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Abstract

In this note we consider non-stationary cluster point processes and we derive their conditional intensity, i.e. the intensity of the process given the locations of one or more events of the process. We then provide some approximations of the conditional intensity.

Conditional intensity; Neyman-Scott process; Point process

1 Introduction

The problem of conditioning is an old problem in the point process theory. The conditional distribution of a spatial point process, say Φ, given a realisation of some events of Φ was introduced by Palm (1943) for stationary point processes on the real line and recently summed up in Cœurjolly et al. (2017) in the general case.

We consider finite point processes Φ defined in a Borel set $S \subset \mathbb{R}^2$ and specified by a density f. We assume that we have observed Φ in $W \subseteq S$ and we denote Φ_W its restriction to the set W. A natural way to predict $\Phi_{S \setminus W}$ is to consider the conditional distribution of $\Phi_{S \setminus W}$ given Φ_W, which is can be expressed in terms of the conditional density (cf Cœurjolly et al. (2017)). Unfortunately, the density of Φ restricted to W is tractable for few usual processes, as Poisson, Gibbs and determinantal processes, but not for Cox and cluster point processes.

In this note, we derive the conditional intensity of non-stationary cluster point processes and provide some approximations for practical applications. Note that Gabriel et al. (2017) and Gabriel et al. (2021) define a "model-free" predictor of the conditional intensity for stationary and non-stationary processes, in sense that it is only related to the first and second-order moments of the point process.

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2 Conditional intensity of non-stationary cluster point processes

Cluster point processes, developed by Neyman and Scott (1958), are formed by a simple procedure, with a homogeneous Poisson process Ψ with intensity κ generating *parent* points at a first step and a random pattern of *offspring* points around each parent point at a second step. The number of offspring points has a Poisson distribution with mean μ and the offspring points are independently and identically distributed with a kernel k depending on the distance from offspring to parent. The cluster point process $\tilde{\Phi}$ is the set of offspring points, regardless their parentage. This process is stationary with intensity $\tilde{\lambda} = \kappa\mu$.

We focus on the $p(x)$−thinned process of $\tilde{\Phi}$, where $p(x)$ is a deterministic function on $\mathbb{R}^2$ with $0 \leq p(x) \leq 1$. If the point x belongs to $\tilde{\Phi}$, it is deleted with probability $1 - p(x)$ and again its deletion is independent of locations and possible deletions of any other points. Let Φ be the $p(x)$−thinned process. The process Φ is second-order intensity reweighted stationary with $\lambda(x) = \kappa\mu p(x)$ (see Chiu et al. (2013)).

Here we want to know the conditional intensity of $\Phi_{S\setminus W}$ given Φ_W. We denote ∂W the border of the observation window W, with width defined by the range of the dispersion kernel k, say r. In other words, $\partial W = W_{\oplus r}\setminus W$. Figure 1 illustrates the different steps of the generating procedure.

(a) Parent process Ψ **(b)** Offspring process $\tilde{\Phi}$ **(c)** Thinned process Φ
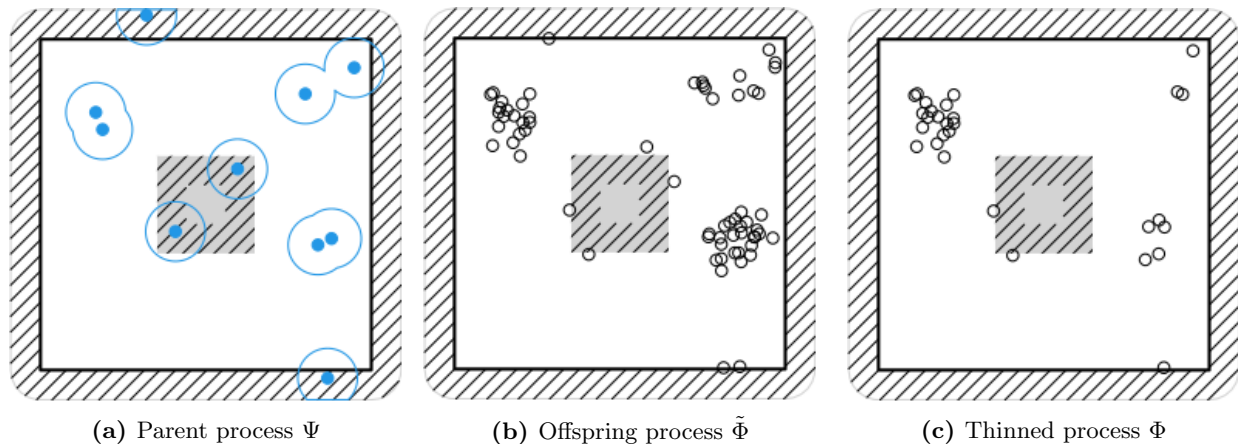
Figure 1: Generating procedure of the $p(x)$-thinned cluster process. Parent points (a) are generated in the union of the observation window W (white area) and ∂W (hatched area). Offspring (b) are generated within the circle around the parent points. The final process (c) is obtained by $p(x)$-thinning. The window of interest S is delineated in black. The prediction window is represented by the grey shaded central square.

Proposition 1. *The conditional intensity of a $p(x)$-thinned cluster process observed in $W \subseteq S$ is:*

$$\lambda(x_o|\Phi_W) = \int \left[\sum_{y \in W \cup \partial W} \mu p(x_o) k(y - x_o) \right.$$

$$\left. + \mu\kappa \int_{b(x_o,r)\backslash(W \cup \partial W)} p(x_o) k(y - x_o)\, dy \right] d\mathbb{P}[\Psi_{W \cup \partial W}|\Phi_W], \quad (1)$$

where $b(x_o, r)$ denotes the disc of centre x_o and radius r.

The proof is rather straightforward as (i) for a cluster process we know the conditional intensity given the realisation of parent points, (ii) the parent process is Poisson and (iii) knowing the offspring points in W is not informative on parent points in $b(x_o, r)\backslash(W \cup \partial W)$.

The conditional intensity (1) depends on the distribution of the parent process in $W \cup \partial W$ given the offspring in W. By Campbell's theorem this equation can be rewritten

$$\lambda(x_o|\Phi_W) = \int_{W \cup \partial W} \mu p(x_o) k(y - x_o) c(y) \rho(y|\Phi_W)\, dy$$

$$+ \mu\kappa \int_{b(x_o,r)\backslash(W \cup \partial W)} p(x_o) k(y - x_o)\, dy, \quad (2)$$

where $\rho(y|\Phi_W)$ is the intensity of parent points in $W \cup \partial W$ given the offspring points in W. Baudin (1983) derived the following formula for such a conditional intensity of parent points for Neyman-Scott processes:

$$\rho(y|\Phi_W) = \kappa G\left(1 - F(W_y)\right) + \sum_{j=1}^{2^n-1} \sum_{b \in \mathcal{B}} b(a_j) \prod_{i=1}^{2^n-1} S(\Phi, W, a_i)^{b(a_i)}$$

$$\times \left[\frac{\kappa G^{|a_j|}\left(1 - F(W_y)\right) \prod_{\ell=1}^{n} k(y_\ell - x_o)^{a_{j\ell}}}{S(\Phi, W, a_j)} \right]^{b(a_j)} \left[\sum_{b \in \mathcal{B}} \prod_{i=1}^{2^n-1} S(\Phi, W, a_i)^{b(a)} \right], \quad (3)$$

where

- G is the probability generating function of the number of points in a cluster,
- $F(\,dx)$ is the probability distribution function of offspring points, with density k,
- $\{x_1, \ldots, x_n\} = \Phi_W$,
- $W_y = -y + W$,
- $a_1 = (0, \ldots, 0, 1)$, $a_2 = (0, \ldots, 0, 1, 0)$, $\ldots$, $a_{2^n-1} = (1, \ldots, 1)$: vectors of length n,
- $\mathcal{B}$ is the set of all functions $b : \{a_1, \ldots, a_{2^n-1}\} \to \{0, 1\}$ such that $\sum_{i=1}^{2^n-1} b(a_i) a_i = (1, \ldots, 1)$, $a_i = (a_{i1}, \ldots, a_{in})$, $|a_i| = a_{i1} + \cdots + a_{in}$,
- $S(\Phi, W, a_i) = \kappa \int G^{|a_i|}(1 - F(W_y)) \prod_{\ell=1}^{n} k(x_l - y)\, dy.$

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However, this conditional intensity is based on combinations and is just too complicated in practice. van Lieshout and Baddeley (1995) interpreted the problem of identifying parent points as a statistical estimation problem with a Bayesian inference based on MCMC methods; see also Lawson and Denison (2002) for similar approaches.

Here we propose to approximate it as follows,

$$\rho(y|\Phi_W) = \frac{1}{\mu p(y)} \sum_{x \in \Phi_W} k(x - y) + \kappa \exp\left(-\mu \int_W p(z) k(y - z) \, \mathrm{d}z\right) \tag{4}$$

where $c(y)$ ensures that $\mathbb{E}\left[\rho(y|\Phi_W)\right] = \kappa$. The first term of Equation (4) is related to observed offspring points in W and the second term to offspring points that have not been observed (due to thinning or their proximity to the boundary of W).

3   Validation procedure

In this section, we aim at quantifying the difference between the distribution of parent points given the offspring points provided in (4) and the observed distribution of the conditional parent process. We thus use complementary statistics to test the interactions (i) between parent points, (ii) between parent and offspring points and (iii) between parent points and the inner (or outer) boundary of W. Let $\Phi(A)$ (resp. $\Phi(B)$) be the number of points of Φ in a Borel set A (resp. of Ψ in B). We denote Ψ_S the observed parent points in S and $\nu(B)$ the Lebesgue measure of B. Then,

- (i) *Interaction statistic between parent points*

  Let $\widehat{H}(d)$ be the empirical cumulative distribution function between observed parent points Ψ_S and $H(d)$ the theoretical one:

$$\widehat{H}(d) = \frac{1}{\Psi(S)} \sum_{y_1, y_2 \in \Psi_S}^{\neq} \mathbb{I}_{\{\|y_1 - y_2\| \leq d\}},$$

$$H(d) = \frac{1}{\int_S \rho(y|\Phi_W) \, \mathrm{d}y} \int_S \int_{b(z,d) \cap S} \rho(y|\Phi_W) \rho(z|\Phi_W) \, \mathrm{d}y \, \mathrm{d}z.$$

- (ii) *Interaction statistic between parent and offspring points*

  Let $\widehat{E}(d)$ be the empirical cumulative distribution function between observed parent points Ψ_S and observed offspring points Φ_W and $E(d)$ the theoretical one:

$$\widehat{E}(d) = \frac{1}{\Phi(W)} \sum_{x \in \Phi_W} \sum_{y \in \Psi_S} \mathbb{I}_{\{\|x - y\| \leq d\}},$$

$$E(d) = \frac{1}{\Phi(W)} \sum_{x \in \Phi_W} \int_{b(z,d) \cap S} \rho(z|\Phi_W) \, \mathrm{d}z.$$

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- *(iii) Interaction statistic between parent points and the boundary b_W of W*

  Let $\widehat{B}(d)$ be the empirical cumulative distribution function between observed parent points Ψ_S and the inner boundary (B^{inner}) or outer boundary (B^{outer}) of W, and $B(d)$ the theoretical one:

$$
\begin{aligned}
\widehat{B}(d) &= \frac{1}{\nu(b_W)} \sum_{\ell \in b_W} \sum_{y \in \Psi_S} \mathbb{I}_{\{\|\ell - y\| \le d\}} = \frac{1}{\nu(b_W)} \sum_{y \in \Psi_S} \nu(b_W \cap b(y, d)), \\
B(d) &= \frac{1}{\nu(b_W)} \int_{b_W} \int_{b(\ell, d) \cap S} \rho(z | \Phi_W) \, \mathrm{d}z \, \mathrm{d}\ell.
\end{aligned}
$$

We illustrate the results for a thinned Matérn cluster process Φ. For this process, k is the uniform distribution on the disc of radius r and the conditional intensity is

$$
\lambda(x_o | \Phi_W) = \frac{\mu p(x_o)}{\pi R^2} \int_{b(x_o, r) \cap (W \cup \partial W)} \rho(y | \Phi_W) \, \mathrm{d}y + \kappa \mu p(x_o) \nu \left(b(x_o, r) \backslash (W \cup \partial W) \right),
$$

with

$$
\rho(y | \Phi_W) = \frac{1}{\mu p(y) \pi R^2} \sum_{x \in \Phi_W} \mathbb{I}_{b(x, r)}(y) + \kappa \exp \left(-\frac{\mu}{\pi r^2} \int_{b(y, r) \cap W} p(z) \, \mathrm{d}z \right).
$$

The non-stationary Matérn Cluster process Φ depends on four parameters: the thinning probability $p(x)$, the intensity of parents κ, the mean number of points per parent μ and the radius of dispersion of the offspring around the parent points r. Here we fix $\kappa = 50$ and $\mu = 40$ and we consider:

- two thinning probabilities: $p_1(x) = p_1(x_1, x_2) = \alpha_1 \mathbb{I}_{\{x_1 \le v\}} + \alpha_2 \mathbb{I}_{\{x_1 > v\}}$, setting $\alpha_1 = 0.8$, $\alpha_2 = 0.2$ and $v = 0.5$, and $p_2(x) = p_2(x_1, x_2) = 1 - x_1$.

- the unit square as study region S. The observation window is $W = S \backslash W_h$, where $W_h = [0.35, 0.65]^2$ when using $p_1(x)$ and $W_h = [0.05, 0.95] \times [0.36, 0.64]$ when using $p_2(x)$.

- $r \in \{0.05, 0.09, 0.13\}$.

For each pair of parameters $(p(x), r)$ we simulate $N = 250$ realisations of the non-stationary Matérn Cluster process and compute all the previous interaction statistics. Because all the theoretical statistics only provide a trend, for each of the N simulations, we generate $n = 100$ simulations of parent points from a Poisson process with intensity $\rho(y | \Phi_W)$ and compute the related empirical cumulative distribution functions, that we denote by $\widehat{H}_{sim}(d)$, $\widehat{E}_{sim}(d)$ and $\widehat{B}_{sim}(d)$. Figure 2 illustrate the 95% envelopes of the empirical cumulative distribution functions computed from the N observed parent points (red hatching) and from the $N \times n$ simulated parent points (blue hatching). The grey envelopes correspond to the theoretical trend. In this figure $p(x) = p_1(x)$ and $r = 0.09$.

Results for all pairs of parameters are very similar. All overlapping envelopes indicate that the empirical cumulative distribution functions between parent points and other parent points / offspring

points and the boundary of the observation window W are similar for the observed parent points Ψ_S and for simulated parent points, which further correspond to the theoretical distribution. This is true at any distances and confirm that the distribution of parent points in S given the offspring points in W provided in (4) is correct.
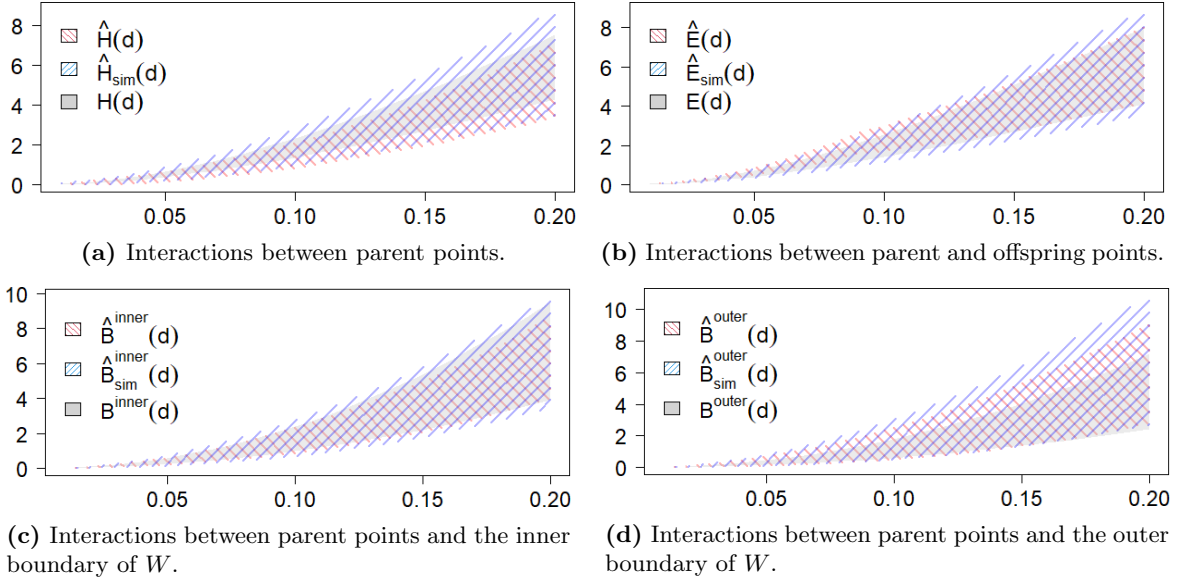

(a) Interactions between parent points.

(b) Interactions between parent and offspring points.

(c) Interactions between parent points and the inner boundary of W.

(d) Interactions between parent points and the outer boundary of W.

Figure 2: 95% envelopes of the cumulative distribution functions calculated for N realisations of the thinned Matérn cluster process (empirical is in red hatching and theoretical is in grey) and for simulated parent points (blue hatching).

For each type of interaction, we computed the coverage rate between the envelopes obtained from the observed distribution of parent points and the enveloped obtained from the approximated distribution of parent points. E.g., denoting by $\mathcal{E}$ the envelopes, the coverage rate for the interaction statistic between parent points is

$$\tau(\widehat{H}, \widehat{H}_{sim}) = \frac{\nu\left(\mathcal{E}(\widehat{H}) \cap \mathcal{E}(\widehat{H}_{sim})\right)}{\nu\left(\mathcal{E}(\widehat{H})\right)}.$$

Results for all combination of parameters $(p(x), r)$ are reported in Table 1. These results show that for any configuration and interaction range the approximation of the conditional intensity of parent points in $W \cup \partial W$ given the offspring points in W is very similar to the one of observed parent points.

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	$\tau(\widehat{H}, \widehat{H}_{sim})$	$\tau(\widehat{E}, \widehat{E}_{sim})$	$\tau(\widehat{B}^{inner}, \widehat{B}^{inner}_{sim})$	$\tau(\widehat{B}^{outer}, \widehat{B}^{outer}_{sim})$
r		$p_1(x)$		
0.05	88.04	97.30	100.00	100
0.09	93.48	97.40	99.86	100
0.13	98.73	99.91	100.00	100
r		$p_2(x)$		
0.05	90.07	97.72	100	100.00
0.09	96.83	96.24	100	100.00
0.13	99.37	90.65	100	98.57

Table 1: Coverage rate between the envelopes of the interaction statistics computed from the observed parent points and from the simulated parent points.

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