The topic of this course is Riemann surfaces.

Riemann surfaces are smooth surfaces with complex coordinates and can be viewed as topological spaces or settings for complex analysis.

Examples come from algebraic geometry usually complex curves over $\mathbb{C}$ but many other settings as well. Meeting point for many mathematical fields.

Prerequisites for the course are:

Complex Analysis MA3B8 and

Topology MA3F1, Hatcher Chapter 1.

The following courses are not prerequisites but there will be a limited amount of overlap:

Manifolds

Topology (Hatcher Chapter 2)

Algebraic geometry.

More specifically we will introduce and use

1 forms and 2 forms
1 dimensional homology and cohomology
surfaces and tangent bundles.
projective curves, elliptic curves.

Relevant texts: Beardon, A primer on Riemann surfaces
Donaldson, Riemann surfaces
McMullen, Online notes for Harvard 213a.

I will post notes and put up example sheets as we go along.

Establish some terminology related to (vector analysis).

Language of tangent bundles (in a simple case).

Let $U \subset \mathbb{R}^n$ be an open set. The idea of the tangent bundle $T(U)$ will be the collection of tangent vectors based at points in U.

We define $T(U)$ as $U \times \mathbb{R}^n$.

If $F: U^{\subset \mathbb{R}^n} \longrightarrow V^{\subset \mathbb{R}^m}$ is a smooth map then we define $DF: T(U) \to T(V)$ by

$DF((p, v)) = (F(p), DF_p(v))$ where

DF_p is the linear map given by

$F(p) = (F_1(p) \ldots F_m(p))$

$$DF_p = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

and the partial derivative are evaluated at p.

In this language the chain rule becomes

$$U^{\subset \mathbb{R}^n} \xrightarrow{G} V^{\subset \mathbb{R}^m} \xrightarrow{F} W^{\subset \mathbb{R}^p}$$

$D(F \circ G) = DF \circ DG$. We write tangent vectors as column vectors so that composition of linear maps corresponds to matrix multiplication.

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and write $(z_1 \ldots z_n)$ as coordinates for $\mathbb{C}^n$ and $(x_1, y_1, x_2, y_2 \ldots x_n, y_n)$ as corresponding coordinates for $\mathbb{R}^{2n}$ where $z_j = x_j + i y_j$.

Consider the case $n = 1$, $f : \overset{\subset \overset{(x,y)}{\mathbb{C}}}{U} \longrightarrow \overset{\subset \overset{(u,v)}{\mathbb{C}}}{V}$.
Let f be a smooth function.

$$f(x,y) = (u(x,y), v(x,y))$$
$$(= u(x,y) + i\, v(x,y))$$

$$Df_p = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Df_p is an $\mathbb{R}$ linear map from the complex vector space T_p to the complex vector space $T_{f(p)}$.

We say that f is <u>holomorphic</u> if this $\mathbb{R}$ linear map is $\mathbb{C}$ linear.
An $\mathbb{R}$ linear map is $\mathbb{C}$ linear if it commutes with the linear map corresponding to multiplication by i. In our coordinates this map is given by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

since
$$i(x+iy) = ix - y$$
$$i \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$
(Using column vector)

Check $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} -c & -d \\ a & b \end{bmatrix} \qquad \begin{bmatrix} b & -a \\ d & -c \end{bmatrix}$$

Interpret the first column as a complex no. CR says that the second column determined by the first.

This condition is equivalent to $b = -c$ and $a = d$ or

$a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These are the Cauchy-Riemann equations.

Equiv. to mult. by $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \cdot I + \ldots$

In higher dimensions we define a map F to be holomorphic if the $\mathbb{R}$ linear map DF is actually $\mathbb{C}$ linear.

If we write $J_n : \mathbb{C}^n \to \mathbb{C}^n$ ($\mathbb{R}^{2n} \to \mathbb{R}^{2n}$) for the matrix corresponding to mult. by i then

$F: U \subset \mathbb{C}^n \longrightarrow V \subset \mathbb{C}^m$. F is holomorphic if at each $p \in U$ we have $DF_p \circ J_n = J_m \circ DF_p$.

$$\begin{matrix} DF_p : \mathbb{C}^n & \longrightarrow & \mathbb{C}^m \\ \wr\wr & & \wr\wr \\ DF_p : \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^{2m} \end{matrix}$$

This gives higher dimensional analogues of the Cauchy-Riemann equations.

Manifold formalism.

Smooth n-dim manifold / Riemann surface

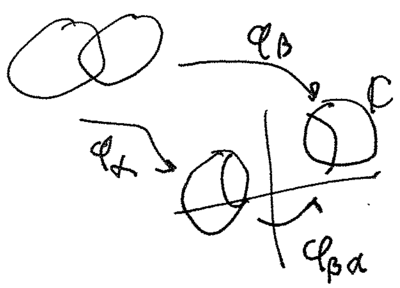
Let X be a topological space (~~say~~ assume Hausdorff)
Let a be an index set (often finite)
Let $\{U_\alpha\}$ be a collection of open sets in X so that

$X = \bigcup_{\alpha \in a} U_\alpha$.

An atlas for X is given by a collection of charts

$\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}$ where φ_α is a homeomorphism and V_α is an open set in $\mathbb{C}$,

and a collection of transition functions:
$\varphi_{\alpha\beta}$ defined when $U_\alpha \cap U_\beta \neq \phi$ so that



$\varphi_{\beta\alpha} \circ \varphi_\alpha = \varphi_\beta$ on $U_\alpha \cap U_\beta$.

and $\varphi_{\beta\alpha}$ is a conformal map. (automorphism?)

Remarks: If we only assume that f ~~has~~ that f is smooth and not conformal then we get a surface.

If we replace $\mathbb{C}$ by $\mathbb{R}^n$ and require trans. maps to be smooth we get a smooth n manifold

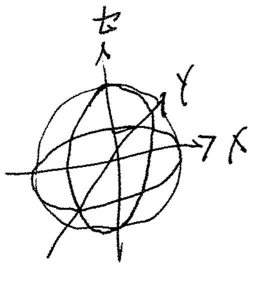
If we replace $\mathbb{C}$ by $\mathbb{C}^n$ and require trans. maps to be holomorphic we get a complex n manifold, 6b15

In week 1 I am trying to lay the foundation for a fairly modern discussion of Riemann surfaces, (facilitates discussion of connections w other fields). Describing the formalism of real & complex manifolds (efficiently with examples of n dimens. (though surfaces are our real interest we also consider paths (real 1 manifolds) $\mathbb{C}^2$ and $\mathbb{CP}^2$ (complex 2 manif. ~~ro~~ general surfaces (real 2 manifolds)), tangent bundles of manifolds and derivatives of maps [Generalized CR equations.]

1-forms and path integrals

Example of a Riemann surface:

Let $U \subset \mathbb{C}$ be an open set. ~~in~~ Then U is a Riemann surface with an atlas consisting of the single chart: $\varphi : U \to \mathbb{C}$ and no change of coordinate maps.



Riemann surface structure on S^2.

Let $S^2 = \{(x,y,t) : x^2+y^2+t^2 = 1\}$.

Let $NP = (0,0,1)$, $SP = (0,0,-1)$.

Use coordinates (x,y,t).

$U_1 = S^2 - NP$, $U_2 = S^2 - SP$.

$V_1 = \mathbb{C}$. Identify $\mathbb{C}$ with the x,y plane in $\mathbb{R}^3$. coordinates by mapping $x+iy$ to $(x,y,0)$

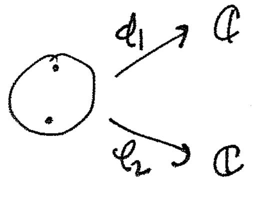
For any pt. $P = (x,y,t) \neq NP$ draw the line from NP to P, intersect with the $t=0$ plane and identify $(x,y,0)$ with $x+iy \in \mathbb{C}$.

Formula for the line $s \mapsto (1-s)(0,0,1) + s(x,y,t)$.

Intersection parameter is s s.t. $(1-s)+st = 0$, $s = \frac{1}{1-t}$. Intersection point is $\left(\frac{x}{1-t}, \frac{y}{1-t}, 0\right)$. Identify this with $\frac{x+iy}{1-t} \in \mathbb{C}$. So $\varphi_1(x,y,t) = \frac{x+iy}{1-t}$.

To define φ_2 we do the same construction starting with SP. In this case we use a different identification of $t=0$ with $\mathbb{C}$, send $(x,y,0)$ to $x-iy$ and get $\varphi_2(x,y,t) = \frac{x-iy}{1+t}$ defined on $U_2 = S^2 - SP$ $V_2 = \mathbb{C}$.

(If we had not done this we would have gotten an orientation reversing angle preserving map.)



(Best to think of two distinct copies of $\mathbb{C}$)

We define the tangent bundle of S^2 as follows. (9a)

$$T(S^2) = \{ (p, v) : p \in S^2 \subset \mathbb{R}^3,\ v \in \mathbb{R}^3 \text{ with } v \cdot p = 0 \}$$

Define the restriction of the tangent bundle to an open subset. $D\varphi_i^{-1}$ identifies the restriction of $T(S^2)$ to U_i with $T(\mathbb{R}^2)$. Give a basis for each tangent space.

We calculate that for $p \overset{(x,y,t)}{=} \in S^2 - \{NP, SP\} = U_1 \cap U_2$

$$\phi_1(p) \cdot \phi_2(p) = \frac{x+iy}{1-t} \cdot \frac{x-iy}{1+t} = \frac{x^2+y^2}{1-t^2} = 1. \quad \left(\begin{smallmatrix}\text{complex}\\ \text{mult.}\end{smallmatrix}\right)$$

since $x^2+y^2+t^2=1$, $x^2+y^2=1-t^2$.

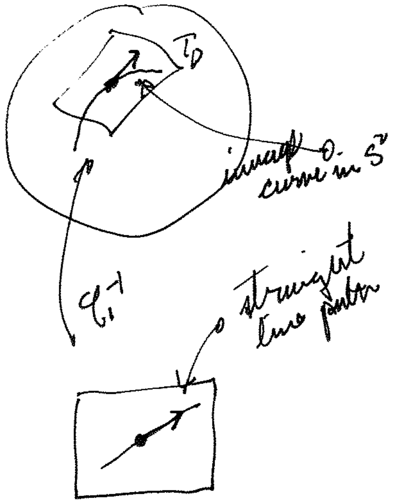
~~Let~~. Now $\phi_{21}(\phi_1(p)) = \phi_2(p) = \dfrac{1}{\phi_1(p)}$.

Solve for ϕ_{21} using the fact that it satisfies $\phi_{21} \circ \phi_1 = \phi_2$.

So ~~ϕ_{21}~~ setting $\phi_{21}(z) = \frac{1}{z}$ we have $\phi_{21} \circ \phi_1 = \phi_2$.
Since $z \mapsto \frac{1}{z}$ is conformal on $\mathbb{C} - \{0\}$ we have a conformal (or holomorphic) atlas.

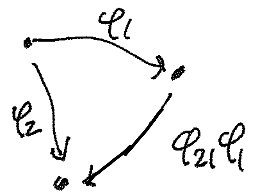
Note that given $p \in U_1 \cap S^2$ we can identify the tangent space at p with the tangent space to $\varphi_1(p) \subset V_1 \subset \mathbb{C}$. using $D\varphi_1^{-1}$.

Geometrically we can think of T_p as the collection of derivatives of smooth paths through p.



Of course distinct paths can have the same derivative so we would have to work harder to turn this into a definition.

Of course we can ~~more the~~ Typically identify p with $\varphi_2(p)$ and think of the T_p as the tangent space at $\varphi_2(p)$. But there are two representations for the same space and $D\varphi_{21}$ takes us from $T_{\varphi_1(p)}$ to $T_{\varphi_2(p)}$.



Using these ideas, for any M we can define the tangent space at $p \in M$ and the tangent bundle which is the union of tangent spaces at points.

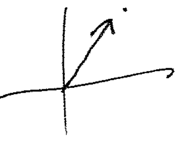
A key feature of a Riemann surface is that each tangent space has a complex structure. This means it is a 1-dimensional vector space over $\mathbb{C}$.

Recall the distinction between ~~being~~ a 1-dim v.s over $\mathbb{C}$ and $\mathbb{C}$ itself. $\mathbb{C}$ has a special vector namely 1 which is a basis. A 1-dim v.s. can be identified with $\mathbb{C}$ but has no canonical identification.

say φ_1, φ_2

Note that each chart φ_1 and φ_2 identifies T_p with $\mathbb{C}$ but these identifications are different but they are both complex linear. The C-R equations imply that the underlying complex structures agree in other words the derivative of the change of coordinate map is complex linear.

Geometrically a complex structure on a 2-dim real vector space is equivalent to a notion of angle plus a choice of orientation.

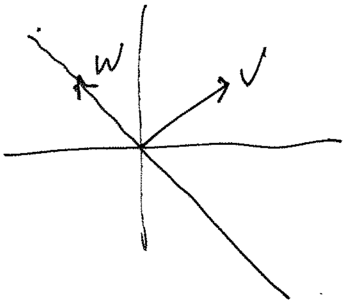


Algebraically it is equivalent to giving a linear transformation J with $J^2 = -I$ (which corresponds to mult. by i).

Given a real vector space V with such a $J: V \to V$ we define the "scalar multiplication" action of $\mathbb{C}$ on V by

$$(a+bi)v = av + b\,J(v).$$

If we know angles between vectors then we know an innerproduct up to scaling. To define $J(v)$ we consider the two perpendicular vectors of the same length as v. Pick the one of these w so that (v, w) gives positively oriented basis and define



$$J(v) = w.$$

at this point we have described 2 complex structures on tangent spaces (2 "almost complex structures"). One comes from the atlas and uses the fact that the transition function φ_{12} is holomorphic (in fact conformal) and one comes from the induced Euclidean metric on the sphere and the orientation. We can describe this second structure in the language of maps J with $J^2 = -1$,

$T(S^2) = \bigcup_{p \in S^2} T_p$. Define $J_p : T_p \to T_p$ by

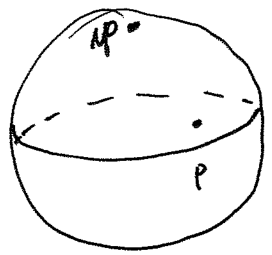
$J_p(v) = p \times v$ cross product

$J_p^2(v) = (p \times v) \times v = -v$,

so $J_p^2 = -I_{T_p}$ gives a complex structure,

Do these two complex structures agree?

Let $v, w \in T_p$. Say $\angle(v,w)=\theta$.



We can find 2-planes H_v and H_w contained in $\mathbb{R}^2$ so that H_v contains NP, p and the vector v

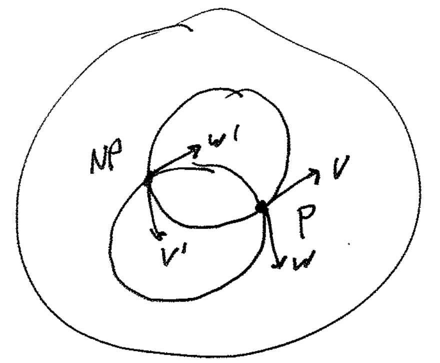
H_w contains NP, p and the vector w.

$H_v \cap S^2$ and $H_w \cap S^2$ are circles tangent to v and w.

~~Let v', w' be~~ These two circles also meet at NP.

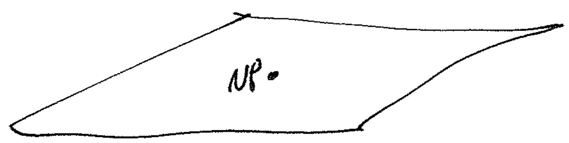
Let v' and w' be the corresponding vectors at NP (well defined up to scaling)

Claim $\measuredangle(v,w) = \measuredangle(v',w')$.



(Can see this by choosing a third ~~f~~ plane through o perp to H_v and H_w and then reflecting.)

Now φ_1 takes $H_v \cap S^2$ to $H_v \cap \mathbb{C}$ and $H_w \cap S^2$ to $H_w \cap \mathbb{C}$.



In particular these circles are mapped to straight lines. Since T_{NP} is parallel to $\mathbb{C}$ the v' is parallel to v'' (in $\mathbb{R}^3$) and w' is parallel to w''.

So $\measuredangle(v'',w'') = \measuredangle(v',w') = \measuredangle(v,w)$.

Let v'' and w'' be the tangent vectors to these lines at the point $\varphi_1(p) \in \mathbb{C}$.

Discussion works for maps from complex n mflds to m mflds though nothing essential is lost if we think n=m=1 ie Riemann surfaces. (1

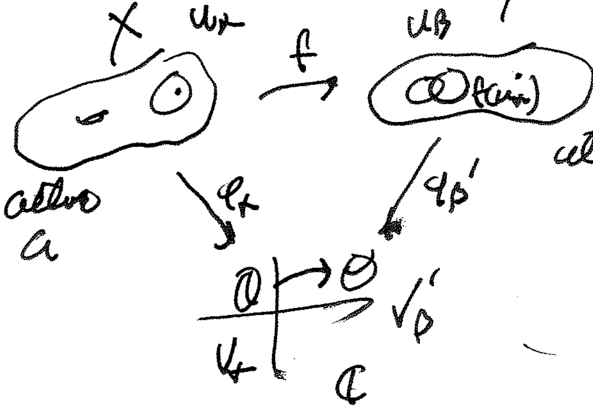
How do we define smooth or holomorphic maps between real or complex manifolds { X and Y

Think ordinary Riemann surfaces though discussion goes through for maps from arbitrary n-mflds to m mflds or smooth mflds.

Let $f: X \to Y$ be a continuous function.

We say that f is (say) holomorphic ~~if for each p~~ with respect to atlases $\mathcal{A}_X$ and $\mathcal{A}_Y$ if for any $p \in X$ and $U_\alpha \in \mathcal{A}_X$ with $p \in U_\alpha$ and any $U'_\beta \in \mathcal{A}_Y$ with $f(p) \in U'_\beta$ ~~there is a~~ then

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ p \in U_\alpha \arrow[u, phantom, "\cup"] \arrow[d, "\varphi_\alpha"] & & f(p) \in U'_\beta \arrow[u, phantom, "\cup"] \arrow[d, "\varphi'_\beta"] \\ V_\alpha \arrow[rr, "\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}"] & & V'_\beta \\ \mathbb{C}^n \arrow[u, phantom, "\cap"] & & \mathbb{C}^m \arrow[u, phantom, "\cap"] \end{tikzcd}$$



the function $\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}$ is (say) holomorphic where it is defined. (Think $n=m=1$.)

Prop. The composition of holomorphic maps is a holomorphic map. } Will give Exercise sheet problem

Reality check: $f: U \to U$ ($U \subset \mathbb{C}$, $U \subset \mathbb{C}$) is new style holomorphic if f is old style holomorphic.

Thus we haven't changed the classic def. we have only extended it. notion of derivative has subtlety, number → ~~matrices~~ linear trans. 1x ...

Given p   Easy to check in practice, (2)

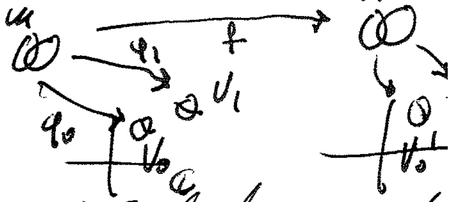
Important point: It suffices to check this property for one chart containing p and one chart containing f(p). It will then hold for <u>any</u> chart containing p and any chart containing f(p).

in $\mathbb{C}$ (or $\mathbb{C}^n$)

Prop. f is hol. at p with respect to one chart iff f is hol. at p with respect to any chart.

$$\begin{tikzcd} V_0 \arrow[r, "\varphi_0'\circ f\circ\varphi_0^{-1}"] & V_0' \arrow[d, "\varphi_{10}'"] \\ V_1 \arrow[u, "\varphi_{01}"] \arrow[r, "\varphi_1'\circ f\circ\varphi_1^{-1}"'] & V_1' \end{tikzcd}$$

$$\begin{tikzcd} V_0 \arrow[r, "\varphi_0'\circ f\circ\varphi_0^{-1}"] & V_0' \\ V_1 \arrow[u, "\varphi_{01}"] \arrow[r, "\varphi_1'\circ f\circ\varphi_0^{-1}"'] & V_1' \arrow[u, "\varphi_{01}'"'] \end{tikzcd}$$



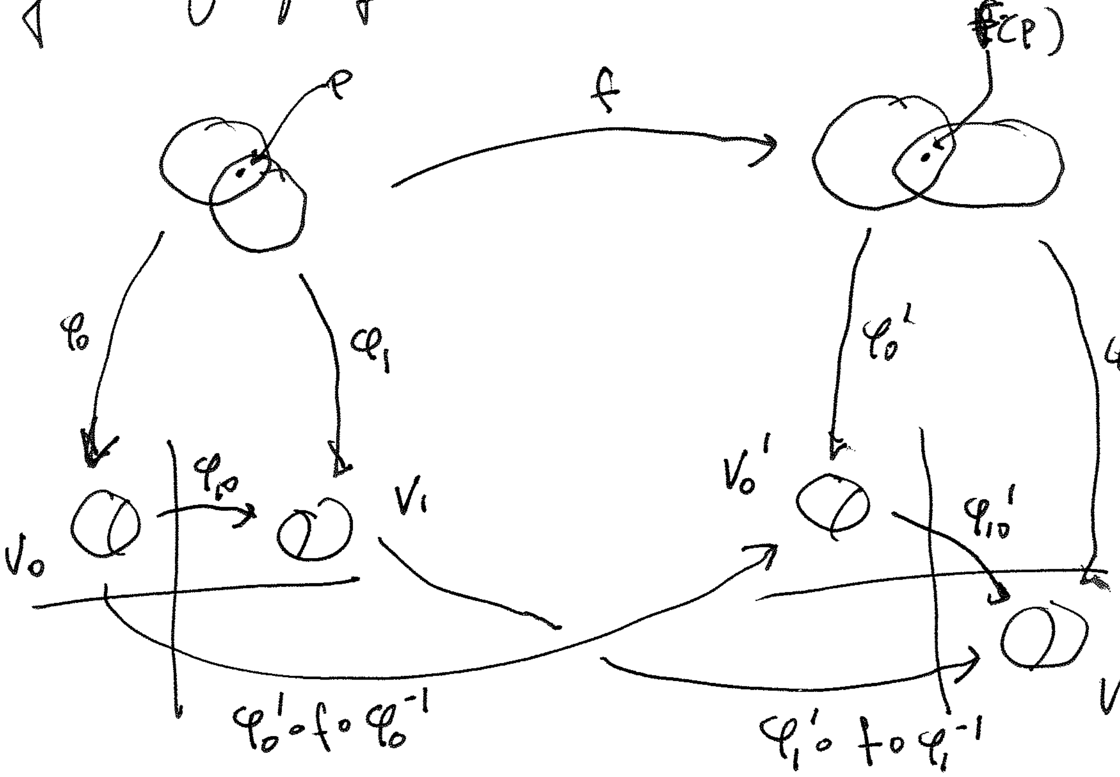
Use $\varphi_1' f \varphi_1^{-1} = \underbrace{\varphi_{10}'\varphi_0'}_{\varphi_1'} \circ f \circ \underbrace{\varphi_0^{-1}\circ\varphi_{01}}_{\varphi_1^{-1}}$

This map is holomorphic at p iff f is holomorphic at p since the composition of holomorphic maps is holomorphic.

To check whether Df is complex linear it suffices to check complex linearity after we pre and post compose by invertible linear maps.

This discussion uses the fact that compositions of holomorphic functions are holomorphic and also that change of coordinate functions are invertible. Works equally well for smooth n-manifolds.

Diagram for page 2.



Want to show that $\varphi_0'\circ f\circ\varphi_0^{-1}$ is holomorphic iff $\varphi_1'\circ f\circ\varphi_1^{-1}$ is holomorphic.

Use $\varphi_1=\varphi_{10}\circ\varphi_0$ $\qquad$ $\varphi_1'=\varphi_{10}'\circ\varphi_0'$

Claim:

Bottom square commutes (where defined)

$$\begin{tikzcd} V_0 \arrow[r, "\varphi_0'\circ f\circ\varphi_0^{-1}"] \arrow[d, "\varphi_{10}"'] & V_0' \arrow[d, "\varphi_{10}'"] \\ V_1 \arrow[r, "\varphi_1'\circ f\circ\varphi_1^{-1}"'] & V_1' \end{tikzcd}$$

$V_0 \to V_1$

Bottom square commutes:

$\varphi_1'\circ f\circ\varphi_1^{-1}\varphi_{10} = \varphi_1' f \varphi_0^{-1}$ and $\varphi_{10}'\varphi_0' f\varphi_0^{-1} = \varphi_1' f\varphi_0^{-1}$

V_0 So $\varphi_1'\circ f\circ\varphi_1^{-1} = \underbrace{\varphi_{10}'}_{\text{holomorphic}}(\varphi_0'\circ f\circ\varphi_0^{-1})\underbrace{\varphi_{10}^{-1}}_{\text{holomorphic}}$

Definition: $f: M \to N$ is a holomorphic diffeomorphism if it has a holomorphic inverse function.

In one dimension we usually say "conformal equivalence" for historical reasons.

What is the classification of Riemann surfaces up to conformal equivalence?

What are conformal invariants?

(Riemann mapping thm.)

Thm. If $U \subset \mathbb{C}$ is simply connected and $U \neq \mathbb{C}$ then U is conformally equivalent to the disk.

Existence thm. Can be hard to construct conformal equivalences concretely.

Examples of holomorphic fcns. between Riemann surfaces.

Write $\mathbb{C}_\infty$.

$$\begin{array}{ccccc} S^2 & = & U_1 & \cup & NP. \\ \wr & & \wr\,\varphi_1 & & \wr \\ \mathbb{C}_\infty & = & \mathbb{C} & \cup & \infty \end{array}$$

$\varphi_1 : U_1 \to \mathbb{C}$ is an isomorphism so we also write

topologically $\mathbb{C}_\infty$ is the one pt. compactification of $\mathbb{C}$.

extended complex plane

$S^2 = \mathbb{C} \cup \infty = \mathbb{C}_\infty$ identifying U_1 with $\mathbb{C}$ and NP with ∞.

$f : M \to S^2$ can also be written as $f : M \to \mathbb{C} \cup \infty$.

Recall that a holomorphic function f in a punctured disk around $z_0 \in \mathbb{C}$ has Laurent expansion.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

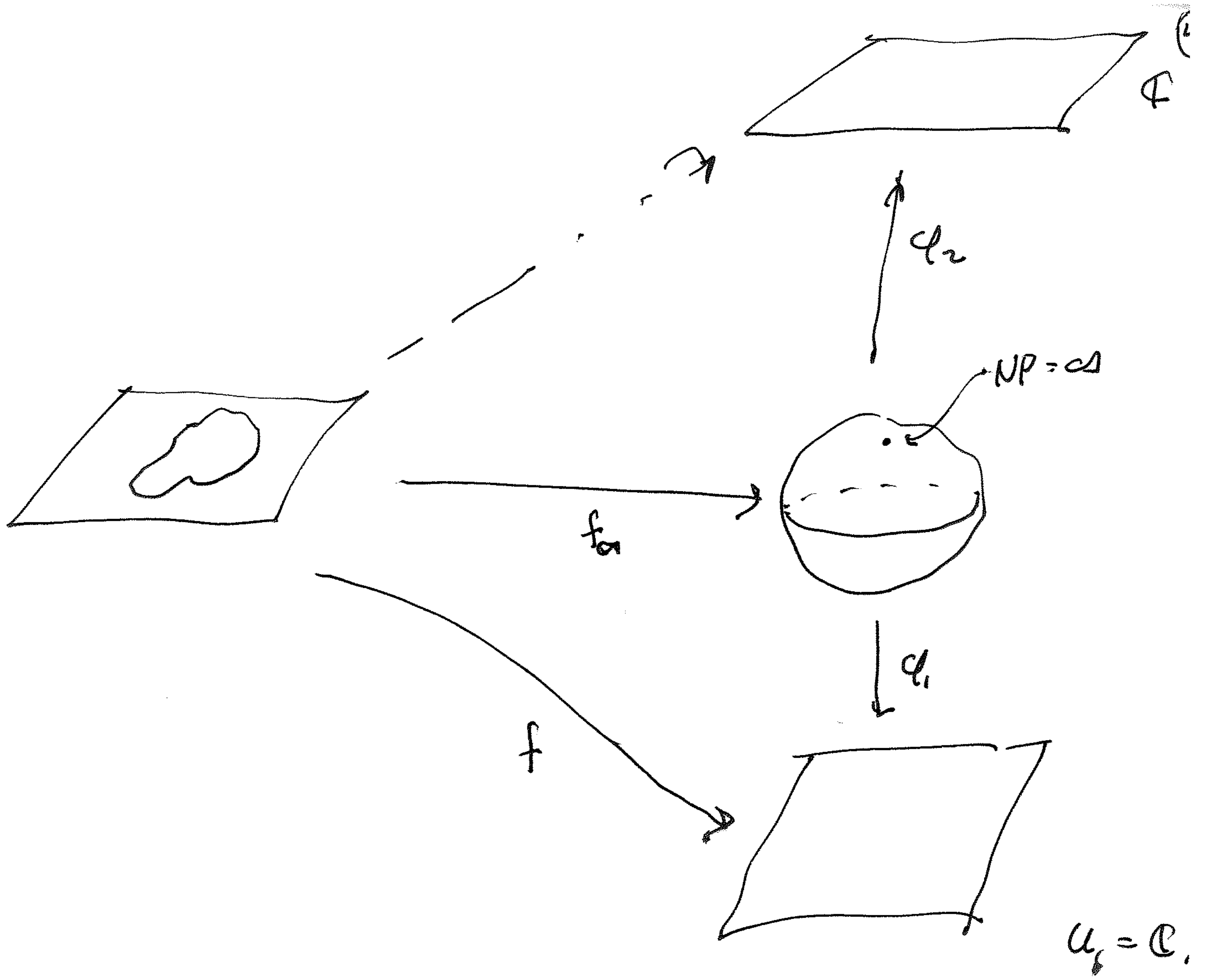
If $a_n = 0$ for $n < 0$ we say f has a removable singularity. If not removable $a_n = 0$ for $n < k$ we say that f has a pole at z_0. Otherwise f has an essential singularity at z_0

define $f^*(z) = f(z)$ for $z \neq z_0$ $f^*(z) = a_0$ for $z = z_0$ f^* is holomorphic. If $a_n = 0$ for $n < k$, $a_k \neq 0$ and $k < 0$ then f has a pole of order k. In this case

$f(z) = \frac{g(z)}{(z-z_0)^k}$ where $g(z) = \sum_{n \geq 0} a_{n+k}(z-z_0)^n$ is holomorphic

Let $f: U \to \mathbb{C}$. We say f is meromorphic if at each $z_0 \in U$ f ~~has at worst~~ ha is holomorphic or has a pole.

We can view a meromorphic function as a function from U to the $\mathbb{C}_\infty$ in the obvious way.



Write f_∞ for the extension of f to $\mathbb{C}_\infty$.

~~It~~ If $z_0 \in U$ is such that $f(z_0) \neq \infty$ then we can check for holomorphicity by using the chart U_1 since $f(z_0) \in U_1$.

Consider $\varphi_1 \circ f_\infty$. This is just f so f_∞ is holomorphic at z_0.

If $f_\infty(z_0) = \infty$ then we can't use the chart U_1 we have to use U_2. $\varphi_2 \circ f_\infty(z) = \varphi_{21} \circ \varphi_1 \circ f_\infty(z) = \varphi_{21} \circ f(z)$ (where $f(z) \neq \infty$ and 0 where $f(z) = \infty$ since $\varphi_2(NP) = 0$, Now $\varphi_{z_1} \circ f(z) = \frac{1}{f(z)}$. Write $f(z) = \frac{g(z)}{(z-z_0)^4}$

so $\varphi_{z_1} \circ f(z) = \frac{(z-z_0)^4}{g(z)}$. Since $g(z_0) \neq 0$ ~~g is~~ and

g is holomorphic $\frac{1}{g(z)} = g^{-1}(z)$ is holomorphic in a nbd.

of z_0 so $\varphi_{z_1} \circ f(z) = (z-z_0)^4 \cdot g^{-1}(z)$ is holomorphic

in a nbd. of z_0.

Def. If R is any connected Riemann surface then we define a meromorphic function on R to be a holomorphic function from R to $\mathbb{C}_\infty$ which is not equal to ∞ at every point.