# PHYS 575, HW #1

Due: 1/29/20

1. **More practice with group theory of the Lorentz group (30 points).**

   (a) Using the definition of the Lorentz group $\eta M^{\mathrm{T}} \eta M = \mathbf{1}$, prove that the Lorentz group is closed; that is, if $M$ and $N$ are in the Lorentz group, so is $MN$.

   (b) Write down the Lorentz matrix $\Lambda$ corresponding to a boost of velocity $\beta \hat{\mathbf{x}}$. Taylor-expand to first-order in $\beta$: $\Lambda = \mathbf{1} + \beta X + \ldots$. Show that up to a factor of $\pm i$ (the sign depends on your conventions for active vs. passive transformations), $X$ is the same as $K_x$ as defined in class.

   (c) Do the same as part (b) for a rotation matrix by an angle $\theta$ around the **x**-axis (Taylor-expand in $\theta$ this time).

   (d) Show by explicit computation that $[K_x, K_y] = -iJ_z$. We can interpret this as follows: two infinitesimal boosts performed along the **x** and **y** axes differ by a rotation about the **z** axis depending on the order of the boosts.

   (e) Consider two Lorentz matrices $M$ and $N$. The quantity $M^{-1}N^{-1}MN$ "measures" how much the two matrices $M$ and $N$ don't commute: if they do commute, this quantity is simply the identity. Write $M = \mathbf{1} + X + X^2/2 + \mathcal{O}(X^3)$, $N = \mathbf{1} + Y + Y^2/2 + \mathcal{O}(Y^3)$, and expand $M^{-1}N^{-1}MN$ to quadratic order in $X$ and $Y$. How does this computation relate to the result in part (d)? (The infinitesimal form for $M$ and $N$ comes from the exponential representation, see problem 4 below.)

2. **Lorentz vectors (10 points).** Show that a 4-vector $V^\mu = (V^0, \vec{V})$ transforms under an infinitesimal Lorentz transformation $\Lambda = \mathbf{1} + iX$ with $X = \vec{\beta} \cdot \vec{K} + \vec{\theta} \cdot \vec{J}$ as:

   $$V^0 \to V^0 + \vec{\beta} \cdot \vec{V}, \qquad \vec{V} \to \vec{V} + \vec{\beta} V^0 - \vec{\theta} \times \vec{V}.$$

   (You may want to check the commutation relations to make sure that your $J_y$ has the correct sign.)

3. **Wigner rotation (20 points).** Peskin Problem 2.4. This explores the situation of 1(d)-1(e) in more detail. (Note there is a typo in part (f) of this problem, which should ask you to expand the matrix of part (e), not of part (d).)

4. **Lie groups from Lie algebras (20 points).** We saw in class that the Lie algebra arises from considering infinitesimal group transformations. We can reverse this process and reconstruct the group from its infinitesimal elements through exponentiation. Define the *matrix exponential* $\exp(X)$ by a formal power series:

$$\exp(X) \equiv \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Compute $\exp(i\alpha_1 J_x)$ and $\exp(i\alpha_2 K_x)$, and relate $\alpha_1$ and $\alpha_2$ to the usual parameterization of rotations and boosts in terms of $\theta$ and $\beta$. *(Hint: consider $J_x^2$ and $K_x^2$ first.)*

5. **Baker-Campbell-Hausdorff Formula (20 points).** Larkoski Problem 3.4.