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# Radial mollifiers, mean value operators and harmonic functions in Dunkl theory

Léonard GALLARDO\* and Chaabane REJEB†

## Abstract

In this paper we show how to use mollifiers to regularise functions relative to a set of Dunkl operators in  $\mathbb{R}^d$  with Coxeter-Weyl group  $W$ , multiplicity function  $k$  and weight function  $\omega_k$ . In particular for  $\Omega$  a  $W$ -invariant open subset of  $\mathbb{R}^d$ , for  $\phi \in \mathcal{D}(\mathbb{R}^d)$  a radial function and  $u \in L^1_{loc}(\Omega, \omega_k(x)dx)$ , we study the Dunkl-convolution product  $u *_k \phi$  and the action of the Dunkl-Laplacian and the volume mean operators on these functions. The results are then applied to obtain an analog of the Weyl lemma for Dunkl-harmonic functions and to characterize them by invariance properties relative to mean value and convolution operators.

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## 1 Introduction

Let  $R$  be a normalized root system in  $\mathbb{R}^d$  and  $k \geq 0$  a multiplicity function on  $R$ . The Dunkl-Laplacian operator associated to  $R$  and  $k$ , acting on  $C^2(\mathbb{R}^d)$ -functions is a differential-difference operator of the form

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left( \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right), \quad (1.1)$$

where  $\Delta$  (resp.  $\nabla$ ) is the usual Laplace (resp. gradient) operator,  $R_+$  is a fixed positive subsystem of  $R$  and  $\sigma_\alpha$  is the reflexion directed by the root  $\alpha$  (see [6]). We recall that  $R$  is normalized if  $\|\alpha\|^2 = 2$  for all  $\alpha \in R$  and that  $k$  is invariant under the action of the Coxeter-Weyl group  $W$  generated by the reflections  $\sigma_\alpha$ ,  $\alpha \in R$ .

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We know that the Dunkl-Laplace operator can be written  $\Delta_k = \sum_{j=1}^d D_{\xi_j}^2$ , for  $\xi_j$ ,  $j = 1, \dots, d$ , an orthonormal basis of  $\mathbb{R}^d$  and where for every  $\xi \in \mathbb{R}^d$ ,  $D_\xi$  is the Dunkl operator defined by

$$D_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad (1.2)$$

with  $\partial_\xi f$  denoting the usual  $\xi$ -directional derivative of  $f$  (see [3] and [6]). These operators are related to partial derivatives by means of the so-called Dunkl intertwining operator  $V_k$  (see [5] or [6]) as follows

$$\forall \xi \in \mathbb{R}^d, \quad D_\xi V_k = V_k \partial_\xi. \quad (1.3)$$

The operator  $V_k$  is a topological isomorphism from the space  $\mathcal{C}^\infty(\mathbb{R}^d)^1$  onto itself satisfying (1.3) and  $V_k(1) = 1$  (see [17]). Furthermore, according to [13] or [14], for every  $x \in \mathbb{R}^d$  there exists a unique probability measure  $\mu_x$  on  $\mathbb{R}^d$  with compact support contained in the convex hull of the orbit of  $x$  under the group  $W$ , such that

$$\forall f \in \mathcal{C}^\infty(\mathbb{R}^d), \quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y). \quad (1.4)$$

For abbreviation and later use, we introduce the weight function

$$\omega_k(x) := \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)} \quad (1.5)$$

and the set  $\mathcal{D}_r^+(\mathbb{R}^d)$ ,  $r > 0$ , of nonnegative radial  $C^\infty$ -functions with compact support contained in the Euclidean closed ball  $B(0, r)$ .

Let  $\Omega \subset \mathbb{R}^d$  be a (nonempty)  $W$ -invariant open set and  $\phi$  a radial mollifier i.e.  $\phi \in \mathcal{D}_r^+(\mathbb{R}^d)$  with  $r > 0$  small enough in order that the open set

$$\Omega_r := \{x \in \Omega; \text{dist}(x, \partial\Omega) > r\} \quad (1.6)$$

is nonempty.

For every  $u \in L_{loc}^1(\Omega, m_k)$ , with  $dm_k = \omega_k(x)dx$ , we define the Dunkl-convolution product

$$u *_k \phi(x) := \int_{\mathbb{R}^d} u(y) \tau_{-x} \phi(y) \omega_k(y) dy, \quad (1.7)$$

where  $\tau_x$ ,  $x \in \mathbb{R}^d$ , are the Dunkl translation operators (see Annex A.2). Note also that a very useful formula for the Dunkl translation has been obtained by M. Rösler ([15]) when  $f \in \mathcal{C}^\infty(\mathbb{R}^d)$  is a radial function. In such case, the operator  $\tau_x$  is given by

$$\forall y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} \tilde{f}(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, z \rangle}) d\mu_y(z), \quad (1.8)$$

where  $\tilde{f}$  is the profile function of  $f$  defined by  $f(x) = \tilde{f}(\|x\|)$ .

This formula shows that the Dunkl translation operators are positivity preserving on the set of radial functions whereas this is not true in general ([12] or [16]).

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<sup>1</sup>carrying its usual Fréchet topology.

We turn now to the content and the organization of this paper. In section 2, we recall the properties of the volume mean value operator  $M_B^r$ ,  $r > 0$ , acting on continuous functions as follows

$$M_B^r(f)(x) := \frac{1}{m_k(B(0, r))} \int_{\mathbb{R}^d} f(y) h_k(r, x, y) \omega_k(y) dy. \quad (1.9)$$

where  $y \mapsto h_k(r, x, y)$  is a compactly supported measurable function (called harmonic kernel, see [8]) given by

$$h_k(r, x, y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0, r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle}) d\mu_y(z). \quad (1.10)$$

In section 3, we study the Dunkl convolution product (1.7). In particular, We will prove that this function is well defined and is of class  $C^\infty$  on  $\Omega_r$ . Moreover, we will show the commutativity relations

$$\Delta_k(u *_k \phi) = u *_k \Delta_k \phi \quad \text{and} \quad M_B^r(u *_k \phi) = M_B^r(u) *_k \phi$$

and the following associativity property

$$(u *_k \phi) *_k \psi = (u *_k \psi) *_k \phi$$

In section 4, we apply the previous results to give some new properties of  $\Delta_k$ -harmonic functions (i.e. functions  $u \in C^2(\Omega)$  such that  $\Delta_k u = 0$ ). As a first result we show that any  $\Delta_k$ -harmonic function on  $\Omega$  is in fact of class  $C^\infty$ . Then we prove that  $\Delta_k$ -harmonic functions can be characterized by the local-volume mean value property i.e.

$$\forall x \in \Omega, \exists r_x > 0, \forall r < r_x, u(x) = M_B^r(u)(x).$$

Finally we will establish the following Weyl's lemma: If  $u \in L_{loc}^1(\Omega, m_k)$  satisfies  $\Delta_k(u\omega_k) = 0$  in distributional sense, then  $u$  coincides almost everywhere with a  $\Delta_k$ -harmonic function on  $\Omega$ .

**Notations:** Let us introduce the following functional spaces and notations which will be used throughout the paper. For  $\Omega$  a (nonempty)  $W$ -invariant open subset of  $\mathbb{R}^d$ , we denote by:

- $L_{k,loc}^1(\Omega) = L_{loc}^1(\Omega, m_k)$  the space of measurable functions  $f : \Omega \rightarrow \mathbb{C}$  such that  $\int_K |f(x)| \omega_k(x) dx < +\infty$  for any compact set  $K \subset \Omega$ .
- $\mathcal{D}(\Omega)$  the space of  $C^\infty$ -functions on  $\Omega$  with compact support.
- $\mathcal{D}'(\Omega)$  the space of distributions on  $\Omega$  (i.e. the topological dual of  $\mathcal{D}(\Omega)$  carrying the Fréchet topology).
- $\mathcal{S}(\mathbb{R}^d)$  the Schwartz space of  $C^\infty$ -functions on  $\mathbb{R}^d$  which are rapidly decreasing together with their derivatives.
- $B(a, \rho)$  (resp.  $\overset{\circ}{B}(a, \rho)$ ) the closed (resp. the open) Euclidean ball centered at  $a$  and with radius  $\rho > 0$ .
- $\mathcal{D}_r^+(\mathbb{R}^d)$ ,  $r > 0$ , the set of radial mollifiers with support contained in  $B(0, r)$ .

## 2 The mean value operators

In this section, we will recall some facts about the volume and the spherical mean operators in Dunkl setting.

Let  $(r, x, y) \mapsto h_k(r, x, y)$  be the harmonic kernel defined by (1.10). We note that when the multiplicity function is the zero function, we have  $\mu_y = \delta_y$ ,  $h_0(r, x, y) = \mathbf{1}_{[0,r]}(\|x-y\|) = \mathbf{1}_{B(x,r)}(y)$  and then our volume mean operator given by (1.9) coincides with the classical one. We note also that if  $f \in C^\infty(\mathbb{R}^d)$ , the volume mean of  $f$  at  $(x, r)$  can also be written by means of the Dunkl translation as follows (see [8]):

$$M_B^r(f)(x) = \frac{1}{m_k(B(0, r))} \int_{B(0,r)} \tau_x f(y) \omega_k(y) dy. \quad (2.1)$$

The harmonic kernel has the following properties (see [8]):

1. For all  $r > 0$  and  $x, y \in \mathbb{R}^d$ ,  $0 \leq h_k(r, x, y) \leq 1$ .
2. For all fixed  $x, y \in \mathbb{R}^d$ , the function  $r \mapsto h_k(r, x, y)$  is right-continuous and non decreasing on  $]0, +\infty[$ .
3. For all fixed  $r > 0$  and  $x \in \mathbb{R}^d$ ,

$$\text{supp } h_k(r, x, \cdot) \subset B^W(x, r) := \cup_{g \in W} B(gx, r). \quad (2.2)$$

4. Let  $r > 0$  and  $x \in \mathbb{R}^d$ . For any sequence  $(\chi_\varepsilon) \subset \mathcal{D}(\mathbb{R}^d)$  of radial functions such that for every  $\varepsilon > 0$ ,

$$0 \leq \chi_\varepsilon \leq 1, \quad \chi_\varepsilon = 1 \text{ on } B(0, r) \quad \text{and} \quad \forall y \in \mathbb{R}^d, \quad \lim_{\varepsilon \rightarrow 0} \chi_\varepsilon(y) = \mathbf{1}_{B(0,r)}(y), \quad (2.3)$$

we have

$$\forall y \in \mathbb{R}^d, \quad h_k(r, x, y) = \lim_{\varepsilon \rightarrow 0} \tau_{-x} \chi_\varepsilon(y). \quad (2.4)$$

5. For all  $r > 0$ , all  $x, y \in \mathbb{R}^d$  and all  $g \in W$ , we have

$$h_k(r, x, y) = h_k(r, y, x) \quad \text{and} \quad h_k(r, gx, y) = h_k(r, x, g^{-1}y). \quad (2.5)$$

6. For all  $r > 0$  and  $x \in \mathbb{R}^d$ , we have

$$\|h_k(r, x, \cdot)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r, x, y) \omega_k(y) dy = m_k(B(0, r)) = \frac{d_k r^{d+2\gamma}}{d+2\gamma}, \quad (2.6)$$

where  $\gamma := \sum_{\alpha \in R_+} k(\alpha)$  and  $d_k$  is the constant

$$d_k := \int_{S^{d-1}} \omega_k(\xi) d\sigma(\xi). \quad (2.7)$$

Here  $d\sigma(\xi)$  is the surface measure of the unit sphere  $S^{d-1}$  of  $\mathbb{R}^d$ .

7. Let  $r > 0$  and  $x \in \mathbb{R}^d$ . Then the function  $h_k(r, x, \cdot)$  is upper semi-continuous on  $\mathbb{R}^d$ .

8. The harmonic kernel satisfies the following geometric inequality: if  $\|a - b\| \leq 2r$  with  $r > 0$ , then

$$\forall \xi \in \mathbb{R}^d, \quad h_k(r, a, \xi) \leq h_k(4r, b, \xi) \quad (2.8)$$

Note that in the classical case (i.e.  $k = 0$ ), this inequality says that if  $\|a - b\| \leq 2r$ , then  $B(a, r) \subset B(b, 4r)$ .

9. Let  $x \in \mathbb{R}^d$ . Then the family of probability measures

$$d\eta_{x,r}^k(y) = \frac{1}{m_k[B(0, r)]} h_k(r, x, y) \omega_k(y) dy \quad (2.9)$$

is an approximation of the Dirac measure  $\delta_x$  as  $r \rightarrow 0$ . That is

$$\forall \alpha > 0, \quad \lim_{r \rightarrow 0} \int_{\|x-y\| > \alpha} d\eta_{x,r}^k(y) = 0 \quad (2.10)$$

and if  $f \in \mathcal{C}(\Omega)$ , then

$$\forall x \in \Omega, \quad \lim_{r \rightarrow 0} \int_{\mathbb{R}^d} f(y) d\eta_{x,r}^k = \lim_{r \rightarrow 0} M_B^r(f)(x) = f(x). \quad (2.11)$$

According to [10], the spherical mean of a  $C^\infty$ -function  $f$  defined on whole  $\mathbb{R}^d$  is given by

$$M_S^r(f)(x) := \frac{1}{d_k} \int_{S^{d-1}} \tau_x f(ry) \omega_k(y) d\sigma(y), \quad (2.12)$$

where  $d_k$  is the constant (2.7) and  $\tau_x$  is the Dunkl translation (see Annex A.2). Moreover, M. Rösler has proved that there exists a compactly supported probability measure  $\sigma_{x,r}^k$  on  $\mathbb{R}^d$  which represents the spherical mean operator (see [15]). More precisely, for  $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ , the spherical mean of  $f$  at  $(x, r) \in \mathbb{R}^d \times \mathbb{R}_+$  is given by

$$M_S^r(f)(x) = \int_{\mathbb{R}^d} f(y) d\sigma_{x,r}^k(y), \quad (2.13)$$

with

$$\text{supp } \sigma_{x,r}^k \subset B^W(x, r) = \cup_{g \in W} B(gx, r). \quad (2.14)$$

Clearly formula (2.13) shows that we can define the spherical mean at  $(x, r)$  of any continuous function on  $B^W(x, r)$ .

As a first link between the spherical and the volume operators, we have

**Proposition 2.1** *Let  $f$  be a continuous function on  $\Omega$ . Then the formula*

$$M_B^r(f)(x) = \frac{d+2\gamma}{r^{d+2\gamma}} \int_0^r M_S^t(f)(x) t^{d+2\gamma-1} dt \quad (2.15)$$

*holds whenever  $B(x, r) \subset \Omega$ .*

*Proof:* When  $f \in C^\infty(\mathbb{R}^d)$ , (2.15) has been proved in [8]. Now if  $f \in \mathcal{C}(\Omega)$ ,  $x \in \Omega$  and  $r > 0$  such that  $B(x, r) \subset \Omega$ , the result follows by using (2.2), (2.13) and an uniform approximation of  $f$  by polynomials on the compact set  $B^W(x, r)$ .  $\square$

At the end of this section, we will give some useful properties of the volume mean operator when it acts on  $L^1_{k,loc}(\Omega)$ -functions. Firstly, note that thanks to (2.2) and to the boundedness of the kernel  $h_k$ , the function  $(x, r) \mapsto M_B^r(f)(x)$  is well defined for any  $f \in L^1_{k,loc}(\Omega)$  whenever  $B(x, r) \subset \Omega$ .

Secondly, we will need the following notations which will be used frequently in this paper:

$$r_\Omega := \sup\{r > 0; \quad \Omega_r \neq \emptyset\}, \quad (2.16)$$

with  $\Omega_r$  the open set defined by (1.6). Clearly, we have  $\Omega_{r_1} \subset \Omega_{r_2}$  whenever  $r_2 \leq r_1$  and  $\Omega = \cup_{r>0}\Omega_r = \cup_{r<r_\Omega}\Omega_r$  (note that, since  $\Omega \neq \emptyset$ , we have  $r_\Omega > 0$ ). Moreover, since

$$\Omega_r = \{x \in \Omega; \quad B(x, r) \subset \Omega\}, \quad (2.17)$$

the open set  $\Omega_r$ ,  $0 < r < r_\Omega$ , is  $W$ -invariant.

**Proposition 2.2** *Let  $f \in L^1_{k,loc}(\Omega)$ .*

1) *Let  $0 < r < r_\Omega$ . Then the function  $M_B^r(f)$  belongs to  $L^1_{k,loc}(\Omega_r)$ .*

2) *Let  $x \in \Omega$ . Then the function  $r \mapsto M_B^r(f)(x)$  is continuous on  $]0, \varrho_x[$  with*

$$\varrho_x := \text{dist}(x, \partial\Omega). \quad (2.18)$$

*Proof:* 1) By compactness, it suffices to prove that  $M_B^r(f)\omega_k \in L^1(B(x_0, R))$  where  $B(x_0, R)$  is an arbitrary closed ball of center  $x_0$  and radius  $R$  included in  $\Omega_r$ . We have

$$\begin{aligned} I &:= \int_{B(x_0, R)} |M_B^r(f)(x)|\omega_k(x)dx \\ &\leq \frac{1}{m_k(B(0, r))} \int_{B(x_0, R)} \left( \int_{B^W(x, r)} |f(y)|h_k(r, x, y)\omega_k(y)dy \right) \omega_k(x)dx \\ &\leq \frac{1}{m_k(B(0, r))} \int_{B(x_0, R)} \left( \int_{B^W(x_0, R+r)} |f(y)|\omega_k(y)dy \right) \omega_k(x)dx \\ &\leq \frac{m_k(B(x_0, R))}{m_k(B(0, r))} \int_{B^W(x_0, R+r)} |f(y)|\omega_k(y)dy < +\infty, \end{aligned}$$

where the second inequality follows from the relation  $h_k(r, x, y) \leq 1$  and from the fact that for every  $x \in B(x_0, R)$  and every  $g \in W$ ,  $B(gx, r) \subset B(gx_0, R+r) \subset \Omega_r$ .

2) By (2.6), it suffices to show that  $\phi : r \mapsto \int_{\mathbb{R}^d} f(y)h_k(r, x, y)\omega_k(y)dy$  is continuous on  $]0, \varrho_x[$ . Since  $r \mapsto h_k(r, x, y)$  is right-continuous, by the dominated convergence theorem, we can see that  $\phi$  is also right-continuous on  $]0, \varrho_x[$ .

Now, fix  $r \in ]0, \varrho_x[$  and  $\eta > 0$  such that  $]r - \eta, r + \eta[ \subset ]0, \varrho_x[$ . Let  $(r_n)$  be a sequence of nonnegative real number such that  $r_n \rightarrow 0$  as  $n \rightarrow +\infty$ .

Using (2.5), (1.10) and applying Fubini's theorem, we obtain

$$\begin{aligned} |\phi(r) - \phi(r - r_n)| &\leq \int_{\mathbb{R}^d} \left( \int_{\Omega} |f(y)|\mathbf{1}_{]r-r_n, r]}(\sqrt{\|y\|^2 + \|x\|^2 - 2\langle y, z \rangle})\omega_k(y)dy \right) d\mu_x(z) \\ &= \int_{\mathbb{R}^d} \left( \int_{A_n} |f(y)|\omega_k(y)dy \right) d\mu_x(z), \end{aligned}$$

where  $A_n = A_n(x, z) := \left\{ y \in \Omega, \quad r - r_n < \sqrt{\|y\|^2 + \|x\|^2 - 2\langle y, z \rangle} \leq r \right\}$ . Since  $\cap_n A_n$  is a hypersurface, by the dominated convergence theorem, we get

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \left( \int_{A_n} |f(y)| \omega_k(y) dy \right) d\mu_x(z) = 0.$$

Hence, by the previous relations, we conclude that  $\phi$  is also left continuous.  $\square$

### 3 Dunkl convolution Product

The Dunkl convolution product has been defined by means of the Dunkl translation operators (see [18] and [14]). So that it has been considered only in some particular cases. Here, we will prove that we can define the Dunkl convolution product of a function  $u \in L^1_{k,loc}(\Omega)$  with a nonnegative and radial function  $f \in \mathcal{D}(\mathbb{R}^d)$  and we will study some properties of this product.

For  $f, g \in \mathcal{S}(\mathbb{R}^d)$ , the Dunkl convolution product is defined by

$$\forall x \in \mathbb{R}^d, \quad f *_k g(x) := \int_{\mathbb{R}^d} \tau_x f(-y) g(y) \omega_k(y) dy. \quad (3.1)$$

We note that it is commutative and satisfies the following property:

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f) \mathcal{F}_k(g), \quad (3.2)$$

where  $\mathcal{F}_k$  is the Dunkl transform (see Annex A.1).

From (3.2), (A.8) and the injectivity of the  $\mathcal{F}_k$  transform, we obtain the following relations

**Lemma 3.1** *Let  $f, g \in \mathcal{S}(\mathbb{R}^d)$ . Then, for every  $x \in \mathbb{R}^d$ , we have*

$$(\tau_x f) *_k g = f *_k (\tau_x g) = \tau_x (f *_k g). \quad (3.3)$$

**Theorem 3.1** *Let  $u \in L^1_{k,loc}(\Omega)$  and  $\phi \in \mathcal{D}^+_{\rho}(\mathbb{R}^d)$  with  $0 < \rho < r_{\Omega}$ . Let*

$$u *_k \phi(x) := \int_{\mathbb{R}^d} u(y) \tau_{-x} \phi(y) \omega_k(y) dy. \quad (3.4)$$

Then

1) *the function  $u *_k \phi$  is well defined on  $\Omega_{\rho}$  and can be written*

$$\forall x \in \Omega_{\rho}, \quad u *_k \phi(x) = \int_{\mathbb{R}^d} u(y) \tau_{-y} \phi(x) \omega_k(y) dy \quad (3.5)$$

$$= \int_{\mathbb{R}^d} u(y) \tau_x \phi(-y) \omega_k(y) dy, \quad (3.6)$$

2)  *$u *_k \phi$  belongs to  $C^{\infty}(\Omega_{\rho})$  and we have*

$$\Delta_k(u *_k \phi) = u *_k \Delta_k \phi, \quad (3.7)$$



3) for all  $B(x, r) \subset \Omega_\rho$ , we have

$$M_B^r(u *_k \phi)(x) = M_B^r(u) *_k \phi(x). \quad (3.8)$$

*Proof:* 1) • For every  $\varepsilon > 0$ , we see that

$$\forall y \in \mathbb{R}^d, \quad 0 \leq \phi(y) \leq \|\phi\|_\infty \mathbf{1}_{B(0, \rho)}(y) \leq \|\phi\|_\infty \varphi_\varepsilon(y),$$

where  $(\varphi_\varepsilon)$  is a sequence satisfying (2.3) (with  $r = \rho$ ). Using the positivity of the Dunkl translation operators on radial functions, we deduce that

$$\forall y \in \mathbb{R}^d, \quad 0 \leq \tau_{-x}\phi(y) \leq \|\phi\|_\infty \tau_{-x}\varphi_\varepsilon(y).$$

Letting  $\varepsilon \rightarrow 0$  and using (2.4), we obtain

$$\forall y \in \mathbb{R}^d, \quad 0 \leq \tau_{-x}\phi(y) \leq \|\phi\|_\infty h_k(\rho, x, y). \quad (3.9)$$

Consequently, from the relations (2.2) and (3.9), we get that

$$\text{supp } \tau_{-x}\phi \subset B^W(x, \rho). \quad (3.10)$$

This implies that for all  $x \in \Omega_\rho$ , the function  $y \mapsto u(y)\tau_{-x}\phi(y)\omega_k(y)$  is integrable on  $\Omega$ .

• The relation (3.5) follows from (A.13) and the relation (3.6) follows from (3.5) and (A.10).

2) Let  $x_0 \in \Omega_\rho$  and  $R > 0$  such that  $B(x_0, R) \subset \Omega_\rho$ . We shall prove that the function  $u *_k \phi$  is of class  $C^\infty$  on  $\mathring{B}(x_0, R)$ .

Define the function  $\Phi$  on  $\mathbb{R}^d \times \mathbb{R}^d$  by (see (A.9))

$$\Phi(x, y) := \tau_{-x}\phi(y) = c_k^{-2} \int_{\mathbb{R}^d} \mathcal{F}_k(\phi)(\xi) E_k(-ix, \xi) E_k(iy, \xi) \omega_k(\xi) d\xi.$$

We see that  $\Phi$  is in  $\mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  and by (A.3) and the inequality  $|E_k(iy, \xi)| \leq 1$ , for every multi-indices  $v \in \mathbb{N}^d$  we get

$$\forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \left| \frac{\partial^v}{\partial x^v} \Phi(x, y) \right| \leq c_k^{-2} \int_{\mathbb{R}^d} |\mathcal{F}_k(\phi)(\xi)| \|\xi\|^{|v|} \omega_k(\xi) d\xi := C_v < +\infty.$$

On the other hand, from (3.10) we have

$$\forall x \in \mathring{B}(x_0, R), \quad \text{supp } \Phi(x, \cdot) \subset B^W(x, \rho) \subset B^W(x_0, R + \rho) \subset \Omega.$$

This implies that we can write

$$\forall x \in \mathring{B}(x_0, R), \quad \forall y \in \mathbb{R}^d, \quad \Phi(x, y) = \Phi(x, y) \mathbf{1}_{B^W(x_0, R + \rho)}(y).$$

Thus, for every multi-indices  $v \in \mathbb{N}^d$ , we deduce that

$$\forall x \in \mathring{B}(x_0, R), \quad \forall y \in \mathbb{R}^d, \quad \left| \frac{\partial^v}{\partial x^v} \Phi(x, y) \right| \leq C_v \mathbf{1}_{B^W(x_0, R + \rho)}(y).$$

Now, since  $u\omega_k$  is locally integrable, this proves that we can differentiate under the integral sign in (3.4) and we obtain the desired result.

Furthermore, using respectively (3.5), (A.11) and (A.13) (here note that we can use the relation (A.13) because  $\Delta_k\phi$  is also a radial function<sup>2</sup> (see [10])), we obtain

$$\begin{aligned}\Delta_k(u *_k \phi)(x) &= \int_{\mathbb{R}^d} u(y)\Delta_k[\tau_{-y}\phi](x)\omega_k(y)dy = \int_{\mathbb{R}^d} u(y)\tau_{-y}[\Delta_k\phi](x)\omega_k(y)dy \\ &= \int_{\mathbb{R}^d} u(y)\tau_{-x}[\Delta_k\phi](y)\omega_k(y)dy = u *_k \Delta_k\phi(x).\end{aligned}$$

This completes the proof of 2).

3) We need the following lemma:

**Lemma 3.2** *Let  $f \in \mathcal{S}(\mathbb{R}^d)$  be radial. Then, for all  $r > 0$  and  $a, b \in \mathbb{R}^d$ , we have*

$$\tau_a\tau_b f = \tau_b\tau_a f \quad (3.11)$$

and

$$M_B^r(\tau_{-a}f)(b) = M_B^r(\tau_{-b}f)(a). \quad (3.12)$$

*Proof of Lemma 3.2*

- We obtain (3.11) from (A.8) and the injectivity of the Dunkl transform on  $\mathcal{S}(\mathbb{R}^d)$ .
- Let  $r > 0$  and  $a, b \in \mathbb{R}^d$ . We have

$$\begin{aligned}M_B^r(\tau_{-a}f)(b) &= \frac{1}{m_k(B(0, r))} \int_{\mathbb{R}^d} \tau_a f(-y)h_k(r, b, y)\omega_k(y)dy \\ &= \frac{1}{m_k(B(0, r))} \int_{\mathbb{R}^d} \tau_b\tau_a f(-y)\mathbf{1}_{B(0, r)}(y)\omega_k(y)dy \\ &= \frac{1}{m_k(B(0, r))} \int_{\mathbb{R}^d} \tau_b f(-y)h_k(r, a, y)\omega_k(y)dy \\ &= M_B^r(\tau_{-b}f)(a),\end{aligned}$$

where we have used respectively the relations (A.13) and (A.10) in the first equality, the relation (2.1) in the second equality, the relations (3.11) and (2.1) in the third equality and the relations (A.10), (A.13) in the last equality.  $\square$

Now, we turn to the proof of (3.8).

Let  $B(x, r) \subset \Omega_\rho$ . By Proposition 2.2- 1), the function  $M_B^r(u)$  belongs to  $L_{k,loc}^1(\Omega_r)$ . This proves, by assertion 1), that the function  $M_B^r(u) *_k \phi$  is well defined on  $\Omega_{\rho+r}$ .

---

<sup>2</sup>More precisely, we have  $\Delta_k\phi(x) = (\frac{d^2}{dr^2} + \frac{d+2\gamma-1}{r}\frac{d}{dr})\tilde{\phi}(r)$ ,  $r = \|x\|$  and  $\tilde{\phi}$  the profile of  $\phi$ .

By 2), the function  $u *_k \phi$  is clearly in  $L^1_{k,loc}(\Omega_\rho)$  and for  $x \in \Omega_{\rho+r}$  we have<sup>3</sup>

$$\begin{aligned}
M_B^r(u *_k \phi)(x) &= \frac{1}{m_k(B(0, r))} \int_{B^W(x, r)} \left( \int_{B^W(x, r+\rho)} u(z) \tau_{-y} \phi(z) \omega_k(z) dz \right) h_k(r, x, y) \omega_k(y) dy \\
&= \frac{1}{m_k(B(0, r))} \int_{B^W(x, r+\rho)} u(z) \left( \int_{B^W(x, r)} \tau_{-z} \phi(y) h_k(r, x, y) \omega_k(y) dy \right) \omega_k(z) dz \\
&= \int_{B^W(x, r+\rho)} u(z) M_B^r(\tau_{-z} \phi)(x) \omega_k(z) dz \\
&= \int_{B^W(x, r+\rho)} u(z) M_B^r(\tau_{-x} \phi)(z) \omega_k(z) dz \\
&= \frac{1}{m_k(B(0, r))} \int_{B^W(x, r+\rho)} u(z) \left( \int_{B^W(x, \rho)} \tau_{-x} \phi(y) h_k(r, z, y) \omega_k(y) dy \right) \omega_k(z) dz \\
&= \frac{1}{m_k(B(0, r))} \int_{B^W(x, \rho)} \tau_{-x} \phi(y) \left( \int_{B^W(x, r+\rho)} u(z) h_k(r, y, z) \omega_k(z) dz \right) \omega_k(y) dy \\
&= M_B^r(u) *_k \phi(x),
\end{aligned}$$

where,

- the first equality follows from the relations (2.2), (3.5), (3.10) and from the fact that

$$\forall y \in B^W(x, r), \quad B^W(y, \rho) \subset B^W(x, r + \rho) \subset \Omega, \quad (3.13)$$

-the second equality follows from Fubini's theorem and (A.13),

-the third equality comes from the relation (1.9),

-the fourth equality follows from (3.12),

-in the fifth equality we have used the relations (1.9) and (3.10),

-in the sixth equality we have used (2.5) and Fubini's theorem. Finally, using (2.2) and (3.13), we obtain the last equality.

This completes the proof of the theorem.  $\square$

**Remark 3.1** *If  $u$  is of class  $C^2$  on  $\Omega$ , then the relation (3.7) can be also written*

$$\forall x \in \Omega_\rho, \quad \Delta_k(u *_k \phi)(x) = (\Delta_k u) *_k \phi(x). \quad (3.14)$$

Indeed, using respectively the relations (3.7), (3.4) and (A.11), we get

$$\Delta_k(u *_k \phi)(x) = \int_{\Omega} u(y) \tau_{-x} [\Delta_k \phi](y) \omega_k(y) dy = \int_{\Omega} u(y) \Delta_k [\tau_{-x} \phi](y) \omega_k(y) dy. \quad (3.15)$$

Now, we obtain the desired result by using the following integration by parts formula (see [5] or [14]): Let  $f, g \in C^1(\Omega)$  such that  $g$  has compact support. Then, for all  $\xi \in \mathbb{R}^d$ , we have

$$\int_{\Omega} D_\xi f(x) g(x) \omega_k(x) dx = - \int_{\Omega} f(x) D_\xi g(x) \omega_k(x) dx, \quad (3.16)$$

with  $D_\xi$  the  $\xi$ -directional Dunkl operator (see (1.2)).  $\square$

<sup>3</sup>Note that, in the integrals below, the consideration of the supports permits to justify the correct application of Fubini's theorem.

**Proposition 3.1** *Let  $u \in L^1_{k,loc}(\Omega)$  and  $\phi \in \mathcal{D}^+(\mathbb{R}^d)$  with  $0 < \rho < r_\Omega$ . If  $u$  is with compact support, then  $u *_k \phi$  is also with compact support and*

$$\text{supp } (u *_k \phi) \subset B(0, \rho) + W.\text{supp } u \subset \Omega, \quad (3.17)$$

with  $W.\text{supp } u := \{gx, (g, x) \in W \times \text{supp } u\}$ .

*Proof:* If  $x \notin B(0, \rho) + W.\text{supp } u$ , then  $x - gy \notin B(0, \rho)$  for every  $(g, y) \in W \times \text{supp } u$ . That is  $\|gx - y\| > \rho$  for all  $y \in \text{supp } u$  and all  $g \in W$ . In other words  $\text{supp } u \cap B^W(x, \rho) = \emptyset$ . Hence, by (3.10), we obtain  $u *_k \phi(x) = 0$ .  $\square$

It is interesting to note that when  $u$  is a continuous function, we can write the Dunkl convolution product in spherical coordinates as follows:

**Proposition 3.2** *Let  $u$  be a continuous function on  $\Omega$  and let  $\phi \in \mathcal{D}^+(\mathbb{R}^d)$  with  $0 < \rho < r_\Omega$  (i.e.  $\Omega_\rho$  is nonempty). Then, for all  $x \in \Omega_\rho$ , we have*

$$u *_k \phi(x) = d_k \int_0^\rho \tilde{\phi}(t) t^{d+2\gamma-1} M_S^t(u)(x) dt, \quad (3.18)$$

where  $\tilde{\phi}$  is the profile function of  $\phi$  and  $d_k$  is the constant given by (2.7).

*Proof:* At first we suppose that  $u \in C^\infty(\mathbb{R}^d)$ . By (A.14), we have

$$u *_k \phi(x) = \int_{\mathbb{R}^d} \phi(y) \tau_x u(y) \omega_k(y) dy.$$

Then, using spherical coordinates, we can write

$$u *_k \phi(x) = \int_0^\rho \tilde{\phi}(t) t^{d+2\gamma-1} \int_{S^{d-1}} \tau_x u(t\xi) \omega_k(\xi) d\sigma(\xi) dt.$$

Therefore, from (2.12) we deduce that the relation (3.18) holds in this case.

Let us now suppose only that  $u$  is a continuous function on  $\Omega$ . Let  $(p_n)$  a sequence of polynomial functions such that  $p_n \rightarrow u$  as  $n \rightarrow +\infty$  uniformly on the compact set  $K := B^W(x, \rho)$ . Since  $\tau_{-x}\phi \geq 0$ , by (A.12) we conclude that

$$|u *_k \phi(x) - p_n *_k \phi(x)| \leq \|\tau_{-x}\phi\|_{L^1_k(\mathbb{R}^d)} \sup_K |p_n(y) - u(y)| = \|\phi\|_{L^1_k(\mathbb{R}^d)} \sup_K |p_n(y) - u(y)|.$$

Hence

$$u *_k \phi(x) = \lim_{n \rightarrow +\infty} p_n *_k \phi(x). \quad (3.19)$$

Furthermore, as the probability measures  $\sigma_{x,t}^k$ ,  $0 < t \leq \rho$ , have compact support contained in  $B^W(x, t) \subset K = B^W(x, \rho)$  (see (2.14)), we deduce

$$\forall t \leq \rho, \quad |M_S^t(p_n - u)(x)| \leq \sup_K |p_n(y) - u(y)|.$$

This implies

$$\lim_{n \rightarrow +\infty} d_k \int_0^\rho \tilde{\phi}(t) t^{d+2\gamma-1} M_S^t(p_n)(x) dt = d_k \int_0^\rho \tilde{\phi}(t) t^{d+2\gamma-1} M_S^t(u)(x) dt. \quad (3.20)$$

From (3.19), (3.20) and the first step, we deduce that the relation (3.18) holds when the function  $u$  is continuous on  $\Omega$ .  $\square$

We have the following associativity result for the Dunkl convolution product:

**Proposition 3.3** *Let  $u \in L_{k,loc}^1(\Omega)$ ,  $\phi \in \mathcal{D}_\rho^+(\mathbb{R}^d)$  and  $\psi \in \mathcal{D}_r^+(\mathbb{R}^d)$  such that  $\Omega_{r+\rho}$  is nonempty (i.e.  $r + \rho < r_\Omega$ ). Then*

$$\forall x \in \Omega_{r+\rho}, \quad (u *_k \phi) *_k \psi(x) = u *_k (\phi *_k \psi)(x) = (u *_k \psi) *_k \phi(x). \quad (3.21)$$

*Proof:* • From Theorem 3.1, the functions  $(u *_k \phi) *_k \psi$  and  $(u *_k \psi) *_k \phi$  are well defined on  $\Omega_{r+\rho}$ .

• We claim that  $\phi *_k \psi$  is a nonnegative  $C^\infty$ -radial function on  $\mathbb{R}^d$  with compact support contained in  $B(0, r + \rho)$  which implies that  $u *_k (\phi *_k \psi)$  is also well defined on  $\Omega_{r+\rho}$ . Indeed, again by Theorem 3.1 we see that  $\phi *_k \psi$  is of class  $C^\infty$  on  $\mathbb{R}^d$  and using (3.17) we obtain  $\text{supp } \phi *_k \psi \subset B(0, r + \rho)$ . Furthermore, by the positivity of Dunkl translation operators on radial functions, we deduce that the function  $\phi *_k \psi$  is nonnegative. Now, using the fact that the Dunkl transform  $\mathcal{F}_k$  is an isomorphism of the Schwartz space onto itself and the relation (3.2), we can write that

$$\phi *_k \psi = \mathcal{F}_k^{-1}(\mathcal{F}_k(\phi)\mathcal{F}_k(\psi)).$$

Therefore, since  $\mathcal{F}_k$  preserves the radial property (see the relation (A.6)), we deduce that  $\phi *_k \psi$  is radial as claimed.

• For  $x \in \Omega_{r+\rho}$  fixed, we have

$$\begin{aligned} (u *_k \phi) *_k \psi(x) &= \int_{B^W(x,r)} (u *_k \phi)(y) \tau_{-x} \psi(y) \omega_k(y) dy \\ &= \int_{B^W(x,r)} \left( \int_{B^W(x,r+\rho)} u(z) \tau_{-y} \phi(z) \omega_k(z) dz \right) \tau_{-x} \psi(y) \omega_k(y) dy \\ &= \int_{B^W(x,r+\rho)} u(z) \left( \int_{B^W(x,r)} \tau_{-y} \phi(z) \tau_{-x} \psi(y) \omega_k(y) dy \right) \omega_k(z) dz \\ &= \int_{B^W(x,r+\rho)} u(z) \left( \int_{B^W(x,r)} \tau_{-z} \phi(y) \tau_{-x} \psi(y) \omega_k(y) dy \right) \omega_k(z) dz \\ &= \int_{B^W(x,r+\rho)} u(z) (\phi *_k \tau_{-x} \psi)(z) \omega_k(z) dz \\ &= \int_{B^W(x,r+\rho)} u(z) \tau_{-x} (\phi *_k \psi)(z) \omega_k(z) dz \\ &= u *_k (\phi *_k \psi)(x). \end{aligned}$$

where we have used

- the relations (3.4) and (3.10) in the first line;
- the same relations in the second line with (3.13);
- Fubini's theorem in the third line: the relation (3.9), the inequality  $h_k(R, a, b) \leq 1$  and the hypothesis  $u \in L_{k,loc}^1(\Omega)$  imply that we can use Fubini's theorem;
- the relation (A.13) in the fourth line;
- the relation (3.4) in the fifth line;
- relation (3.3) in the sixth line;
- the above properties of the function  $\phi *_k \psi$  and (3.10) in the last line.

Now, changing the role of  $\phi$  and  $\psi$ , we obtain

$$(u *_k \psi) *_k \phi(x) = u *_k (\psi *_k \phi)(x).$$

Finally, by the commutativity of the Dunkl convolution product (see (3.2)), we conclude the last equality in (3.21).  $\square$

## 4 Applications to $\Delta_k$ -harmonic functions

In this section, we will give some properties of Dunkl harmonic functions. Let us introduce the space  $\mathcal{H}_k(\Omega)$  of  $\Delta_k$ -harmonic functions on  $\Omega$ .

In order to give further study of Dunkl harmonic functions, we first need some lemmata. Let us consider the following radial function

$$\varphi(x) := a \exp\left(-\frac{1}{1-\|x\|^2}\right) \mathbf{1}_{B(0,1)}(x), \quad x \in \mathbb{R}^d, \quad (4.1)$$

where  $a$  is a constant such that  $x \mapsto \varphi(x)\omega_k(x)$  is a probability density.

For  $\varepsilon > 0$ , define the function

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^{d+2\gamma}} \varphi\left(\frac{x}{\varepsilon}\right). \quad (4.2)$$

It is well known that  $\varphi_\varepsilon$  is a radial mollifier i.e.  $\varphi_\varepsilon \in \mathcal{D}_\varepsilon^+(\mathbb{R}^d)$ . We begin by the following preparatory result:

**Lemma 4.1** *Let  $u$  be a continuous function on  $\Omega$ . For  $0 < \varepsilon < r_\Omega$ , define the function  $u_\varepsilon$  by*

$$\forall x \in \Omega_\varepsilon, \quad u_\varepsilon(x) := u *_k \varphi_\varepsilon(x) := \int_{\mathbb{R}^d} u(y) \tau_{-x} \varphi_\varepsilon(y) \omega_k(y) dy. \quad (4.3)$$

*Then the sequence  $(u_\varepsilon)_{0 < \varepsilon < r_\Omega}$  satisfies*

- i)** *for every  $0 < \varepsilon < r_\Omega$ , the function  $u_\varepsilon$  is in  $C^\infty(\Omega_\varepsilon)$ ,*
- ii)** *for every  $x \in \Omega$ ,  $u_\varepsilon(x) \rightarrow u(x)$  as  $\varepsilon \rightarrow 0$ .*
- iii)** *for every  $0 < r < r_\Omega$  and every  $x \in \Omega_r$ , we have  $M_B^r(u *_k \varphi_\varepsilon)(x) \rightarrow M_B^r(u)(x)$  and  $M_S^r(u *_k \varphi_\varepsilon)(x) \rightarrow M_S^r(u)(x)$  as  $\varepsilon \rightarrow 0$ .*

*Proof:* The first assertion follows immediately from Theorem 3.1.

ii) Let  $x \in \Omega$ . Applying (3.9) with  $\phi = \varphi_\varepsilon$  we get

$$\forall y \in \mathbb{R}^d, \quad 0 \leq \tau_{-x}\varphi_\varepsilon(y) \leq a\varepsilon^{-d-2\gamma}h_k(\varepsilon, x, y).$$

Using (2.6), we can write

$$\forall y \in \mathbb{R}^d, \quad 0 \leq \tau_{-x}\varphi_\varepsilon(y) \leq a \frac{d_k}{d+2\gamma} \frac{1}{m_k[B(0, \varepsilon)]} h_k(\varepsilon, x, y). \quad (4.4)$$

Consequently, for every  $x \in \Omega$  and every  $\varepsilon > 0$  small enough, we have by (A.12) and (4.4)

$$\begin{aligned} |u_\varepsilon(x) - u(x)| &\leq \int_{\mathbb{R}^d} \tau_{-x}\varphi_\varepsilon(y) |u(y) - u(x)| \omega_k(y) dy \\ &\leq a \frac{d_k}{d+2\gamma} \frac{1}{m_k[B(0, \varepsilon)]} \int_{\mathbb{R}^d} |u(y) - u(x)| h_k(\varepsilon, x, y) \omega_k(y) dy. \end{aligned}$$

This can be rewritten in the following form

$$|u_\varepsilon(x) - u(x)| \leq a \frac{d_k}{d+2\gamma} M_B^\varepsilon(|u - u(x)|)(x). \quad (4.5)$$

Thus from (2.11), we conclude the result.

iii) Let  $x \in \Omega_r$ . There exists  $\varepsilon_0 > 0$  such that  $x \in \Omega_{r+\varepsilon_0}$ . For  $\varepsilon \in ]0, \varepsilon_0[$ , we have

$$\begin{aligned} \forall y \in B^W(x, r), \quad |u * \varphi_\varepsilon(y)| &\leq \int_{B^W(y, \varepsilon)} |u(z)| \tau_{-y}\varphi_\varepsilon(z) \omega_k(z) dz \\ &\leq \int_{B^W(x, r+\varepsilon_0)} |u(z)| \tau_{-y}\varphi_\varepsilon(z) \omega_k(z) dz \\ &\leq \sup_{B^W(x, r+\varepsilon_0)} |u(y)|, \end{aligned}$$

where in the last line we have used the relation (A.12).

According to (2.2), (2.14) and to the statement ii), the previous inequality implies that we can apply the dominated convergence theorem to obtain the results of assertion iii).  $\square$

**Lemma 4.2** *Let  $u$  be a  $C^2$ -function on  $\Omega$ . Then, for any  $0 < r < r_\Omega$  and any  $x \in \Omega_r$ , we have*

$$M_S^r(u)(x) = u(x) + \frac{1}{d+2\gamma} \int_0^r M_B^t(\Delta_k u)(x) t dt, \quad (4.6)$$

and

$$M_B^r(u)(x) = u(x) + \frac{1}{r^{d+2\gamma}} \int_0^r \int_0^\rho M_B^t(\Delta_k u)(x) t dt \rho^{d+2\gamma-1} d\rho. \quad (4.7)$$

**Remark 4.1** *We have obtained in [8] that the relations (4.6) and (4.7) hold when  $u \in C^\infty(\mathbb{R}^d)$ . Furthermore, by polynomial approximation, we have extended them to any function  $u$  of class  $C^2$  on an arbitrary  $W$ -invariant open set  $\Omega \subset \mathbb{R}^d$  and any  $x \in \Omega$  but with the condition  $r \in ]0, \varrho_x/3[$ ,  $\varrho_x$  being defined by (2.18).*

*Proof: Step 1:* Suppose that  $u$  is of class  $C^\infty$  on  $\Omega$ .

Let  $x \in \Omega$  and  $r \in ]0, \rho_x[$  and let  $\epsilon > 0$  such that  $B(x, r + \epsilon) \subset \Omega$ . We can find  $\phi \in \mathcal{D}(\mathbb{R}^d)$  such that

1.  $\phi = 1$  on the compact set  $B^W(x, r + \epsilon/2)$ ,
2.  $\text{supp } \phi \subset B^W(x, r + \epsilon)$ ,
3.  $0 \leq \phi \leq 1$ .

Therefore, the function  $f = u\phi$  is in  $C^\infty(\mathbb{R}^d)$ ,  $\text{supp } f \subset B^W(x, r + \epsilon)$  and  $f = 1$  on  $B^W(x, r + \epsilon/2)$ . According to Remark 4.1 and noting that the relations (4.6) and (4.7) only involve the compact set  $B^W(x, r)$  (through the supports of  $h_k(r, x, \cdot)$  and  $\sigma_{x,r}^k$ ), we can replace  $f$  by  $u$  in these formulas.

**Step 2:** Here, we will suppose that  $u \in \mathcal{C}^2(\Omega)$ . By (2.15), it is enough to prove (4.6). Fix  $0 < r < r_\Omega$ ,  $x \in \Omega_r$  and  $\epsilon_0 > 0$  such that  $x \in \Omega_{r+\epsilon_0}$ . By step 1, for every  $0 < \epsilon < \epsilon_0$  we have

$$M_S^r(u *_k \varphi_\epsilon)(x) = u *_k \varphi_\epsilon(x) + \frac{1}{d + 2\gamma} \int_0^r M_B^t(\Delta_k[u *_k \varphi_\epsilon])(x) t dt. \quad (4.8)$$

Now, using (3.14) and Lemma 4.1, we get

$$\lim_{\epsilon \rightarrow 0} M_B^t(\Delta_k[u *_k \varphi_\epsilon])(x) = \lim_{\epsilon \rightarrow 0} M_B^t([\Delta_k u] *_k \varphi_\epsilon)(x) = M_B^t(\Delta_k u)(x),$$

and following the proof of the statement iii) of Lemma 4.1, for every  $0 < \epsilon < \epsilon_0$  we obtain

$$\forall t \leq r, \quad |M_B^t(\Delta_k[u *_k \varphi_\epsilon])(x)| \leq \sup_{B^W(x, r+\epsilon_0)} |\Delta_k u(y)|.$$

Therefore, we can use the dominated convergence theorem in the integral of (4.8) and using again Lemma 4.1, we obtain the result by letting  $\epsilon \rightarrow 0$ .  $\square$

In the following result, we will characterize the  $\Delta_k$ -harmonicity by the global mean value property.

**Proposition 4.1** *Let  $u \in \mathcal{C}^2(\Omega)$ . The following statements are equivalent*

- i)  $u \in \mathcal{H}_k(\Omega)$ ,
- ii)  $u(x) = M_S^r(u)(x)$  whenever  $B(x, r) \subset \Omega$ ,
- iii)  $u(x) = M_B^r(u)(x)$  whenever  $B(x, r) \subset \Omega$ ,
- iv)  $M_B^r(u)(x) = M_S^r(u)(x)$  whenever  $B(x, r) \subset \Omega$ .

*Proof: i)  $\Rightarrow$  ii)* It follows from (4.6).

*ii)  $\Rightarrow$  iii)* By (2.15), we obtain immediately the result.

*iii)  $\Rightarrow$  iv)* Let  $f \in \mathcal{C}^2(\Omega)$  and let  $x \in \Omega$  be fixed. By (4.6) and (2.15) the function  $r \mapsto M_B^r(f)(x)$  is of class  $C^2$  on  $]0, \rho_x[$  and we have

$$\frac{d}{dr} M_B^r(f)(x) = \frac{d + 2\gamma}{r} \left( M_S^r(f)(x) - M_B^r(f)(x) \right). \quad (4.9)$$



Now, from (4.9) clearly iii) implies iv).

**iv)  $\Rightarrow$  i)** Let  $x \in \Omega$ . Using (4.9) and (2.11), we deduce that

$$\forall r \in [0, \rho_x[, \quad M_B^r(u)(x) = u(x).$$

Now, if we use (4.7) et we differentiate two times with respect to  $r$ , we get  $M_B^r(\Delta_k u)(x) = 0$  for every  $r \in ]0, \rho_x[$ . Finally, by (2.11), we obtain  $\Delta_k u(x) = 0$ .  $\square$

**Corollary 4.1** *Every  $\Delta_k$ -harmonic function on  $\Omega$  is of class  $C^\infty$ .*

*Proof:* Let  $u$  be a  $\Delta_k$ -harmonic function on  $\Omega$ . As  $u$  satisfies the spherical mean property, by (3.18), we see that  $u = u *_k \varphi_\varepsilon$  on  $\Omega_\varepsilon$ . Therefore,  $u$  is of class  $C^\infty$  on  $\Omega_\varepsilon$  for every  $\varepsilon > 0$  arbitrary small.  $\square$

**Corollary 4.2** *Every  $\Delta_k$ -harmonic function on  $\mathbb{R}^d$  is real analytic.*

*Proof:* Let  $f$  be a  $\Delta_k$ -harmonic function on  $\mathbb{R}^d$ . Since  $f \in \mathcal{C}^\infty(\mathbb{R}^d)$  (by Corollary 4.1) and  $V_k : \mathcal{C}^\infty(\mathbb{R}^d) \rightarrow \mathcal{C}^\infty(\mathbb{R}^d)$  is a topological isomorphism, the function  $g := V_k^{-1}(f)$  is harmonic on  $\mathbb{R}^d$  in the usual sense (i.e.  $\Delta g = 0$ ) as an immediate consequence of the intertwining relation<sup>4</sup>  $\Delta_k V_k = V_k \Delta$ . It is well known that  $g$  is real analytic (see [1]) and thus, using multi-indices  $v = (v_1, \dots, v_d) \in \mathbb{N}^d$ , can be written

$$g(x) = \sum_v a_v x^v, \quad x \in \mathbb{R}^d,$$

where  $a_v$  are real coefficients. If  $g_N$  ( $N \in \mathbb{N}$ ) denotes the partial sum  $g_N(x) := \sum_{|v| \leq N} a_v x^v$  (with  $|v| = v_1 + \dots + v_d$ ), then  $g_N \rightarrow g$  as  $N \rightarrow +\infty$  in the Fréchet topology of  $\mathcal{C}^\infty(\mathbb{R}^d)$ . Therefore  $V_k(g_N) \rightarrow V_k(g) = f$  in the Fréchet topology. In particular,  $f$  is real analytic as being the uniform limit of the polynomials<sup>5</sup>  $V_k(g_N)$  on each compact subset of  $\mathbb{R}^d$ .  $\square$

**Corollary 4.3** *Let  $u$  be a function defined on  $\Omega$ . Then  $u \in \mathcal{H}_k(\Omega)$  if and only if  $u \in L_{k,loc}^1(\Omega)$  and satisfies*

$$u *_k \phi = u \quad \text{on } \Omega_r \tag{4.10}$$

*whenever  $0 < r < r_\Omega$  and  $\phi \in \mathcal{D}_r^+(\mathbb{R}^d)$  is such that  $\phi \omega_k$  is a probability density.*

*Proof:* Using the spherical mean value property and the relation (3.18) we see that any  $\Delta_k$ -harmonic function on  $\Omega$  satisfies (4.10). Conversely, let  $u \in L_{k,loc}^1(\Omega)$  satisfying (4.10). At first, by Theorem 3.1, clearly the function  $u$  is of class  $C^\infty$  on  $\Omega$ . On the other hand, let  $x \in \Omega$  and  $0 < r < \rho_x$  be fixed. Let  $(\varphi_\varepsilon)$  be a sequence as in (2.3) (i.e. such that  $\tau_{-x}\varphi_\varepsilon(y) \rightarrow h_k(r, x, y)$  as  $\varepsilon \rightarrow 0$ ). So, by the dominated convergence theorem, the hypothesis

$$u *_k \frac{\varphi_\varepsilon(x)}{\|\varphi_\varepsilon\|_{L^1(\mathbb{R}^d, m_k)}} = u(x)$$

implies that  $M_B^r(u)(x) = u(x)$ . Thus,  $u \in \mathcal{H}_k(\Omega)$  by the volume mean value property.  $\square$

In the following result, we will characterize de  $\Delta_k$ -harmonicity by the local mean value property:

<sup>4</sup>which follows clearly from (1.3).

<sup>5</sup> $V_k$  is a bijection of the space of polynomials of degree  $\leq n$  onto itself ([6]).

**Theorem 4.1** *Let  $u \in \mathcal{C}^2(\Omega)$ . The following statements are equivalent*

i)  $u \in \mathcal{H}_k(\Omega)$ ,

ii)  $u$  satisfies the local spherical mean property i.e. for every  $x \in \Omega$ , there exists  $r_x > 0$  such that  $\forall r < r_x$ ,  $u(x) = M_S^r(u)(x)$ ,

iii)  $u$  satisfies the local volume mean property i.e. for every  $x \in \Omega$ , there exists  $r_x > 0$  such that  $\forall r < r_x$ ,  $u(x) = M_B^r(u)(x)$ .

*Proof:* i)  $\implies$  ii) We know that the  $\Delta_k$ -harmonic function  $u$  satisfies the global spherical mean property and then it satisfies also the local one.

ii)  $\implies$  iii) It is a direct consequence of (2.15).

iii)  $\implies$  i) Let  $x \in \Omega$ . The local volume property and the relation (4.7) imply

$$\forall r < r_x, \quad \int_0^r \int_0^\rho M_B^t(\Delta_k u)(x) t dt \rho^{d+2\gamma-1} d\rho = 0.$$

If we differentiate two times with respect to  $r$ , we obtain

$$\forall r < r_x, \quad M_B^r(\Delta_k u)(x) = 0.$$

Hence, according to (2.11), we get  $\Delta_k u(x) = 0$ . □

In order to prove a Weyl type lemma for Dunkl harmonic functions, we will clarify some facts about the action of Dunkl operators on distributions. For a distribution  $T \in \mathcal{D}'(\Omega)$ , we define the weak Dunkl  $\xi$ -directional derivative of  $T$  ( $\xi \in \mathbb{R}^d$ ) by

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle D_\xi T, \phi \rangle = -\langle T, D_\xi \phi \rangle.$$

Note that by the intertwining relation (1.3), the operator  $D_\xi = V_k \partial_\xi V_k^{-1} : \mathcal{C}^\infty(\mathbb{R}^d) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^d)$  is continuous for the Fréchet topology. Moreover, since  $D_\xi$  leaves the space  $\mathcal{D}(\Omega)$  invariant, we deduce that  $D_\xi : \mathcal{D}(\Omega) \longrightarrow \mathcal{D}(\Omega)$  is also continuous for the Fréchet topology. This justifies that  $D_\xi T$  is well defined as an element of  $\mathcal{D}'(\Omega)$ .

In particular, if  $f \in L_{k,loc}^1(\Omega)$  i.e.  $f\omega_k \in L_{loc}^1(\Omega)$ , the weak Dunkl-Laplacian of  $f\omega_k$  is given by

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle \Delta_k(f\omega_k), \phi \rangle = \langle f\omega_k, \Delta_k \phi \rangle = \int_\Omega f(x) \Delta_k \phi(x) \omega_k(x) dx. \quad (4.11)$$

**Theorem 4.2** *Let  $u \in L_{k,loc}^1(\Omega)$  such that  $\Delta_k(u\omega_k) = 0$  in  $\mathcal{D}'(\Omega)$ . Then, there exists a Dunkl harmonic function  $h$  on  $\Omega$  such that  $u = h$  a.e. on  $\Omega$ .*

*Proof:* Let  $0 < \varepsilon < r_\Omega$  and let  $\varphi_\varepsilon$  the function defined by (4.2). From Theorem 3.1, we know that the function

$$u_\varepsilon(x) := u *_k \varphi_\varepsilon(x) := \int_{\mathbb{R}^d} u(y) \tau_{-x} \varphi_\varepsilon(y) \omega_k(y) dy.$$

is well defined and is of class  $C^\infty$  on  $\Omega_\varepsilon$ . Furthermore, using (3.15) and the fact that  $u\omega_k$  is weakly  $\Delta_k$ -harmonic on  $\Omega$ , we can see that the function  $u_\varepsilon$  is (strongly)  $\Delta_k$ -harmonic in  $\Omega_\varepsilon$ .

• Let us now fix  $r$  such that  $0 < r < r_\Omega$ . We claim that

$$u_{\varepsilon_1}(x) = u_{\varepsilon_2}(x) \quad \text{whenever } x \in \Omega_r \quad \text{and } \varepsilon_1 + \varepsilon_2 < r \quad (4.12)$$

Indeed, since  $u_{\varepsilon_1} \in \mathcal{H}_k(\Omega_r)$ , by the spherical mean property and (3.18), we have

$$u_{\varepsilon_1} = u_{\varepsilon_1} *_k \varphi_{\varepsilon_2} = (u *_k \varphi_{\varepsilon_1}) *_k \varphi_{\varepsilon_2} \quad \text{on } \Omega_r.$$

If we change the role of  $\varepsilon_1$  and  $\varepsilon_2$ , we also obtain

$$u_{\varepsilon_2} = u_{\varepsilon_2} *_k \varphi_{\varepsilon_1} = (u *_k \varphi_{\varepsilon_2}) *_k \varphi_{\varepsilon_1} \quad \text{on } \Omega_r.$$

Thus the relation (4.12) follows from the associativity property (3.21).

• Now, we will use the following lemma which follows from (4.5) and Lebesgue's differentiation theorem (see [11]):

**Lemma 4.3** *For almost every  $x \in \Omega$ ,  $u_\varepsilon(x) \rightarrow u(x)$  as  $\varepsilon \rightarrow 0$ .*

Letting  $\varepsilon_2 \rightarrow 0$  in the relation (4.12) and using Lemma 4.3, we deduce that  $u$  coincides almost everywhere with the  $\Delta_k$ -harmonic function  $h := u_{\varepsilon_1}$  on  $\Omega_r$ . Since  $r$  can be taken arbitrarily small, the proof of the theorem is complete.  $\square$

**Remark 4.2** *A version of Weyl's lemma has been proved in [2] for  $\Delta_k$ -harmonic functions  $f$  on whole  $\mathbb{R}^d$  and under the additional assumption that the function  $f$  is locally bounded.*

Now, we will characterize the strong  $\Delta_k$ -harmonicity by means of weak  $\Delta_k$ -harmonicity. More precisely, we have

**Corollary 4.4** *Let  $u$  be a function defined on  $\Omega$ . Then  $u \in \mathcal{H}_k(\Omega)$ , if and only if  $u$  satisfies :  $u \in L^1_{k,loc}(\Omega)$ ,  $\Delta_k(u\omega_k) = 0$  in  $\mathcal{D}'(\Omega)$  and  $u(x) = \lim_{r \rightarrow 0} M_B^r(u)(x)$  for every  $x \in \Omega$ .*

*Proof:* By integration by parts formula (3.16) and (2.11), we obtain the 'only if' part. Let us now prove the 'if' part. By Theorem 4.2, there exists  $h \in \mathcal{H}_k(\Omega)$  such that  $u = h$  a.e. on  $\Omega$ . Therefore, for all  $x \in \Omega$ , we have

$$\forall r \in ]0, \varrho_x[, \quad M_B^r(u)(x) = M_B^r(h)(x) = h(x),$$

where  $\varrho_x$  is defined by (2.18). We obtain  $u = h$  on  $\Omega$  by letting  $r \rightarrow 0$ .  $\square$

**Corollary 4.5** *The space  $\mathcal{H}_k(\Omega)$  is closed for the  $L^1_{k,loc}(\Omega)$ -topology.*

*Proof:* Let  $(u_n)$  be a sequence of  $\Delta_k$ -harmonic functions on  $\Omega$  such that  $u_n \rightarrow u$  in  $L^1_{k,loc}(\Omega)$ . As the functions  $u_n\omega_k$  and  $u\omega_k$  are in  $L^1_{loc}(\Omega)$ , we have also  $u_n\omega_k \rightarrow u\omega_k$  in  $\mathcal{D}'(\Omega)$ . In particular,  $\Delta_k(u_n\omega_k) \rightarrow \Delta_k(u\omega_k)$  in  $\mathcal{D}'(\Omega)$ . Now, since  $u_n \in \mathcal{H}_k(\Omega)$ , we deduce that  $\Delta_k(u\omega_k) = 0$  in  $\mathcal{D}'(\Omega)$ .

Finally, by Theorem 4.2 there exists a  $\Delta_k$ -harmonic function  $h$  on  $\Omega$  such that  $u = h$  a.e. in  $\Omega$ . Thus  $u = h$  in  $L^1_{k,loc}(\Omega)$  and the result is proved.  $\square$

As a last result, we will improve Proposition 4.1 as follows

**Corollary 4.6** *Let  $u$  be a function defined on  $\Omega$ . The following statements are equivalent*

- i)**  $u \in \mathcal{H}_k(\Omega)$ ,
- ii)**  $u$  is continuous on  $\Omega$  and  $u(x) = M_S^r(u)(x)$  whenever  $B(x, r) \subset \Omega$ ,
- iii)**  $u$  is continuous on  $\Omega$  and  $u(x) = M_B^r(u)(x)$  whenever  $B(x, r) \subset \Omega$ .

*Proof:* **i)  $\Rightarrow$  ii)** It is obvious.

**ii)  $\Rightarrow$  iii)** By (2.15), clearly we obtain the result.

**iii)  $\Rightarrow$  i)** Let  $\varepsilon_0 > 0$  small enough. Since  $u$  satisfies the volume mean value property, from (3.8) we see that the  $C^\infty$ -function  $u *_k \varphi_\varepsilon$  ( $\varepsilon < \varepsilon_0$ ) defined by (4.3) satisfies also

$$u *_k \varphi_\varepsilon = M_B^r(u *_k \varphi_\varepsilon) \quad \text{on} \quad \Omega_{r+\varepsilon_0},$$

for any  $r$  such that  $0 < r < r + \varepsilon_0 < r_\Omega$  with  $r_\Omega$  given by (2.16). Thus, according to Proposition 4.1,  $u *_k \varphi_\varepsilon \in \mathcal{H}_k(\Omega_{\varepsilon_0})$ .

On the other hand, let  $K$  be a compact subset of  $\Omega_{\varepsilon_0}$ . Noting that for all  $0 < \varepsilon < \varepsilon_0$  and all  $x \in K$ ,  $\text{supp } \tau_{-x}\varphi_\varepsilon \subset B^W(x, \varepsilon) \subset B^W(x, \varepsilon_0) \subset K^W := \cup_{g \in W} g.K + B(0, \varepsilon_0)$  and using (A.12) and the fact that  $\tau_{-x}\varphi_\varepsilon \geq 0$ , we can see that

$$\forall x \in K, \forall \varepsilon < \varepsilon_0, \quad |u *_k \varphi_\varepsilon(x)| \leq \int_{K^W} |u(y)| \tau_{-x}\varphi_\varepsilon(y) \omega_k(y) dy \leq \sup_{y \in K^W} |u(y)|.$$

From this inequality and from the dominated convergence theorem, we deduce that when  $\varepsilon$  tends to zero,  $u *_k \varphi_\varepsilon \rightarrow u$  in  $L^1(K, m_k)$ . In other words,  $u *_k \varphi_\varepsilon \rightarrow u$  in  $L_{k,loc}^1(\Omega_{\varepsilon_0})$ . Finally, Corollary 4.5 and the continuity of  $u$  imply that  $u$  coincides with a Dunkl harmonic function  $h$  on  $\Omega_{\varepsilon_0}$ . As  $\varepsilon_0 > 0$  can be chosen arbitrary small, the result is proved.  $\square$

## A Annex

### A.1 The Dunkl transform

In this Annex we recall some properties of the Dunkl transform (see [9] and [14]).

- The Dunkl transform of a function  $f \in L^1(\mathbb{R}^d, m_k)$  is defined by

$$\mathcal{F}_k(f)(\lambda) := \int_{\mathbb{R}^d} f(x) E_k(-i\lambda, x) \omega_k(x) dx, \quad \lambda \in \mathbb{R}^d, \quad (\text{A.1})$$

where  $E_k(x, y) := V_k(e^{(x, \cdot)})(y)$ ,  $x, y \in \mathbb{R}^d$ , is the Dunkl kernel which is analytically extendable to  $\mathbb{C}^d \times \mathbb{C}^d$  and satisfies the following properties (see [4], [6], [9] and [14])

1. for all  $x, y \in \mathbb{C}^d$ , all  $\lambda \in \mathbb{C}$  and all  $g \in W$ ,

$$E_k(x, y) = E_k(y, x), \quad E_k(x, \lambda y) = E_k(\lambda x, y), \quad E_k(gx, gy) = E_k(x, y), \quad (\text{A.2})$$

2. for all  $x \in \mathbb{R}^d$ ,  $y \in \mathbb{C}^d$  and all multi-indices  $v \in \mathbb{N}^d$ ,

$$\left| \frac{\partial^v}{\partial y^v} E_k(x, y) \right| \leq \|x\|^{|v|} \max_{g \in W} e^{\operatorname{Re} \langle gx, y \rangle}. \quad (\text{A.3})$$

• It is well known (see [9]) that the Dunkl transform  $\mathcal{F}_k$  is an isomorphism of  $\mathcal{S}(\mathbb{R}^d)$  onto itself and its inverse is given by

$$\mathcal{F}_k^{-1}(f)(x) = c_k^{-2} \int_{\mathbb{R}^d} f(\lambda) E_k(ix, \lambda) \omega_k(\lambda) d\lambda, \quad x \in \mathbb{R}^d, \quad (\text{A.4})$$

where  $c_k$  is the Macdonald-Mehta constant given by  $c_k = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x) dx$  (see [7]). Moreover, the following Plancherel theorem holds (see [9]): The transformation  $c_k^{-1} \mathcal{F}_k$  extends uniquely to an isometric isomorphism of  $L^2(\mathbb{R}^d, m_k)$  and we have:

$$\forall f \in L^2(\mathbb{R}^d, m_k), \quad \|c_k^{-1} \mathcal{F}_k(f)\|_{L^2(\mathbb{R}^d, m_k)} = \|f\|_{L^2(\mathbb{R}^d, m_k)}. \quad (\text{A.5})$$

• It is useful to note that if  $f \in L^1(\mathbb{R}^d, m_k)$  is radial,  $\mathcal{F}_k(f)$  is also radial. Precisely, using spherical coordinates and Corollary 2.5 of ([15]), we have

$$\mathcal{F}_k(f)(\lambda) = d_k \int_0^{+\infty} \tilde{f}(r) j_{\gamma+\frac{d}{2}-1}(r\|\lambda\|) r^{2\gamma+d-1} dr, \quad \lambda \in \mathbb{R}^d, \quad (\text{A.6})$$

where  $d_k$  is defined by the relation (2.7) and for  $\lambda \geq -1/2$ ,  $j_\lambda$  is the normalized Bessel function given by

$$j_\lambda(z) = \Gamma(\lambda + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! \Gamma(n + \lambda + 1)} \left(\frac{z}{2}\right)^{2n}.$$

## A.2 Dunkl's translation operators

The Dunkl translation operators  $\tau_x$ ,  $x \in \mathbb{R}^d$ , are defined on  $\mathcal{C}^\infty(\mathbb{R}^d)$  by (see [18])

$$\forall y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f)(y) d\mu_x(z), \quad (\text{A.7})$$

where  $T_x$  is the classical translation operator given by  $T_x f(y) = f(x + y)$ . If  $f \in \mathcal{S}(\mathbb{R}^d)$ ,  $\tau_x f \in \mathcal{S}(\mathbb{R}^d)$  and using the Dunkl transform we have (see [18]):

$$\forall y \in \mathbb{R}^d, \quad \tau_x f(y) = \mathcal{F}_k^{-1}[E_k(ix, \cdot) \mathcal{F}_k(f)](y) \quad (\text{A.8})$$

$$= \frac{1}{c_k^2} \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\lambda) E_k(ix, \lambda) E_k(iy, \lambda) \omega_k(\lambda) d\lambda. \quad (\text{A.9})$$

In particular, the relations (A.9) and (A.3) show that  $(x, y) \mapsto \tau_x f(y)$  is of class  $C^\infty$  on  $\mathbb{R}^d \times \mathbb{R}^d$ .

The operators  $\tau_x$ ,  $x \in \mathbb{R}^d$ , satisfy the following properties:

1) For all  $x \in \mathbb{R}^d$ , the operator  $\tau_x$  is continuous from  $\mathcal{C}^\infty(\mathbb{R}^d)$  into itself.

2) For all  $f \in \mathcal{C}^\infty(\mathbb{R}^d)$  and  $y \in \mathbb{R}^d$ , the function  $x \mapsto \tau_x f(y)$  is of class  $\mathcal{C}^\infty$  on  $\mathbb{R}^d$ .

3) For all  $f \in \mathcal{C}^\infty(\mathbb{R}^d)$  and all  $x, y \in \mathbb{R}^d$ , we have

$$\tau_x f(0) = f(x), \quad \tau_x f(y) = \tau_y f(x). \quad (\text{A.10})$$

4) The Dunkl-Laplace operator  $\Delta_k$  commutes with the Dunkl translations i.e

$$\tau_x(\Delta_k f) = \Delta_k(\tau_x f), \quad x \in \mathbb{R}^d, \quad f \in \mathcal{C}^\infty(\mathbb{R}^d). \quad (\text{A.11})$$

5) for all  $f \in \mathcal{D}(\mathbb{R}^d)$ , we have

$$\forall y \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} \tau_x f(y) \omega_k(y) dy = \int_{\mathbb{R}^d} f(y) \omega_k(y) dy, \quad (\text{A.12})$$

6) Let  $f \in \mathcal{S}(\mathbb{R}^d)$  be radial. Then we have (see [8], Lemme 3.1)

$$\tau_{-x} f(y) = \tau_{-y} f(x) \quad (\text{A.13})$$

The following duality formula has been established by the authors (see [8], Proposition 2.1):

7) Let  $f \in \mathcal{C}^\infty(\mathbb{R}^d)$  and  $g \in \mathcal{D}(\mathbb{R}^d)$ . Then, for all  $x \in \mathbb{R}^d$ , we have

$$\int_{\mathbb{R}^d} \tau_x f(y) g(y) \omega_k(y) dy = \int_{\mathbb{R}^d} f(y) \tau_{-x} g(y) \omega_k(y) dy. \quad (\text{A.14})$$

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