Section 7.2: Almost Linear Systems

Main Topics:

- **Linear approximation of nonlinear systems**
- **Perturbations of eigenvalues**
- **The dumped oscillating pendulum.**

Solving nonlinear differential systems is hard and often out of reach. One could try to approximate, in a suitable sense, nonlinear systems by linear ones.

Such an approximation makes sense for nonlinear systems which are “not too far” from being linear.

For them, one could also try to deduce information on the stability of critical points out the corresponding properties for linear systems.

Almost Linear Systems

Consider a system of differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$$

where $\mathbf{A}$ is an invertible 2×2 real matrix and $\mathbf{g}$ is a vector function.
If $\mathbf{g}$ is nonzero then the system is nonlinear.

Assume that $\mathbf{x} = \mathbf{0}$ is an **isolated critical point** of this nonlinear system. This means that there is some circle about the origin within which there are no other critical points.

Remark: since $\det(\mathbf{A}) \neq 0$, then $\mathbf{x} = \mathbf{0}$ is the only critical point (and therefore isolated) for the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. (Note that $\mathbf{A}$ cannot have a zero eigenvalue.)

Definition

Suppose that $\mathbf{x} = \mathbf{0}$ is an isolated critical point of $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$. We say that the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is **an approximation** of the nonlinear system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$ if

- the components of $\mathbf{g}$ have continuous first partial derivatives, and
- $\lim_{\mathbf{x}\to\mathbf{0}} \frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = 0$.

In this case, the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$ is said to be **almost linear** in the neighborhood of the critical point $\mathbf{x} = \mathbf{0}$.

Remark: write $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix}$. Writing in polar coordinates $x = r\cos\phi$ and $y = r\sin\phi$, we have

$$r = \|\mathbf{x}\| = \sqrt{x^2+y^2} \qquad \text{and} \qquad \|\mathbf{g}(\mathbf{x})\| = \sqrt{g_1(x,y)^2 + g_2(x,y)^2}$$

In particular,

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = 0 \Longleftrightarrow \begin{cases} \lim_{r\to 0} \dfrac{g_1(x,y)}{r} = 0 \\ \lim_{r\to 0} \dfrac{g_2(x,y)}{r} = 0 \end{cases}$$

Example:

Consider the nonlinear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^2 - xy \\ -0.75xy - 0.25y^2 \end{pmatrix}$$

- Find the critical points of this system.
- Determine if this system is almost linear in the neighborhood of the origin.

Remark: By replacing $\mathbf{x}$ with $\mathbf{x} - \mathbf{x}_0$, the notion of almost linear system extends to the neighborhood of any isolated critical point $\mathbf{x}_0$ of a nonlinear system.

Linear approximations of nonlinear systems

Consider the nonlinear system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, i.e.

$$\begin{aligned} x' &= F(x, y) \\ y' &= G(x, y) \end{aligned}$$

Assume that (x_0, y_0) is an isolated critical point of the system. Suppose that F and G have continuous partial derivatives up to order 2.
Then the system is almost linear in the neighborhood of (x_0, y_0).

Indeed:
Use Taylor expansions of F and G about the critical point (x_0, y_0):

$$\begin{aligned} F(x, y) &= F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + \eta_1(x, y) \\ G(x, y) &= G(x_0, y_0) + G_x(x_0, y_0)(x - x_0) + G_y(x_0, y_0)(y - y_0) + \eta_2(x, y) \end{aligned}$$

where

$$\frac{\eta_1(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \longrightarrow 0 \quad \text{for} \quad (x, y) \to (x_0, y_0)$$

$$\frac{\eta_2(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \longrightarrow 0 \quad \text{for} \quad (x, y) \to (x_0, y_0)$$

For us, $F(x_0, y_0) = 0$ and $G(x_0, y_0) = 0$ because (x_0, y_0) is a critical point.

Also, $\dfrac{dx}{dt} = \dfrac{d(x - x_0)}{dt}$ and $\dfrac{dy}{dt} = \dfrac{d(y - y_0)}{dt}$

The system

$$\begin{aligned} x' &= F(x, y) \\ y' &= G(x, y) \end{aligned}$$

can be written as

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} \eta_1(x, y) \\ \eta_2(x, y) \end{pmatrix}$$

By definition of η_1 and η_2, this system is almost linear in the neighborhood (x_0, y_0) and its linear approximation is the system

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

The matrix $\mathbf{J} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$ is the **Jacobian matrix** of the system.

If $\det(\mathbf{J}(x_0, y_0)) \neq 0$, then (x_0, y_0) is also an isolated critical point for the linear system.

Example:

The motion of the oscillating pendulum is described by the nonlinear system of DE:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\omega^2 \sin x - \gamma y \end{cases}$$

- Show that this system almost linear near $(0, 0)$ and $(\pi, 0)$.
- Determine the corresponding linear approximation.

Small perturbations: the linear case

From Chapter 3:

Consider the system of two homogeneous linear DE with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

If $\det(\mathbf{A}) \neq 0$, then $\mathbf{x} = \mathbf{0}$ is the unique equilibrium solution (or critical point).

The stability properties of this critical point depend on the nature of the roots λ_1, λ_2 of the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, as in the following table:

Stability properties of linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$.

Eigenvalues	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or improper node	Asymptotically stable
$\lambda_1, \lambda_2 = \mu \pm i\nu$		
$\mu > 0$	Spiral point	Unstable
$\mu < 0$	Spiral point	Asymptotically stable
$\mu = 0$	Center	Stable

Suppose that the coefficients of a 2×2-matrix $\mathbf{A}'$ are *small* perturbations of the coefficients of $\mathbf{A}$.

Then:

- $\det(\mathbf{A}')$ is a *small* perturbation of $\det(\mathbf{A})$. So, $\det(\mathbf{A}') \neq 0$ if $\det(\mathbf{A}) \neq 0$.
- The coefficients of the characteristic equation $\det(\mathbf{A}' - \lambda \mathbf{I}) = 0$ are also *small* perturbations of those of $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.
- The eigenvalues λ_1', λ_2' of $\mathbf{A}'$ are hence *small* perturbations of the eigenvalues λ_1, λ_2 of $\mathbf{A}$.

Recall that $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ is a quadratic equation.
Let Δ be its discriminant.

▷ Suppose $\Delta \neq 0$. Then

- either $\Delta > 0$ (distinct real roots λ_1, λ_2)
- or $\Delta < 0$ (complex conjugate roots $\lambda_2 = \overline{\lambda_1}$).

Δ' (the discriminant of the characteristic equation of $\mathbf{A}'$) is a perturbation of Δ.
If the perturbation is small enough, we will have

$$\Delta > 0 \Rightarrow \Delta' > 0 \quad \text{and} \quad \Delta < 0 \Rightarrow \Delta' < 0\,.$$

▷ Suppose $\Delta > 0$ (so $\Delta' > 0$ too).

The roots of $\lambda^2 + \alpha\lambda + \beta = 0$ are $\dfrac{-\alpha \pm \sqrt{\Delta}}{2}$.
Then, for instance:

$\lambda_1 < \lambda_2 < 0$ means that the biggest root is positive, i.e. $-\alpha + \sqrt{\Delta} < 0$.
For small enough perturbations, we can obtain the same relations for $\mathbf{A}'$, i.e. $\lambda_1' < \lambda_2' < 0$.

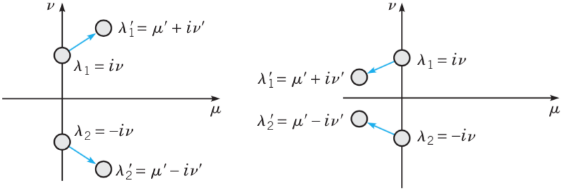
This gives:
If the critical point $\mathbf{0}$ is a node and asymptotically stable for $\mathbf{x}' = \mathbf{A}\mathbf{x}$, the same is true for $\mathbf{x}' = \mathbf{A}'\mathbf{x}$.

▷ A similar argument works for all cases for λ_1, λ_2 in previous table where one has conditions given by strict inequalities.

▷ The sensitive cases are those where there are equality conditions:

- $\lambda_1 = \lambda_2$ real (proper or improper node)
- $\lambda_1 = i\nu$ (i.e. complex case $\mu \pm i\nu$ with $\nu = 0$)

Perturbation of $\lambda_1 = i\nu$ and $\lambda_2 = -i\nu$



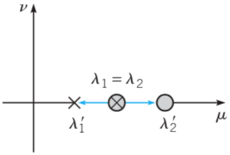
Before perturbation, the critical point $\mathbf{x} = \mathbf{0}$ is a center and it is stable.

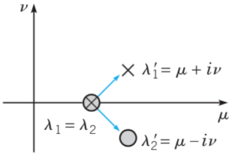
After perturbation, the critical point $\mathbf{x} = \mathbf{0}$:

- remains a center and is stable if $\mu' = 0$,
- becomes a spiral point and is unstable if $\mu' > 0$,
- becomes a spiral point and is asymptotically stable if $\mu' < 0$.

Perturbation of $\lambda_1 = \lambda_2$
Before perturbation, the critical point $\mathbf{x} = \mathbf{0}$ is a node.
It is unstable if $\lambda_1 = \lambda_2 > 0$, and asymptotically stable is $\lambda_1 = \lambda_2 < 0$.

Suppose for instance $\lambda_1 = \lambda_2 > 0$:





After perturbation, the critical point:

- remains a node and is unstable if $\lambda_2' > \lambda_1' > 0$,
- becomes a spiral point and is unstable if λ_1' and λ_2' are complex conjugate (and $\mu > 0$, since $\lambda_1 = \lambda_2 > 0$).

Small perturbations: almost linear systems

Theorem (Theorem 7.2.2)

Let λ_1 and λ_2 be the eigenvalues of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, corresponding to the almost linear system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$. Suppose $\mathbf{x} = \mathbf{0}$ is an isolated critical point of both systems. Then its type and stability are as follows:

Stability and instability properties of linear and almost linear systems.

	Linear System		Almost Linear System	
λ_1, λ_2	Type	Stability	Type	Stability
$\lambda_1 > \lambda_2 > 0$	N	Unstable	N	Unstable
$\lambda_1 < \lambda_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	SP	Unstable	SP	Unstable
$\lambda_1 = \lambda_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$\lambda_1 = \lambda_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$\lambda_1, \lambda_2 = \mu \pm i\nu$				
$\mu > 0$	SpP	Unstable	SpP	Unstable
$\mu < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$\mu = 0$	C	Stable	C or SpP	Indeterminate

Note: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

Remark: as in the linear case, small perturbations do not alter the type and the stability of the critical point except for the two sensitive cases.

Remark: the previous analysis considers the case where the system is almost linear near an isolated critical point $\mathbf{x} = \mathbf{0}$. It extends in the same way at every other isolated critical point $\mathbf{x} = \mathbf{x}_0$ near which the system is almost linear.

Example:

The linear approximation of the system for the motion of the damped oscillating pendulum at $(0, \pi)$ is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{(with } \gamma > 0\text{)}$$

Its characteristic equation is

$$\det\begin{pmatrix} -\lambda & 1 \\ \omega^2 & -\gamma - \lambda \end{pmatrix} = \lambda^2 - \gamma\lambda - \omega^2 = 0\,.$$

So the eigenvalues are

$$\lambda_1, \lambda_2 = \frac{\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}\,.$$

They are both real: one is positive, the other one is negative.
Thus $(0, \pi)$ is a saddle point and is unstable for both the almost linear and linear systems.

The linear approximation of the system for the motion of the damped oscillating pendulum at $(0,0)$ is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{(with } \gamma > 0\text{)}$$

Its characteristic equation is

$$\det\begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\gamma - \lambda \end{pmatrix} = \lambda^2 + \gamma\lambda + \omega^2 = 0\,.$$

So the eigenvalues are

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}\,.$$

The nature of the critical point $(0,0)$ depends on the sign of $\Delta = \gamma^2 - 4\omega^2$:

- $\Delta > 0$: in this case $\lambda_1 < \lambda_2 < 0$: node and asymptotically stable critical point for both linear and almost linear system.
- $\Delta = 0$: in this case $\lambda_1 = \lambda_2 < 0$: $(0,0)$ is an asymptotically stable node for the linear system; for the pendulum, it can can be either an asymptotically stable node or an asymptotically stable spiral point.
- $\Delta < 0$: in this case λ_1, λ_2 are complex conjugates with negative real part μ: asymptotically stable spiral point for both linear and almost linear system.