The neighbouring lattice method

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The method: History

Origin: Martin Kneser: Klassenzahlen definiter quadratischer Formen, Arch. Math. 8, 1957

Goal: Classify unimodular positive definite lattices in small dimensions

Underlying basic idea: Arithmetically indefinite lattices (Martin Eichler: Ähnlichkeitsklassen indefiniter Gitter, Math. Zeitschrift 55, 1952)

Strong approximation

Theorem (Strong approximation theorem)

F a global field, S a finite set of places of F, G a simply connected semisimple linear algebraic group, defined over F, absolutely almost simple, $G_S = \prod_{v \in S} G(F_v)$ not compact.
Then G has the strong approximation property with respect to S, i.e., $G(F)G_S$ is dense in $G(\mathbb{A})$, hence for $\infty \subseteq S$:

Given a finite set S' of places with $S \cap S' = \emptyset$ and $h_v \in G(F_v), t_v \in \mathbb{N}_0$ for all $v \in S'$, there is $g \in G(F)$ with

$$g \equiv h_v \bmod {}^{t_v}_{v} \text{ for all } v \in S', \quad g \in G(_v) \text{ for all } v \notin S \cup S'.$$

Notice: If (V, Q) is a quadratic space over F, the group $G = SO_V$ is not simply connected, but $G_1 = \mathrm{Spin}_V$ is.

Spinor genus

(V, Q) a quadratic space over F (a number field),
$G = SO_V, G_1 = \mathrm{Spin}_V, \pi : G_1 \to G$ the covering map,
$O'_V = (\pi) = \{\sigma \in O_V \mid \theta(\sigma) = 1\}$, with θ the spinor norm
$(\theta(\prod \tau_{y_i}) = \prod Q(y_i)(F^\times)^2)$, L an -lattice on V.
The spinor genus $\mathrm{spn}(L)$ is the $O_V(F)O'_V(\mathbb{A})$-orbit of L.

Corollary

Let $S \supseteq \infty$ be a set of places, $V \otimes F_v$ isotropic for at least one $v \in S$. Then every class in the spinor genus of L has a representative M in the G_S-orbit of L (the $_S$-class of L), i.e., $M \otimes_v = L \otimes_v$ for all $v \notin S$.

In particular:

1. ($S = \infty$) If V is indefinite, spinor genus and class coincide
2. ($S \supsetneq \infty$) If V is definite, but arithmetically indefinite at some finite place w (i.e., $V \otimes F_w$ is isotropic), every class in the spinor genus of L has a representative which differs from L only at the place w.

Notations

From now on: F a number field.

- $b(x,y) = Q(x+y) - Q(x) - Q(y), b(x) := b(x,x), B = \frac{1}{2}b$
- $L^{\#} = \{y \in V \mid b(y,L) \subseteq\}$ (dual lattice), $L^{\#,S}$ (S-dual, where S is a finite set of places of F) is the lattice with $L_v^{\#,S} = L_v$ for $v \notin S$, $L_w^{\#,S} = L_w^{\#}$ for $w \in S$.
- (L,b) (bilinear module) is integral if $L \subseteq L^{\#}$, even if $2Q(L) = b(L) \subseteq 2\mathbb{Z}$, odd if (L,b) is integral but $2Q(L)$ contains odd integers.
- L unimodular if $L = L^{\#}$, L is S-unimodular if $L = L^{\#,S}$ (equivalently: $L\otimes_w$ is unimodular for all $w \in S$).
- The quadratic module (L,Q) is classically integral if $L \subseteq 2L^{\#}$.

The method

From now on: Let w be a finite place of F, $S = \infty \cup \{w\}$.

Definition

Let L, M be lattices on V with $L\otimes_v = M\otimes_v$ for all $v \neq w$. Lattices L, M on V are $_w$-neighbours if $(L : M \cap L) = (M : M \cap L) = q_w = N(_w)$. Equivalently: L, M have elementary divisors $_w^{-1}, 1, \ldots, 1, _w$ in each other.

From now on: Let L, M be $\{w\}$-unimodular lattices on V with $L\otimes_v = M\otimes_v$ for all $v \neq w$, assume $\dim(V) \geq 3$.

Lemma

For $x \in_w^{-1} L \setminus L$ with $b(x,x) \in_w$ put

$$L_x := \{y \in L \mid b(y,x) \in_w\} = (L+x)^{\#,S}, \quad L(x) := L_x + x.$$

Then $L(x)$ is a $\{w\}$-unimodular lattice in $_SL$ and $L, L(x)$ are $_w$-neighbours. If L is a free -lattice, so is $L(x)$.

The method, (2)

Instead of $L_x := \{y \in L \mid b(y,x) \in_w\}$ with $x \in_w^{-1} L \setminus L$, $b(x,x) \in_w$ we can also pick $a \in$ with $\mathrm{ord}_w(a) = 1$ and consider $z := ax \in L \setminus_w L$ with $b(z,z) \in_w^2$. We have then $L_x := \{y \in L \mid b(y,z) \in_w\}$ and $L(x) = L_x +_w^{-1} z =: \tilde{L}(z)$.

Lemma

Let L_w be even, $x \in_w^{-1} L \setminus L$ with $Q(x) \in_w$ and $L(x)$ be as above.

- *There is $\sigma \in SO_V(F_w)$ with $\sigma(L_w) = L(x)_w$ and $\theta(\sigma) = \pi_w$.*
- *$L(x)$ is even and in the genus of L, it is in $\mathrm{spn}^+(L)$ (proper spinor genus, i.e., $SO_v(F)O'_V(\mathbb{A})$-orbit) if and only if $\pi_w \in \theta(SO_v(F))\theta(SO_L(\mathbb{A}))(\mathbb{A}^\times)^2$ holds.*
- *For each proper spinor genus $\mathrm{spn}^+(K)$ in the genus of L there are infinitely many places w such that the $_w$-neighbours of L are in $\mathrm{spn}^+(K)$.*

The method, (3)

Lemma

Let L, M, w be as above.
There is $x \in_w^{-1} L \setminus L$ with $b(x,x) \in_w$ (with $Q(x) \in_w$ if M_w is even) such that $(M : L(x) \cap M) < (M : L \cap M)$.
In particular, all $_w$-neighbours of L are of the form $L(x)$ for some x.

Proof.

Pick $y \in M \setminus L$. Then $y \in_w^{-r} L$ for some $r \in \mathbb{N}$, hence there is $a \in$ with $x := ay \in_w^{-1} L \setminus L$. Since $x \in M$, we have $M \cap L \subseteq L_x$, so $x \notin L$ implies $M \cap L \subsetneq M \cap L(x)$. □

Theorem

Any two lattices L, M as above can be connected by a chain of $_w$-neighbouring lattices. If M_w, L_w are even, all lattices in the chain can be chosen to be even at w. Any two lattices at an even distance in this chain are in the same proper spinor genus.

Choice of vectors

Proposition

Assume that w is non-dyadic, i.e., $2 \notin_w$*, or that* L_w *is even (*$Q(L) \subseteq_w$*) with* $Q(x) \in_w$ *for the x used. Then*

- *All* $_w$*-neighbours of L belong to the genus of L.*
- *let* $z' \in L$ *with* $Q(z') \in_w$*. Then there is* $z \in z' +_w L$ *with* $Q(z) \in_w^2$*, and for any such* $z \in L \setminus_w L$ *the neighbouring lattice* $\tilde{L}(z)$ *depends only on the line through* $z +_w L$ *in the* $_w/_w$*-space* $L/_w L$*.*
  That is, the $_w$*-neighbours of L correspond bijectively to the isotropic lines in the* $_w/_w$*-space* $L/_w L$ *(quadratic space!).*
- *If* z_1', z_2' *are as above and there exists* $\sigma \in O_L$ *mapping the line through* $z_1' +_w L$ *onto the line through* $z_2' +_w L$ *in the* $_w/_w$*-space* $L/_w L$*, the* $_w$*-neighbours of L corresponding to these lines as above are isometric.*

Classification algorithm for a genus

L given with L_w even and $Q(x) \in_w$ for the vectors $x \in_w^{-1} L \setminus L$ used.
Assume $\pi_w \in \theta(SO_V(F))\theta(SO_L(\mathbb{A}))(\mathbb{A}^\times)^2$ (in particular true if the genus of L coincides with its proper spinor genus).
Start the list of known classes of lattices with L.

1. For each M in the list not marked as done find representatives of all isotropic lines in the quadratic $_w/_w$-space $M/_w M$.
2. Compute for each M the O_M-orbits of the isotropic lines as above, pick (for each M) one representative from each orbit.
3. For each M and each of the representatives z from the previous step compute the neighbouring lattice $M(z)$, discard those that belong to isometry classes already listed (**Note: This needs a fast equivalence test**), mark M as done, add the new classes to the list.
4. If the last step obtained new classes, return to step 1, otherwise terminate and output the list of classes.

Classification algorithm, (2)

If the assumption $\pi_w \in \theta(SO_V(F))\theta(SO_L(\mathbb{A}))(\mathbb{A}^\times)^2$ does not hold, the above algorithm gives representatives of all classes in two of the proper spinor genera in the genus of L. By varying $_w$ one can obtain all proper spinor genera.

Example 1 (Kneser, 1957)

The original example. Start with $L = I_n \quad (Q(x_1, \dots, x_n) = \frac{1}{2}\sum x_i^2)$ over $\mathbb{Z}$.
Notice:

1. An odd unimodular $\mathbb{Z}$-lattice has precisely two even neighbours, it intersects with both of them in its even sublattice.
2. If $x \in 2^{-1}L \setminus L$ is given and there is $y \equiv x \bmod L$ with $2b(x,y) \equiv 1 \bmod 2$ and $b(y,y) = 1$, the neighbour $L(x)$ is isometric to L (one has $\tau_y(x) \in L$ and $\tau_y(L_x) = L_x$).

The action of O_L allows to reduce to $x = x_m := \frac{1}{2}\sum_{i=1}^m e_i$ with $m \leq n$ and $m \equiv 0 \bmod 4$ because of $b(x,x) \in \mathbb{Z}$.
Denote $I_m(x) = K_m$ for this x_m.
For $m = 4$ one obtains $L(x) = K_4 \perp I_{n-4}$ with K_4 the (unique) neighbour class of I_4. But $K_4 \cong I_4$ (take $y = \frac{1}{2}(e_1 + e_2 + e_3 - e_4)$.
Consequence: for $n \leq 7$ there is only one class of unimodular lattices, that of I_n.

Example 1 (Kneser, 1957), (2)

In general one has $I_n(x_m) \cong K_m \perp I_{n-m}$.

K_8 is even, it is the E_8-lattice, and for all $m \geq 8$ we have $K_m \not\cong I_m$, since $1 \notin 2Q(K_m)$.
Discussion of the neighbours of $K_8 = E_8$ is more tricky (read yourself!), result: All neighbours are isometric to I_8 or to K_8, so E_8 is the only even unimodular class and I_8 the only odd one.

For the further discussion restrict to indecomposable lattices.
A tricky, but short discussion (the paper has 10 pages) gives the complete list in dimensions ≤ 16:
$I_1, K_8, K_{12}, M_{14}, M_{15}, K_{16}, L_{16,8}$ (with K_8, K_{16} even, the others odd), notation as in Kneser's article.

Example 2: Niemeier lattices

Niemeier (1973), a doctoral student of Kneser, classified the even unimodular lattices in dimension 24 with the neighbouring lattice method by hand (published version 36 pages, very tricky).

Example 3: Neighbourhood graph

S-P (Thesis, 1979): Neighbourhood graph as Bruhat-Tits tree of the spin group in the case $\mathrm{rk}(L) = 3$.

Example 4: Scharlau-Hemkemeier 1998

Exposition of the theory for lattices over number fields, classification of several genera of $\mathbb{Z}$-lattices of rank n, prime level ℓ and determinant $\ell^{n/2}$ by computer calculation (algorithm tn, two-neighbours).

- $n = 14, \ell = 3$: class number 29, existence of a unique extremal modular lattice (minimum of $2Q(x)$ is 4).
- Non existence of an extremal modular lattice (minimum 6) for $n = 12, \ell = 7$, class number 395.

For further recent work in Scharlau's group: Ask him!

Anzahlmatrix: Counting neighbours

It is not much extra work to keep track of the numbers $N(L, M_{,w})$ of ${}_w$-neighbours of L in the isometry class of M. The matrix obtained by collecting these is called the Anzahlmatrix (Eichler, [QFOG]) of the elementary divisor system $({}_w^{-1}, 1, \ldots, 1, {}_w)$.

One has $N(L, M_{,w})|O_M| = N(M, L_{,w})|O_L|$ (Eichler, relate to numbers of embeddings).

More generally, Eichler defines an Anzahlmatrix associated to any elementary divisor system. Even more general: Consider similitude genera, count (e.g.) for some p and unimodular L, M the number of lattices contained in L and similar with similitude norm p to M. Special case: Brandt matrices.

Anzahlmatrix: Counting neighbours, (2)

Eichler shows multiplicativity relations for the Anzahlmatrices similar to those of the Hecke operators for modular forms, shows (for integral weight): The Anzahlmatrices (summed over all elementary divisor systems belonging to the same similitude norm) describe the action of the Hecke operators on theta series (of one variable $z \in H$).

In modern language: Study Hecke operators on the space of automorphic forms on the orthogonal group, modulo the kernel of the theta lifting in degree one.

Generalized to half integral weight (S-P), Siegel modular forms (Andrianov, Zhuravlev, Yoshida), automorphic representations (Rallis, Kudla: Eichler commutation relations).

Depth of the algorithm

Question: How many neighbour steps are there between two classes?

- $\mathrm{rk}(L) = 3$: S-P (Nagoya J. 1986): For all large enough primes, the neighbours of L are equidistributed among the eligible classes. The Anzahlmatrix is essentially a Brandt matrix, so its entries are representation numbers of quaternary quadratic forms in a fixed genus, hence asymptotically equal.
- Hsia/Jöchner (Inventiones 1997): Almost strong approximation (ASAP) for spin groups, same result for arbitrary rank, new proof for rank 3. No asymptotics.
- Chan/Hsia (J. Algebra 2002) ASAP for classical groups of orthogonal/hermitian type.

Algebraic modular forms in Japan

Automorphic forms on (compact) forms of orthogonal, unitary, symplectic groups have been studied in Japan since the mid sixties, usually in connection with trace formulas or theta liftings.
Some examples are:

- Shimizu (1965, 1972), Hijikata/Saito 1973: Quaternion algebras
- Ihara (1964, J.Math.Soc.Jap.): Compact form of $Sp_2 \subseteq GL_4$
- Hashimoto, Ibukiyama (1982, Sem. Th. des Nombres, 1985, Adv.Studies Pure Math. 7): Compact form of $Sp_2 \subseteq GL_4$
- H. Yoshida (1980-1985): Definite orthogonal groups (relate Hecke operators of orthogonal groups to Hecke operators acting on theta series of degree n), quaternion algebras (Yoshida lifting $O(4) \to Sp_2$)

Hermitian case

The case of hermitian forms has been studied by Hoffmann (1991) and Schiemann (1998).
The method works in essentially the same way, with some technical difficulties. But: Neighbours have the same Steinitz class only modulo squares in the ideal class group.
The basis of the method is here the strong approximation theorem for SU.