Seminar

Catastrophe Theory/Katastrophentheorie

(Singularitäten differenzierbarer Abbildungen)

Unfoldings of Germs and Catastrophes

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In other words, an unfolding of $[f]$ is a smooth family of germs $\{[F_u = F(-, u)]\}_u$ whose **center of organization** is $[f]$ at the origin $u = \mathbf{0} \in \mathbb{R}^r$, i.e., a deformation of the germ $[f]$.

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Let $[F] \in \mathscr{E}_{n+r}$ be an unfolding of $[f]$. The development of the singularities of $[f]$ along the unfolding $[F]$ is described by the following subsets:

- **Catastrophe surface**: $M_F = \{(x, u) \mid DF_u(x) = 0\}$.
- **Catastrophe set**: $C_F = \{(x, u) \in M_F \mid \text{x is a degenerate critical point of } F_u\} \subseteq M_F$.
- **Birufication set**: $B_F = \{u \mid \text{there is x such that } (x, u) \in C_F\}$.

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B_F describes the region of the (u,v)-plane at which the nature of the critical points of the germs changes from 3 non-degenerate critical points ($\Delta < 0$) to 1 non-degenerate critical point ($\Delta > 0$). For the values of (u,v) in B_F itself, the germs have 1 degenerate critical point and 1 non-degenerate critical point (except for the case $u = v = 0$).

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An unfolding $[F] \in \mathscr{E}_{n+r}$ of a germ $[f] \in \mathscr{E}_n$ is **versal** if every unfolding $[G] \in \mathscr{E}_{n+s}$ of $[f]$ is induced by $[F]$. If the number r of external parameters is minimal, then $[F]$ is a **universal unfolding**.

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(2) (**Transitivity**) Let $[F] \in \mathscr{E}_{n+r}$, $[G] \in \mathscr{E}_{n+s}$, and $[H] \in \mathscr{E}_{n+t}$ be unfoldings of $[f] \in \mathscr{E}_n$ and suppose that $[H]$ is induced by $[G]$ and $[G]$ is induced by $[F]$.

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Properties of Induced Unfoldings (ctd.)

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- (**Versal vs. k-transversal unfoldings**) A versal unfolding $[F] \in \mathscr{E}_{n+r}$ of $[f] \in \mathscr{m}_n^2$ is k-transversal for every $k > 0$.

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- (**Existence of transversal unfoldings**) Let $[f] \in \mathscr{m}_n^2$ and $k > 0$. There is a k-transversal unfolding $[F]$ of $[f]$ constructed as follows.

  Note that the vector space $\mathscr{m}_n/\mathscr{J}[f] + \mathscr{m}_n^{k+1}$ is finite dimensional. Let $[g_j]$, $j = 1, \cdots, r$, be germs in $\mathscr{m}_n$ which define a basis of $\mathscr{m}_n/\mathscr{J}[f] + \mathscr{m}_n^{k+1}$. Then the germ of

  $$F(x, u) = f(x) + u_1 g_1(x) + \cdots + u_r g_r(x)$$

  defines a k-transversal unfolding of $[f]$.

  (Proof. Note that $\frac{\partial}{\partial u_j} F(x, 0) - \frac{\partial}{\partial u_j} F(0, 0) = g_j(x)$.)

- (**Versal vs. k-transversal unfoldings**) A versal unfolding $[F] \in \mathscr{E}_{n+r}$ of $[f] \in \mathscr{m}_n^2$ is k-transversal for every $k > 0$.

  (Proof. Let $[G]$ be a k-transversal unfolding of $[f]$. Since $[G]$ is induced by $[F]$, it follows that $\mathscr{V}[G] \subseteq \mathscr{J}[f] + \mathscr{V}[F]$. Then $\mathscr{m}_n = \mathscr{J}[f] + \mathscr{V}[F] + \mathscr{m}_n^{k+1}$ as required.)

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We consider the following smooth function for small $(a, u) \in \mathbb{R}^n \times \mathbb{R}^r$ and $x \in \mathbb{R}^n$,

$$F_u^a(x) :\ = F(a + x, u) - F(a, u).$$

The family $\{[F_u^a]\}$ consists essentially of the germs nearby $[f]$ in the unfolding $[F]$.

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$$j^k_e[F] \colon W \to \mathbb{R}^N, \ (a, u) \mapsto j^k[F^a_u] \ \ (k\text{-}\textbf{th extension of } [F])$$

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An elementary calculation shows that the vector space

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Let $T_{j^k[f]}(j^k[f] \cdot \mathscr{G}_n^k)$ be the tangent space at $j^k[f]$ of the orbit of $j^k[f]$ with respect to the action of the group of (k-jets of) local diffeomorphisms $\mathscr{G}_n^k$ preserving the origin.

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Using the identification $T_{j^k[f]}(j^k[f] \cdot \mathscr{G}_n^k) \cong j^k(\mathscr{m}_n \mathscr{J}[f])$ and the equality of vector spaces

$$\mathscr{m}_n = \mathscr{m}_n \mathscr{J}[f] + \langle D_1 f \rangle_{\mathbb{R}} + \cdots + \langle D_n f \rangle_{\mathbb{R}}$$

we conclude:

Theorem

$[F]$ is a k-transversal unfolding of $[f]$ if and only if

$$j^k(\mathscr{m}_n) = T_{j^k[f]}(j^k[f] \cdot \mathscr{G}_n^k) + \mathrm{Im}\big(Dj_e^k[F](\mathbf{0})\big).$$

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Let $[f]$ be a k-determined germ in $\mathcal{m}_n^2$. Then any two r-parameter k-transversal unfoldings of $[f]$ are equivalent.

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The proof can be viewed as a family version of the proof of Mather's Sufficient Criterion for the determinacy of germs and makes use of the **Malgrange–Mather Preparation Theorem** – the significance of this theorem in the proof of the Main Lemma is that it allows us to apply Nakayama's Lemma.
Let $\phi\colon U \to \mathbb{R}^s$ be a smooth function on an open neighborhood of $\mathbf{0} \in \mathbb{R}^n$ with $\phi(\mathbf{0}) = \mathbf{0}$. There is a ring homomorphism $\phi^*\colon \mathscr{E}_s \to \mathscr{E}_n$, $[f] \mapsto [f][\phi] = [f \circ \phi]$.

Theorem (Malgrange–Mather Preparation Theorem)

Let M be a finitely generated module over $\mathscr{E}_n$. Then M is finitely generated as an $\mathscr{E}_s$-module (with respect to the module structure defined by ϕ^*) if and only if the vector space $M/\langle \mathcal{m}_s M\rangle$ is finite dimensional.

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Let $[F] \in \mathscr{E}_{n+r}$ *be an unfolding of* $[f] \in m_n^2$*. The following are equivalent:*

(a) $[F]$ *is a versal unfolding.*

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(c) $[F]$ *is* k*-transversal for some* $k > r + 1$*.*

(d) $m_n = \mathscr{J}[f] + \mathscr{V}[F]$*.*

These imply that $\mathrm{cod}[f] \leq r$ *and* $[f]$ *is* $(r+2)$*-determined. Moreover, (a)-(d) are also equivalent to:*

(e) $[f]$ *is finitely determined and* $[F]$ *is* k*-transversal for* $k = \det[f]$*.*

Proof (Sketch) We have already discussed (a) $\Rightarrow$ (b). (b) $\Rightarrow$ (c) is obvious.

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$$m_n = \mathscr{J}[f] + m_n \supseteq \mathscr{J}[f] + m_n^2 \supseteq \cdots \supseteq \mathscr{J}[f] + m_n^{r+1} \supseteq \mathscr{J}[f] + m_n^{r+2} \supseteq \cdots$$

must stabilize after at most r steps because $\dim_{\mathbb{R}} \mathscr{V}[F] \leq r$.

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$$\mathrm{cod}[f] = \dim_{\mathbb{R}}(m_n/\mathscr{J}[f]) = \dim_{\mathbb{R}}(m_n/\mathscr{J}[f] + m_n^{r+1}) \leq \dim_{\mathbb{R}}\mathscr{V}[F] \leq r.$$

(d) also follows because $m_n = \mathscr{J}[f] + \mathscr{V}[F] + m_n^{r+1} = \mathscr{J}[f] + \mathscr{V}[F]$.

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(b) *$[F]$ is k-transversal for all $k > 0$.*

(c) *$[F]$ is k-transversal for some $k > r+1$.*

(d) $m_n = \mathscr{J}[f] + \mathscr{V}[F]$.

These imply that $\operatorname{cod}[f] \leq r$ and $[f]$ is $(r+2)$-determined. They are also equivalent to:

(e) *$[f]$ is finitely determined and $[F]$ is k-transversal for $k = \det[f]$.*

Proof (Sketch) (d) $\Rightarrow$ (e) is obvious. (e) $\Rightarrow$ (a): Let $[G] \in \mathscr{E}_{n+s}$ be an unfolding of $[f]$ and consider the sum $[H] = [F] + [G] \in \mathscr{E}_{n+r+s}$ of the unfoldings. $[H]$ is k-transversal because $[F]$ is so. We may consider $[F]$ as an $(r+s)$-parameter unfolding, denoted by $[\widetilde{F}]$, which is constant in the s variables. Note that $[\widetilde{F}]$ is again k-transversal. According to the Main Lemma, $[H]$ and $[\widetilde{F}]$ are equivalent.

Characterizations of Versal Unfoldings (ctd.)

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