Perturbation Analysis of Parametric Resonance

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Article Outline

1 Glossary

Coexistence The special case when all the independent solutions of a linear, T-periodic ODE are T-periodic.
Hill's equation A second order ODE of the form $\ddot{x} + p(t)x = 0$, with $p(t)$ T-periodic.
Instability pockets Finite domains, usually intersections of instability tongues, where the trivial solution of linear, T-periodic ODEs is unstable.
Instability tongues Domains in parameter space where the trivial solution of linear, T-periodic ODEs is unstable.
Mathieu equation An ODE of the form $\ddot{x} + (a + b\cos(t))x = 0$.
Parametric resonance Resonance excitation arising for special values of coefficients, frequencies and other parameters in T-periodic ODEs.
Quasi-periodic A function of the form $\sum_{i=1}^{n} f_i(t)$ with $f_i(t)$ T_i-periodic, n finite, and the periods T_i independent over $\mathbb{R}$.
Sum resonance A parametric resonance arising in the case of at least three frequencies in a T-periodic ODE.

2 Definition of subject

Parametric resonance arises in mechanics in systems with external sources of energy and for certain parameter values. Typical examples are the pendulum with oscillating support and a more specific linearization of this pendulum, the Mathieu equation in the form

$$\ddot{x} + (a + b\cos(t))x = 0.$$

The time-dependent term represents the excitation. Tradition has it that parametric resonance is usually not considered in the context of systems with *external* excitation of the form $\dot{x} = f(x) + \phi(t)$, but for systems where time-dependence arises in the coefficients of the equation. Mechanically this means usually periodically varying stiffness, mass or load, in fluid or plasma mechanics one can think of frequency modulation or density fluctuation, in mathematical biology of periodic environmental changes. The term 'parametric' refers to the dependence on parameters and certain resonances arising for special values of the parameters. In the case of the Mathieu equation, the parameters are the frequency ω ($a = \omega^2$) of the equation without time-dependence and the excitation amplitude b; see section 5 for an explicit demonstration of resonance phenomena in this two parameters system.
Mathematically the subject is concerned with ODEs with periodic coefficients. The study of linear dynamics of this type gave rise to a large amount of literature in the first half of the 20th century and this highly technical, classical material is still accessible in textbooks. The standard equations are Hill's equation and the Mathieu equation (see section 5.1). We will summarize a number of basic aspects. The reader is also referred to the article 'Dynamics of parametric excitation' by Alan Champneys in this Encyclopaedia (vol. Non-Linear ODEs and Dynamical Systems).
Recently, the interest in nonlinear dynamics, new applications and the need to explore higher dimensional problems has revived the subject. Also structural stability and persistence problems have been investigated. Such problems arise as follows. Suppose that we have found a number of interesting phenomena for a certain equation and suppose we imbed this equation in a family of equations by adding parameters or perturbations. Do the 'interesting phenomena' persist in the family of equations? If not, we will call the original equation structurally unstable. A simple example of structural instability is the harmonic equation which shows qualitative different behaviour on adding damping. In general, Hamiltonian systems are structurally unstable in the wider context of dissipative dynamical systems.

3 Introduction

Parametric resonance produces interesting mathematical challenges and plays an important part in many applications . The linear dynamics is already nontrivial whereas the nonlinear dynamics of such systems is extremely rich and largely unexplored. The role of symmetries is essential, both in linear and in nonlinear analysis.
A classical example of parametric excitation is the swinging pendulum with oscillating support. The equation of motion describing the model is

$$\ddot{x} + (\omega_0^2 + p(t))\sin x = 0, \tag{1}$$

where $p(t)$ is a periodic function. Upon linearization - replacing $\sin x$ by x - we obtain Hill's equation (section 5.1):

$$\ddot{x} + (\omega_0^2 + p(t))x = 0.$$

This equation was formulated around 1900 in the perturbation theory of periodic solutions in celestial mechanics. If we choose $p(t) = \cos \omega t$, Hill's equation becomes the Mathieu equation. It is well-known that special tuning of the frequency ω_0 and the period of excitation (of $p(t)$) produces interesting instability phenomena (resonance).
More generally we may study nonlinear parametric equations of the form

$$\ddot{x} + k\dot{x} + (\omega_0^2 + p(t))F(x) = 0, \tag{2}$$

where $k > 0$ is the damping coefficient, $F(x) = x + bx^2 + cx^3 + \cdots$ and time is scaled so that $p(t)$ is a π-periodic function with zero average. We may also take for $p(t)$ a quasi-periodic or almost-periodic function.
The books [49] cover most of the classical theory, but for a nice introduction see [39]. In [37], emphasis is placed on the part played by parameters, it contains a rich survey of bifurcations of eigenvalues and various applications. There are many open questions for eqs (1) and (2); we shall discuss aspects of the classical theory, recent theoretical results and a few applications.

As noted before, in parametric excitation we have an oscillator with an independent source of energy. In examples, the oscillator is often described by a one degree of freedom system but of course many more degrees of freedom may play a part; see for instance in section 7 the case of coupled Mathieu-equations as studied in [33]. In what follows, ε will always be a small, positive parameter.

4 Perturbation techniques

In this section we review the basic techniques to handle parametric perturbation problems. In the case of Poincaré-Lindstedt series which apply to periodic solutions, the expansions are in integer powers of ε. It should be noted that in general, other order functions of ε may play a part; see section 5.1 and [47].

4.1 Poincaré-Lindstedt series

One of the oldest techniques is to approximate a periodic solution by the construction of a convergent series in terms of the small parameter ε. The method can be used for equations of the form

$$\dot{x} = f(t, x) + \varepsilon g(t, x) + \varepsilon^2 \cdots ,$$

with $x \in \mathbb{R}^n$ and (usually) assuming that the 'unperturbed' problem $\dot{y} = f(t, y)$ is understood and can be solved. Note that the method can also be applied to perturbed maps and difference equations.
Suppose that the unperturbed problem contains a periodic solution, under what conditions can this solution be continued for $\varepsilon > 0$? The answer is given by the conditions set by the implicit function theorem, see for formulations and theorems [30] and [45].
Usually we can associate with our perturbation problem a parameter space and one of the questions is then to find the domains of stability and instability. The common boundary of these domains is often characterized by the existence of periodic solutions and this is where Poincaré-Lindstedt series are useful. We will demonstrate this in the next section.

4.2 Averaging

Averaging is a normalization method. In general, the term "normalization" is used whenever an expression or quantity is put in a simpler, standardized form. For instance, a $n \times n$-matrix with constant coefficients can be put in Jordan normal form by a suitable transformation. When the eigenvalues are distinct, this is a diagonal matrix.
Introductions to normalization can be found in [1], [15], [19] and [13]. For the relation between averaging and normalization in general the reader is referred to [34] and [45].
For averaging in the so-called standard form it is assumed that we can put the perturbation problem in the form

$$\dot{x} = \varepsilon F(t,x) + \varepsilon^2 \cdots ,$$

and that we have the existence of the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(t,x)dt = F^0(x).$$

The analysis of the averaged equation $\dot{y} = F^0(y)$ produces asymptotic approximations of the solutions of the original equation on a long timescale; see [34]. Also, under certain conditions, critical points of the averaged equation correspond with periodic solutions in the original system.
The choice to use Poincaré-Lindstedt series or the averaging method is determined by the amount of information one wishes to obtain. To find the location of stability and instability domains (the boundaries), Poincaré-Lindstedt series are very efficient. On the other hand, with somewhat more efforts, the averaging method will also supply this information with in addition the behaviour of the solutions within the domains. For an illustration see section 5.1.

4.3 Resonance

Assume that $x = 0$ is a critical point of the differential equation and write the system as:

$$\dot{x} = Ax + f(t,x,\varepsilon), \tag{3}$$

with $x \in \mathbb{R}^n$, A a constant $n \times n$-matrix; $f(t,x,\varepsilon)$ can be expanded in a Taylor series with respect to ε and in homogeneous vector polynomials in x starting with quadratic terms. Normalization of eq. (3) means that by successive transformation we remove as many terms of eq. (3) as possible. It would be ideal if we could remove all the nonlinear terms, i.e. linearize eq. (3) by transformation. In general, however, some nonlinearities will be left and this is where resonance comes in.
The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix A are resonant if for some $i \in \{1, 2, \ldots, n\}$ one has:

$$\sum_{j=1}^{n} m_j \lambda_j = \lambda_i, \tag{4}$$

with $m_j \geq 0$ integers and $m_1 + m_2 + \cdots + m_n \geq 2$.
If the eigenvalues of A are non-resonant, we can remove all the nonlinear terms and so linearize the system. However, this is less useful than it appears, as in general the sequence of successive transformations to carry out the normalization will be divergent. The usefulness of normalization lies in removing nonresonant terms to a certain degree to simplify the analysis.

4.4 Normalization of time-dependent vectorfields

In problems involving parametric resonance, we have time-dependent systems such as equations perturbing the Mathieu equation. Details of proofs and methods to compute the normal form coefficients in such cases can be found in [1], [18] and [34]. We summarize some aspects. Consider the following parameter and time dependent equation:

$$\dot{x} = F(x, \mu, t), \tag{5}$$

with $x \in \mathbb{R}^m$ and the parameters $\mu \in \mathbb{R}^p$.
Here $F(x, \mu, t) : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^m$ is C^∞ in x and μ and T-periodic in the t-variable. We assume that $x = 0$ is a solution, so $F(0, \mu, t) = 0$ and, moreover assume that the linear part of the vectorfield $D_x F(0, 0, t)$ is time-independent for all $t \in \mathbb{R}$. We will write $L_0 = D_x F(0, 0, t)$. Expanding $F(x, \mu, t)$ in a Taylor series with respect to x and μ yields the equation:

$$\dot{x} = L_0 x + \sum_{n=2}^{k} F_n(x, \mu, t) + O(|(x, \mu)|^{k+1}), \tag{6}$$

where the $F_n(x, \mu, t)$ are homogeneous polynomials in x and μ of degree n with T-periodic coefficients.

Theorem 4.1
Let $k \in \mathbb{N}$. There exists a (parameter- and time-dependent) transformation

$$x = \hat{x} + \sum_{n=2}^{k} P_n(\hat{x}, \mu, t),$$

where $P_n(\hat{x}, \mu, t)$ are homogeneous polynomials in x and μ of degree n with T-periodic coefficients, such that eq. (6) takes the form (dropping the hat):

$$\begin{aligned} \dot{x} &= L_0 x + \sum_{n=2}^{k} \tilde{F}_n(x, \mu, t) + O(|(x, \mu)|^{k+1}) \quad , \\ \dot{\mu} &= 0. \end{aligned} \tag{7}$$

The truncated vectorfield:

$$\dot{x} = L_0 x + \sum_{n=2}^{k} \tilde{F}_n(x, \mu, t) = \tilde{F}(x, \mu, t) \quad , \tag{8}$$

which will be called the normal form of eq. (5), has the following properties:

1. $\frac{d}{dt} e^{L_0^* t} \tilde{F}(e^{-L_0^* t} x, \mu, t) = 0$, for all $(x, \mu) \in \mathbf{R}^{m+p}, t \in \mathbb{R}$.
2. If eq. (5) is invariant under an involution (i.e. $SF(x, \mu, t) = F(Sx, \mu, t)$ with S an invertible linear operator such that $S^2 = I$), then the truncated normal form (8) is also invariant under S. Similarly, if eq. (5) is reversible under an involution R (i.e. $RF(x, \mu, t) = -F(Rx, \mu, t)$), then the truncated normal form (8) is also reversible under R.

For a proof, see [18].

The theorem will be applied to situations where L_0 is semi-simple and has only purely imaginary eigenvalues. We take $L_0 = diag\{i\lambda_1, \ldots, i\lambda_m)\}$. In our applications, $m = 2l$ is even and $\lambda_{l+j} = -\lambda_j$ for $j = 1, \ldots, l$. The variable x is then often written as $x = (z_1, \ldots, z_l, \bar{z}_1, \ldots, \bar{z}_l)$. Assume $L_0 = diag\{i\lambda_1, \ldots, i\lambda_m)\}$ then:

- A term $x_1^{\gamma_1} \ldots x_m^{\gamma_n} e^{i\frac{2\pi}{T}kt}$ is in the j-th component of the Taylor-Fourier series of $\tilde{F}(x, \mu, t)$ if:

$$-\lambda_j + \frac{2\pi}{T}k + \gamma_1\lambda_1 + \cdots + \gamma_m\lambda_m = 0 \quad . \tag{9}$$

  This is known as the resonance condition.

- Transforming the normal form through $x = e^{L_0 t}w$ leads to an autonomous equation for w:

$$\dot{w} = \sum_{n=2}^{k} \tilde{F}_n(w, \mu, 0) \quad . \tag{10}$$

- An important result is this:
  If eq. (5) is invariant (respectively reversible) under an involution S, then this also holds for eq. (10).

- The autonomous normal form (10) is invariant under the action of the group $\mathcal{G} = \{g \mid gx = e^{jL_0T}x \ , \ j \in \mathbb{Z}\}$, generated by e^{L_0T}. Note that this group is discrete if the ratios of the λ_i are rational and continuous otherwise.

For a proof of the last two statements see [31].

By this procedure we can make the system autonomous. This is very effective as the autonomous normal form (10) can be used to prove the existence of periodic solutions and invariant tori of eq. (5) near $x = 0$. We have:

Theorem 4.2
Let $\varepsilon > 0$, sufficiently small, be given. Scale $w = \varepsilon\hat{w}$.

1. If $\hat{w}_0$ is a hyperbolic fixed point of the (scaled) eq. (10), then eq. (5) has a hyperbolic periodic solution $x(t) = \varepsilon\hat{w}_0 + O(\varepsilon^{k+1})$.

2. If the scaled eq. (10) has a hyperbolic closed orbit, then eq. (5) has a hyperbolic invariant torus.

These results are related to earlier theorems in [3], see also the survey [46]. Later we shall discuss normalization in the context of the so-called sum-resonance.

4.5 Remarks on limit sets

In studying a dynamical system the behaviour of the solutions is for a large part determined by the limit sets of the system. The classical limit sets are equilibria and periodic orbits. Even when restricting to autonomous equations of dimension three, we have no complete classification of possible limit sets and this makes the recognition and description of non-classical limit sets important. In parametrically excited systems, the following limit sets, apart from the classical ones, are of interest:

- Chaotic attractors. Various scenarios were found, see [31], [32], [43].
- Strange attractors without chaos, see [27]. The natural presence of various forcing periods in real-life models make their occurrence quite plausible.
- Attracting tori. These limit sets are not difficult to find; they arise for instance as a consequence of a Neimark-Sacker bifurcation of a periodic solution, see [19].
- Attracting heteroclinic cycles, see [20].

A large number of these phenomena can be studied both by numerics *and* by perturbation theory; using the methods simultaneously gives additional insight.

5 Parametric excitation of linear systems

As we have seen in the introduction, parametric excitation leads to the study of second order equations with periodic coefficients. More in general such equations arise from linearization near T-periodic solutions of T-periodic equations of the form $\dot{y} = f(t, y)$. Suppose $y = \phi(t)$ is a T-periodic solution; putting $y = \phi(t) + x$ produces upon linearization the T-periodic equation

$$\dot{x} = f_x(t, \phi(t))x. \tag{11}$$

This equation often takes the form

$$\dot{x} = Ax + \varepsilon B(t)x, \tag{12}$$

in which $x \in \mathbb{R}^m$; A is a constant $m \times m$-matrix, $B(t)$ is a continuous, T-periodic $m \times m$-matrix, ε is a small parameter.
For elementary studies of such an equation, the Poincaré-Lindstedt method or continuation method is quite efficient. The method applies to nonlinear equations of arbitrary dimension, but we shall demonstrate its use for equations of Mathieu type.

5.1 Elementary theory

Floquet theory tells us that the solutions of eq. (12) can be written as:

$$x(t) = \Phi(t, \varepsilon)e^{C(\varepsilon)t} \quad , \tag{13}$$

with $\Phi(t, \varepsilon)$ a T-periodic $m \times m$-matrix, $C(\varepsilon)$ a constant $m \times m$-matrix and both matrices having an expansion in order functions of ε. The determination of $C(\varepsilon)$ provides us with the

stability behaviour of the solutions.
A particular case of eq. (12) is Hill's equation:

$$\ddot{x} + b(t,\varepsilon)x = 0 \quad , \tag{14}$$

which is of second order; $b(t,\varepsilon)$ is a scalar T-periodic function. A number of cases of Hill's equation are studied in [23]. A particular case of eq. (14) which arises frequently in applications is the Mathieu equation:

$$\ddot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0, \quad \omega > 0, \tag{15}$$

which is reversible. (In [23] one also finds Lamé's, Ince's, Hermite's, Whittaker-Hill and other Hill equations.) A typical question is: for which values of ω and ε in (ω^2, ε)-parameter space is the trivial solution $x = \dot{x} = 0$ stable?
Solutions of eq. (15) can be written in the Floquet form (13), where in this case $\Phi(t,\varepsilon)$ will be π-periodic. The eigenvalues λ_1, λ_2 of C, which are called characteristic exponents and are ε-dependent, determine the stability of the trivial solution. For the characteristic exponents of eq. (12) we have:

$$\sum_{i=1}^{n} \lambda_i = \frac{1}{T}\int_0^T Tr(A + \varepsilon B(t))dt \quad , \tag{16}$$

see theorem 6.6 in [45].
So in the case of eq. (15) we have:

$$\lambda_1 + \lambda_2 = 0. \tag{17}$$

The exponents are functions of ε, $\lambda_1 = \lambda_1(\varepsilon)$, $\lambda_2 = \lambda_2(\varepsilon)$ and clearly $\lambda_1(0) = i\omega$, $\lambda_2(0) = -i\omega$. As $\lambda_1(\varepsilon) = -\lambda_2(\varepsilon)$, the characteristic exponents, which are complex conjugate, are purely imaginary or real. The implication is that if $\omega^2 \neq n^2$, $n = 1, 2, \ldots$ the characteristic exponents are purely imaginary and $x = 0$ is stable near $\varepsilon = 0$. If $\omega^2 = n^2$ for some $n \in \mathbb{N}$, however, the imaginary part of $\exp(C(\varepsilon)t)$ can be absorbed into $\Phi(t,\varepsilon)$ and the characteristic exponents may be real. We assume now that $\omega^2 = n^2$ for some $n \in \mathbb{N}$, or near this value, and we shall look for periodic solutions of $x(t)$ of eq. (15) as these solutions define the boundaries between stable and unstable solutions. We put:

$$\omega^2 = n^2 - \varepsilon\beta, \tag{18}$$

with β a constant, and we apply the Poincaré-Lindstedt method to find the periodic solutions; see [45], appendix 2.
We find that periodic solutions exist for $n = 1$ if:

$$\omega^2 = 1 \pm \frac{1}{2}\varepsilon + \mathcal{O}(\varepsilon^2).$$

In the case $n = 2$, periodic solutions exist if:

$$\begin{aligned} \omega^2 &= 4 - \frac{1}{48}\varepsilon^2 + \mathcal{O}(\varepsilon^4) \quad , \\ \omega^2 &= 4 + \frac{5}{48}\varepsilon^2 + \mathcal{O}(\varepsilon^4) \quad . \end{aligned} \tag{19}$$

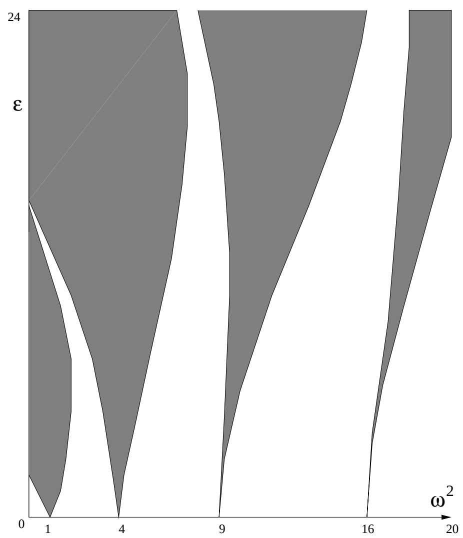

Figure 1: Floquet tongues of the Mathieu eq. (15); the instability domains are shaded.

The corresponding instability domains are called Floquet tongues, instability tongues or resonance tongues, see fig. 1.
On considering higher values of n, we have to calculate to a higher order of ε. At $n = 1$ the boundary curves are intersecting at positive angles at $\varepsilon = 0$, at $n = 2$ ($\omega^2 = 4$) they are tangent; the order of tangency increases as $n - 1$ (contact of order n), making instability domains more and more narrow with increasing resonance number n.

Higher order approximation and an unexpected timescale
The instability tongue of the Mathieu equation at $n = 1$ can be determined with more precision by Poincaré expansion. On using averaging, one also characterizes the flow outside the tongue boundary and this results in a surprise. Consider eq. (15) in the form

$$\ddot{x} + (1 + \varepsilon a + \varepsilon^2 b + \varepsilon \cos 2t)x = 0,$$

where we can choose $a = \pm\frac{1}{2}$ to put the frequency with first order precision at the tongue boundary. The eigenvalues of the trivial solution are from first order averaging

$$\lambda_{1,2} = \pm\frac{1}{2}\sqrt{\frac{1}{4} - a^2},$$

which agrees with Poincaré expansion; $a^2 > \frac{1}{4}$ gives stability, the $<$ inequality instability. The transition value $a^2 = \frac{1}{4}$ gives the tongue location. Take for instance the $+$ sign. Second order averaging, see [47], produces for the eigenvalues of the trivial solution

$$\lambda_{1,2} = \pm\sqrt{-\frac{1}{4}\left(\frac{1}{32} + b\right)\varepsilon^3 + \left(\frac{1}{64} + \frac{1}{2}b\right)\left(\frac{7}{64} - \frac{1}{2}b\right)\varepsilon^4}.$$

So, if $\frac{1}{32} + b > 0$ we have stability, if $\frac{1}{32} + b < 0$ instability; at $b = -\frac{1}{32}$ we have the second order approximation of this tongue boundary. Note that near this boundary the solutions are characterized by eigenvalues of $O(\varepsilon^{\frac{3}{2}})$ and accordingly the time-dependence by timescale $\varepsilon^{\frac{3}{2}}t$.

The Mathieu equation with viscous damping
In real-life applications there is always the presence of damping. We shall consider the effect of its simplest form, small viscous damping. Eq. (15) is extended by adding a linear damping term:

$$\ddot{x} + \kappa\dot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0, \quad a, \kappa > 0 \quad . \tag{20}$$

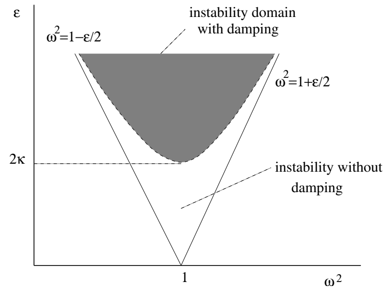

Figure 2: First order approximation of instability domains without and with damping for eq. (20) near $\omega^2 = 1$.

We assume that the damping coefficient is small, $\kappa = \varepsilon\kappa_0$, and again we put $\omega^2 = n^2 - \varepsilon\beta$ to apply the Poincaré-Lindstedt method.
We find periodic solutions in the case $n = 1$ if:

$$\omega^2 = 1 \pm \sqrt{\frac{1}{4}\varepsilon^2 - \kappa^2} \quad . \tag{21}$$

Relation (21) corresponds with the curve of periodic solutions, which in (ω^2, ε)-parameter space separates stable and unstable solutions. We observe the following phenomena. If $0 < \kappa < \frac{1}{2}\varepsilon$, we have an instability domain which by damping has been lifted from the ω^2-axis; also *the width has shrunk*. If $\kappa > \frac{1}{2}\varepsilon$ the instability domain has vanished. For an illustration see fig. 2.
Repeating the calculations for $n \geq 2$ we find no instability domains at all; damping of $O(\varepsilon)$ stabilizes the system for ε small. To find an instability domain we have to decrease the damping, for instance if $n = 2$ we have to take $\kappa = \varepsilon^2\kappa_0$.

Coexistence
Linear periodic equations of the form (12) have m independent solutions and it is possible that all the independent solutions are periodic. This is called 'coexistence' and one of the consequences is that the instability tongues vanish. An example is Ince's equation:

$$(1 + a\cos t)\ddot{x} + \kappa \sin t\dot{x} + (\omega^2 + \varepsilon \cos t)x = 0,$$

see [23]. An interesting question is whether this phenomenon persists under nonlinear perturbations; we return to this question in subsection 6.3.

5.2 More general classical results

The picture presented by the Mathieu equation resulting in resonance tongues in the ω, ε-parameter space, stability and instability intervals as parametrized by ω shown in fig. 1, has

been studied for more general types of Hill's equation. The older literature can be found in [40], see also [44].
Consider Hill's equation in the form

$$\ddot{x} + (\omega^2 + \varepsilon f(t))x = 0, \tag{22}$$

with $f(t)$ periodic and represented by a Fourier series. Along the ω^2-axis there exist instability intervals of size L_m, where m indicates the m^{th} instability interval. In the case of the Mathieu equation, we have from [16]

$$L_m = O(\varepsilon^m).$$

The resonance tongues become increasingly narrow.
For general periodic $f(t)$ we have weak estimates, like $L_m = O(\varepsilon)$, but if we assume that the Fourier series is finite, the estimates can be improved. Put

$$f(t) = \sum_{j=0}^{s} f_j \cos 2jt,$$

so $f(t)$ is even and π-periodic. From [22] we have the following estimates:

- If we can write $m = sp$ with $p \in \mathbb{N}$, we have

$$L_m = \frac{8s^2}{((p-1)!)^2}\left(\frac{|f_s \varepsilon|}{8s^2}\right)^p + O(\varepsilon^{p+1}).$$

- If we can not decompose m like this and $sp < m < s(p+1)$, we have

$$L_m = O(\varepsilon^{p+1}).$$

In the case of eq. (22) we have no dissipation and then it can be useful to introduce canonical transformations and Poincaré maps. In this case, for example, put

$$\dot{x} = y, \;\; \dot{y} = -\frac{\partial H}{\partial x},$$

with Hamiltonian function

$$H(x, y, t) = \frac{1}{2}y^2 + \frac{1}{2}(\omega^2 + \varepsilon f(t))x^2.$$

We can split $H = H_0 + \varepsilon H_1$ with $H_0 = \frac{1}{2}(y^2 + \omega^2 x^2)$ and apply canonical perturbation theory. Examples of this line of research can be found in [6] and [10]. Interesting conclusions can be drawn with respect to the geometry of the resonance tongues, crossings of tongues and as a possible consequence the presence of so-called instability pockets. In this context, the classical Mathieu equation turns out to be quite degenerate.
Hill's equation in the case of damping was considered in [36]; see also [37] where an arbitrary number of degrees of freedom is discussed.

5.3 Quasi-periodic excitation

Equations of the form

$$\ddot{x} + (\omega^2 + \varepsilon p(t))x = 0, \tag{23}$$

with parametric excitation $p(t)$ quasi-periodic or almost-periodic, arise often in applications. Floquet theory does not apply in this case but we can still use perturbation theory. A typical example would be two rationally independent frequencies:

$$p(t) = \cos t + \cos \gamma t,$$

with γ irrational. As an interesting example, in [7], $\gamma = \frac{1}{2}(1 + \sqrt{5})$ was chosen, the golden number. It will be no surprise that many more complications arise for large values of ε, but for ε small (the assumption in this article), the analysis runs along similar lines producing resonance tongues, crossings of tongues and instability pockets. See also extensions in [11]. Detailed perturbation expansions are presented in [50] with a comparison of Poincaré expansion, the harmonic balance method and numerics; there is good agreement between the methods. Real-life models contain dissipation which inspired the authors of [50] to consider the equation

$$\ddot{x} + 2\mu\dot{x} + (\omega^2 + \varepsilon(\cos t + \cos \gamma t))x = 0, \ \mu > 0,$$

γ irrational. They conclude that

- The instability tongues become thinner and recede into the ω-axis as μ increases.
- High-order resonance tongues seem to be more affected by dissipation than low-order ones producing a dramatic loss of 'fine detail', even for small μ.
- The results of varying the parameter μ certainly needs more investigation.

5.4 Parametrically forced oscillators in sum resonance

In applications where more than one degree of freedom plays a part, many more resonances are possible. For a number of interesting cases and additional literature see [37]. An important case is the so-called sum resonance. In [17] a geometrical explanation is presented for the phenomena in this case using 'all' the parameters as unfolding parameters. It will turn out that four parameters are needed to give a complete description. Fortunately three suffice to visualize the situation.

Consider the following type of differential equation with three frequencies

$$\dot{z} = Az + \varepsilon f(z, \omega_0 t; \lambda), \ z \in \mathbb{R}^4, \ \lambda \in \mathbb{R}^p, \tag{24}$$

which describes a system of two parametrically forced coupled oscillators. Here A is a 4×4 matrix, containing parameters, and with purely imaginary eigenvalues $\pm i\omega_1$ and $\pm i\omega_2$. The vector valued function f is 2π-periodic in $\omega_0 t$ and $f(0, \omega_0 t; \lambda) = 0$ for all t and λ. Eq. (24) can be resonant in many different ways. We consider the sum resonance

$$\omega_1 + \omega_2 = \omega_0,$$

where the system may exhibit instability. The parameter λ is used to control detuning $\delta = (\delta_1, \delta_2)$ of the frequencies (ω_1, ω_2) from resonance and damping $\mu = (\mu_1, \mu_2)$. We summarize the analysis from [17].

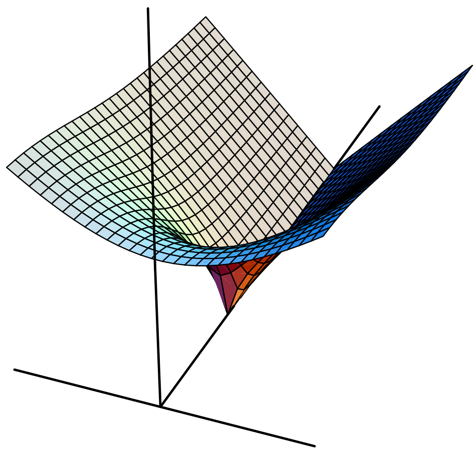

Figure 3: The critical surface in (μ_+, μ_-, δ_+) space for eq. (24). $\mu_+ = \mu_1 + \mu_2$, $\mu_- = \mu_1 - \mu_2$, $\delta_+ = \delta_1 + \delta_2$. Only the part $\mu_+ > 0$ and $\delta_+ > 0$ is shown. The parameters δ_1, δ_2 control the detuning of the frequencies, the parameters μ_1, μ_2 the damping of the oscillators (vertical direction). The base of the umbrella lies along the δ_+-axis.

- The first step is to put eq. (24) into normal form by normalization or averaging. In the normalized equation the time-dependence appears only in the higher order terms. But the autonomous part of this equation contains enough information to determine the stability regions of the origin. The linear part of the normal form is $\dot{z} = A(\delta, \mu)z$ with

$$A(\delta, \mu) = \begin{pmatrix} B(\delta, \mu) & 0 \\ 0 & \overline{B}(\delta, \mu) \end{pmatrix}, \tag{25}$$

  and

$$B(\delta, \mu) = \begin{pmatrix} i\delta_1 - \mu_1 & \alpha_1 \\ \overline{\alpha}_2 & -i\delta_2 - \mu_2 \end{pmatrix}. \tag{26}$$

  Since $A(\delta, \mu)$ is the complexification of a real matrix, it commutes with complex conjugation. Furthermore, according to the normal form theorem 4.1 and if ω_1 and ω_2 are independent over the integers, the normal form of eq. (24) has a continuous symmetry group.

- The second step is to test the linear part $A(\delta, \mu)$ of the normalized equation for structural stability i.e. to answer the question whether there exist open sets in parameter space where the dynamics is qualitatively the same. The family of matrices $A(\delta, \mu)$ is parametrized by the detuning δ and the damping μ. We first identify the most degenerate member N of this family and then show that $A(\delta, \mu)$ is its versal unfolding in the sense of [1]. The family $A(\delta, \mu)$ is equivalent to a versal unfolding $U(\lambda)$ of the degenerate member N.

- Put differently, the family $A(\delta, \mu)$ is structurally stable for $\delta, \mu > 0$, whereas $A(\delta, 0)$ is not. This has interesting consequences in applications as small damping and zero damping may exhibit very different behaviour, see section 7.2. In parameter space, the stability regions of the trivial solution are separated by a *critical surface* which is the hypersurface where $A(\delta, \mu)$ has at least one pair of purely imaginary complex conjugate eigenvalues. This critical surface is diffeomorphic to the *Whitney umbrella*, see fig. 3 and for references [17]. It is the singularity of the Whitney umbrella that causes the discontinuous behaviour of the stability diagram in section 7.2. The structural stability argument guarantees that the results are 'universally valid', i.e. they qualitatively hold for generic systems in sum resonance.

6 Nonlinear parametric excitation

Adding nonlinear effects to parametric excitation strongly complicates the dynamics. We start with adding nonlinear terms to the (generalized) Mathieu equation. Consider the following equation that includes dissipation:

$$\ddot{x} + \kappa\dot{x} + (\omega^2 + \varepsilon p(t))f(x) = 0, \tag{27}$$

where $\kappa > 0$ is the damping coefficient, $f(x) = x + bx^2 + cx^3 + \cdots$, and time is scaled so that:

$$p(t) = \sum_{l \in Z} a_{2l} e^{2ilt}, \quad a_0 = 0, \quad a_{-2l} = \bar{a}_{2l}, \tag{28}$$

is an even π-periodic function with zero average. As we have seen in section 5, the trivial solution $x = 0$ is unstable when $\kappa = 0$ and $\omega^2 = n^2$, for all $n \in \mathbb{N}$. Fix a specific $n \in \mathbb{N}$ and assume that ω^2 is close to n^2. We will study the bifurcations from the solution $x = 0$ in the case of primary resonance, which by definition occurs when the Fourier expansion of $p(t)$ contains nonzero terms $a_{2n}e^{2int}$ and $a_{-2n}e^{-2int}$. The bifurcation parameters in this problem are the detuning $\sigma = \omega^2 - n^2$, the damping coefficient κ and the Fourier coefficients of $p(t)$, in particular a_{2n}. The Fourier coefficients are assumed to be of equal order of magnitude.

6.1 The conservative case, $\kappa = 0$

An early paper is [21] in which eq. (27) for $\kappa = 0$ is associated with the Hamiltonian

$$H(x, \dot{x}, t) = \frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2 + p(t)\int_0^x f(s)ds.$$

After transformation of the Hamiltonian, Lie transforms are implemented by MACSYMA to produce normal form approximations to $O(\varepsilon^2)$. A number of examples show interesting bifurcations.

A related approach can be found in [5]; as $p(t)$ is even, the equation is time-reversible. After construction of the Poincaré (timeperiodic) map, normal forms are obtained by equivariant transformations. This leads to a classification of integrable normal forms that are approximations of the family of Poincaré maps, a family as the map is parametrized by ω and the coefficients of $p(t)$.
Interestingly, the nonlinearity αx^3 is combined with the quasi-periodic Mathieu equation in [51] where global phenomena are described like resonance bands and chaos.

6.2 Adding dissipation, $\kappa > 0$

Again time-periodic normal form calculations are used to approximate the dynamics; see [31], also [32] and the monograph [43]. The reflection symmetry in the normal form equations implies that all fixed points come in pairs, and that bifurcations of the origin will be symmetric (such as pitchfork bifurcations). We observe that the normal form equations show additional symmetrices if either $f(x)$ in eq. (27) is odd in x or if n is odd. The general normal form can be seen as a non-symmetric perturbation of the symmetric case. One finds pitchfork and saddle-node bifurcations, in fact all codimension one bifurcations; for details and pictures see [43], ch. 9.

6.3 Coexistence under nonlinear perturbation

A model describing free vibrations of an elastica is described in [26]:

$$(1 - \frac{\varepsilon}{2}\cos 2t)\ddot{x} + \varepsilon \sin 2t\dot{x} + cx + \varepsilon\alpha x^2 = 0.$$

For $\alpha = 0$, the equation shows the phenomenon of coexistence. It is shown by second order averaging in [26] that for $\alpha \neq 0$ there exist open sets of parameter values for which the trivial solution is unstable.
An application to the stability problem of a family of periodic solutions in a Hamiltonian system is given in [29].

6.4 Other nonlinearities

In applications various nonlinear terms play a part. In [25] one considers

$$\ddot{x} + (\omega^2 + \varepsilon\cos(t)) + \varepsilon(Ax^3 + Bx^2\dot{x} + Cx\dot{x}^2 + D\dot{x}^3) = 0,$$

where averaging is applied near the 2 : 1-resonance. If $B, D < 0$ the corresponding terms can be interpreted as progressive damping. It turns out that for a correct description of the bifurcations second-order averaging is needed.
Nonlinear damping can be of practical interest. The equation

$$\ddot{x} + (\omega^2 + \varepsilon\cos(t)) + mu|\dot{x}|\dot{x} = 0,$$

is studied with μ also a small parameter. A special feature is that an acceptable description of the phenomena can be obtained in a semi-analytical way by using Mathieu-functions as starting point. The analysis involves the use of Padé-approximants, see [28].

7 Applications

There are many applications of parametric resonance, in particular in engineering. In this section we consider a number of significant applications, but of course without any attempt at completeness. See also [37] and the references in the additional literature.

7.1 The parametrically excited pendulum

Choosing the pendulum case $f(x) = \sin(x)$ in eq. (27) we have

$$\ddot{x} + \kappa\dot{x} + (\omega^2 + \varepsilon p(t))\sin(x) = 0.$$

It is natural,because of the sin periodicity, to analyze the Poincaré map on the cylindrical section $t = 0 \bmod 2\pi\mathbb{Z}$. This map has both a spatial and a temporal symmetry. As we know from the preceding section, perturbation theory applied near the equilibria $x = 0, x = \pi$, produces integrable normal forms. For larger excitation (larger values of ε), the system exhibits the usual picture of Hamiltonian chaos; for details see [24], [12].
The inverted case is intriguing. It is wellknown that the upper equilibrium of an undamped pendulum can be stabilized by vertical oscillations of the suspension point with certain frequencies. See for references [8], [9] and [38]. In [8] the genericity of the classical result is studied for (conservative) perturbations respecting the symmetries of the equation. In [9] genericity is studied for (conservative) perturbations where the spatial symmetry is broken, replacing $\sin x$ by more general 2π-periodic functions. Stabilization is still possible but the dynamics is more complicated.

7.2 Rotor dynamics

When adding linear damping to a system there can be a striking discontinuity in the bifurcational behaviour. Phenomena like this have already been observed and described in for instance [49] or [41]. The discontinuity is a fundamental structural instability in linear gyroscopic systems with at least two degrees of freedom and with linear damping. The following example is based on [42] and [33].
Consider a rigid rotor consisting of a heavy disk of mass M which is rotating with rotation-speed Ω around an axis. The axis of rotation is elastically mounted on a foundation; the connections which are holding the rotor in an upright position are also elastic. To describe the position of the rotor we have the axial displacement u in the vertical direction (positive upwards), the angle of the axis of rotation *with respect to* the z-axis and *around* the z-axis. Instead of these two angles we will use the projection of the centre of gravity motion on the horizontal (x, y)-plane, see fig. 4. Assuming small oscillations in the upright (u) position, frequency 2η, the equations of motion become:

$$\begin{aligned} \ddot{x} + 2\alpha\dot{y} + (1 + 4\varepsilon\eta^2\cos 2\eta t)x &= 0, \\ \ddot{y} - 2\alpha\dot{x} + (1 + 4\varepsilon\eta^2\cos 2\eta t)y &= 0. \end{aligned} \tag{29}$$

System (29) constitutes a system of Mathieu-like equations, where we have neglected the effects of damping. Abbreviating $P(t) = 4\eta^2\cos 2\eta t$, the corresponding Hamiltonian is:

$$H = \frac{1}{2}(1 + \alpha^2 + \varepsilon P(t))x^2 + \frac{1}{2}p_x^2 + \frac{1}{2}(1 + \alpha^2 + \varepsilon P(t))y^2 + \frac{1}{2}p_y^2 + \alpha x p_y - \alpha y p_x,$$

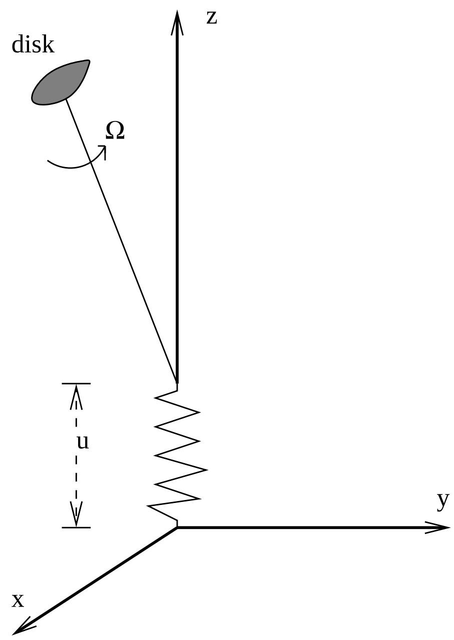

Figure 4: Rotor with diskmass M, elastically mounted with axial (u) and lateral directions.

where p_x, p_y are the momenta. The natural frequencies of the unperturbed system (29), $\varepsilon = 0$, are $\omega_1 = \sqrt{\alpha^2+1} + \alpha$ and $\omega_2 = \sqrt{\alpha^2+1} - \alpha$. By putting $z = x + iy$, system (29) can be written as:

$$\ddot{z} - 2\alpha i\dot{z} + (1 + 4\varepsilon\eta^2 \cos 2\eta t)z = 0 \quad . \tag{30}$$

Introducing the new variable:

$$v = e^{-i\alpha t} z \quad , \tag{31}$$

and putting $\eta t = \tau$, we obtain:

$$v'' + \left(\frac{1+\alpha^2}{\eta^2} + 4\varepsilon \cos 2\tau\right) v = 0, \tag{32}$$

where the prime denotes differentiation with respect to τ. By writing down the real and imaginary parts of this equation, we get two identical Mathieu equations.
We conclude that the trivial solution is stable for ε small enough, providing that $\sqrt{1+\alpha^2}$ is not close to $n\eta$, for some $n = 1, 2, 3, ...$ The first-order interval of instability, $n = 1$, arises if:

$$\sqrt{1+\alpha^2} \approx \eta. \tag{33}$$

If condition (33) is satisfied, the trivial solution of equation (32) is unstable. Therefore, the trivial solution of system (29) is also unstable. Note that this instability arises when:

$$\omega_1 + \omega_2 = 2\eta,$$

i.e. when the sum of the eigenfrequencies of the unperturbed system equals the excitation frequency 2η. This is known as a sum resonance of first order. The domain of instability can be calculated as in section 5.1; we find for the boundaries:

$$\eta_b = \sqrt{1+\alpha^2}\ (1 \pm \varepsilon) + O(\varepsilon^2) \quad . \tag{34}$$

The second order interval of instability of eq. (32), $n = 2$, arises when:

$$\sqrt{1+\alpha^2} \approx 2\eta \quad , \tag{35}$$

i.e. $\omega_1 + \omega_2 \approx \eta$. This is known as a sum resonance of second order. As above, we find the boundaries of the domains of instability:

$$\begin{aligned} 2\eta &= \sqrt{1+\alpha^2} \quad \left(1 + \tfrac{1}{24}\varepsilon^2\right) + O(\varepsilon^4) \quad , \\ 2\eta &= \sqrt{1+\alpha^2} \quad \left(1 - \tfrac{5}{24}\varepsilon^2\right) + O(\varepsilon^4) \quad . \end{aligned} \tag{36}$$

Higher order combination resonances can be studied in the same way; the domains of instability in parameter space continue to narrow as n increases. It should be noted that the parameter α is proportional to the rotation speed Ω of the disk and to the ratio of the moments of inertia.

7.2.1 Instability by damping

We add small linear damping to system (29), with positive damping parameter $\mu = 2\varepsilon\kappa$. This leads to the equation:

$$\ddot{z} - 2\alpha i\dot{z} + \left(1 + 4\varepsilon\eta^2 \cos 2\eta t\right) z + 2\varepsilon\kappa\dot{z} = 0. \tag{37}$$

Because of the damping term, we can no longer reduce the complex eq. (37) to two identical second order real equations, as we did in the previous section.
In the sum resonance of the first order, we have $\omega_1 + \omega_2 \approx 2\eta$ and the solution of the unperturbed ($\varepsilon = 0$) equation can be written as:

$$z(t) = z_1 e^{i\omega_1 t} + z_2 e^{-i\omega_2 t}, \qquad z_1, z_2 \in \mathbb{C}, \tag{38}$$

with $\omega_1 = \sqrt{\alpha^2+1} + \alpha, \quad \omega_2 = \sqrt{\alpha^2+1} - \alpha$.
Applying variation of constants leads to equations for z_1 and z_2:

$$\begin{aligned} \dot{z}_1 &= \frac{i\varepsilon}{\omega_1+\omega_2}(2\kappa(i\omega_1 z_1 - i\omega_2 z_2 e^{-i(\omega_1+\omega_2)t}) + \\ &\qquad 4\eta^2 \cos 2\eta t(z_1 + z_2 e^{-i(\omega_1+\omega_2)t})), \\ \dot{z}_2 &= \frac{-i\varepsilon}{\omega_1+\omega_2}(2\kappa(i\omega_1 z_1 e^{i(\omega_1+\omega_2)t} - i\omega_2 z_2) + \\ &\qquad 4\eta^2 \cos 2\eta t(z_1 e^{i(\omega_1+\omega_2)t} + z_2)). \end{aligned} \tag{39}$$

To calculate the instability interval around the value $\eta_0 = \frac{1}{2}(\omega_1 + \omega_2) = \sqrt{\alpha^2+1}$, we apply perturbation theory to find for the stability boundary:

$$\begin{aligned} \eta_b &= \sqrt{1+\alpha^2}\left(1 \pm \varepsilon\sqrt{1+\alpha^2 - \frac{\kappa^2}{\eta_0^2}} + \ldots.\right) \quad , \\ &= \sqrt{1+\alpha^2}\left(1 \pm \sqrt{(1+\alpha^2)\varepsilon^2 - \left(\frac{\mu}{\eta_0}\right)^2} + \ldots.\right) \quad . \end{aligned} \tag{40}$$

It follows that the domain of instability actually becomes *larger* when damping is introduced. The most unusual aspect of the above expression for the instability interval, however, is that there is a discontinuity at $\kappa = 0$. If $\kappa \to 0$, then the boundaries of the instability domain tend to the limits $\eta_b \to \sqrt{1+\alpha^2}(1 \pm \varepsilon\sqrt{1+\alpha^2})$ which differs from the result we found when $\kappa = 0$: $\eta_b = \sqrt{1+\alpha^2}(1 \pm \varepsilon)$.
In mechanical terms, the broadening of the instability-domain is caused by the coupling between the two degrees of freedom of the rotor in lateral directions which arises in the presence of damping. Such phenomena are typical for gyroscopic systems and have been noted earlier in the literature; see [2], [4] and [37]. The explanation of the discontinuity and its genericity in [17], see subsection 5.4, is new.
For hysteresis and phase-locking phenomena in this problem, the reader is referred to [33]

7.3 Autoparametric excitation

In [43], autoparametric systems are characterized as vibrating systems which consist of at least two constituing subsystems that are coupled. One is a Primary System that can be in normal mode vibration. In the instability (parameter) intervals of the normal mode solution in the full, coupled system, we have *autoparametric resonance*. The vibrations of the Primary System act as parametric excitation of the Secondary System which will no longer remain at rest. An example is presented in fig. 5.

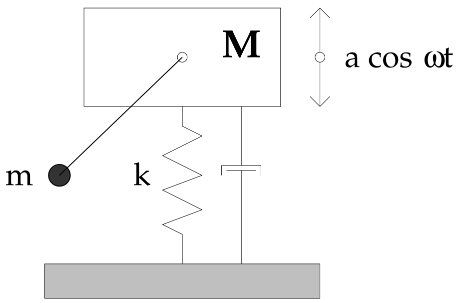

Figure 5: Two coupled oscillators with vertical oscillations as Primary System and parametric excitation of the coupled pendulum (Secondary System).

In actual engineering problems, we wish sometimes to diminish the vibration amplitudes of the Primary System; sometimes this is called 'quenching of vibrations'. In other cases we have a coupled Secondary System which we would like to keep at rest.
As an example we consider the following autoparametric system studied in [14]:

$$\begin{aligned} x'' + x + \varepsilon(k_1 x' + \sigma_1 x + a\cos 2\tau x + \frac{4}{3}x^3 + c_1 y^2 x) &= 0 \\ y'' + y + \varepsilon(k_2 y' + \sigma_2 y + c_2 x^2 y + \frac{4}{3}y^3) &= 0 \end{aligned} \tag{41}$$

where σ_1 and σ_2 are the detunings from the 1 : 1-resonance of the oscillators. In this system, $y(t) = \dot{y}(t) = 0$ corresponds with a normal mode of the x-oscillator.
The system (41) is invariant under $(x, y) \to (x, -y)$, $(x, y) \to (-x, y)$, and $(x, y) \to (-x, -y)$. Using the method of averaging as a normalization procedure we investigate the stability of

solutions of system (41). To give an explicit example we follow [14] in more detail. Introduce the usual variation of constants transformation:

$$x = u_1 \cos\tau + v_1 \sin\tau \quad ; \quad x' = -u_1 \sin\tau + v_1 \cos\tau \tag{42}$$

$$y = u_2 \cos\tau + v_2 \sin\tau \quad ; \quad y' = -u_2 \sin\tau + v_2 \cos\tau \tag{43}$$

After rescaling $\tau = \frac{\varepsilon}{2}\tilde{\tau}$ the averaged system of (41) becomes:

$$\begin{aligned}
u_1' &= -k_1 u_1 + (\sigma_1 - \frac{1}{2}a)v_1 + v_1(u_1^2 + v_1^2) + \frac{1}{4}c_1 u_2^2 v_1 + \frac{3}{4}c_1 v_2^2 v_1 + \frac{1}{2}c_1 u_2 v_2 u_1 \\
v_1' &= -k_1 v_1 - (\sigma_1 + \frac{1}{2}a)u_1 - u_1(u_1^2 + v_1^2) - \frac{3}{4}c_1 u_2^2 u_1 - \frac{1}{4}c_1 v_2^2 u_1 - \frac{1}{2}c_1 u_2 v_2 v_1 \\
u_2' &= -k_2 u_2 + \sigma_2 v_2 + v_2(u_2^2 + v_2^2) + \frac{1}{4}c_2 u_1^2 v_2 + \frac{3}{4}c_2 v_1^2 v_2 + \frac{1}{2}c_2 u_1 v_1 u_2 \\
v_2' &= -k_2 v_2 - \sigma_2 u_2 - u_2(u_2^2 + v_2^2) - \frac{3}{4}c_2 u_1^2 u_2 - \frac{1}{4}c_2 v_1^2 u_2 - \frac{1}{2}c_2 u_1 v_1 v_2.
\end{aligned} \tag{44}$$

This system is analyzed for critical points, periodic and quasi-periodic solutions, producing existence and stability diagrams in parameter space. The system also contains a sequence of period-doubling bifurcations leading to chaotic solutions, see fig. 6.

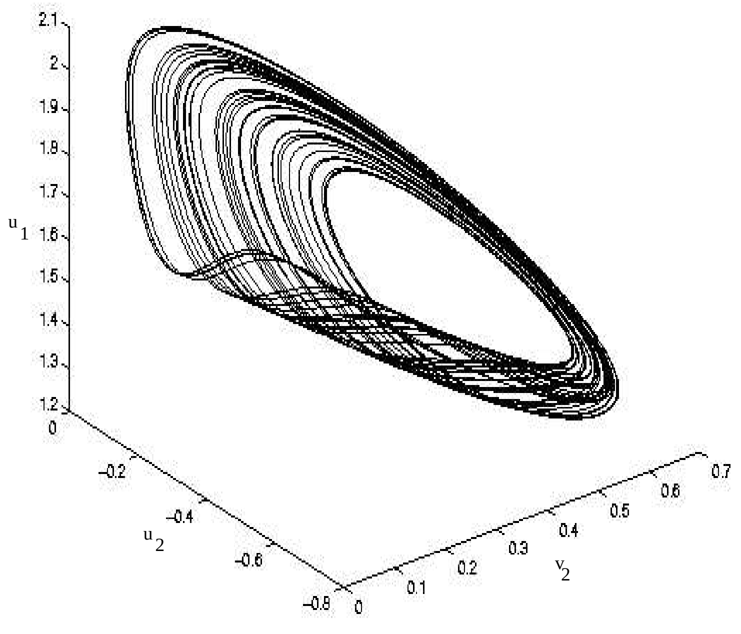

Figure 6: The strange attractor of the averaged system (44). The phase-portraits in the (u_2, v_2, u_1)-space for $c_2 < 0$ at the value $\sigma_2 = 5.3$. The Kaplan-Yorke dimension for $\sigma_2 = 5.3$ is 2.29.

To prove the presence of chaos involves an application of higher dimensional Melnikov theory developed in [48]. A rather technical analysis in [14] shows the existence of a Šilnikov orbit in the averaged equation, which implies chaotic dynamics, also for the original system.

8 Future directions

Ongoing research in dynamical systems includes nonlinear systems with parametric resonance, but there are a number of special features as these systems are non-autonomous. This complicates the dynamics from the outset. For instance a two degrees of freedom system with parametric resonance involves at least three frequencies, producing many possible resonances. The analysis of such higher dimensional systems with many more combination resonances, has begun recently, producing interesting limit sets and invariant manifolds. Also the analysis of PDEs with periodic coefficients will play a part in the near future. These lines of research are of great interest.
In the conservative case, the association with Hamiltonian systems, KAM theory etc. gives a natural approach. This has already produced important results. In real-life modeling, there will always be dissipation and it is important to include this effect. Preliminary results suggest that the impact of damping on for instance quasi-periodic systems, is quite dramatic. This certainly merits more research.
Finally, applications are needed to solve actual problems *and* to inspire new, theoretical research.

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A number of improvements and clarifications were suggested by the editor, Giuseppe Gaeta. Additional references were obtained from Henk Broer and Fadi Dohnal.

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