mathematical problems.[115]

Another problem instead concerns the characterization of symbolic schemes in higher dimensional case ($n \geq 4$). Specifically, which combinatorial conditions have to satisfy the process of division in a general n-dimensional continuous manifold. Weyl deals with this issue in *Análisis Situs Combinatorio* published for the *Revista Matematica Hispano-Americana* in 1923.[116] Weyl is well aware of the mathematical difficulties involved and he proposes a provisional answer in the hope that a better solution will be found:

> A difficult arises with higher dimensional numbers [...] it has not yet been possible to determine the combinatorial conditions for $n \geq 4$. However, since combinatorial topology can only ever be developed as far as such conditions have been found, I have proposed an axiomatic path [...] reserving the right to supplement this system of axioms in the course of historical development until one day, hopefully, [...] lead to a clear determination.[117]

Weyl in fact seems to be quite satisfied with respect to this line of research. He declares that the concept of continuous manifold may be considered as fairly clarified from the mathematical point of view according to the previous investigations.[118]

Coming back to his approach involving infinitesimal quantities, we can now make a few remarks. The mathematical approach in *Raum-Zeit-Materie* aims to approach the continuum by means of real numbers. It dissolves it into isolated points and it ascribes to any of them a set of co-ordinates. Physical quantities then are ascribed to this mathematical conceptualization, which describes the main features of the real world by means of the algebraic structure of these quantities. The notion of infinitesimal

[115]Weyl (2012, p. 79). This mathematical problem is usually known as the *Hauptvermutung* in the theory of topological manifolds.

[116]Weyl (1923a, 1924). See also Weyl (1923b).

[117]"Eine Schwierigkeit tritt bei höherer Dimensionszahl auf [...] Und es ist bisher nicht gelungen, für $n \geq 4$ die kombinatorischen Bedingungen dafür zu ermitteln. Da sich die kombinatorische Topologie jedoch immer nur so weit wird entwickeln lassen, als solche Bedingungen ausfindig gemacht worden sind, habe ich einen axiomatischen Weg vorgeschlagen [...] wobei man sich vorbehält, dieses Axiomensystem im Laufe der historischen Entwicklung fortgesetzt zu ergänzen, bis hoffentlich eines Tages [...] zu einer eindeutigen Bestimmung führen" (Weyl 1988, 10, *my translation*).

[118]Weyl (1988, p. 11).

quantities therefore is used within this framework. Weyl's investigations within combinatorial topology instead approach the continuum in a radical different way. Firstly, as we have seen they attempt a mathematical conceptualization of the continuum whose idealization is more adequately intuitively ascertained. Secondly, this approach does not make any reference to real numbers. The localization of individual points is realised by means of symbolic sequences. These symbolic sequences belong to a general symbolic construction that attempts to describe the continuous manifold as a whole. Since the notion of real number is not involved within this framework the notion of infinitesimal quantity does not seem to be needed. It was needed in order to formulate a mathematical conceptualization of all the quantities involved in the infinitesimal small. Indeed, the infinitesimal quantities were needed to differentiate between a local property and a global property.[119] Within the context of combinatorial topology instead the situation seems to be different. Weyl's works on combinatorial topology do not appear very clear on this point, but the following considerations seems to be a fair interpretation of his intentions. In light of our reconstruction indeed it seems reasonable that Weyl aim to deal with the behaviour in the infinitesimal small by referring to the process of division involved in this symbolic construction. All the quantities needed to describe the real world are indeed associated with the symbolic sequences. We are then able to investigate the algebraic structure of these quantities, both locally and globally, by referring to the features that characterize this symbolic construction. Weyl seems to go in such a direction when he refers to the main ideas underlying Riemann's Analysis situs:

> The analysis situs, according to Riemann, "form the general theory of magnitudes, which are independent of dimensional determinations, where the quantities are not regarded as being independent of the situation and can not be expressed as a unity, but are regarded as fields in a manifold; one can only compare in it two quantities, if one is a part of the other, and then only decide the more or less, not the how much". [...] A posthumous fragment deals with the beginnings of n-dimensional analysis situs. For the strict formulation of the terms one must start from the division scheme Σ. All quantities are first explained by this division scheme, and it is to be shown that they do not change their value by replacing Σ by a scheme that results from it by subdivision.[120]

[119]Although global properties in *Raum-Zeit-Materie* are always relate to local co-ordinates. A proper analysis of the continuous manifold as a whole indeed is still lacking in this work.

[120]"Die Analysis situs bildet nach Riemann "einen allgemeinen von Maßbestimmungen unabhängigen Teil der Größenlehre, wo die Größen nicht als unabhängig von der Lage existierend und nicht als durch eine Einheit ausdrückbar, sondern als Gebiete in einer Mannigfaltigkeit betrachtet werden; man kann in ihr zwei Größen nur vergleichen, wenn die eine ein Teil der andern ist, und auch dann nur das Mehr oder Weniger, nicht das Wieviel entscheiden. [...] Ein nachgelassenes Fragment behandelt die Anfänge der n-dimensionalen Analysis situs. Zur strengen Fassung der Begriffe mußman von dem Teilungsschema Σ ausgehen. Alle Größen werden zunächst an diesem Teilungsschema erklärt, und es ist der Nachweis zu führen, daßsie ihren Wert nicht ändern, wenn man Σ durch ein Schema ersetzt, das aus ihm durch Unterteilung hervorgeht" (Weyl 1988, 13, *my translation*).

Moreover, some other remarks concerning differentiability within a manifold seem to support this interpretation. In more than one occasion Weyl expresses his concerns with respect to the notion of differentiable manifold. A proper understanding of its real meaning is still lacking:

> It remains a problem how to formulate this fact [the limitation to differentiable manifold] in its real meaning. It must be admitted that for this question the meaning of the differential calculus in its application to reality is still almost nothing.[121]

Nevertheless we need this notion in order to formulate a proper mathematical understanding of the real world:

> We do not yet understand the inner meaning of the restriction to differentiable manifolds. Perhaps one day physics will be able to discard it. At present it seems indispensable since the laws of transformation of most physical quantities are intimately connected with that of the differential dx_i.[122]

Then it seems that Weyl deals with the notion of differentiable manifold although he wish to discard it in future.[123] Combinatorial topology seems to Weyl a possible way out to this problem:

> In order to account for the nature of a manifold as a whole, topology had to develop combinatorial schemes of a more general nature. By this combinatorial approach it also got rid of the restriction to differentiable manifolds.[124]

Weyl's worries about the use of infinitesimal quantities would then seem to be resolved within this context. However, as we have seen the main worries concerns their use within calculus. Especially, how to connect local properties, involving infinitesimal quantities, and global properties such a those involving integrals. They do not regard the possibility of developing a non-Archimedean field of quantities. Therefore, Weyl might be agree in using infinitesimal quantities as far as we deal with them and all other quantities from an algebraic point of view.

Therefore we can conclude that Weyl's investigations in combinatorial topology aim to overcome the main difficulties encountered in the mathematical conceptualization in *Raum-Zeit-Materie*. They concern the first level of analysis, i.e. the continuum *in the sense of Analysis situs*. As we have seen Weyl's mathematical approach shouldn't

[121] "Es bleibt ein Problem, wie dieser Sachverhalt in seiner realen Bedeutung präzis zu formulieren ist. Es mußzugegeben werden, daßfür diese Frage nach der Bedeutung der Differentialrechnung in ihrer Anwendung auf die Wirklichkeit noch fast nichts geleistet ist" (Weyl 1988, p. 12).

[122] Weyl (1949, p. 86).

[123] Observe that he explicitly makes reference to the infinitesimal quantity dx_i as a key notion.

[124] Weyl (1949, p. 89).

be understood in the light of modern differential geometry. It has instead to be understood as an attempt to formulate a mathematical conceptualization of the continuum identifying the algebraic structure of a domain of quantities that properly describes the abstract schemata underlying this phenomenon. Both the mathematics involving infinitesimal quantities in *Raum-Zeit-Materie* and symbolic construction developed within the context of combinatorial topology follow this path. The latter however attempts to establish a more faithful connection between a descriptive analysis of the continuum and its exact determination following some phenomenological guidelines. For this reason Weyl claims for it as a fair attempt to improve our mathematical understanding of the real world.

3 Towards a Phenomenological Clarification

Of course, we need history too.[...] in order to let the philosophies themselves, in accord with their spiritual content, work on us as an inspiration. [...] But it is not through philosophies that we became philosophers. Remaining immersed in the historical, forcing oneself to work therein in historico-critical activity [...] all that leads to nothing but hopeless efforts. The impulse to research must proceed not from philosophies but from things and from the problems connected with them.

E. Husserl[1]

The previous chapters have attempted to shed light on Weyl's thought both on the mathematical and philosophical aspects. They can be regarded as a fair attempt on that direction. They have clarified Weyl's studies showing how they are related with other mathematical and philosophical issues, especially with the phenomenological framework of Husserl's philosophy. Despite this, the theoretical proposal revealed by them is not so easy to understand. That issue seems to be shared by many other contemporary studies. The relevant literature on this author dealing with a phenomenological interpretation seems often to be hardly understandable. I'm going to outline the main problems involved in this field of research and how they are related with the peculiarity of Husserl's framework. I will then suggest a way to improve these studies. Specifically, I will attempt a phenomenological clarification of Weyl's writings. To this aim, I will argue for an approach that makes use of Husserl's writings as a sort of "analytic tools" so that a sort of phenomenologically-informed reconstruction of Weyl's thought can be achieved. I will finally consider Weyl's notion of surface as a case study to show a concrete example of this kind of reconstruction.

[1]Husserl (1965, p. 146).

3.1 Hermann Weyl and Phenomenology: a Problematic Field of Research

The secondary literature related to the phenomenologically-oriented works of Hermann Weyl seems to be often hardly understandable. Several important studies have shown interesting connections between Weyl's mathematical works and Husserl's phenomenology. Thanks to that they have made Weyl's writings much more understandable. Nevertheless, the overall result is not always so easy to understand. Previous chapters follow this pattern. In the first chapter, for instance, the variety of Weyl's foundational studies along the years 1917-1927 have became more understandable within the phenomenological framework we have outlined. Specifically, his desire to connect a *morphological description* of what is intuitively given with its mathematical formulation *constructed in a logical conceptual way* has been clarified shedding light on the Husserlian distinction between *descriptive* and *exact* sciences, and how they might be related. We have outlined how this distinction is carried out within the Husserlian framework. To this aim, several notions have been introduced, such as *morphological essence*, *inexact concept*, *limiting idea*, *definite manifold* and many others. Some of them have been introduced by referring to other more familiar notions.[2] Other instead have just been introduced putting them within the context of Husserl's phenomenological philosophy.[3] Of course, it has always been the case of a middle way between these two situations.[4] We have then pointed out all the relevant connections between Weyl's writings and Husserl's philosophy.[5] In this way we have argued that Weyl's view follows that philosophical framework. In the light of that, Weyl's theoretical proposal has became more understandable. In a similar way other issues have been enlightened by Husserl's philosophy along the way.

In the relevant literature we may recall Da Silva (1997). In this paper Weyl's remarks on *meaningful/meaningless propositions* are clarified by reference to Husserl's theory of meaning.[6] Da Silva clearly states that he aims to "analyze Weyl's philosophical ideas in connexion with the work of Husserl, in particular *Logische Untersuchungen* and *Ideen I*" (Da Silva 1997, p. 277). He then introduces Husserl's theory of truth as follows: "According to Husserl no judgment can be true if its content is not fulfilled by

[2]For instance, I have introduced the notion of *definite manifold* as follows: "a field of inquiry is articulated as a definite manifold if, out of a few basic concepts and a given set of axioms, it is possible to derive the totality of all possible formations concerning that field". This expression seems to be reasonably understandable since the notions involved are quite well-known.

[3]For instance, the notion of *exact concept* has been introduced in the following way: "[exact science] is an axiomatic science that operates with exact concepts, which express ideal essences". Both the notion of exact concept and of ideal essence haven't be defined in any way. We have just assumed their meaning clear enough and outlined how they take place within Husserl's philosophy. That is, "exact concepts" express "ideal essences" and they both pertain to the domain of exact sciences.

[4]Observe, for instance, that the notion of "exact science" has been introduced referring to the well-known expression "axiomatic science", but its meaning has been associated with the notion of "ideal essence" that hasn't be further clarified.

[5]They include, of course, also all the relevant remarks that explicitly make reference to Husserl and all historical facts that support our Husserlian reading of Weyl's writings.

[6]In the first chapter we have outlined some of these remarks. Weyl deals with that especially in Weyl (1994) and Weyl (1949).

intuition, that is, if what it expresses is not given *as expressed* in an intuitive experience" (Da Silva 1997, p. 285). Several other Husserlian notions follow, as well as some important Husserl's conceptual distinctions. For instance, Da Silva stresses on the distinction between meaning with respect to *form* and meaning with respect to *matter*. This Husserlian framework allow him to shed light on several issues. Weyl's unclear remarks on paradoxes, for instance, are enlightened by this framework since they are interpreted as "a form of 'reduction of size' with respect to the extension of predicates (and relations), although not *ad hoc* reduction arbitrarily devised [...] but, rather, one determined by a clear-cut notion of *material* sense" (Da Silva 1997, p. 285). So that he can conclude that "Weyl's theory of (material) meaning is teleologically oriented by the ideal of adequate fulfilling of intentions expressed by judgments by appropriate intuitions" (Da Silva 1997, p. 286). We face a similar situation as before. Although Weyl's theory of meaning has got some clarity by means of these considerations, several Husserlian notions have just been introduced putting them within the context of Husserl's theory of meaning. The term "fulfilled", for instance, is introduced referring to something that is "expressed in an intuitive experience". However, the notion "expression" is a peculiar Husserlian notion itself. Da Silva has properly associated these two notions speaking of "fulfillment" in terms of "expression", but the term "fulfilled" has not been further clarified.

Similar considerations applies also in Tieszen (2000), van Atten et al. (2002) and Ryckman (2005). Tieszen aims to clarify Weyl's critical remarks against the classical notion of *real number*. He sketches Husserl's notion of *constitutive processes grounded in sensory experience*. He declares that according to Husserl "mathematical knowledge is distinct from knowledge of sensory objects but it is founded on sensory experience" (Tieszen 2000, p. 278). The notion of founding and founded structures in mathematical cognition is "a fundamental part of Husserl's view" (Tieszen 2000, p. 284). Weyl is following this framework when he considers the category of *natural numbers* as properly founded on the notion of *iteration*. From this founding structure he can then develop his own construction of real numbers in substantial agreement with the Husserlian notion of constitutive process. The classical notion of real number instead is not founded on sensory experience and for this reason it should be dismissed. As before Weyl's dismissal of the classical notion of real number has got some clarity by means of these consideration. However, the meaning of several Husserlian notions have been assumed to be clear enough without being further clarified. The notion of "foundation on sensory experience", for instance, is introduced without any clarifications on the peculiar notion of "grounding" or "foundation" within Husserl's framework. Similarly, van Atten *et al.* aims to shed light on Weyl's remarks on *lawless sequences*. He quotes a passage from Weyl and he comments as follows (in square brackets): "This is not the place to discuss how such insights into the essence of a developing sequence are to be gained [Here Weyl thinks of Husserl's *Wesenschau*]. Yet only this sort of insight provides a justification for the fact that, when given a law ϕ by someone, we can, without examination, reply: The sequence determined in infinitum by this law does not possess the property *E*" (van Atten et al. 2002, p. 219). Again Weyl's remark is enlightened by reference to Husserl's notion of eidetic vision, but this complex notion is not further clarified. Lastly, Ryckman considers the rather unclear Weyl's remark on *coordinate system* that see it as the "unavoidable residuum of the ego's annihilation in that geometrico-physical world". Ryckman introduces Husserl's notion of *phenomenological reduction* and *pure consciousness* among other related ideas. He then declares:

> Just as "pure consciousness" remains when the natural attitude's posit of the reality of a "transcendent Nature" is "put out of action" through the phenomenological reduction, so also *within* that posited reality, namely, the "geometrico-physical world" of physical science, and so beginning *within the natural attitude*, reference to a coordinate system is a reminder that this "worldly transcendency" has been constituted from "pure consciousness".[7]

Therefore, the odd feature ascribed to the coordinate system of being a "residuum of ego's annihilation" becomes more understandable by reference to Husserl's notion of "pure consciousness residuum". The coordinate system recalls us that our objective construction of the world is unavoidably constituted starting from the subjective nature of pure ego. Anyway, also in this case Ryckman does not explore the complex notion of "epoché" or "bracketing" in greater detail.

These are just few examples, but many others can be found in literature.[8] A question then arises: why we are facing such a situation? One of the main reasons seems to rely on the terminological complexity of Husserl's writings. Husserl's phenomenology aims to develop a radical new philosophical approach that should have lead philosophy to became a rigorous science. At the very beginning of *Ideen I* he declares:

> Pure phenomenology, the way to which we seek here, the unique position of which relative to all other sciences we shall characterize and show to be the science fundamental to philosophy, is *an essentially new science* which, in consequence of its most *radical essential peculiarity*, is remote from natural thinking and therefore only in our days presses toward development.[9]

The radicalism of phenomenology is reflected also in the terminology involved. Husserl makes few remarks on this issue at the end of his introduction of *Ideen*

[7]Ryckman (2005, p. 131).

[8]See, for instance, Bell (2000), Bell (2004), Folina (2008) and Mancosu (2010b). That situation, honestly, seems to be shared by many other Husserlian-oriented studies. They might be part of the so-called phenomenological tradition broadly conceived or be properly Husserlian studies. It is not uncommon that secondary literature of this kind has been under criticism, especially from the so-called analytic-oriented tradition. Husserlian scholars are often criticized to use their own way of expression, to keep their philosophical theorizing within the Husserlian terminological framework. Their reluctant to attempt a reformulation of Husserl's terminology is often seen as an excuse to avoid a serious discussion. They are then accused to not engage a philosophical investigation in a serious way. Although I believe analytic-oriented philosophers' criticism being often naive, I do not think we are allowed to not taken seriously their criticism. On the contrary, we should be able first to explore this issue and then to face it.

[9]Husserl (1982, XVII, *my emphasis*).

I, before starting the exposition of his philosophical proposal. He remarks that as he did in *Logische Untersuchungen* he will avoid the expressions "a priori" and "a posteriori" because "of the confusing obscurities and many significations clinging to them in general use" (Husserl 1982, XXII). They instead will be used "only as equivalents of other terms which are joined to them and on which we have conferred clear and univocal significations" (Husserl 1982, XXII). A new meaning to the notions of "idea" [*Idee*] and "ideal" [*Ideal*] has also to be ascribed in order to avoid the frequent misinterpretations that have undergone the previous *Logische Untersuchungen*. For this reason Husserl remarks that he will make use of the terminologically unspoiled foreign word "eidos" or equivalently, the German word "Wesen". Therefore a new terminology is needed within this radically new philosophical perspective. However, Husserl frankly admits that it is not possible to choose technical expressions that fall entirely outside the frame of historically given philosophical language, as it might happen in mathematics. In philosophy, the fundamental philosophical concepts "are not to be defined by means of firm concepts identifiable at all times on the basis of immediately accessible intuitions" (Husserl 1982, XXIII). One then has to start from the frame of historically given philosophical language and usually only after long investigations a proper determinations of these concepts is achieved. He explicitly states:

> [...] in general *long investigations must precede their definitive clarifications and determinations*: combined ways of speaking are therefore frequently indispensable which arrange together a *plurality* of expressions of common discourse which are in use in approximately the same sense and which give terminological pre-eminence to single expressions of this sort.[10]

After this deliberative process the terminological expressions involved is to receive its fixed sense. These deliberations play a fundamental role since a "word can be equivocal and as word demand various significations" so that "a clarification also has the function of giving old words *a newly constituted sense*" (Husserl 1980, 88, *my emphasis*). The goal of clarification can be then understood "*as that of producing anew, as it were, the concept already given*, nourishing it from the primal source of conceptual validity, i.e., intuition" (Husserl 1980, 88, *my emphasis*).[11]

Furthermore it may happen that unconventional expressions are needed to properly fix the results of this long deliberative process of clarification. They indicate us "the direction" of the underlying intuitions of these concepts:

> What makes them [the new concepts] valuable is not verbal univocalness, obtained by differentiating the empty word-significations already at hand, but rather the adaptation to the essences separated out in Intuition by

[10] Husserl (1982, XXIII, *the first emphasis is mine*).

[11] In what follows I will better clarify the distinction between words, concepts and underlying intuitions. For now observe that the idea that a concept receive its validity when its underlying intuition is offered, pertains to the so-called Husserl's theory of meaning.

> analytical study and differentiated for everything interwoven with them. A great number of new concepts is obtained, and the connection to words of the language has only *the function of indicating the approximate direction* in which they lie, or *of making easier by means of pictorial expressions* the retention (and for the learner the grasping) of eidetic moments that are seen.[12]

The use of pictorial expressions then turns out to be very useful to "grasp" some insights in the phenomena being object of investigation. After that process, they can then be replaced by new concepts. These last having now received a determined sense that expresses the "eidetic moments that are seen". In this sense the use of words of our language, even if unconventional expressions, has "only the function of indicating the approximate direction" upon which the sense of these new concepts can be properly determined. For this reason the rather specific phenomenological terminology might need these long chain of expressions to be properly understood in its own sense.

Nevertheless, this terminology is not arbitrarily chosen. We are not dealing with arbitrary definitions as it might be the case in mathematics. These new concepts should express their underlying intuitions in an adequate way. Once we have grasped a certain intuitive state of affairs in the phenomena, not every concept is able to express it in a proper way. We should identify which concept best suits in this case. However, these concepts have already a meaning in our ordinary language. For this reason Husserl speaks in terms of "producing anew the concept already given" or similarly of giving "old words a newly constituted sense". After this long deliberative process of clarification old words acquire a new sense. That is, they now express an intuitive state of affair that might be different from the old one. In *Logische Untersuchungen*, for instance, Husserl considers the term "imagination" (*Einbildung*):

> The word 'imagination', likewise, normally means a non-positing act, but we should have to *extend its original meaning* beyond the sphere of sensuous imagination, so as to cover all possible counterparts of affirmations.[13]

The word "imagination" should then acquire a new sense according to Husserl.[14] The above remarks is put forward after a long discussion concerning the notion of *objectifying act*. In this long process of clarification several distinctions are pointed out. A fundamental distinction concerns the qualitatively differentiation of an objectifying act into a *positing act* and a *non-positing act*. The former acts are regarded as *affirmative acts* (*fürwahrhaltende Akte*) and they concern not only the case of judgments. Husserl speaks in terms of positing acts also in the domain of perception. He then asks himself whether we can speak of non-positing acts both in the case of judgments and in the

[12] Husserl (1980, 48, *my emphasis*).

[13] Husserl (1970a, vol II, 165, *my emphasis*).

[14] Husserl makes use of inverted commas in the introduction of the word "imagination". Husserl seems often to do that whenever he ascribes a new meaning to a word. After that, he continues to use this word without any inverted commas.

case of perceptions. The answer is affirmative. In the case of judgments we deal with non-positing acts when reading a novel "we understand narrations without decision as to their truth or falsity" (Husserl 1970a, vol II, 165). In the domain of perception instead we deal with such kind of acts when we face "sensible appearances, e.g. stereoscopic phenomena, which one can treat, like aesthetic objects, as 'mere phenomena', without adopting an existential stance" (Husserl 1970a, vol II, 166). In the light of these observations Husserl remarks that the word "imagination" should be extended from its original meaning as non-positing act in the sphere of perception to mean non-positing act also in the domain of judgments. So that we can "cover all possible counterparts of affirmations". In this sense Husserl says that when we read a novel we deal with judgments that "lack the character of genuine judgments", we deals with "mere imaginings" (Husserl 1970a, vol II, 165).[15]

These considerations then show how the word "imagination' has acquired a new sense, and a rather specific one, in this framework. Observe further that a proper understanding of the notions of "objectifying act", "positing act" and "non-positing act" is required to properly grasp this new sense of "imagination". And they in turn are the results of a rather long deliberative process of clarification.[16] Finally, observe that notions such as "objectifying act" or "positing act" are easily recognizable as Husserlian terminology. The notion of "imagination" instead sounds more familiar and for this reason its new sense can be easily misunderstood in secondary literature. Especially if this word is used in a studies that is not focus on the issue of imagination. In that case this Husserlian notion is likely to be used without a proper clarification of its sense.

A similar example is offered by Husserl's clarification of the notion of "sense" (*Sinn*). In *Ideen I* Husserl claims:

> Corresponding in every case to the multiplicity of Data pertaining to the really inherent noetic content, there is a multiplicity of Data demonstrable in actual pure intuition, in a correlative "noematic content" or, in short, in the "noema" — terms which we shall continue to use form now on. Perception, for example, has its noema, most basically its perceptual sense, i.e., the *perceived as perceived* [...] the judging has the *judged as judged* [...] the noematic correlate, *which is called "sense" (*Sinn*) here (in a very extended signification)* is to be taken precisely as it inheres "immanentally" in the mental process of perceiving, of judging.[17]

After having introduced the notions of *noetic moment* and *noematic moment*, Husserl proposes an extension of the meaning of the word "sense" (*Sinn*). It now refers to

[15] In what follows we will shed some light on how these considerations are connected with a clarification of the word "imagination" starting from the word itself and not ascribing to it a new sense as we have shown in the above case.

[16] We have not actually clarified the meaning of "imagination". We should have clarified all the notions involved for this purpose. These remarks aim only to show that some words acquire a new sense in the framework of Husserl's philosophy and that their comprehension depends substantially from previous investigations.

[17] Husserl (1982, p. 214).

what Husserl calls the *noematic content*, or *noema*, and it concerns both the domain of perception and of judgment. In the case of perceptions it refers to the "perceived as perceived", whereas in the case of judgments it refers to the "judged as judged". In both cases Husserl speaks in terms of "sense".[18] He further remarks that we are dealing with an "extended signification" of this word. Later he adds:

> [...] "signifying" [*Bedeuten*] and "signification" [*Bedeutung*]. Originally, these words concerned only the linguistic sphere, that of "expressing" [*Ausdrückens*]. But one can scarcely avoid and, at the same time, take an important cognitive step, *extending the signification of these words and suitably modifying them* so that they can find application of a certain kind to the whole noetic-noematic sphere: thus application to all acts, be they now combined with expressive acts or not. Thus we have continued to speak of "sense" [*Sinn*] in the case of all intentive mental processes – a word which is used in general as equivalent to "signification" [*Bedeutung*].[19]

These considerations then show again another case where a word has acquired a new rather specific sense. Also in this case, moreover, a proper understanding of this notion requires the comprehension of other notions, such as "noetic moment" and "noematic moment" which in turn are the results of a long deliberative process of clarification. For this reason the Husserlian notion of "sense", whose meaning sounds quite familiar, is likely to be misunderstood if it is used without a proper clarification. That might the case of a secondary literature whose focus does not regard this Husserlian notion although it makes reference to it.[20]

These considerations regard two specific examples, but many others can be found in Husserl's writings. We have focused on two cases where to a word, whose meaning might sound familiar, is ascribed a new constituted sense. However, Husserl often

[18] The notion of *noema* is far more complex, but it is not our aim to explore this issue here. For further details on this notion and its correlated notion of *noesi*, see Husserl (1982, §87-§96).

[19] Husserl (1982, 294, *my emphasis*).

[20] It should be noted that the meaning of "sense" [*Sinn*] and of "signification" [*Bedeutung*] changes from *Logische Untersuchungen* to *Ideen I*. In the first publication these two notions are used deliberately as synonyms. He clearly claims: "'Meaning' [*Bedeutung*] is further used by us as synonymous with 'sense' [*Sinn*]. It is agreeable to have parallel, interchangeable terms in the case of this concept, particularly since the sense of the term 'meaning' is itself to be investigated. A further consideration is our ingrained tendency to use the two words as synonymous, a circumstance which makes it seem rather a dubious step if their meanings are differentiated" (Husserl 1970a, vol I, 201). In *Ideen I* instead their meaning is distinguished. The word *Sinn*, as we have shown above, acquires an extended signification, whereas the word *Bedeutung* refers to those acts that are expressive, "logical" in the specific sense. He says: "For the sake of distinctness we shall prefer the term *signification* for the old concept and, in particular, in the complex locution of "*logical*" or "*expressive*" *signification*" (Husserl 1982, p. 294). This semantic shift anyway shouldn't be understood as a change of Husserl's theoretical proposal. It instead should be understood as a finer clarification of this issue. We will shed some light on that claim in what follows. For the moment, observe that Husserl invites the reader to compare the notion of "perceptual sense" (see above the first quotation) with the notion of "fulfilling meaning" used in *Logische Untersuchungen*. Indeed, in the first logical investigations Husserl makes a distinction between *intending meaning* (*intendierende Bedeutung*) and *fulfilling meaning* (*erfüllende Bedeutung*) (see Husserl (1970a, vol I, 200)). We can better understand Husserl's remark if we observe that the two notions aim to express the same intuition to a certain degree. In the light of a finer clarification of the underlying intuitions, Husserl then ascribes a new sense to the notions of *Sinn* and of *Bedeutung*.

makes use of unspoiled words taken from Greek, such as *eidos*, *noesi* or *noema*. In this way he aims to avoid as far as possible misunderstanding. Nevertheless, many common notions undergo this process of clarification. That the reason why we face such a terminological complexity in Husserl's writings, which often turns out to be hardly understandable without a proper clarification of its meaning.[21]

Coming back to Weyl's literature, we can then better understand why it might be so problematic. Usually authors introduce an Husserlian notion making reference to other notions, but these last should again be clarified. The meaning of Husserl's terminology often relies on a rather long process of clarification. In some cases, moreover, an Husserlian notion is introduced by reference to other more familiar notions and it might be even worst. That because, as we have seen, many common notions acquire a rather specific and often unconventional sense. Several misunderstanding are likely to happen in these cases. Weyl's phenomenologically-oriented studies then turn out to be a problematic field of research and we should deal with this situation. That issue indeed shouldn't be underestimated. It might be desirable that also readers not having an expertise in Husserl's philosophy will be able to judges to what extent the theoretical proposal revealed by this kind of studies is tenable. One might say that such a problem is inevitable and the readers should be aware of Husserl's philosophy at least with respect to the basic notions. Anyway, such a situation does not seem to be easily achievable as we have seen. A proper understanding of the basic notions seems often to presuppose a careful reading of Husserl's writings.[22] Any secondary literature has then to make a choice. Either it repeats (or at most rephrases) the lengthy Husserlian investigations or it overlooks some details. The former case doesn't seem to be a good choice. The outcome would be a bad secondary literature. Usually this kind of studies indeed follows the latter case. Authors are used to explore some issues through all Husserl's writings. In such a manner they are able to contextualize them within the framework of Husserl's phenomenology so that a clearer picture of them can be pointed out. Therefore, if we take seriously Husserl's obsession for a rigorous clarification of the terminology involved then such a secondary literature seems to be very helpful to readers who have already been acquainted of Husserl's writings. However, it is not obvious that people not having such an expertise can fully understand this kind of literature.[23]

[21]I am not supporting any claim on the strength of Husserl's approach. I would like to remain neutral on that issue for the moment. My goal is just to focus the attention of this issue and try to shed some light on its nature.

[22]The notion of categorical intuition, for instance, is properly introduced in the sixth logical investigation, but many notions involved in its determination are introduced in the previous logical investigations. The notion of *non-independent moment*, for instance, is needed to understand the notion of categorical intuition and it is introduced in the third logical investigation. For this reason a proper understanding of this notion seems to need a careful reading of the previous lengthy investigations (all the six investigations amounts to about five hundred pages). One might think that many of these considerations were redundant and so we can simplify these investigations without compromising their content. However, Husserl was rather persuaded that any of these details were redundant. On the contrary he was hardly convinced that although a fair analysis was achieved it should be further improved. These investigations were just a starting point for a never ending process of clarification.

[23]That does not apply to that literature aiming at a clarification of some Husserlian issues that can be properly analyzed by means of formal (mathematical) methods. In this case proper definitions can be

Therefore, one might simply claim that a reader should be aware of Husserl's philosophy, at least with respect those Husserlian issues involved in that specific literature on Hermann Weyl. Anyway, even in that case we still face a problem. A poor handling of this terminology indeed might lead to serious misunderstandings. For instance, if we speak in terms of *intuition* we might mean a *sensible* or a *categorial* intuition. Moreover, even if we make explicit such a difference and if we use the expression "sensible intuition" we might refer to *perception* or *imagination* and in both cases again we might mean the *intentional act* or the *object* given in such an act. Similarly, if we speak in terms of "categorial intuition" then we might mean the *general founded act* or the specific case of *universal intuition*. These are just a few examples, but many other distinctions can be made. As we have shown each meaning has its own peculiarity and Husserl is really careful in handling such distinctions in order to clarify their differences. This process of clarification is never ending but along the way we should be able to determine our terminology with increasing care, fixing adequately the underlying intuition for each meaning. I do not mean that using the expression "intuition" in the proper context of sensible intuition (without specifying it) is wrong. I'm just pointing out that Husserlian terminology is stratified and a notion can undergo many further specifications. What really matters is that we express properly the underlying intuition that we are referring. If a scholar makes use of the word "categorial intuition" meaning "the general founded act" but without specifying it, then it might be the case that a reader understands this word in the sense of "universal intuition".[24] It may not be a problem or instead it may lead to a serious misunderstanding.[25] In both cases further specifications can be clearly developed, but it does not represent an issue as long as the reader is grasping the proper underlying intuition. The following considerations might shed some light on this issue. In *Ideen III* Husserl remarks:

> The process of clarification, therefore, means two things: making a concept *clear by recourse to fulfilling intuition*, and, second, a process of clarification *executed in the sphere of intuition itself*: the meant object (the intuition "means", too, it also has a noema which is the possible member of noematic manifolds in which the noematic object stands out more and more perfectly) must be brought to ever greater clarity, must be brought ever nearer, must be brought in the process of clarification to perfect self-givenness.[26]

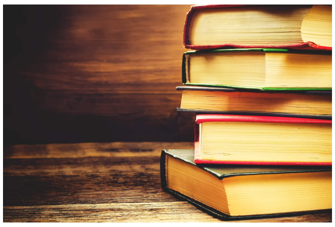
set out so that the exposition seems to be fairly understandable to anyone. In this case, however, we have narrowed our research since most of Husserlian investigations cannot be understood in such a way as we will see.

[24]The scholar might use the word "categorial intuition" also without being aware of the specific case he is intending (and only after further investigations he becomes aware of that). Also in this case a communication of the intended underlying intuition is relevant in order to avoid misunderstanding.

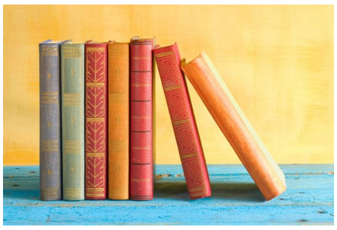
[25]It wouldn't be a problem of course if the scholar makes use of this word meaning indifferently both cases. In that case he simply means all the categorial intuitions without any specification. However, that should be clearly stated.

[26]Husserl (1980, 89, *my emphasis*).

The process of clarification can then refer to two different things. The first regards the identification of the underlying intuition ascribed to a given meaning. The second instead refers to the never ending process of clarification of such an intuition. The latter case is not a serious problem as long as the underlying intuition is properly identified. To understand better this last claim, I suggest the following metaphor.[27] Let suppose that a person *A* is looking at the picture below.



He then uses the expression "bunch of books on a table" to refer to the general state of affairs represented by those books.[28] He then writes this expression on a book that is going to be read by another person, say *B*. Let suppose now that *B* reads this expression and she believes that it represents the general state of affairs depicted in the following picture.



If *A* aims at a better representation of what he was intending, then he might try to explore the relations between the books. He might say that the books "touch each other". Then *B* can be convinced of that, but she will understand such a claim in a different way. The spatial relations defining the expression "touch each other" will be different for the two of them. This discrepancy may be lead to a misunderstanding. A different situation would be if *A* first shows the picture and then describes it as a concrete example of a "bunch of books on a table". In this case *B* will better understand the expression "touch each other", and from here all the further specifications that might be pointed out by a finer analysis of this general state of affairs. For example, both of them can recognize that the books do not touch each other in the same way, and so on. What really matters is the proper recognition of what we aim to express by the words "bunch of books on a table". Coming back to our previous considerations, what really matters is the recognition of the underlying intuition that we aim to express by a given meaning. Husserl's use of concrete examples should be understood as

[27]It could be also understood as a proper phenomenological example, but further details should be added. For our purpose it can be regarded only as a metaphor.

[28]We could also say that he aims to express the abstract state of affairs, and not this very specific concrete one.

a sort "practices" that help us the recognition of intuitions.[29] By means of concrete examples he leads the reader as close as possible to the underlying intuition he is aiming to express by a given meaning.[30] In the previous metaphor, *A*'s use of the first picture has to be understood as a way to lead *B* to the recognition of the general state of affairs he was aiming to express. Without such a concrete picture, there might be some misunderstanding. For example, that would happened if *B* had intended the general state of affairs represented in the second picture.

This remark aims to give a better understanding of the situation we face with Husserl's phenomenology.[31] If a scholar does not pay adequate attention to communicate the underlying intuition of a given meaning, then some misunderstanding are likely to occur. For this reason, the use of concrete examples are useful to avoid as far as possible these problems.[32]

To conclude, therefore, previous considerations have shown that the terminological complexity of Husserl's writings shouldn't be underestimated. Many further details should be added to give a clearer picture of this situation, especially with respect to the so-called Husserl's theory of meaning.[33] However, I hope that previous considerations have shown at least that we should seriously take care of this issue. In the next section I'm going to suggest a possible solution.

I would just add a further remark. Not taking care of such an issue seems to be quite unfair to the *leitmotiv* of all Husserl's writings which is notoriously characterized by a continuous obsession for a rigorous clarification of his analysis. That view couldn't be better represented by the following remark taken from his personal notebook:

> In the first place I name the general task that I must solve if I am ever to be able to call myself a philosopher. I mean a *critique of reason*. [...] Without getting clear in general terms on the sense, essence, methods, main points of a critique of reason, without having thought out, drafted, established, and founded a general outline for them, I cannot live truly

[29]We use the term "practices" but we should be aware that Husserl wouldn't be completely agree with that. Anyway it helps us to understand how we can actually perform such a recognition.

[30]In what follows I will better clarify this claim. For now observe that Husserl's use of examples should be understood within the framework of *eidetic vision* (*Wesenserschauung*). In *Erfahrung und Urteil* he claims: "Let us attempt to get a first concept of this operation [the acquisition of concepts of essences]. It is based on the modification of an experienced or imagined objectivity, turning it into an *arbitrary example* which, at the same time, *receives the character of a guiding "model"*, a point of departure for the production of an infinitely open multiplicity of variants" (Husserl 1973, 340, *my emphasis*). He further adds: "In this multiplicity [...] is grounded as a higher level *the true seeing of the universal as eidos*. Preceding this seeing, there is the transition from *the initial example, which gives direction*" (Husserl 1973, 342, the latter emphasis is *mine*). Husserl's use of example is then an essential part of the true seeing of an essence (or eidos), which is an intuitive object (categorical object) and it should be expressed by an adequate concept of essence.

[31]Husserl's investigations always proceed in such a way, starting from a description and then improving it by a finer description. In this process, however, what he means by an expression is kept fixed while its meaning is clarified determining its underlying intuition with increasing care. To be clear, that refers to Husserl's ultimate ideal. In fact, the most part of his published works but not all of them respect such an ideal.

[32]To be clear, I do not mean that Weyl's scholars do not use properly this terminology. I'm just saying that we should take seriously the case that some readers do not immediately grasp which intuition others are referring.

[33]In what follows I will better clarify this situation.

> and truthfully. The agonies of the lack of clarity, of wavering in doubt, I have savored enough. I must achieve an inner firmness. [...] I simply cannot live without clarity. [...] Here I am not striving for honors and fame; I don't want to be admired; I don't think of others, nor of my outward advancement. *Only one thing can fulfill me: I must gain clarity, otherwise I cannot live.*[34]

3.2 Proposal for a Solution

In the previous chapters we have considered several Weyl's writings and have shown many connections between them and Husserl's phenomenological researches.[35] Although we have gained a better understanding of Weyl's proposal in doing this, our focus was on the connections that can be revealed between these two authors. One of its greatest benefit was to provide a general framework that helped us to better understand the underlying motivations of Weyl's studies and shed some light on them. However, as we have observed the phenomenologically-oriented works of Hermann Weyl seems to be often hardly understandable. For this reason, I wish now to provide a better clarification of some of these studies.[36] However, as we have observed in the previous section, it does not seem to be easily achievable. One of the main reasons seems to rely on the complexity of Husserl's terminology. Since this issue shouldn't be underestimated, the demand for a solution arises.

Previous considerations have pointed out that readers seems to need an Husserlian background (at least with respect those issues involved in that specific literature on Weyl). However, it does not always very clear to what extent one has to know Husserl's philosophy. Moreover, it seems to be wise that also readers not having such an expertise can understand this kind of literature. For this reason I wish to suggest a kind of compromise solution. On one side I will assume that a reader is not completely unaware of Husserl's philosophy. On the other side I will take seriously the task of clarifying as better as I can the terminology involved. Specifically, I would make use of Husserl's writings in order to clarify the terminology involved. In this sense I might say that I'm going to use Husserl's writings as a kind of "analytic tools".[37] Of course, I will not clarify everything. A great amount of Husserl's philosophy will need to be

[34] Dated September 25, 1906. The latter emphasis *is mine*. See Hopkins and Crowell (2001, p. 321).

[35] Of course, we do not show only connections between Weyl and Husserl, but also among Weyl's writings themselves.

[36] We might refer to the previous chapters as following a sort of *synthetic approach* whose value is to reveal unexpected connections among writings in the history of ideas. What we are suggesting now instead can be said to follow a sort of *analytic approach*. I would add a brief remark with this respect. I've decided to follow both approaches because a detailed clarification seems to be truly meaningful when it becomes part of a more comprehensive framework and it is not just a self-seeking exercise. In this sense an analytic approach carried out in the light of a more comprehensive synthetic approach seems to be a fair attempt to made such an approach more valuable.

[37] Previous chapters allow us to justify the use of Husserl's writings in this way. As we have argued Weyl's studies can be better understood within the phenomenological framework of Husserl's philosophy. Specifically, we have shown that several notions occurring in Weyl's writings should be properly understood by reference to Husserl's phenomenological framework. For this reason, we are allowed to clarify Weyl's terminology in terms of Husserl's terminology.

presupposed. It would be impossible to do differently. For instance, let consider the case we aim to clarify Weyl's account of affine geometry, which involves the notion of spatial point. Since Weyl refers to it as a categorial object, then we should clarify the notion of categorical intuition in that specific case. To shed light on this notion we need to clarify the terminology involved in this notion, such as the notion of abstract moment. This process, however, cannot be carried out beyond a certain point. The notion of intentional act and how it is related with the notion of *cogito*, for instance, cannot be clarified in all its details. Otherwise it would be impossible to proceed in a clarification of Weyl's account of affine geometry. For this reason, the clarification I'm going to develop has to be ideally regarded as a partial work that should be further developed. Since I will focus on some details and not others, my outcome will need to be taken by other scholars and improved. In this sense I would say that my work has to be considered as a partial work in a continuous process of clarification among the scholars of this field of research. To what extent it would be actually achievable it does not concern me at the present stage. It aims just to define what it is the ultimate ideal of my approach.[38] In the light of that, I will add Husserl's quotations as footnotes. I need them to claim that the clarification of Weyl I'm suggesting is based on the use of Husserl's writings. However, since my goal aims to be a *clarification of Weyl's studies by means of Husserl's writings* and not a clarification of the comparison between Weyl and Husserl, I will add these quotations as footnotes. Anyway, I will enrich my analysis with remarks concerning a comparison between these two authors other than relevant historical background whenever it will be useful. Finally, I will add in the footnotes any further reference useful for the improvement of this continuous process of clarification.

A further remarks should be added. In doing this kind of clarification I will pay attention on another important issue. As we have shown in the previous section, some misunderstanding might occur even if readers are aware of Husserl's philosophy. For this reason I will make use of examples in order to lead the reader as close as possible to the underlying intuition I'm aiming to express by a given meaning.

This approach, however, has to deal with some issues. Husserl's terminology is rather accurate and such a clarification of Weyl's studies may give rise to some problems. It is not obvious that Weyl was so accurate from a philosophical point of view. Even if his mathematical works seem to be strongly influenced by his readings of Husserl, it doesn't mean he was always so careful of all phenomenological distinctions

[38]This approach may appear a bit odd. However, it is not so far from what it actually happens in the scientific community. Any scientific paper has to presuppose a certain amount of background. Take, for instance, a mathematical proof of a given theorem. The mathematician does not clearly explain every notion that is involved, but the comprehension of the proof in the research community relies on the fact that some knowledge is mutually shared. A step in the proof, for instance, might involve a notion that requires a "long deliberative process" of definition to be understood and it is clearly omitted. Also in this case some problems might arise and a continuous process of "clarification" in the community aims to face them. The outcome of this research process is the constitution of a *manual*. In the case of Weyl's scholars the situation is far from being so optimistic. Although a common knowledge might be shared, it is by no means sure that everyone agrees on that knowledge. Anyway, if we take at face value Husserl's phenomenology we should at least attempt to approach in that way.

that characterize Husserl's researches. The more we are careful in handling Husserl's terminology, the more our clarification is likely to be unfaithful to Weyl's works. Therefore, our analysis might not be able to fit properly Weyl's writings in some cases. However, I argue that Weyl would have agreed with this approach if he had been aware. Indeed, previous chapters established his adherence to Husserl's phenomenology. Moreover, Weyl is well aware that his mathematical studies do not avoid philosophical questions although they are primarily mathematical treatises. On one side he deals seriously with the philosophical aspects of his works but on the other side he does not care too much to give an exhaustive philosophical analysis. He explicitly supports this point of view in several occasions. In *Das Kontinuum* referring to his considerations on the nature of continuum he affirms:

> The reflections contained in this sections are, of course, only a slightly illuminating surrogate for a genuine philosophy of the continuum. [...] *our task is mathematical rather than epistemological*.[39]

Similarly, in the preface of *Raum-Zeit-Materie* he states:

> Philosophy, mathematics, and physics have each a share in the problems presented here. We shall, however, be concerned above all with the mathematical and physical aspects of these questions. I shall only touch lightly on the philosophical implications for the simple reason that in this direction nothing final has yet been reached, and that for my own part *I am not in a position to give such answers to the epistemological questions* involved as my conscience would allow me to uphold.[40]

Furthermore, in a lecture given at the Swiss Society of Gymnasium Teachers in the summer of 1931 Weyl speaks about the process of understanding and he admits that it strikes him "as very difficult to give a precise analysis of the relevant mental acts" (Weyl 1995, p. 454).

These remarks suggest that Weyl's studies have been developed taking account of Husserl's phenomenological framework, but they should be enlightened by deeper phenomenological analysis. Therefore, a phenomenological clarification of Weyl's writings seems to be appropriate even if our analysis might give rise to some discrepancy in some cases. In these cases my goal would be to improve Weyl's studies by an adequate phenomenological analysis and our considerations suggest that Weyl would have agreed if he had been aware. In this sense I might say that I'm suggesting a kind of *phenomenologically-informed reconstruction* of Weyl's thought.[41]

[39] Weyl (1994, 97, *my emphasis*).

[40] Weyl (1952, 2, *my emphasis*).

[41] It might seem odd such an approach, but it is nothing different to those studies that attempt a modern reconstruction of some mathematical works of a given author from the past. They aim at a clarification of these writings but their reconstruction is often unable to fit them completely. However, these studies claim to be a fair elaboration of the implicit intentions of such an author. For an important work of this kind related to Weyl's writings see, for instance, Scholz (2001).

In the next section I will attempt a phenomenologically-informed reconstruction of Weyl's *concept of surface*, but before that I would add a brief remark on the Husserlian texts I'm going to consider.[42] Our phenomenological clarifications will be based on the philosophical framework underlying the two main texts that Weyl quoted in the preface of *Das Kontinuum*, that is *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie I* and *Logische Untersuchungen*. He explicitly claims to have read these texts and to have been influenced by them.[43] For these reasons, a reconstruction based on these texts enhances the legitimacy of our approach. Anyway, it might happen that I will also consider other texts. In this case I will add few remarks case by case. Finally, it should be noted that Weyl quotes the second edition of *Logische Untersuchungen*, but it did not contain the revision of the sixth logical investigation. This part indeed appeared few years later in 1922. Husserl did not publish this section at that time because he had planned a complete revision of it. This project anyway turned out to be too complex and Husserl gave up. For this reason he published a revision of the sixth logical investigation in the same manner he did for the other logical investigations in 1922.[44] The revision is then an improvement of the first edition but it keeps its main structure as Husserl himself declares in the preface of the edition of 1922. For this reason the two editions can be considered equivalent with respect the main aspects. Moreover, Weyl read the second revised version of the sixth logical investigation and he expresses his admiration for this work in a letter sent to Husserl in 1921.[45] Therefore, I'm going to use the second edition of *Logische Untersuchungen* which contains the revision of the first five logical investigations published on 1913 and the sixth logical investigation published on 1922.

3.3 The Concept of Surface

The concept of surface discussed in *Das Kontinuum* represents an important case study among Weyl's attempts to establish a connection between a descriptive analysis of phenomena and their exact determination. In *Das Kontinuum* he writes:

[42]Observe that previous considerations themselves undergo the problems we have outlined, since we did not clarify all the Husserlian terminology involved. However, our goal has been to outline the main problems and call for the need for a solution. We haven't aimed at a proper clarification of the problems themselves. Our analysis of Weyl's concept of surface instead aims to follow the guidelines suggested by a phenomenologically-informed reconstruction.

[43]As already pointed out previously he claims: "Concerning the epistemological side of logic, I agree with the conceptions which underlie Husserl's *Logische Untersuchungen* (2d ed. Halle). The reader should also consult the deepened presentation in Husserl's *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie* which places the logical within the framework of a comprehensive philosophy" (Weyl 1994, p. 2).

[44]For further details, see Melle (2002).

[45]On the occasion of Husserl's gift of the second edition of the sixth logical investigation to Weyl and his wife, he writes: "You have made me and my wife very happy with the last volume of the *Logical Investigations*; and we thank you with admiration for this present. [. . .] Despite all the faults you attribute to the Logical Investigations from your present standpoint, I find the conclusive results of this work, which has rendered such an enormous service to the spirit of pure objectivity in epistemology, the decisive insights on evidence and truth, and the recognition that "intuition" extends beyond sensual intuition, established with great clarity and conciseness" (Weyl to Husserl, March 26-27, 1921). Translated from German by P. Mancosu and T. Ryckman in Mancosu (2010b, p. 280).

> As an example of how analytic concepts enable us to formulate geometric presentations in an exact way, we wish to conclude these investigations into the continuum by discussing the concepts "two-dimensional curve" and "spatial surface".[46]

In the first chapter we have shown that this kind of researches should be interpreted within the philosophical framework of Husserl's phenomenology. For this reason a phenomenological clarification of this example is rather appropriate.

We need first to clarify a few notions. Weyl speaks of a geometric presentation (geometrische Vorstellung) and he refers to some object given in intuition (Anschauung).[47] Nevertheless it is not so clear the nature of this intuition. Weyl indeed uses the property of being immediately given in an ambiguous way. He ascribes this property both to individual objects and to ideal objects. An individual object is, for instance, 'this red pen' or 'this table'; whereas an ideal object is, for instance, 'a red pen' or 'a table'. In *Philosophy of Mathematics and Natural Science*, for instance, he speaks about "point" as a category of object (Gegenstandskategorie) that is intuitively given:

> In Euclidean geometry we are concerned with three categories of objects, *points*, *lines*, and *planes*, which are not defined but assumed to be intuitively given.[48]

Weyl is clearly referring to an ideal object, a spatial point, and not to some specific spatial point given in an act of perception (if this last case makes sense at all). He emphasises this point in *Das Kontinuum*:

> Let a definite category of object (e.g., "spatial point") be "immediately given" (i.e. exhibited in intuition).[49]

In order to better understand this issue, we have to consider Husserl's distinction between the intentional object given in an individual (or sensuous) intuition (individuelle/sinnliche Anschauung) and the intentional object given in an universal intuition (allgemeinen Anschauung).[50] These two kinds of intuition are very different and now we are going to clarify in which sense. However, before we deal with that, we have to examine Husserl's notion of *abstract moment* (abstraktes Moment) (or *non-independent moment* - unselbständiges Moment). An abstract moment is a part of a larger whole that can't be presented by itself, but it has to be presented with other parts.[51] Some

[46] Weyl (1994, p. 101).

[47] Recall that Weyl's researches concern "the relations between what is intuitively given and the analytic concepts through which we seek to construct the given in geometry" (Weyl 1994, p. 2). In this case what is intuitively given is represented by the geometric presentation.

[48] Weyl (1949, p. 3).

[49] Weyl (1994, p. 8)

[50] It actually represents a specific kind of categorial intuition (kategoriale Anschauung). We are going to deal with it later. For the moment, note only that the intentional object given in an universal intuition is an essence (Wesen).

[51] "[...] they are parts which only exist as parts, that cannot be thought of as existing by themselves. The colour of this paper is a non-independent 'moment' of the paper. [...] a colour in general and purely as such can exist only as a 'moment' in a coloured thing" (Husserl 1970a, vol II, 12).

examples are the followings: a sensuous quality necessarily refers to a certain spread (e.g. the redness of this apple is given with some space that it covers); an auditory enhancement of a sound necessarily refers to a certain quality of sound (e.g. the increase of this music is given with some type of music).[52]

We now need to clarify the notion of individual intuition. The intentional object given in an individual intuition is immediately given in a straightforward manner.[53] For instance, consider the case when we turn our glance toward a bottle of wine. We perceive it and such a object is given to us in a simple act of perception. We can clearly recognizes several abstract moments in this unitary perception. We can grasp, for instance, the color of the bottle or its spatial shape. They are two different abstract moments, but the bottle does not appear before us as the mere sum of them or, more generally, as the mere sum of its abstract moments. We can indeed distinguish an abstract moment but we have to take note of it only in the ever complete, unified object. Further analysis can show the complexity of this perception, but we do not live such a complexity in the simple act of perception. A perception of a bottle of wine gives such an object in a straightforward manner. Let consider now a different situation. The situation when we first turn our glance toward a bottle of wine, then we focus on its spatial shape (abstract moment) and finally by another intentional act we brought to consciousness its universal, i.e. such a spatial shape *in general*. In this case we are dealing with an intentional object (a spatial shape in general) given in an universal intuition.[54]

In the light of these remarks we can clarify Weyl's use of the property of being immediately given. When he ascribes this property to individual objects, he is speaking of intentional objects given in individual intuitions. When he instead ascribes this property to ideal objects, he is speaking of intentional objects given in universal intuitions. Therefore we can better understand Weyl's notion of geometric presentation (geometrische Vorstellung). He refers to an intentional object given in an universal intuition. That means he is referring to the universal ascribed to an abstract moment that can be recognised in an intentional object given in an individual intuition. In order now to understand which universal intuition we are speaking about, let consider the following remark:

[52]Husserl speaks extensively on the relations between whole and parts in the third logical investigation. The notion of abstract moment has to be considered in the broader context of regional ontologies. Further analysis would be required in order to clarify this fundamental issue, but it is not our purpose here. For instance, see Husserl (1970a, vol II, §1-§25) and Husserl (1982, §9-§16).

[53]"In sense-perception, the 'external' thing appears 'in one blow', as soon as our glance falls upon it. The manner in which it makes the thing appear present is *straightforward* [...]" (Husserl 1970a, vol II, §47).

[54]"Here we have the field of the *universal intuition* [...] abstraction gets to work on a basis of primary intuitions, and with it a new categorial act-character emerges, in which a new style of objectivity becomes apparent, an objectivity which can only became apparent - whether given as 'real' or as 'merely imagined' - in just such a founded act. Naturally I do not here mean 'abstraction' merely in the sense of a setting-in-relief of some non-independent moment in a sensible object, but Ideational Abstraction, where no such non-independent moment, but its Idea, its Universal, is brought to consciousness, and achieve *actual givenness*" (Husserl 1970a, vol II, 292). Further analysis would be required in order to clarify this notion of universal intuition, but it is not our purpose here.

> We are interested in that concept of surface which is analogous to "curve" rather than "line".[55]

Weyl indeed makes a distinction between two different geometric presentations in plane geometry:

> In plane geometry we must distinguish two entirely different presentations which are usually both designed by the word curve. In order to keep them separate, I use the expressions "line" and "curve". Roughly stated, we are concerned with the distinction between the roadway system of a city or a street-car "line" on the one hand and, on the other hand, the route (= "curve") which a pedestrian traverses in the streets of this city (and which is *in statu nascendi* during the time of the stroll) or, respectively, the path which a moving street-car describes. "Lines" appear as, e.g., *boundaries* of parts of a region of the plane; a "curve" is the *path* of a moving point. [...] line (which can be called the "trace" or "track" of the movement) must be not be identified with the path of the point. If the tracks on which a freight train is to run are given, the train can still traverse very different paths, in particular, paths of very different lengths [depending on the course it takes at the switching points and on whether it moves both backwards and forwards].[56]

A curve then has to be distinguished from a line. A line is the geometric presentation, given in an universal intuition, that is the universal ascribed to the spatial shape[57] (abstract moment) of a line on a flat surface, given in an individual intuition (e.g. the perception of a line drawn on a paper or the perception of the trace of a movement of a colour spot on the cover of a book[58]). A curve instead is the geometric presentation, given in an universal intuition, that is the universal ascribed to the spatial moment (abstract moment) of the movement of a moving point on a flat surface, given in an individual intuition (e.g. the perception of a movement of a colour spot on the cover of a book[59]). Weyl indeed declares:

> It is essential to a "curve" [...] that it be exhibited only in a movement - as an abstract (non-independent) moment (abstraktes/unselbständiges Moment) of it.[60]

[55]Weyl (1994, p. 103).

[56]Weyl (1994, p. 101).

[57]It might be better to speaks of planar shape or boundaries shape, which can be then generalized as spatial shape. Husserl outlines some guidelines that would provide a complete classification of what he calls hierarchy of generality and specificity. Further analysis would be clearly required in order to clarify this issue, but it is not our purpose here. See, for instance, Husserl (1982, §12).

[58]Someone might wonder if we can actually experience the case of a moving colour spot. We can remark that at least it is possible to imagine it. Husserl indeed deals extensively on the interconnection between perception and imagination and for this reason such an issue can be handled. Nevertheless, it is enough to remark that we can actually perceive it with by means of digital technologies.

[59]Similar remarks on this kind of experience apply also in this case.

[60]Weyl (1994, p. 102).

Therefore, a curve cannot be identified with a line since the spatial moment (of a movement) has to be distinguished from the spatial shape.[61] The former indeed "contains" information on the stages of the movement, so to speak. The latter instead lacks such an information. Weyl claims in this respect:[62]

> Each path-point is in a definite place, i.e., coincides with a definite point of the plane, but is not itself this point of the plane. The path-points, as "stages" of the movement, stand in the relations "earlier" and "later" to one another.[63]

A more careful phenomenological analysis can point out that we can also recognize the universal ascribed to the shape of movement (abstract moment) of the movement.[64] The spatial moment then will turn out to be an abstract moment of such a shape of movement (considered now as a whole). In same way as the temporal moment is an abstract moment of it.[65] The shape of movement then would be an overlapping continua, i.e. the overlapping of a temporal extension by a linear spatial one. This interpretation is also supported by the following remark:

> [...] overlapping continua (a particular instance of which is given, e.g., by a moving point; i.e., the overlapping of a temporal continuum by a linear spatial one.[66]

To be precise, Weyl is actually referring to the shape of movement (in general) of a moving point in the space, but our considerations in two-dimensional space can be clearly extended in three-dimensional space. In this respect, we better understand the reason why Weyl deals extensively with the phenomenal time in *Das Kontinuum*. As well as providing a fundamental example of continuum, it is essential to investigate the physical phenomenon of motion.

We can then come back to the concept of surface. As seen before, the concept of surface is analogous to curve. It might mean that a surface is the geometric presentation, given in an universal intuition, that is the universal ascribed to the spatial moment (abstract moment) of the movement of a moving line in the space, given in an individual intuition (e.g. the perception of a movement of a moving deformable bar in the space). It has to be distinguished from the geometric presentation, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of a

[61]Recall that both have to be regarded *in general*.

[62]In this passage Weyl is already referring to the exact conceptualization of the spatial moment. He indeed speaks of *path-points*. Nevertheless we do not need to deal with this issue right now, whereas we are interested in pointing out the difference between a curve and a line.

[63]Weyl (1994, p. 102).

[64]Husserl makes reference to the notion of *shape of movement* in *Ideen I* as the abstract object (universal ascribed to the abstract moment) that can be found in a given regional ontology: "the province is made up either of concrete objects (as in the case of the eidetics of Nature) or else of abstract objects (such as spatial shapes, temporal shapes, or the shapes of movements)" (Husserl 1982, p. 162).

[65]Further clarifications on the relations between whole, parts, parts of parts, and the like, would be required. It is not our concern here anyway. See, for instance, Husserl (1970a, vol II, §1-§25).

[66]Weyl (1994, p. 93).

surface, given in an individual intuition (e.g. the perception of the cover of a book or the spatial extension of a shirt). The former "contains" information on the stages of the movement, so to speak. The latter instead lacks such an information. In analogy with before, Weyl claims that a *surface-point* has to be distinguished from the *space-point* in which the former is located. Moreover, a more careful phenomenological analysis can point out that also in this case we can recognize the universal ascribed to the shape of movement (abstract moment) of the movement. The spatial and temporal moments then will turn out to be abstract moments of such a shape of movement (considered now as a whole). Again, it would be an overlapping continua, i.e. the overlapping of a temporal extension by a spatial one. In this case anyway, the spatial extension (i.e. the surface) forms a continuum spread out in a dual way:

> [the surface consists] of "surface-points", i.e., *sui generis* elements which form a continuum spread out in a dual way.[67]

Several issues have needed to clarify in which sense Weyl speaks of a geometric presentation of a surface. I'm now going to clarify in which sense Weyl speaks of a formulation of such a geometric presentation by means of exact concepts. Again, we need to refer to Husserl's writings. A first distinction that Husserl outlines is between *concept*, conceived as meaning (Bedeutung)[68], and *essence* (Wesen). As seen before, the notion of essence has to be understood within the context of eidetic vision. Nevertheless, at this stage we can speak of essences as intentional objects given in universal intuitions.[69] A concept instead has to be understood within the context of meaning-intentions (Bedeutungsintentionen), but at this stage we can refer to it as the usual notion of concept. An essence then finds its expression in his corresponding concept.[70] For instance, the spatial shape of the first bottle of wine (essence)[71] finds its expression in the concept "Bordeuax". Or, the spatial shape of the second bottle of wine (essence)[72] finds its expression in the concept "Burgundy".

[67] Weyl (1994, p. 103).

[68] I translate "Bedeutung" as "meaning", but observe that "signification" is also used sometimes. I will not stress on this point unless I have to.

[69] See footnote 50.

[70] "we can understand, on the one hand, the concepts in the sense of significations but, on the other hand the [...] essences themselves which find their expression in those significations [...] things that must be separated throughout: "signification" and that which can undergo "expression" by signification [...] one can distinguish between [...] *concepts* (as significations) and [...] *essences*" (Husserl 1982, p. 22).

[71] The intentional object, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of the first bottle of wine, given in an individual intuition.

[72] The intentional object, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of the second bottle of wine, given in an individual intuition.

Further analysis would be required in order to clarify the very complex issue of how concepts properly express essences, but it not our purpose here.[73] Two related distinctions have to be addressed. The first concerns essences, between *morphological essences* (morphologischen Wesen) and *ideal essences* (idealen Wesen). The second concerns their respective concepts, between *descriptive concepts* (deskriptiven Begriffe) and *exact concepts* (Exakte Begriffe). Morphological essences find their expression in descriptive concepts, whereas ideal essences find their expression in exact concepts. Descriptive concepts are essential needed to express what is given exactly as it is given. The spatial shape of a coast[74], for instance, is properly expressed by the descriptive concept "jagged spatial shape". Similarly, the spatial shape of the border of a knife[75] is properly expressed by the descriptive concept "serrated spatial shape". The vagueness of these concepts does not make them defective since in these spheres of knowledge they are absolutely indispensable. They are essentially, rather than accidentally, inexact. They are the only legitimate concepts in these spheres of knowledge.[76]

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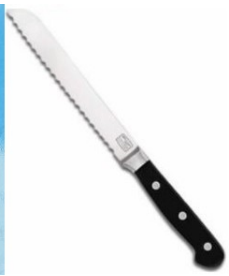
[73]"Extraordinarily difficult problems are related to the phenomena subsumed under the headings of "signifying" and "signification". (in footnote: as can be seen from the second volume of the *Logische Untersuchungen* where they form a major theme) [...] the problems of expression and signification are the most immediate for philosophers and psychologists guided by universal logical interests; and they are, therefore, the first to require a phenomenological inquiry into essence as soon as one seriously comes to seek out their ground. (in footnote: in fact, this was the way in which the *Logische Untersuchungen* endeavored to penetrate into phenomenology. A second way, starting from the opposite side, namely from the side of experience and sensuous givenness followed by the author since the beginning of the 1890's, was not fully expressed in that work" (Husserl 1982, p. 296). Husserl indeed deals extensively with this issue in *Logische Untersuchungen*, especially in the sixth investigation. In this book the notions of meaning-intention, meaning-fulfilment, adequation and other fundamental notions are carefully addressed.

[74]The intentional object, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of the coast, given in an individual intuition.

[75]The intentional object, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of the border of the knife, given in an individual intuition.

[76]"The geometer is not interested in *de facto* sensuously intuitable shapes, as the descriptive natural scientist is. He does not, like the latter, fashion *morphological concepts* of vague configurational types which are directly seized upon on the basis of sensuous intuition and which, in their vagueness, became conceptually and terminologically fixed. The *vagueness* of such concepts, the circumstance that their spheres of application are fluid, does not make them defective; for in the spheres of knowledge where they are used they are absolute indispensable, or in those spheres they are the only legitimate concepts. If the aim is to give appropriate conceptual expression to the intuitionally given essential characteristics of intuitionally given physical things, that means precisely that the latter must be taken as they are given. And they are given precisely as fluid; and typical essences can become seized upon as exemplified in them only in immediately analytic eidetic intuition. The most perfect geometry and the most perfect practical mastery of it cannot enable the descriptive natural scientist to express (in exact geometrical concepts) what he expresses in such a simple, understandable, and completely appropriate manner by the words "notches", "scalloped", "lens-shaped", "umbelliform", and the like - all to them concepts

Exact concepts, in contrast to descriptive concepts, express something that cannot be "seen". These concepts indeed have as their correlates ideal essences, which are grasped by *ideation* as ideal "limits" that cannot be found in any sensuous intuition. Morphological essences approaches them more or less closely without ever reaching them.[77] A sphere, for instance, is the ideal limit of the spatial shape of an orange[78]. Similarly, a parallelepiped is the ideal limit of the spatial shape of a box.[79]

This ideation, which yields ideal essences, has not to be confused with the case in which the universal ascribed to a specific abstract moment is brought to consciousness,

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which are *essentially, rather than accidentally, inexact* and *consequently* also non-mathematical. [...] we find *morphological essences* as the correlate of descriptive concepts" (Husserl 1982, p. 166).

[77]"in ordinary life, one speaks of sharp points and corners as opposed to blunt or even rounded ones. Plainly the essential forms of all intuitive data are not in principle to be brought under 'exact' or 'ideal' notions, such as we have in mathematics. The spatial shapes of the perceived tree as such, taken precisely as a 'moment' found in the relevant percept's intentional object, is no geometric shape, no ideal or exact shape in the sense of exact geometry. Just so a seen colour as such is no ideal colour, whose Species occupies an ideal point in the colour-pyramid. The essences which direct ideation elicits from intuitive data are 'inexact essences', they may not be confused with the 'exact' essences which are Ideas in the Kantian sense, and which (like an 'ideal point', an ideal surface or solid, or ideal Species of colour in the ideal colour-pyramid) arise through a peculiar 'idealization'. The descriptive concepts of all pure description, i.e. of description adapted to intuition immediately and with truth and so of all phenomenological description, differ in principle from those which dominate objective science. To clear up these matters is a phenomenological task never yet seriously undertaken and not carried out in relation to our present distinction" (Husserl 1970a, vol II, 15). A similar remark can be found in *Ideen*: "Geometrical concepts are *"ideal" concepts*, expressing something which cannot be "seen"; their "origin" and therefore their content are essentially other than those of *descriptive concepts*; as concepts they express, not "ideals", but essences drawn immediately from intuition simpliciter. Exact concepts have as their correlates essences which have the characteristic of *"ideas" in the Kantian sense*. Contrasted with these ideas, or ideal essences, we find *morphological essences* as the correlates of descriptive concepts" (Husserl 1982, p. 166).

[78]The intentional object, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of the orange, given in an individual intuition.

[79]The intentional object, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of the box, given in an individual intuition.

in an universal intuition, as something essentially vague.[80] That was indeed the case of our previous examples, as the spatial shape of a bottle of wine, of a coast or of the border of a knife. Idealization is a very complex issue and Husserl himself sketched only few guidelines in *Ideen*.[81] We are not going to deal extensively with it. Nevertheless some considerations on the outcome of such idealization, i.e. on the notion of ideal essence, is needed. Husserl refers to ideal essences as ideal "limits" or *ideas in the Kantian sense*.[82] These objects cannot be given in complete determinedness or intuitiveness in a closed consciousness, but they are properly given as an absolutely determined system of endless processes of continuous appearings approximating the same object which is always more precisely determined in a way in which all these processes are governed throughout by a fixed set of eidetic laws. Although these processes represents a *continuum of appearances*, the idea of this continuum is not itself an infinity and it is perfectly given.[83] A parallelepiped, for instance, is the ideal limit (idea in the Kantian sense) of the spatial shape of a box.[84] Such a parallelepiped cannot be given in any intuition, whereas it is given as an absolutely determined

[80]"That ideation which yields ideal essences [. . .] is fundamentally different in its essence from the seizing upon an essence by simple "abstraction" in which a salient "moment" is raised into the region of essences as something essentially vague, as something typical" (Husserl 1982, p. 167). Husserl refers to it also as Ideational Abstraction, see footnote 54.

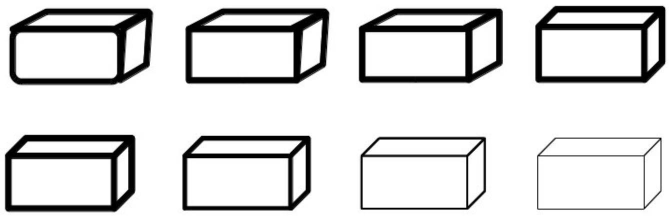
[81]"the fundamental and still unsolved problems pertaining to an essentially necessary clarification of the relationship between *"description"*, with its *"descriptive concept"*, and "unambiguous determination" or *"exact determination"*, with its *"ideal concepts"*, and, parallel with that, a clarification of the so-little understood relationship between "descriptive" and "explanatory" science. An attempt to deal with these problems will be communicated in the sequel to these investigations" (Husserl 1982, p. 165). Moreover, in a remarked dated circa 1914, Husserl declares: "here central difficulties arise, and we must not omit to consider how far they have been resolved".

[82]This notion has not be regarded properly as Kant's notion of *idea*. The reference to Kant instead has to understood in connection with Husserl's belief that his phenomenology represents a clarification of Kant's philosophy. In the second edition of the sixth logical investigation, for instance, he claims: "All the main obscurities of the Kantian critique of reason depend ultimately on the fact that Kant never made clear to himself the peculiar character of pure Ideation, the adequate survey of conceptual essences, and of the laws of universal validity rooted in those essences. He accordingly lacked the phenomenologically correct concept of the *a priori*. For this reason he could never rise to adopting the only possible aim of a strictly scientific critique of reason: the investigation of the pure, essential laws which govern acts as *intentional* experiences, in all their modes of sense-giving objectivation, and their fulfilling constitution of 'true being'. Only a perspicuous knowledge of these laws of essence could provide us with and absolutely adequate answer to all the questions regarding our understanding, questions which can be meaningfully raised in regard to the 'possibility of knowledge' " (Husserl 1970a, p. 319). I believe him to be right, but it is not my purpose to claim for it here. Nevertheless this remark makes sense of Husserl's reference to Kant.

[83]"*perfect givenness is nevertheless predesignated as "Idea"* (in the Kantian sense) — as a system which, in its eidetic type, is an absolutely determined system of endless processes of continuous appearings, or as a field of these processes, an a priori determined *continuum of appearances* with different, but determined, dimensions, and governed throughout by a fixed set of eidetic laws. This continuum is determined more precisely as infinite on all sides, consisting of appearances in all its phases of the same determinable X so ordered in its concatenations and so determined with respect to the essential contents that any of its *lines* yields, in its continuous course, a harmonious concatenation (which itself is to be designated as a unity of mobile appearances) in which the X, given always as one and the same, is more precisely and never "otherwise" continuously-harmoniously determined. [. . .] The idea of an infinity motivated in conformity with its essence is not itself an infinity; seeing intellectually that this infinity of necessity cannot be given does not exclude, but rather requires, the intellectually seen givenness of the *idea* of this infinity" (Husserl 1982, p. 342).

[84]The intentional object, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of the box, given in an individual intuition.

system of endless processes of continuous appearings approximating the same object, as sketched in the picture below.



Each of these appearance represents the spatial shape of a box[85] which comes closest to the ideal limit, i.e. the parallelepiped. Such a parallelepiped is always more precisely determined and this determination is governed by some eidetic laws, as is clear from the picture.

So far, we have dealt with essences as intentional objects given in universal intuitions. However, an universal intuition is a specific kind of categorial intuition (kategoriale Anschauung). For this reason we can speak of essences (intentional objects) given in a categorial intuition. As we have seen we face with an universal intuition, for instance, when we first turn our glance toward a bottle of wine, then on its spatial shape and finally by another intentional act we brought to consciousness its universal. In this case an act of perception (the perception of the bottle of wine) serves as a basic act for new acts which bring to consciousness a new awareness of objects (the spatial shape in general) which is essentially related to what appears in the basic act (the bottle of wine). Its manner of appearance is essentially determined by this relation. We deal with categorial intuitions when we are in this kind of situation, i.e. in the case of new acts founded on other acts that bring other objects to perception. They can be, for instance, acts of conjunction, of disjunction, relation knowledge or of generalization (as in the case of universal intuition). Such acts set up new objects which were not given and could not have been given in the grounding acts alone. Nevertheless their manner of appearance is essentially related to what appears in the grounding acts.[86] For instance, consider the case when we first turn our glance toward a blue pen on a table and then toward the red pen next to it. Either of these two perceptions gives us a relation between the two pens. We have to focus our perception

[85]The intentional object, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of the box, given in an individual intuition.

[86]"Each straightforward act of perception, by itself or together with other acts, can serve as basic act for new acts which at times include it, at time merely presuppose it, acts which in their new mode of consciousness likewise bring to maturity *a new awareness of objects which essentially presupposes the old*. When the new acts of conjunction, of disjunction, of definite and indefinite individual apprehension (that - something), of generalization, of straightforward, relational and connective knowledge, arise, we do not then have *any* sort of subjective experiences, nor just acts connected with the original ones. What we have are acts which, as we said, *set up new objects*, acts in which something *appears as actual and self-given*, which was not given, and could not have been given, as what it now appears to be, in these foundational acts alone. *On the other hand, the new objects are based on the older ones, they are related to what appears in the basic acts*. Their manner of appearance is essentially determined by this relation. We are here dealing with a sphere of objects, *which can only show themselves 'in person' in such founded acts*" (Husserl 1970a, vol II, 282).

of one of them, say, the red pen. Once the red pen has became the "main theme" of our attention, we can wander from it to the other pen but still retaining the red pen as the main theme of our attention. This lays the foundations of a further intentional acts that can bring to consciousness a new essence (intentional object), i.e. the state of affairs 'the red pen lies on the right of the blue one'.[87]

This kind of states of affairs falls under the general type of relation of part to parts within a whole.[88] For this reason a better phenomenological clarification of this situation would required a proper examination of the general relations between whole and parts. Anyway, a complete clarification of the categorial intuition of this state of affairs does not concern us at this stage. For the moment, we need just to make clear that a categorial intuition of essences refers to any intentional act that is grounded on other acts, which can be individual (as in the case shown above) or categorial intuitions.[89] These essences, moreover, are intentional objects in a broad sense, i.e. any subject of possible predications.[90]

Any essence given in a categorial intuition finds its expression in his corresponding concept. Our previous considerations on this issue clearly applies here, but now that we have dealt with objects in a broader sense, as the case of state of affairs, a few remarks seems to be needed. Let consider the picture below. The state of affairs 'with respect to the blue pen, the red pen is closer than the green one' is a morphological states

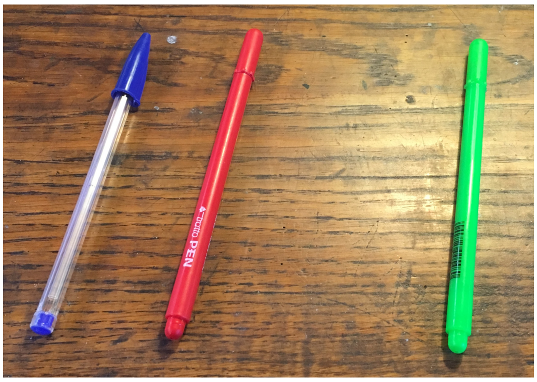
[87]As we have seen an essence is an object in a broad sense, i.e. any subject of possible predications. It is possible to state, for instance, that the state of affairs 'the red pen lies on the right of the blue one' is true or is expressed by the proposition "'the red pen lies on the right of the blue one".

[88]"in the case of *external* relations, from which predications such as 'A is to the right of B', 'A is larger, brighter, louder than B etc.', take their rise. Wherever sensible objects - directly and independently perceptible - are brought together, despite their mutual exclusion, into more or less intimate unities, into what fundamentally are more comprehensive objects, then a possibility of such external relations arises. They all fall under the general type of the relation of *part to parts within a whole*" (Husserl 1970a, vol II, 288).

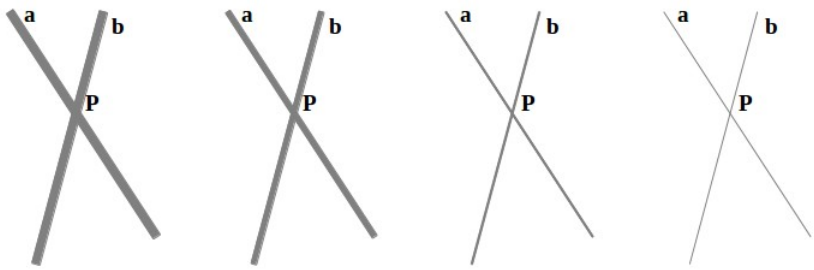
[89]"The varied forms of founded acts where, instead of straightforward, sensuously-intuitive objects, categorially formed and synthetically connected objects are constituted, permit manifold complications into new forms: in consequence of certain *a priori* categorial laws, categorial unities may again and again become the objects of new connecting, relating or ideating acts. Universal objects, e.g., can be collectively connected, the collections thus formed can in their turn be collectively connected with other collections of similar or different type, and so on *in infinitum*. [...] one can unify states of affairs in new states of affairs, pursue an indefinitely extended search for internal and external relations among all such possible unities, use the results of such discovery as terms for novel relations etc" (Husserl 1970a, vol II, 306).

[90]Further analysis would be required in order to clarify the notion of categorial intuition and how it is connected with eidetic vision. Anyway, it is not our purpose here. Husserl deals extensively with the notion of categorial intuition in Husserl (1970a, vol II, §40-§66).

of affairs (morphological essence)[91] and it is properly expressed by the descriptive concept "with respect to the blue pen, the red pen is closer than the green one".[92] This concept is essentially, rather than accidentally, inexact.[93]



In the picture below, instead, the state of affairs 'the lines a and b intersect at the point P' is an ideal states of affairs (ideal essence), that is, the ideal limit of the state of affairs 'the lines a and b drawn on the paper intersect at the point P drawn on the paper'.[94] And it is properly expressed by the exact concept "the lines a and b intersect at the point P".



Now we need to point out that the exactness of exact concepts has not be confused with the exactness of concepts that express formal essences.[95] Formal essences are given in categorial intuitions and they pertain to the eidetic science of any object whatever. They are, for instance, any state of affairs whatever, any plurality whatever, any number whatever, and the like. They are expressed by the concepts "state of affairs",

[91] The intentional object, given in a categorial intuition which is grounded on other acts that lead back to sense perception.

[92] Observe that any statement is an act of meaning. Following our previous considerations, we can then refer to it as a concept.

[93] I believe that a further clarification of this issue would consider Husserl's investigations on *pure theory of semantic forms*. It would enable us to clarify the connection between the whole statement and its parts. For instance, how the descriptive connotation of the concept "closer" is related to the descriptive connotation of the whole statement. Anyway, it is not our purpose here.

[94] The intentional object, given in a categorial intuition which is grounded on other acts that lead back to sense perception.

[95] "must not be confused with [...] the exactness of concepts of formal ontology which are exact as such". Husserl wrote down this remark on a copy of *Ideen*. He was used to revise and make notes on his first publications of the 1913. This note can be fairly dated around 1913-1915.

"plurality", "number", and the like.[96] Formal essences, moreover, are characterized by the fact that they are given in particular categorial intuitions. These intuitions indeed are grounded on other categorial intuitions (which, in turn, will be grounded on other acts) and they exclude all sensibility.[97] For instance, an intentional act grounded on a categorial intuition of the relation 'the first ruler is longer than the second one' can brought to consciousness the formal essence any relation (between two objects) whatever, so that everything sensuous is excluded.[98] The categorial intuition of formal essences is a very complex issue. Anyway, a complete phenomenological clarification is not our purpose here. For the moment, we need just to make clear that such an intuition allows us to obtain formal objects starting from objects that are connected with sensibility. These objects pertain to the eidetic science of any object whatever, also called *formal ontology* or *mathesis universalis*.

We can come back to Weyl and clarify in which sense he speaks of a formulation of a geometric presentation by means of exact concepts. We have seen that Weyl's geometric presentation of a surface has to be understood as the intentional object, given in an universal intuition, that is the universal ascribed to the spatial moment (abstract moment) of the movement of a moving line in the space, given in an individual intuition (e.g. the perception of a movement of a moving deformable bar in the space). Such a geometric presentation is clearly a morphological essence and it is properly expressed by the descriptive concept 'spatial moment (of a movement)' or 'spatial extension (of a movement)'. A fundamental question then arises. Is it possible to express such a geometric presentation by means of an exact concept? To be clear, formulating geometric presentations in an exact way does not mean that we have to search for some exact concepts that express those morphological essences. This would be impossible. Morphological essences find their expression only in descriptive concepts. The general problem raised by Weyl is instead the following: *it is possible an idealizing procedure which substitutes ideal essences for morphological essences?*[99]

[96]"formal ontology [...] which, as we know, is the eidetic science of any object whatever. [...] We define now as *logical categories* [...] the concepts by means of which, in the total set of axioms, the logical essence of any object whatever becomes determined, or the concepts which express the unconditionally necessary and constituent determinations of an object as object, of anything whatever in so far as it can be something it all. Accordingly, concepts such as property, relative determination, predicatively formed affair-complex, relationship, identity, equality, aggregate (collection), cardinal number, whole and part, genus and species, and the like, are examples of logical categories" (Husserl 1982, p. 21).

[97]"intuitions which not merely exclude all individuality, but also all sensibility from their intentional purview. [...] find their immediate basis in the data of categorial intuition, purely with regard to the categorial form of the whole categorially formed object" (Husserl 1970a, vol II, 306). Observe that universal intuitions exclude all individuality (this very spatial shape of the bottle of wine regarded as an abstract moment of this very bottle of wine), but they do not exclude all sensibility. What is given in the intuition indeed is its universal which does not pertain to the eidetic science of any object whatever. It pertains to the eidetic science concerning the material thing.

[98]"If, e.g., the intuition of a relation underlies an abstraction, the abstractive consciousness may direct itself to the relational form *in specie*, so that everything sensuous in what underlies the relation is discounted. So arise *categories*" (Husserl 1970a, vol II, 307).

[99]This interpretation is supported by Husserl's remark: "As for phenomenology, it is concerned to be a descriptive eidetic doctrine of transcendentally pure mental processes as viewed in the phenomenological attitude; and, like any other descriptive, non-substructing and non-idealizing discipline, it has its inherent legitimacy" (Husserl 1982, p. 167). Later he further remarks: "[...] In the eidetic province of reduced phenomena (either as a whole or in some partial province), this admittedly does not answer the pressing

In the case of a surface it would mean we should be able to substitute such a morphological essence with an ideal essence. An ideal essence should be an ideal "limit" of a surface. As it might be the case of a sphere with respect to the spatial shape of an orange. Or a parallelepiped with respect to the spatial shape of a box. Since ideal essences are ideas in the Kantian sense, we will be able then to establish a proper connection between the two kinds of essences. Nevertheless it is not the case. There do not exist any ideal "limit" of a surface, or we could also say that there do not exist any exact concept that can be correlated to the descriptive concept 'spatial extension (of a movement)' as a "limit". This fact is essentially due to the peculiar nature of continuum that characterizes the surface. For this reason Weyl has to follow another path. He first considers the surface as a whole, i.e. a system of parts, in relation to one another, that determine a interconnected form. We need to better clarify this point. For this reason we consider again Husserl's investigations. In act of meaning we have to distinguish between the *quality* (Qualität) and the *matter* (Materie) (or *interpretative sense* - Auffassungssinn). Any of them has not to be confused with the intentional object of the act. The quality represents that particular mode of relation of an act to its object that marks an act as judgemental, emotional or desiderative.[100] For instance, we can identify the same quality within the following groups of acts of meaning, whereas between the groups there exist a different quality:

- *the bottle of wine is empty*;
- *there are four orange in the box*;

- *I wish the bottle of wine isn't empty*;
- *I wish the market is still open*;

- *is the bottle of wine empty?*;
- *is it still raining?*.

Each act of meaning of the first group is qualified as an assertion, of the second group as a desire and of the third group as a question. The matter instead represents that particular mode of relation of an act to its object that fixes the way in which the object

question of whether, besides the descriptive procedure, one might not follow - as a counterpart to descriptive phenomenology - an idealizing procedure which substitutes pure and strict ideals for intuited data" (Husserl 1982, p. 169).

[100]"the distinction between the general act-character, which stamps an act as merely presentative, judgemental, emotional, desiderative etc., and its 'content' which stamps it as presenting *this*, as judging *that* etc. etc. The two assertions '$2 \times 2 = 4$' and 'Ibsen is the principal founder of modern dramatic realism', are both, *qua* assertions, of one kind; each is qualified as an assertion, and their common feature is their *judgement-quality*" (Husserl 1970a, vol II, 119).

is meant.[101] For instance, in each of the following groups, we can identify a different matter between the acts of meaning. They both refer to the same object, but they mean it in a different way:

- *it is a equilateral triangle*;
- *it is a equiangular triangle*;

- *it is a length of $a + b$ units*;
- *it is a length of $b + a$ units*.

Observe that a difference in matter does not coincide with a difference in quality. We can keep, for instance, the matter and vary the quality:

- *it is a equilateral triangle*;
- *is it a equilateral triangle?*.

Or we can keep the quality and vary the matter:

- *is it a equilateral triangle?*;
- *is it a equiangular triangle?*.

Finally, observe that both quality and matter has not to be confused with the intentional object of the act. For instance, the two acts of meaning:

- *the bottle of wine is empty*,
- *is the bottle of wine empty?*,

refer to the same state of affairs, but they have different quality. Similarly, in the case of matter. A special case of varying the matter occurs when an essence finds its expression in a new act of meaning that ascribes to it a role of being a whole in relation

[101] "The matter, therefore, must be *that element in an act which first gives it reference to an object, and reference so wholly definite that it not merely fixes the object meant in a general way, but also the precise way in which it is meant*. The matter - to carry clearness a little further . is that peculiar side of an act's phenomenological content that not only determines *that* it grasps the object but also *as what* it grasps it, the properties, relations, categorial forms, that it itself attributes to it. It is the act's matter that makes its object count as this object and no other, it is the *objective, the interpretative sense* which serves as basis for the act's quality (while indifferent to such qualitative differences)" (Husserl 1970a, vol II, 121).

with its parts.[102] For instance, a square[103] is properly expressed by means of the concept 'square'. But such an essence can also find its expression in the concept 'the geometric figure that consists of four equal sides, pairwise parallel and orthogonal'. We have then changed the matter, i.e. the way in which the object is meant. Specifically, now the square is regarded as being a whole in relation with its parts.[104] Coming back to the surface we can then say that considering a surface as a system of parts means that we aim to express such an essence by a concept (act of meaning) with a different matter. The surface then finds its expression in the concept 'a system of parts, in relation to one another, that determine a interconnected form'. Before going on, let's observe that such a change does not modify the essence itself. The object is given as it was before given, the new concept (act of meaning) ascribes to the essences just a role, it puts it just into relation.[105]

Weyl aims then to express a surface in this way, but not only. He aims to express it by means of exact concepts, i.e. parts and their mutual relations have to formulated in an exact way. That is, parts and their mutual relations have to be ideal essences. Nevertheless, aiming to express the surface in this way does not mean, of course, that it is achievable. It should be possible indeed to exhibit in immediate intuition that these essences are proper ideal "limits" traceable within the surface. It shouldn't be a matter of free choice.[106] In order to uncover to what extent it is possible, a more detailed phenomenological description of surface, regarded as a whole, is needed. The surface is an *extended whole*. That means it be fragmented into a plurality of mutually exclusive parts, which are of the same genus of the whole. We call these parts *extended*

[102]"*Objectifying acts which exist purely 'on their own', and 'the same' objectifying acts serving to constitute the terms of some relation or other*, are not really the same acts: they *differ phenomenologically, and differ in respect of what we have called their intentional matter*. Their *interpretative sense* has changed, and hence the *changed meaning* of their adequate expression. [...] The object does not appear before us with new real (*realen*) properties; it stands before us as this same object, but in a *new manner*. Its fitting into its categorial context gives it a definite place and role in this context, the role of a *relatum*" (Husserl 1970a, vol II, 289). He further adds: "relations as that of whole and part" (Husserl 1970a, vol II, 290).

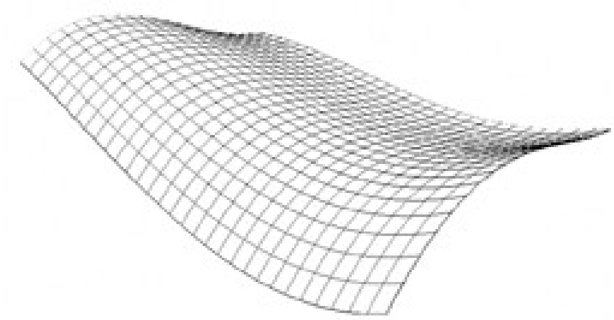
[103]That is the ideal limit (idea in the Kantian sense) of the intentional object, given in an universal intuition, that is the universal ascribed to the spatial shape (abstract moment) of a squared box, given in an individual intuition.

[104]Of course, a better understanding of this situation should clarify which kinds of relations are involved and so on.

[105]Husserl actually struggles with this issue, but he claims that there are not phenomenological evidences for any change in the object itself: "The gradual constitution of the object has been completed, as a finished object it becomes a term in a relation: it keeps, it seems, its constitutive sense quite unaltered. One can certainly say that the phenomenological change in sense made by entry into a relational act is at first masked by the very fact that the new form includes the whole previous interpretative sense in itself, to which it only imparts the new sense of a 'role'. Perception remains perception, the object is given as it was before given, 'only' it is 'put into relation' " (Husserl 1970a, vol II, 290).

[106]"A highly significant problem pertaining to the theory of science is that of becoming completely clear about all the relevant essential questions involved here [...] One necessary condition is exactness in "*conception-formation*", which is by no means a matter of free choice and logical technique; rather, in the case of the supposedly axiomatic concepts which, after all, must be demonstrable in immediate intuition, it presupposes *exactness in the essences themselves which are seized upon* (Husserl 1982, p. 165).

parts.[107] We can indeed fragment the surface into several smaller surfaces.[108]



Observe further that these parts are not isolated. That means they still have a common identical abstract moment. Two adjoining parts of a surface indeed share a common boundary, which is an abstract moment of both parts. We can recognize it observing that a boundary is a part of any of two parts of surface (considered now as whole) that can't be presented by itself but it has to be presented as a "limit" of the continuous extension. In the light of these considerations ideal essences do not seem to be easily recognizable. A smaller surface indeed is still a surface, i.e. a morphological essence for which does not exist any ideal "limit". Similarly, the relations between these smaller surfaces do not seem to be easily replaceable with ideal essences.

This represents a very rough phenomenological description of the surface, but it is useful to grasp some essential features. From this very starting point Weyl's mathematical investigations properly begins. His research aims to identify a suitable idealizing procedure that is able to account of the geometric presentation of a surface. Weyl begins with a strong assumption. That is possible to place within the surface an isolated spatial point. He explicitly states it in the case of phenomenal time:

> In order to have some hope of connecting phenomenal time with the world of mathematical concepts, let us grant the ideal possibility that a rigidly punctual "now" can be placed within this species of time and that time-points can be exhibited.[109]

Anyway the continuum of spatial extension (of a movement) is analogous to the continuum of phenomenal time:

> Corresponding remarks [on phenomenal time] apply to every intuitively given continuum; in particular, to the continuum of spatial extension.[110]

[107] "When a whole permits the sort of 'piecing' in which the pieces essentially belong to the same lowest Genus as is determined by the undivided whole, we speak of it as an *extended whole*, and of its pieces as *extended parts*. Here belong, e.g., the division of an extent into extents, in particular of a spatial stretch into spatial stretches, of a temporal stretch into temporal stretches etc" (Husserl 1970a, vol II, 29).

[108] These smaller surfaces are the intentional objects, given in an universal intuition, that is the universal ascribed to the spatial moment (abstract moment) of the fragmented movement of a moving line in the space, given in an individual intuition (e.g. the perception of a movement of a moving deformable bar in the space). Observe that the picture has to be understood in this sense.

[109] Weyl (1994, p. 88).

[110] Weyl (1994, p. 92).

For this reason a similar remark is meant also for the spatial case. In this sense we can understand Weyl's remark in *Raum-Zeit-Materie*:

> [...] we fix an exact "here", a point in space, as the first element of continuous spatial extension, which like time, is infinitely divisible.[111]

Therefore we can speak of a surface as an extension of isolated spatial points. They are clearly ideal essences and they form a continuum spread out in a dual way. Weyl refers to it as a *surface in itself* (Fläche an sich):

> [the surface consists] of "surface-points", i.e., *sui generis* elements which form a continuum spread out in a dual way - a continuum which we shall call the "surface in itself".[112]

Given this assumption we now deal with an extension of isolated spatial points. The next step is then to identify which kinds of relations occur between these objects. This however is not an easy task. One of the main problems can be well-represented by the following example.

The state of affairs, say A, 'with respect to P, Q is closer than T' is a morphological essence that cannot be easily substitutable by an ideal essence.[113] Moreover, we would be able to express in exact way few issues. For instance given the state of affairs, say B, 'with respect to P, O is closer than T',

then we might want to express the state of affairs 'the difference between A and B is that O is closer than Q to P'. It is a morphological essence and again it does not seem easily substitutable by an ideal essence. For this reason Weyl faces this issue making use of the so-called *transfer principle* (Übertragungsprinzip). He declares:

> When we give the mathematical formulation, we replace the "surface in itself" by a set S (appearing in pure number theory) of objects whose category one may freely choose; the elements of this set are the surface-points. (We shall not discuss the "transfer principle" (Übertragungsprinzip) which, on the basis of intrinsic relations existing between the surface-points, leads us from the latter to these objects of pure analysis, much as the concept of coordinates leads us from the space-points to triples of real numbers).[114]

[111] Weyl (1952, p. 11).
[112] Weyl (1994, p. 103).
[113] Observe that although the point P, Q and T are ideal essences, the state of affairs is not. Similar remarks apply in the following cases.
[114] Weyl (1994, p. 103).

The main idea is then to transfer any insight gained into the "surface in itself" to another suitable field of inquiry, appearing in pure number theory. The last one is a domain of exact essences and we can deal with it as we were dealing with a "surface in itself". In this way we have reached our goal to substitute ideal essences for morphological essences, i.e. we have formulated the geometric presentation of a surface by means of exact concepts.[115]

We stop our analysis here, but of course it should be continued and further improved in a never-ending process of clarification. Anyway, I wish that the phenomenologically-oriented reconstruction of Weyl's concept of surface proposed in this section provides a good example of how a phenomenological clarification of Weyl's studies can be carried out.

[115]As far as I know Husserl does not claim for such an approach. Although some connections can be found in Husserl's notion of *pure theory of manifold* (reine Mannigfaltigkeitslehre), he does not explicitly claim for any transfer principle. Anyway our goal is a clarification of Weyl's thought by means of Husserl's philosophy and not a comparison between them. For this reason we do not deal with this issue.

Conclusion

Previous chapters have attempted to shed light on Weyl's mathematical and philosophical works. We have seen how these studies should be understood within the phenomenological framework of Husserl's philosophy, especially with respect Husserl's distinction between a descriptive analysis of a field of inquiry and its exact determination. Several efforts are made by Weyl to provide a mathematical account of the phenomenal continuum of space and time whose meaning can be phenomenologically clarified within the domain of our experience. These studies, however, are not restricted to the analysis of continuum but they concerns the general issue of developing a phenomenologically meaningful mathematical understanding of real world and just not a formal account whose connection with the real world is lost.

Each chapter is valuable for at least one reason. The first chapter provides us the hermeneutical framework according to which we should interpret other Weyl's studies. In that respect a comparison between Weyl's and Becker's works on these subject matters may be very fruitful. The second chapter instead sets the stage for further historical investigations. Weyl's use of infinitesimal quantities raises several issues and further studies might shed light on this rich period in the history of mathematics. Specifically, a deeper historical investigation on those mathematicians dealing with the arising differential geometry might point out some philosophical aspects that have been usually overlooked by historians and philosophers of mathematics of the beginning of 20th century, traditionally focused on calculus and the arising mathematical logic. Finally, I believe the phenomenologically-informed reconstruction proposed in the last chapter is a promising starting point to develop a genuinely phenomenological philosophy of science. The phenomenologically-informed reconstruction of Weyl's notion of surface seems indeed to be a good starting point for a critical evaluation of Weyl's approach and a comparison with other philosophical views. Starting from here we might also be able to better understand to what extent Husserl's phenomenology can provide a foundational framework for our mathematical understanding of real world.

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