Quantum Field Theory I assignment IV

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1. **Follow the more elaborate formulation of Noether's theorem. And show how all this works for the Lagrangian in equation (3)**

   Here we could see that the under the infinitesimal rotation of the $SO(N)$ symmetry, we are performing transformation:

   $$\varphi_a \to \varphi_a + \epsilon\omega^i_{ab}\varphi_b \tag{1}$$

   where ω^i are the generator matrices of $SO(N)$, since we know now that under this transformation $\delta\varphi_a = \omega^i_{ab}\varphi_b$. Our original definition of the conservation current is given as:

   $$J^{i\mu} = \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\varphi_a = \left(\partial_\mu\varphi_a\right)\omega^i_{ab}\varphi_b \tag{2}$$

   Now if we take a look at the method given in the question, we could see that the Lagrangian $\mathcal{L} = \left((\partial_\mu\vec{\varphi})^2 - m^2\vec{\varphi}^2\right)/2 + \lambda\vec{\varphi}^4/4!$, should transform as something like:

   $$\mathcal{L}' = \frac{1}{2}\left(\left(\partial_\mu\left(\vec{\varphi} + \epsilon\omega^i\vec{\varphi}\right)\right)^2 - m^2\left(\vec{\varphi} + \epsilon\omega^i\vec{\varphi}\right)^2\right) + \frac{\lambda}{4!}\left(\vec{\varphi} + \epsilon\omega^i\vec{\varphi}\right)^4 \tag{3}$$

   And the supposed Noether's current could be determined by the partial variation of $\mathcal{L}$ by $\partial_\mu\epsilon$, by treating ϵ as another free field in the Lagrangian. Since the term $\partial_\mu\epsilon$ only appears in the first term of the Lagrangian, it is not hard to see that:

   $$\frac{\delta L}{\delta\partial_\mu\epsilon} = \left(\partial_\mu\varphi_a\right)\omega^i_{ab}\varphi_b \tag{4}$$

   Thus we could see that the two methods gives use the identical Noether's current.

2. **Show that $D_{ab}(x)$ must be proportional to δ_{ab} is stated in the text.**

   We could see that since the measure of the path integral and the action is said to be invariant under internal $SO(N)$ transformation, we could see that the only terms that at left is the two fields. And thus under transformation, we get the result:

   $$iD'_{ab}(x) = \int \mathcal{D}\vec{\varphi}\, e^{iS}\, \varphi'_a(x)\varphi'_b(0) \tag{5}$$

Under the infinitesimal transformation in $SO(N)$ space we could get the relation:

$$\varphi'_a = R_{aa'}\varphi_{a'}\ ; \quad \varphi'_b = R_{bb'}\varphi_{b'} \tag{6}$$

where R is an arbitrary rotation matrix. Using this we could get the relation between the transformed D and the original D to be:

$$iD'_{ab} = \int \mathcal{D}\vec{\varphi}\ e^{iS}\ R_{aa'}R_{bb'}\varphi_{a'}\varphi_{b'} = iR_{aa'}R_{bb'}D_{a'b'} \tag{7}$$

Since the integral factor is only a dummy variable, we could transform our integral variable by a rotation, and thus we could see that D should be invariant under the transformation. We could see that if $D' = D$, the only solution is that if D is proportional to δ_{ab}

3. **Write the Lagrangian for an $SO(3)$ invariant theory containing a Lorentz scalar field in the 5-dimensional representation up to the quadratic term. (Write φ as a 3 by 3 symmetric traceless matrix)**

Here we are going to write the components of the scalar field as:

$$\varphi(x) = \phi_{(a)}(x)\epsilon^{(a)}_{ij} \tag{8}$$

With $\epsilon^{(a)}$ being the 5 basis tensors for the $SO(3)$ symmetry with the traceless condition of $\sum_{ij}\delta_{ij}\epsilon^{(a)}_{ij} = 0$. More compactly, we could contract the dummy index (a) and simply write down the $\varphi_{ij}(x)$ and the internal transformation is like that of a transformation of a three tensor (not a Lorentz tensor):

$$\varphi_{ij}(x) \rightarrow \varphi'_{ij}(x) = R_{ik}R_{jl}\varphi_{lk}(x) \tag{9}$$

Under these restrictions, and obvious way of constructing the desired Lorentz invariant required to construct the Lagrangian, and that is $\varphi_{ij}\varphi_{ij}$, and if we are including the differential terms $\left(\partial_\mu\varphi_{ij}(x)\right)^2$, thus we guess the form of the Lagrangian containing up to a quartic term to be something like:

$$\mathcal{L} = \frac{1}{2}\sum_{ij}\left(\left(\partial_\mu\varphi_{ij}\right)^2 - m^2\varphi^2_{ij}\right) - \frac{\lambda}{4!}\left(\sum_{ij}\varphi^2_{ij}\right)^2 \tag{10}$$

4. **Add a Lorentz scalar field η transforming as a vector under $SO(3)$ to the Lagrangian in the previous exercise, maintaining the SO(3) invariance. Determine the Noether currents in this theory. Using the equations of motion, check that the currents are conserved.**

The task of adding an additional field is simply enough, we guess that the Lagrangian should be something like:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\sum_{ij}\left(\left(\partial_\mu\varphi_{ij}\right)^2 - m^2\varphi_{ij}^2\right) - \frac{\lambda}{4!}\left(\sum_{ij}\varphi_{ij}^2\right)^2 \\ &+\frac{1}{2}\sum_{i}\left(\left(\partial_\mu\eta_{i}\right)^2 - m^2\eta_{i}^2\right) - \frac{\lambda}{4!}\left(\sum_{i}\eta_{i}^2\right)^2\end{aligned}$$

While the $SO(N)$ symmetry simultaneously transforms both η and ϕ. Under the infinitesimal transformation, we could see that to the first order of ϵ, the transformation of the field should be something like:

$$\begin{aligned}\varphi'_{ij} &= \left(\delta_{ik} + \epsilon\omega^\alpha_{ik}\right)\left(\delta_{jl} + \epsilon\omega^\alpha_{jl}\right)\varphi_{kl} \\ &= \varphi_{ij} + \epsilon\omega^\alpha_{ik}\varphi_{kj} + \epsilon\omega^\alpha_{jl}\varphi_{il} \quad (11) \\ \eta'_i &= \eta_i + \epsilon\omega^\alpha_{ij}\eta_j \quad (12)\end{aligned}$$

Again, ω^α are the three generators for $SO(3)$ symmetry. Taking in the definition of Noethers current, we could get the Noethers current to be:

$$\begin{aligned}J^{\mu\alpha} &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi_{ij})}\delta\varphi_{ij} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\eta_i)}\delta\eta_i \\ &= \left(\partial_\mu\varphi_{ij}\right)\left(\omega^\alpha_{ik}\varphi_{kj} + \omega^\alpha_{jl}\varphi_{il}\right) + \left(\partial_\mu\eta_i\right)\omega^\alpha_{ij}\eta_j \quad (13)\end{aligned}$$

Using the equation of motion:

$$\partial^2\varphi_{ij} + m^2\varphi_{ij} + 4\frac{\lambda}{4!}\varphi_{ij}\left(\sum_{kl}\varphi_{ij}^2\right) = 0 \quad (14)$$

$$\partial^2\eta_i + m^2\eta_i + 4\frac{\lambda'}{4!}\eta_i\left(\sum_{j}\eta_j^2\right) = 0 \quad (15)$$

We could see that the total derivative of the conservative current could be given as:

$$\begin{aligned}\partial_\mu J^{\mu\alpha} &= \left(\partial^2\varphi_{ij}\right)\left(\omega^\alpha_{ik}\varphi_{kj} + \omega^\alpha_{jl}\varphi_{il}\right) + \left(\partial^2\eta_i\right)\omega^\alpha_{ij}\eta_j \\ &+ \left(\partial_\mu\varphi_{ij}\right)\left(\omega^\alpha_{ik}\partial^\mu\varphi_{kj} + \omega^\alpha_{jl}\partial^\mu\varphi_{il}\right) + \left(\partial_\mu\eta_i\right)\omega^\alpha_{ij}\partial^\mu\eta_j \\ &= 0 \quad (16)\end{aligned}$$

Which verifies that the current is indeed conserved.

5. **Show that the following bilinears in the spinor field $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$, $\psi\sigma^{\mu\nu}$, $\bar{\psi}\gamma^\mu\gamma^5\psi$, $\bar{\psi}\gamma^5\psi$ transform under Lorentz group and parity as a scalar, a vector, a tensor, a pseudo-vector (or axial vector), and a pseudo-scalar respectively.**

First lets consider the transformation of the five bilinears under a Lorentz transformation. Since the γ matrices themselves don't change under the Lorentz transformation, we could see that all of the transformation dependence comes from the transformation of the spinors.

Since the transformation of the should be given as:

$$\psi \to e^{\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi; \quad \bar{\psi} \to \bar{\psi}e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \tag{17}$$

The first kind of the transformation is simple $\bar{\psi}\psi \to \bar{\psi}\psi$. The others are a bit trickier, and we are going to use the trick of first assuming an infinitesimal transformation to reduce the form of the exponential, then push the limit to a finite transformation.

We could see that under and infinitesimal transformation, we get the relation:

$$\begin{aligned}
\bar{\psi}\gamma^\alpha\psi \to \quad & \bar{\psi}(1 + \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu})\gamma^\alpha(1 - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu})\psi \\
= \quad & \bar{\psi}\left(\gamma^\alpha + \frac{i}{4}\left[\omega_{\mu\nu}\sigma^{\mu\nu}, \gamma^\alpha\right]\right)\psi \\
= \quad & \bar{\psi}\left(\gamma^\alpha - \frac{1}{8}\omega_{\mu\nu}\left[\left[\gamma^\mu, \gamma^\nu\right], \gamma^\alpha\right]\right)\psi
\end{aligned}$$

Here we are going to set the use the fact that if $\mu = \nu$, when $\omega_{\mu\nu} = 0$, and thus the term is insignificant when performing the contraction, and thus we could focus of the terms where $\mu \neq \nu$ and thus $\{\gamma^\mu, \gamma^\nu\} = 0$ or in other words

$$\omega_{\mu\nu}\left[\gamma^\mu, \gamma^\nu\right] = 2\omega_{\mu\nu}\gamma^\mu\gamma^\nu \tag{18}$$

We could thus contract the second term in the relation to something like:

$$\frac{1}{4}\omega_{\mu\nu}\left(\gamma^\mu\gamma^\nu\gamma^\alpha - \gamma^\alpha\gamma^\mu\gamma^\nu\right) \tag{19}$$

Using repeated use of the commutation relation, we could get:

$$\begin{aligned}
\gamma^\mu\gamma^\nu\gamma^\alpha &= \gamma^\mu\left(\eta^{\nu\alpha} - \gamma^\alpha\gamma^\nu\right) \\
&= \gamma^\mu\eta^{\nu\alpha} - \eta^{\mu\alpha}\gamma^\nu + \gamma^\alpha\gamma^\mu\gamma^\nu
\end{aligned} \tag{20}$$

Contracting the indexes, and using the anti-symmetry property of the ω matrix, we could see that the transformation is reduced down to:

$$\bar{\psi}\gamma^\alpha\psi \to \bar{\psi}\left(\gamma^\alpha + \omega^\alpha_\nu\gamma^\nu\right)\psi = \bar{\psi}\Lambda^\alpha_\nu\gamma^\nu\psi \tag{21}$$

Since linear transformations could be combined, we expect this relation to hold even under finite transformations. Thus we claim, under Lorentz transformation, $\bar{\psi}\gamma^{\nu}\psi$ transforms like a four vector.

Next, let us look at the transformation of γ^5. Following the idea of the argument, we could see that after the commutation relation expansion, we are left with:

$$\frac{1}{4}\omega\left(\gamma^{\mu}\gamma^{\nu}\gamma^{5}-\gamma^{5}\gamma^{\mu}\gamma^{\nu}\right) \tag{22}$$

It is not hard to see that since γ^5 anti-commutes with any of the other γ matrices, We could shift the γ^5 in the first term to the front without changing any of the signs. And thus we could see that $\bar{\psi}\gamma^5\psi$ transforms like a scalar under Lorentz transformation, and by similar arguments, $\bar{\psi}\gamma^5\gamma^{\mu}\psi$ should transform like a four vector. Here we are going to take a look at the parity transformation of the bilinears containing, γ^5.

We could see that under parity transformation, $\bar{\psi}$ and ψ both pick up a factor of $\eta\gamma^0$, with $\eta^2 = 1$. We could see that $\gamma^0\gamma^5\gamma^0 = \gamma^5$, and thus we say that $\bar{\psi}\gamma^5\psi$ is a pseudo-scalar. In the case of $\gamma^0\gamma^{\mu}\gamma^5\gamma^0$, due to the fact that γ^0 commutes with all of the γ^{μ}, we could see that $\gamma^0\gamma^{\mu}\gamma^5\gamma^0 = -\gamma^{\mu}\gamma^5$. And thus we say that $\bar{\psi}\gamma^5\gamma^{\mu}\psi$ is a pseudo-vector.

Finally, we are ready for the computation of the supposed tensor bilinear $\bar{\psi}\sigma^{\mu\nu}\psi$. When solving the vector, we have already see that:

$$S(\Lambda)^{-1}\gamma^{\mu}S(\Lambda) = \Lambda^{\nu}_{\mu}\gamma^{\mu} \tag{23}$$

Or a more useful relation we need here:

$$S(\Lambda)^{\dagger}\gamma^{\mu} = \Lambda^{\nu}_{\mu}\gamma^{\mu}S(\Lambda)^{-1} = \Lambda^{\nu}_{\mu}\gamma^{\mu}S(\Lambda)^{\dagger} \tag{24}$$

And thus we could solve the transformation as something like:

$$S(\Lambda)^{\dagger}\sigma^{\mu\nu}S(\Lambda) = \frac{i}{2}S(\Lambda)^{\dagger}\left[\gamma^{\mu}, \gamma^{\nu}\right]S(\Lambda) = \frac{i}{2}\Lambda^{\sigma}_{\mu}\Lambda^{\lambda}_{\nu}\left[\gamma^{\mu}, \gamma^{\nu}\right] \tag{25}$$

And thus we could safely say that $\bar{\psi}\sigma^{\mu\nu}\psi$ transforms like a Lorentz tensor.

6. **Write down all the bilinears in the previous exercise in terms of ψ_L and ψ_R.**

In the Weyl basis, we could see that the spinors have the nice property of:

$$\psi_L = \begin{pmatrix}\phi_L \\ 0\end{pmatrix} \; ; \; \psi_R = \begin{pmatrix}0 \\ \phi_R\end{pmatrix} \; ; \; \gamma^0 = \begin{pmatrix}0 & I \\ I & 0\end{pmatrix} \tag{26}$$

and due to these properties:

$$\psi_L^\dagger \gamma^\mu \psi_L = 0 \tag{27}$$

$$\psi_L^\dagger \gamma^\mu \psi_L = 0 \tag{28}$$

since all of the γ matrices in the Weyl basis are only significant in the off diagonal blocks. This is also very handy to expand to other matrices placed between the $\psi_i^\dagger$ and ψ_i: for the general case of: $\psi_i^\dagger \mathbf{M} \psi_j$, we could see that for block-diagonal $\mathbf{M}$s would leave the terms where $i = j$, while the off block-diagonal $\mathbf{M}$, would leave the terms where $i \neq j$.

Now using these, we could start rewriting the Dirac bilinears.

First, the scalar, which is simple (since γ^0 is off block-diagonal):

$$\begin{aligned} \bar{\psi}\psi &= \left(\psi_L^\dagger + \psi_R^\dagger\right)\gamma^0\left(\psi_L + \psi_R\right) \\ &= \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \end{aligned} \tag{29}$$

Next the, vector, which is also simple, since $\gamma^0\gamma^\mu$ is guaranteed to be block diagonal.

$$\begin{aligned} \bar{\psi}\gamma^\mu\psi &= \left(\psi_L^\dagger + \psi_R^\dagger\right)\gamma^0\gamma^\mu\left(\psi_L + \psi_R\right) \\ &= \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R \end{aligned} \tag{30}$$

Keeping in mind, that γ^5 is block diagonal, we could see that:

$$\bar{\psi}\sigma^{\mu\nu}\psi = \bar{\psi}_L\sigma^{\mu\nu}\psi_R + \bar{\psi}_R\sigma^{\mu\nu}\psi_L \tag{31}$$

$$\bar{\psi}\gamma^5\gamma^\mu\psi = \bar{\psi}_L\gamma^5\gamma^\mu\psi_L + \bar{\psi}_R\gamma^5\gamma^\mu\psi_R \tag{32}$$

$$\bar{\psi}\gamma^5\psi = \bar{\psi}_L\gamma^5\psi_R + \bar{\psi}_R\gamma^5\psi_L \tag{33}$$

7. **Solve $(\not{p} - m)\psi(p) = 0$ explicitly. Verify that indeed χ is much smaller than ϕ for a slow moving electron. What happens for a fast moving electron?**

Solving the equation in the Dirac basis, we could see that:

$$\begin{pmatrix} p^0 - m & -p^i\sigma^i \\ p^i\sigma^i & -p^0 - m \end{pmatrix}\begin{pmatrix}\phi \\ \chi\end{pmatrix} = 0 \tag{34}$$

Expanding out this expression, we could get

$$\begin{pmatrix} p^0 - m & 0 \\ 0 & p^0 - m \end{pmatrix}\phi - \begin{pmatrix} p_z & p_x + ip_y \\ p_x + ip_y & p_z \end{pmatrix}\chi = 0 \tag{35}$$

$$\begin{pmatrix} p_z & p_x + ip_y \\ p_x + ip_y & p_z \end{pmatrix}\phi - \begin{pmatrix} p^0 + m & 0 \\ 0 & p^0 + m \end{pmatrix}\chi = 0 \tag{36}$$

The first of these two equation tell us:

$$\phi = \frac{1}{p_0 - m} \begin{pmatrix} p_z & p_x + ip_y \\ p_x + ip_y & p_z \end{pmatrix} \chi \tag{37}$$

With the relation that $p_0 = \sqrt{m^2 + p^2} \approx m\left(1 + \frac{1}{2}\left(\frac{p}{m}\right)^2\right)$, we could see that this equation tell us that:

$$\phi \approx \frac{m}{p}\chi \tag{38}$$

In the small p limit, meaning that ϕ is indeed a lot larger than χ.

The second of the two equation tells us that:

$$\chi = \frac{1}{p^0 + m} \begin{pmatrix} p_z & p_x + ip_y \\ p_x + ip_y & p_z \end{pmatrix} \phi \tag{39}$$

which also tells us that up $\chi \approx \frac{p}{m}\phi$ in the small p limit.

In the high p limit, we could see that since $p^0 \approx |\vec{p}|$, both equation would say that the the numeral factor between ϕ and χ approaches 1 (aka, ϕ and χ are approximately the same order in magnitude)

8. **Exploiting that fact that χ is much smaller than ϕ for a slowly moving electron, find the approximation equation satisfied by ϕ.**

If we write down the Dirac equation as something like:

$$(i\gamma^0\partial_0 - m)\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \tag{40}$$

Or, we we once again use different notations for time and space derivatives, we could write:

$$i\gamma^0\partial_t\psi = (i\boldsymbol{\gamma}\cdot\nabla - m)\,\psi \tag{41}$$

And since γ^0 is a self inverse matrix, we could multiply both sides by γ^0 and we recover the form of:

$$i\partial_t\psi = (i\boldsymbol{\alpha}\cdot\nabla - m\gamma^0)\psi \tag{42}$$

where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$.

Here we could start to expand out the solution to obtain the low energy approximation for the equation of motion. If we take the solution of ψ to be $e^{ipx}\,(\phi,\chi)^T$, with $p = (E, \vec{p})^T$ and $E = m + \epsilon$ then our solution equation becomes:

$$\begin{aligned} (m+\epsilon)\begin{pmatrix}\phi\\ \chi\end{pmatrix} &= \begin{pmatrix}0 & \vec{p}\cdot\vec{\sigma}\\ \vec{p}\cdot\vec{\sigma} & -0\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} + \begin{pmatrix}m\phi\\ -m\chi\end{pmatrix} \\ \epsilon\begin{pmatrix}\phi\\ \chi\end{pmatrix} &= \begin{pmatrix}(\vec{p}\cdot\vec{\sigma})\chi\\ (\vec{p}\cdot\vec{\sigma})\phi\end{pmatrix} - \begin{pmatrix}0\\ 2m\chi\end{pmatrix} \end{aligned} \tag{43}$$

The second of the equations above can tell us that:

$$\chi = \frac{\vec{p}\cdot\vec{\sigma}}{\epsilon + 2m}\phi$$

which, just to verify, is of the same order that we obtained in the previous equation and taking this into the first of the equations we could get:

$$\epsilon\phi = \frac{(\vec{p}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma})}{\epsilon + 2m}\phi \tag{44}$$

if we take the approximation of $\epsilon + 2m \approx 2m$ to the first order of p/m the above reduces to:

$$\epsilon\phi = \frac{\vec{p}^2}{2m}\phi \tag{45}$$

This for is similar to the Schrodinger's equation, except for that the interpretation is still not the same. Here we are just writing down the equation of motion for a field, whereas the interpretation of the field is not yet associated with the probability distribution of a single particle.

9. **For a relativistic electron moving along the z-axis, perform a rotation around the z axis. In other words, study the effects of $e^{-i\omega\sigma^{12}/4}$ on $\psi(p)$ and verify the assertion in the text regarding ψ_L and ψ_R.**

Borrowing the relation between ϕ and χ from the seventh equation in this assignment, we could see that if the particle was moving in the z direction then the the relation required to be satisfied would simply be:

$$\chi = \frac{p_z}{p^0 + m}\phi \tag{46}$$

Thus, our spinor would take on a much simpler form, with χ and ϕ being simply proportional to each other.

Now, for the rotation, we are going to perform the infinitesimal rotation, first, before cranking up the scale of the transformation to see the more general case. The transformation could simply be written as:

$$\begin{aligned} \psi' &= \left(1 + \frac{1}{2}\omega_{12}\begin{pmatrix}\sigma_z & 0 \\ 0 & \sigma_z\end{pmatrix}\right)\begin{pmatrix}\phi \\ \kappa\phi\end{pmatrix} \\ &= \begin{pmatrix}\phi + \frac{1}{2}\omega\sigma_z\phi \\ \kappa\left(\phi + \frac{1}{2}\omega\sigma_z\phi\right)\end{pmatrix} \end{aligned} \tag{47}$$

We could see that the two Pauli spinor components of the Dirac spinor simply transforms like the Pauli spinors did in Quantum mechanics.

10. **Solve the massless Dirac equation.**

The massless Dirac equation is given as:

$$\gamma^\mu \partial_\mu \psi = 0 \tag{48}$$

Assuming we could still write down $\psi = e^{ipx}(\phi, \chi)^T$, then the Dirac equation in momentum space could be given as:

$$\begin{pmatrix} p^0 & \sigma^i p^i \\ -\sigma^i p^i & -p^0 \end{pmatrix}\begin{pmatrix}\phi \\ \chi\end{pmatrix} = 0$$

$$\begin{aligned} p^0\phi + \sigma^i p^i \chi &= 0 \\ -\sigma^i p^i \phi - p^0 \chi &= 0 \end{aligned} \tag{49}$$

we could clearly see that ϕ and χ are entirely on equal footing. Substituting one equation into the other, we could get the relation:

$$\begin{aligned} p^0\phi + \sigma^i p^i \left(\frac{-\sigma^i p^i}{p^0}\phi\right) &= 0 \\ \left((p_0)^2 - \sum p_i^2\right)\phi &= 0 \end{aligned}$$

But the first part of the final equation is simply the energy-momentum relation. Thus we could see that ϕ and χ are completely decoupled from each other, and the Dirac equation tell us nothing more than the fact that the particle should obey the energy-momentum relation of massless particles.

11. **Show explicitly that equation (25) violates parity**

We could see that in the Weyl basis, we could see that under parity transformation the Weyl spinors transforms like the way depicted in the textbook (equation

18 of chapter II.1). Using this we could see that using this sort of transformation in equation (25):

$$\frac{\mathcal{L}}{G} = \bar{\psi}_{1L}\gamma^{\mu}\psi_{2L}\bar{\psi}_{3L}\gamma_{\mu}\psi_{4L} \tag{50}$$

we get the transformed Lagrangian of:

$$\begin{aligned} \left(\frac{\mathcal{L}}{G}\right)' &= \left(\bar{\psi}_{1L}{\gamma^{0}}^{\dagger}\eta^{*}\right)\gamma^{\mu}\left(\eta\gamma^{0}\psi_{2L}\right)\left(\bar{\psi}_{3L}{\gamma^{0}}^{\dagger}\eta^{*}\right)\gamma_{\mu}\left(\eta\gamma^{0}\psi_{4L}\right) \\ &= \bar{\psi}'_{1R}\gamma^{\mu}\psi'_{2R}\bar{\psi}'_{3R}\gamma_{\mu}\psi'_{4R} \neq \frac{\mathcal{L}}{G} \end{aligned} \tag{51}$$

12. **The defining equation of C evidently fixes C only up to an overall constant. Show that this constant is fixed by requiring $(\psi_C)_C = \psi$**

    If we define the supposed charge conjugation operator to be the form in the textbook $C = \gamma^2\gamma^0$, and the supposed "complete" charge conjugate operator to be in the form of $C_R = \xi C$, where ξ is a complex number.

    Now we could see hat if we make the requirement that $(\psi_C)_C = \psi$, by taking in the supposed C_R, we could get the relation:

$$\begin{aligned} (\psi_C)_C &= \xi C\gamma^0\left(\xi C\gamma\psi^*\right)^* \\ &= |\xi|^2\gamma^2{\gamma^2}^*\psi \\ &= \xi^2\psi \end{aligned} \tag{52}$$

    Using this, we could see that there are the choice of ξ is not unique, and thus the charge conjugation operator is only unique up to a phase.

13. **Show that the charge conjugate of a left handed side field is right handed and vice versa.**

    We could see that the charge conjugate transformation on a left handed field is given as:

$$C\gamma^0\left(\left(1-\gamma^5\right)\psi\right)^* \tag{53}$$

    Here we are working with the Weyl basis, and since γ^5 is real, we could see that the above expands to:

$$\begin{aligned} &\gamma^2\left(1-\gamma^5\right)\psi^* \\ =\ &\left(1+\gamma^5\right)\gamma^2\psi^* \\ =\ &\left(1+\gamma^5\right)\psi' \end{aligned} \tag{54}$$

We have seen that the charge conjugate transformation has turn the left handed field into a right handed one. Since the argument is entirely base on the fact that γ^2 and γ^5 anti-commute, we could see that the argument should hold for the right handed field as well.

14. **Show that $\psi C\psi$ is a Lorentz scalar.**

This sort of an operation is not allowed, we are going to assume that the textbook's question should be given as $\psi^T C\psi$, where T is the transpose of the Dirac spinor. Since the transpose is simply the complex conjugate of the hermitian, the transpose should transform simply like:

$$\psi(x)^T \rightarrow \psi(\Lambda^{-1}x)\left(S^{-1}(\Lambda)\right)^* = \psi(x')S(\Lambda) \tag{55}$$

And thus, the transformation of interest is simply given as:

$$\psi^T C\psi \rightarrow \psi^T S^T(\Lambda)\gamma^2\gamma^0 S(\Lambda)\psi$$

Under infinitesimal transformation, we get the relation of:

$$S(\Lambda) \approx 1 + \frac{1}{8}\omega_{\mu\nu}\left[\gamma^\mu, \gamma^\nu\right] \tag{56}$$

So under, infinitesimal transformation to the first order, we could get the relation:

$$\begin{aligned} \psi^T C\psi &\rightarrow \psi^T C\psi + \frac{1}{8}\omega_{\mu\nu}\psi^T\left(C\left[\gamma^\mu, \gamma^\nu\right] + \left[\gamma^\mu, \gamma^\nu\right]^T C\right)\psi \\ &= \psi^T C\psi + \frac{1}{8}\omega_{\mu\nu}\psi^T\left(C\left[\gamma^\mu, \gamma^\nu\right] - \left[(\gamma^\mu)^T, (\gamma^\nu)^T\right] C\right)\psi \end{aligned} \tag{57}$$

we could see that in both the Weyl and Dirac basis, we have the relation that $\left(\gamma^2\right)^T = -\gamma^2$ and all others are unchanged under transpose. Also, with the computation of the Dirac bilinears in some previous question, we could focus on the discussion where $\mu \neq \nu$.

We could see that if $(\mu, \nu) = (2, \nu)$ $(\nu \neq 2)$, then the term between ψ^T and ψ becomes:

$$\begin{aligned} & C\left[\gamma^2, \gamma^\nu\right] + \left[\gamma^2, \gamma^\nu\right] C \\ =\ & C\gamma^2\gamma^\nu - C\gamma^\nu\gamma^2 + \gamma^2\gamma^\nu\gamma^2\gamma^0 - \gamma^2\gamma^\nu\gamma^2\gamma^0 \end{aligned}$$

And since, γ^2 anti-commutes with γ^ν, we get the relation:

$$C\gamma^2\gamma^\nu - C\gamma^\nu\gamma^2 - \gamma^2\gamma^2\gamma^\nu\gamma^0 + \gamma^2\gamma^2\gamma^\nu\gamma^0 \tag{58}$$

Where we could see that whether ν is equal to 0 or not, we could use the commutation of the relation to get:

$$C\gamma^2\gamma^\nu - C\gamma^\nu\gamma^2 - C\gamma^2\gamma^\nu + C\gamma^2\gamma^\nu = 0 \tag{59}$$

Since the expression, is anti symmetric under $\nu \leftrightarrow \mu$, we could see that the case of $(\mu\nu) = (\mu, 2)$ is also 0.

So now, the only terms we need to consider is where $\mu\nu$ are both not 2, and thus the center expression could be given as:

$$\begin{aligned} & \gamma^2\gamma^0\left[\gamma^\mu, \gamma^\nu\right] - \left[\gamma^\mu, \gamma^\nu\right]\gamma^2\gamma^0 \\ = \ & \gamma^2\gamma^0\gamma^\mu\gamma^\nu - \gamma^2\gamma^0\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\gamma^2\gamma^0 + \gamma^\nu\gamma^\mu\gamma^2\gamma^0 \\ = \ & \gamma^2\gamma^0\gamma^\mu\gamma^\nu - \gamma^2\gamma^0\gamma^\nu\gamma^\mu - \gamma^2\gamma^\mu\gamma^\nu\gamma^0 + \gamma^2\gamma^\mu\gamma^\nu\gamma^0 \end{aligned} \tag{60}$$

Now we could see that if only one of μ ν is equal to 0, on attempting to move the γ^0 in the last two terms to behind γ^2, both would experience a change in sign. Meaning the overall value would be zero. If neither of μ, ν is 0, the no change of signs would occur during the same operation. Thus we could now safely conclude that no matter what the parameters are, under a legal transformation:

$$\omega_{\mu\nu}\left(C\left[\gamma^\mu, \gamma^\nu\right] - \left[(\gamma^\mu)^T, (\gamma^\nu)^T\right]C\right) = 0\psi'^T C\psi' = \psi^T C\psi \tag{61}$$

And thus we could at least conclude that $\psi^T C\psi$ is either the scalar or a pseudo-scalar.

For the parity check, we could see that the transformation should be given as:

$$\psi^T C\psi \rightarrow \psi^T\left(\gamma^0\right)^T\gamma^2\gamma^0\gamma^0\psi \tag{62}$$

Since in both basis, $\left(\gamma^0\right)^T = \gamma^0$, we could the that the transformation is simply:

$$\begin{aligned} \psi^T C\psi \ \rightarrow \ & \psi^T\gamma^0\gamma^2\gamma^0\gamma^0\psi \\ & = \psi^T\gamma^0\gamma^2\psi \\ & = -\psi^T C\psi \end{aligned} \tag{63}$$

we could see that $\psi^T C\psi$ is actually a pseudo-scalar

15. **Work out the Dirac equation in a (1+1) dimensional space-time.**

Working out the Dirac equation is simple, it should still be something like:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{64}$$

and since it should still satisfy the one dimensional Klein-Gordon equation, the γ matrices should still satisfy the relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{65}$$

Or after expanding out the terms of η:

$$\gamma^0\gamma^0 = I\,;\ \gamma^1\gamma^1 = -1\,;\ \gamma^0\gamma^1 = -\gamma^1\gamma^0; \tag{66}$$

Let us attempt to find the simplest representation of γ^μ simply be assuming that γ^μ could be written as a 2×2 matrix:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ;\ \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{67}$$

Proving that these two matrices satisfies the required relation is simple. In fact, and even more general choice of the matrices exists is we take a look at the anti-commutation relation of the Pauli matrices:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{68}$$

Thus a general choice of of the γ matrix would simply be $\gamma^0 = \sigma_\alpha$, $\gamma^1 = i\sigma_\beta$, with restrictions that $\alpha \neq \beta$.

16. **Work out the Dirac equation in a (2+1) dimensional space-time. Show that the apparently innocuous mass term violates parity and time reversal. (The three γ matrices should just be the Pauli matrices with the appropriate factors of i.)**

Following the work above, we are going to find the three matrices that satisfies the relation of:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{69}$$

Using the hint of the text, we are going to try out the guess that the three matrices are proportional to the Pauli matrices.

Since the Pauli matrices all satisfy the anti-commutation relation of $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, we could see that all of the all of the desired relation could be satisfied, if we choose $\gamma^0 = \sigma_a$, and $\gamma^i = \pm i\sigma_b$, where $a \neq b$. For working out the properties of the two dimensional Dirac equation, we are going to use the basis that is most similar to the text (at least in terms of γ^0):

$$\gamma^0 = \sigma_z\,;\ \gamma^i = i\sigma_i \tag{70}$$

For the case of parity transformation, we must note that $(x^0, \vec{x}) \to (x^0, -\vec{x})$ is not the parity transformation! It is in fact a rotation. Thus we are compelled to find a transformation P such that:

$$\left(i\gamma^\mu \partial'_\mu - m\right) P\psi = 0 \tag{71}$$

where, $\partial'_0 = \partial_0$, $\partial'_2 = \partial_2$ but $\partial'_1 = -\partial_1$, we could guess that P is an matrix that commute with γ^0 and γ^2 but anti-commutes with γ_1. We could see that there is no way for construct a P that satisfies this constraint. Likewise, we could not perform a time reversal operator.

BUT, it is possible to construct such operators if we could remove the mass term. By the algebra of the gamma matrices, we could see that

$$\gamma^1\gamma^0 = -\gamma^0\gamma^1 \; \gamma^1\gamma^1 = \gamma^1\gamma^1 \; \gamma^1\gamma^2 = -\gamma^2\gamma^1 \tag{72}$$

And thus if we multiply the massless Dirac equation by γ^1:

$$\gamma^1 i\gamma^\mu \partial_\mu \psi = 0 = -i\gamma^\mu \partial'_\mu \gamma^1 \psi \tag{73}$$

If the mass term was still to exists however, we could see that:

$$\gamma^1 \left(i\gamma^\mu \partial_\mu - m\right) \psi = 0 = -\left(i\gamma^\mu \partial'_\mu + m\right) \gamma^1 \psi \tag{74}$$

we could be forced to invert the mass of the field. Thus parity transformation is not allowed in the massive 2 dimensional Dirac equation but is allowed in the 2 dimensional *massless* Dirac equation! Even better, we could see that if we simply exchange γ^1 for γ^0, we could get entirely the same argument for time reversal!