TELL ME A PSEUDO-ANOSOV†

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Anosov linear homeomorphisms, and more generally Anosov flows as well as their hyperbolic analogues, have played an important role in the theory of dynamical systems [Ano67, Sma67, Arn80] [1].

Their cousins, the *pseudo-Anosov* homeomorphisms, although interesting and important as well, seem to be less well known. In opposite to the theory of Anosov flows, for which we know their contours rather well, there are several fundamental questions about pseudo-Anosov homeomorphisms that remains widely open so far.

1. An instructive example

Let us start with a naive example that is, in some sense, more than an example. Any matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ acts linearly on the plane $\mathbb{R}^2$. The induced dynamics is not very interesting (the orbits are either circles or escape to infinity). A way of getting it richer is to "pass to the quotient": since A bijectively preserves the $\mathbb{Z}^2$ lattice, that is $A(\mathbb{Z}^2) = \mathbb{Z}^2$, it induces a diffeomorphism ψ of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by $\psi((x,y)+\mathbb{Z}^2) = A(x,y)+\mathbb{Z}^2$.

The dynamics of ψ is governed by the eigenvalues λ, λ^{-1} of A. There are three possibilities:

(1) λ and λ^{-1} are complex conjugate ($\lambda \neq \pm 1$): ψ is of finite order.
(2) $\lambda = \lambda^{-1} = \pm 1$: ψ is reducible, that is: it preserves a closed curve on the torus.
(3) λ and λ^{-1} are distinct irrationals numbers: ψ is of Anosov type.

The second case (parabolic) implies that ψ arises from a map on a simpler surface (in this case an annulus).

The last case (hyperbolic) is by far the one having a richer dynamics (ψ has many periodic points, many points of dense orbits, etc.) The cat map $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, for which $\lambda = (3+\sqrt{5})/2$, is a nice illustration of this situation (see [Ghy94]). These maps, although very simple, capture many properties of elements in an open subset of the set of diffeomorphisms of the torus T^2:

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[1] One can consult the article by A. Bufetov and A. Klimenko in the *Gazette des Mathématiciens* (No. **143**, January 2015)

this is the famous Anosov [Ano67] result on structural stability. It states that any diffeomorphism ϕ sufficiently close to an hyperbolic diffeomorphism ψ in the C^1 topology is topologically conjugated to ψ: there exists a homeomorphism $h \in \mathrm{Homeo}(T^2)$ such that $\phi = h \circ \psi \circ h^{-1}$. Hence ϕ and ψ are the same up to a change of coordinates.

Thus, these Anosov diffeomorphisms provide important informations on large open subsets of the group $\mathrm{Diff}^+(T^2)$. Their hyperbolic counterparts have since then occupied mathematicians: they are the main actors of $\mathrm{Diff}^+(S_g)$, the group of diffeomorphisms of a genus g surface S_g.

These Anosov diffeomorphisms are so important that they are also the actors of another family of groups: the modular groups. In the 1970s, Thurston [Thu88] generalized to the case of compact surfaces the analysis done on a torus, thus extending the notion of Anosov maps to that of pseudo-Anosov ones.

2. Foliations and pseudo-Anosov homeomorphisms

2.1. **Measured foliation.** An important feature of a linear Anosov of the torus is that it leaves invariant the two foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ of "straight lines" of constant slopes (parallel to the directions of the eigenvectors associated to λ and λ^{-1}). These foliations also come with an additional structure: they are integrable in the sense that we can define them globally as the kernel of a closed 1-form $d\nu$.

Hence, we have a measure μ_s defined on arcs α transverse to the leaves of $\mathcal{F}^s$, measuring the total variation of α in the orthogonal direction: $\mu_s(\alpha) = \int_\alpha d\nu_s$.

The measure is invariant in the sense that if we change the extremities of α in the same leaf, the measure remains unchanged. The data $(\mathcal{F}^s, \mu_s)$ is a *measured foliation*. Of course our Anosov preserves these leaves and expands/contracts the measures: we can think that ψ expand by a factor λ in the direction of $\mathcal{F}^u$ and contracts by the same factor in the direction of $\mathcal{F}^s$.

On a surface of higher genus the notion of measured foliations also exists but the Gauß–Bonnet formula forces us to extend them to singular foliations. For pairs of transverse measured foliations there is a very elegant way of doing this with the help of half-translation structures.

If $\Sigma \subset S_g$ is a finite set, a half-translation structure on (S_g, Σ) is an atlas of charts $\omega = (U_\alpha, z_\alpha)$ of $S \backslash \Sigma$ for which the changes of charts are of the form $z \mapsto \pm z + \mathrm{const}$ and such that each point of Σ has a neighbourhood isometric to a finite cover of $\mathbb{R}^2 \backslash \{0\}$. The pullback of the horizontal and vertical leaves of $\mathbb{R}^2$ thus defines a pair of transverse measured foliations on S_g (the measure are dy and dx, respectively).

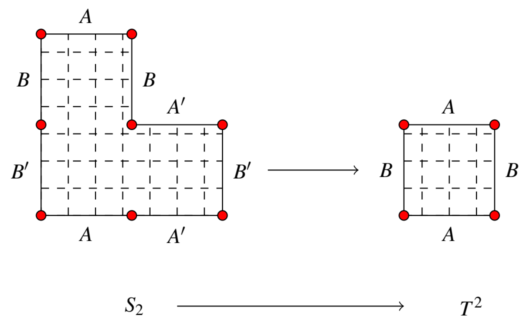

FIGURE 1. Triple cover of the standard torus: surface with three tiles.

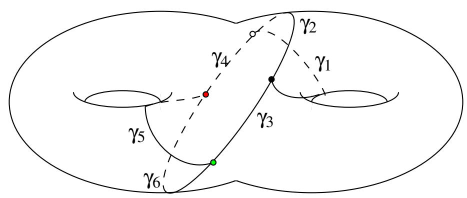

FIGURE 2. Measured foliation on a surface of genus two with four singularities (according to Hubbard–Masur).

Example 2.1. *Figure 1 represents, on the left, a half-translation structure on the surface S_2: we glue together the sides having the same label. We can verify that the vertices of the L shaped polygon represent a single point in S_2, which is singular. It has two obvious measured foliations (horizontal and vertical) with transverse measures dy and dx, respectively.*

Warning! There are measured foliations that are *not* arising from this construction (and so not admitting a transverse measured foliation). In the following example (after Hubbard-Masur), we glue two cylinders, foliated by cercles, according to Figure 2: the boundaries of the first cylinder are the arcs γ_1, γ_2 and $\gamma_1, \gamma_3, \gamma_4, \gamma_6$ and those of the second cylinder are γ_5, γ_6 and $\gamma_2, \gamma_3, \gamma_4, \gamma_5$. The transverse measure is given by the "height function". We can observe that a transverse foliation does not exist otherwise the cylinders would have boundaries with equal lengths, but this does not occur since the linear system

$$\begin{cases} |\gamma_1| + |\gamma_2| & = & |\gamma_1| + |\gamma_3| + |\gamma_4| + |\gamma_6| \\ |\gamma_5| + |\gamma_6| & = & |\gamma_2| + |\gamma_3| + |\gamma_4| + |\gamma_5| \end{cases}$$

does not admit any strictly positive solutions.

2.2. **Pseudo-Anosov homeomorphisms.** A homeomorphism $\psi : S \to S$ is a pseudo-Anosov homeomorphism if there exist a pair of measured transverse foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ on S_g, called unstable and stable, respectively, and a number $\lambda > 1$ (the expansion factor of ψ) such that

$$\psi \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda \cdot \mu_u), \text{ and}$$
$$\psi \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1} \cdot \mu_s).$$

An equivalent way to formulate this is to say that ψ is an affine diffeomorphism on $S \backslash \Sigma$ for the euclidian metric defined above and that its differential $D\psi = \left(\begin{smallmatrix} \pm\lambda & 0 \\ 0 & \pm\lambda^{-1} \end{smallmatrix}\right)$ is hyperbolic, that is $|\mathrm{tr}(D\psi)| > 2$ (in general ψ is not differentiable at the points of Σ). The group formed by all differentials $D\psi$ with ψ affine for the atlas ω is called the *Veech group* $\mathrm{SL}(S, \omega) \subset \mathrm{PSL}(2, \mathbb{R})$.

Although rather natural, it is not an easy task to construct examples satisfying this definition (at least in genus different from 1). A way of achieving it is to lift linear Anosov maps on the torus to coverings.

Example 2.2. *The linear Anosov on the torus* $\psi : T^2 \to T^2$*, with differential* $A = \left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$*, lifts (see Example 2.1) to a pseudo-Anosov* $\widetilde{\psi} : S_2 \to S_2$ *such that* $D\widetilde{\psi} = A$*, as we will explain in Section 4.*

3. Modular group

The pseudo-Anosov homeomorphisms are the elementary bricks for the study of modular groups of surfaces. The group in question is always the group $\mathrm{Diff}^+(S_g)$, but this time up to continuous deformation (we shall say up to isotopy). More precisely, the modular group is the quotient group $\mathrm{Diff}^+(S_g)$ by the group $\mathrm{Diff}(S_g)_0$ of diffeomorphisms isotopic to the identity: $\mathrm{Mod}(S_g) = \mathrm{Diff}^+(S_g)/\mathrm{Diff}(S_g)_0$.

Sometimes, definitions differ from one source to another: group of diffeomorphisms, group of homeomorphisms. It does not matter: the quotient groups are all isomorphic (even if the groups $\mathrm{Diff}^+(S_g)$ and $\mathrm{Homeo}^+(S_g)$ are very different!).

3.1. **Nielsen–Thurston classification.** We are now able to state the classification theorem of surface homeomorphisms, which is very close to the one on the torus. Any $f \in \mathrm{Homeo}^+(S_g)$ is, up to isotopy, either:

(1) periodic: there exists m such that $f^m = \mathrm{Id}$.
(2) reducible: f preserves a family of simple closed curves.
(3) a pseudo-Anosov map.

In the second case some iterate of f preserves a subsurface (with boundaries). As we can again apply the theorem to this subsurface, the third case is by far the most interesting one!

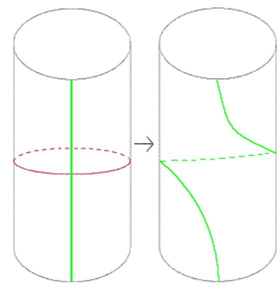

FIGURE 3. Dehn twist along a curve.

3.2. **Classical modular groups.** The modular group of the closed disk is rather simple to describe (here our surface has a boundary: we require the homeomorphism to be the identity map on the boundary).

Such a map ϕ defined on $\overline{D(0,1)}$ can easily be deformed by an isotopy acting like ϕ on the small disk of radius $t < 1$ and being the identity outside. In coordinates this is

$$F(z,t) = \begin{cases} t\phi(z/t), \text{ if } z \in D(0,t) \text{ and } t \neq 0 \\ z, \text{ otherwise.} \end{cases}$$

We have $F(\cdot,0) = \mathrm{Id}$ and $F(\cdot,1) = \psi$. With this idea we easily prove that the modular groups of the disk and of the sphere are trivial.

Although somewhat simplistic, this approach is fundamental: Magnus remarked in 1934 that the action of the isotopies on the punctures allows to connect two *a priori* distinct groups: the modular group on the disk with n punctures and the braid group on n strands.

The first non trivial example of modular group is the one of the flat cylinder $\mathcal{C}$. If γ is an oriented simple closed curve linking the two components of the boundary of $\mathcal{C}$, then the homeomorphism T_γ that twist the cylinder along γ is nontrivial in $\mathrm{Mod}(\mathcal{C}) = \langle T_\gamma \rangle \simeq \mathbb{Z}$. The homeomorphism T_γ has a very simple expression in the parametrization $\mathcal{C} = \mathbb{R}/w\mathbb{Z} \times [0;h]$:

$$T_\gamma(x,y) = (x + w/h \cdot y, y) = (x + \mu^{-1} y, y)$$

where $\mu = h/w$ is the modulus of the cylinder $\mathcal{C}$. It is actually a diffeomorphism and $DT_\gamma = \begin{pmatrix} 1 & \mu^{-1} \\ 0 & 1 \end{pmatrix}$.

Furthermore, since any surface S_g contains an annulus $\mathcal{C}$, we can define by analogy $T_\gamma \in \mathrm{Mod}(S_g)$ along a simple closed curve γ (since T_γ is the identity on the boundary of the annulus). These elements take an important place in the study of the modular group: we call them *Dehn twists*.

3.3. **Modular group of the torus.** Writing $T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ we can define two Dehn twists along the two curves $\alpha = (1,0)$ and $\beta = (0,1)$: this provides a "large" subgroup of $\mathrm{Mod}(T^2)$: $\langle T_\alpha, T_\gamma \rangle = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \rangle = \mathrm{SL}(2,\mathbb{Z})$ (we identify here a Dehn twist with its differential).

In fact, by letting a homeomorphism of T^2 acting on the homology $H_1(T^2,\mathbb{Z}) = \langle \alpha, \beta \rangle \simeq \mathbb{Z}^2$ we obtain an isomorphism

$$\mathrm{Mod}(T^2) \simeq \mathrm{SL}(2,\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^2)$$

that provide us with a rather precise description of the modular group of genus one surfaces.

3.4. **Modular group of a surface.** Like we understand $\mathrm{Mod}(T^2)$ with the help of action on curves, we can study $\mathrm{Mod}(S_g)$ through the action of $\mathrm{Diff}^+(S_g)$ on simple closed curves of S_g. This time this is more complicated than it seems because such a curve can be extremely complicated.

By letting the homeomorphisms acting on the homology $H_1(S_g,\mathbb{Z})$, we obtain a first "linear" approach of the modular group (choosing a symplectic basis for the intersection form):

$$\mathrm{Mod}(S_g) \to \mathrm{Sp}(2g,\mathbb{Z}).$$

This homeomorphism is onto (in fact, every element of $\mathrm{Sp}(2g,\mathbb{Z})$ can be realized by a pseudo-Anosov map, even if we not always know how to characterize those which fix an orientable measured foliation). On the other hand, if $g \geq 2$ its kernel (the Torelli group) is rather large.

We end this section with a result analogous to the well known fact that $\mathrm{SL}(n,\mathbb{Z})$ is generated by transvection matrices:

The group $\mathrm{Mod}(S_g)$ is generated by a finite number of Dehn twists (Dehn, 1922).

The (optimal) number of generators is $2g+1$ (Humphries, 1977).

4. SEVERAL CONSTRUCTIONS

It is not an easy task to construct pseudo-Anosov homeomorphisms.

Let us give a simple and fruitful idea. An affine Dehn twist T_γ possesses a parabolic differential, $|\mathrm{tr}(DT_\gamma)| = 2$. By applying the motto

"a product of parabolic elements is 'generally' an hyperbolic element",

it is possible to show, for well chosen curves γ and η, that $|\mathrm{tr}(DT_\gamma T_\eta)| > 2$, that is $T_\gamma \circ T_\eta$ is pseudo-Anosov. This is the Thurston–Veech construction, popularized on the occasion of a talk by John Hubbard at C.I.R.M. in Marseille in 2003. Since then this construction is sometimes called the *bouillabaisse* construction.

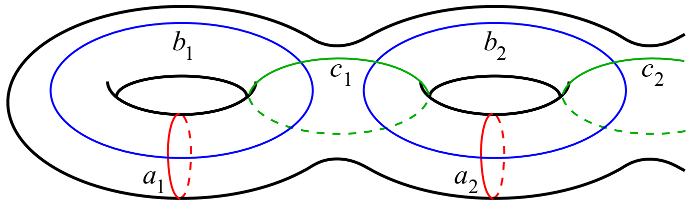

FIGURE 4. *Bouillabaisse* construction.

Example 4.1. *In Example 2.1 the left surface S_2 is horizontally cut along two cylinders of height* 1 *having cores α_1, α_2, with lengths* 1 *and* 2. *Thus $DT_{\alpha_1} = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $DT_{\alpha_2} = \left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$. Since each Dehn twist T_{α_i} is equal to the identity on the boundaries of the cylinders, the "multi-twist" $T_h = T^2_{\alpha_1} \circ T_{\alpha_2}$ is a diffeomorphism on $S_2 \backslash \Sigma$ whose differential is* constant *and equal to $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$. By symmetry reasons, the vertical multi-twist $T_v = T_{\beta_1} \circ T^2_{\beta_2}$ is also affine and has a differential equals to $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$.*

We then check that $D(T_h \circ T_v) = \left(\begin{smallmatrix}5&2\\2&1\end{smallmatrix}\right)$. This is our Example 2.2 which is pseudo-Anosov!

Example 4.2 (A more subtle example)**.** *Let us consider on a genus* 2 *surface the multi-curves $\alpha = \{2a_1, a_2, c_1\}$ and $\beta = \{b_1, b_2\}$ (represented in Figure 4). The product of the two multi-twists $T_\alpha \circ T_\beta$ where*

$$T_\alpha = T^2_{a_1} \circ T_{a_2} \circ T_{c_1} \qquad \textit{and} \qquad T_\beta = T_{b_1} \circ T_{b_2}$$

is an element ψ of pseudo-Anosov type. Its expansion factor $\lambda(\psi)$ is the largest real root ($\simeq 1.72$) of the polynomial $X^4 - X^3 - X^2 - X + 1$.

This idea produces a lot of pseudo-Anosov diffeomorphisms. A beautiful theorem of A. Fathi gives a quantitative version of this motto. Let us consider a family of distinct curves (up to isotopy) $\{\gamma_1, \dots, \gamma_n\}$ filling S ($S \backslash \cup_i \gamma_i$ is a union of disks). Then

$$\exists N \in \mathbb{N}, \forall (n_1, \dots, n_k) \in \mathbb{Z}^k, \text{ if } |n_i| \geq N, \forall i \text{ then}$$
$$T_{\gamma_1} \circ \cdots \circ T_{\gamma_k} \text{ is isotopic to a pseudo-Anosov map.}$$

A surprising corollary is that if ψ is a pseudo-Anosov and γ is a simple closed curve then $T^n_\gamma \circ \psi$ is isotopic to a pseudo-Anosov, for any non negative integer n, with the possible exception of at most 7 consecutive values of n!

There are other constructions, that we do not have time to explain, which are algorithmic and, in certain cases, allow us to describe all the pseudo-Anosov maps. Here are a few of them:

(1) Train track induction.
(2) The Rauzy–Veech induction.

(3) Sections of flows on hyperbolic 3-manifolds.

The first induction has been extensively studied by Papadopoulos and Penner.

5. Abundance

We are tempted to say that most of elements of $\mathrm{Mod}(S_g)$ are of pseudo-Anosov type. This intuition arises from what happens in genus 1: if we choose a "random" matrix in $\mathrm{Mod}(S_1) = \mathrm{SL}(2,\mathbb{Z})$, it has a strong probability to be hyperbolic (the absolute value of its trace is larger than 2). However, we need to precisely formulate the word "random" since all these groups are discrete groups.

A reasonable way to define this is to fix a set of generators of $\mathrm{Mod}(S_g)$ (for instance: the Dehn twists) and to look at bounded length words (or a ball of radius N centered at the identity in the Cayley graph).

For some modular groups, and some generating sets, we can show that the proportion of pseudo-Anosov elements in the ball of radius N tends exponentially fast to 1 as N tends to infinity (see the work by Caruso-Wiest). There are also versions of this result using the tool of random walks.

6. Counting

Another way to show the abundance of pseudo-Anosov diffeomorphisms is to count them. Let us introduce

$$\mathcal{G}_g(T) = \{\text{conjugation classes of } \psi \mid \psi \text{ is pseudo-Anosov and } \log(\lambda(\psi)) < T\}.$$

Veech was the first to study the asymptotic behavior of $|\mathcal{G}_g(T)|$ as T tends to infinity. His work, started in 1986, eventually culminated with the Eskin–Mirzakhani formula:

$$|\mathcal{G}_g(T)| \sim_{T\to\infty} \frac{e^{(6g-6)T}}{(6g-6)T}.$$

This formula has been generalized later by Eskin-Mirzakhani-Rafi and Hamenstädt. The dynamical techniques that were employed used properties of the geodesic flow on the moduli space $\mathcal{M}_g$, inspired by the work of Margulis.

The key point is to make a parallel between the conjugacy class of ψ and a closed curve on $\mathcal{M}_g$; the number $\log(\lambda(\psi))$ being then the length of this curve for some metric (the Teichmüller metric).

7. Expansion factors

Surprisingly, we do not know much about the expansion factors of pseudo-Anosov homeomorphisms.

7.1. **Realisations of algebraic numbers as expansion factors.** Looking at the action on homology (for a suitable cover), we easily deduce that λ is an eigenvalue of a matrix with integer entries. It is thus an algebraic number (that is, the root of an irreducible polynomial $P \in \mathbb{Q}[X]$) of degree bounded by $3g-3$. In fact, Thurston has shown that it is a bi-*Perron* number:

$$\forall \alpha \neq \lambda, \lambda^{-1},\ P(\alpha) = 0 \implies \lambda^{-1} < \alpha < \lambda.$$

The converse (that is, if a bi-Perron number is an expansion factor) is an open problem.
This is the subject of one of the last manuscripts of Thurston [Thu08].

7.2. **Minimization.** There are plenty of conjectures on this topic. The easiest ones to state are often about λ. For a fixed g, an easy argument that relates roots and coefficients shows that the set

$$\mathrm{Spec}_g = \left\{\lambda(\psi),\ \psi : S_g \to S_g \text{ is pseudo-Anosov}\right\} \subset \mathbb{R}$$

is a discrete subset. What is its smallest element

$$\delta_g = \min(\mathrm{Spec}_g)?$$

This is also an open problem! We know that $\delta_1 = \frac{3+\sqrt{5}}{2}$ and $\delta_2 =$ the largest root of $X^4 - X^3 - X^2 - X + 1 \simeq 1.72$ (compare with Example 4.2), but computing δ_3 is already an open problem. It is not difficult to get an upper bound for δ_g (finding an example is sufficient). It is a little more subtle to get a lower bound. For all $g \geq 2$:

$$\frac{\log(2)}{6} \leq |\chi(S_g)| \cdot \log(\delta_g) \leq 2 \cdot \log(\frac{3+\sqrt{5}}{2}) \tag{1}$$

where $\chi(S_g) = 2 - 2g$. We easily deduce that

$$\limsup_{g\to\infty} g \log(\delta_g) \leq \log(\frac{3+\sqrt{5}}{2}).$$

McMullen conjectured that $(g \log(\delta_g))_g$ converges, but so far there is no proof of this. For a positive answer, one needs a better lower bound (on $g \log(\delta_g)$) than (1).

We present a recent result on matrices that goes in this direction and, surprisingly, that was not known before. McMullen [McM14] has shown that, for all $g \geq 1$, the smallest possible value of the spectral radius $\rho(A)$ of a primitive matrix $A \in \mathrm{Sp}_{2g}(\mathbb{Z})$ (that is, one for which there exists n such that all entries of A^n are strictly positive) is given by the largest root of the polynomial

$$X^{2g} - X^g(1 + X + X^{-1}) + 1.$$

In particular $\rho(A)^g \geq \frac{3+\sqrt{5}}{2}$. Even if this problem is closely related to the previous one, it does not (yet) provide a positive solution to the problem...

The discussions in the previous sections evoke a connection between these problems, of geometric nature, and the problem of minimizing the eigenvalues of a matrix, of algebraic nature.

7.3. **Eigendirections of pseudo-Anosov homeomorphisms.** All the above questions are about eigenvalues of matrices (the expansion factor λ). What about the eigendirections associated to the eigenvectors? This is a very short section since we know almost nothing about it! It seems very difficult to characterize these directions at the moment, even if there are some partial results for genus 2 surfaces and Prym surfaces.

8. Lonely guy conjecture

If we choose a "random" flat metric ω on a surface S_g (with respect to some probability measure on the moduli spaces) what kind of group of symmetries $\mathrm{SL}(S_g,\omega)$ could we expect? The answer that we guess is the trivial group. This is indeed the case (except perhaps if the surface has obvious non-trivial symmetry such as the hyperelliptic involution).

And now if we again choose a "random" flat metric ω *among* surfaces already having a symmetry? Again the answer we expect is that generically the Veech group is cyclic. Surprisingly this is not the case if the genus of S_g is two! McMullen gave a quantitative version of this: the group $\mathrm{SL}(S_g,\omega)$ is very large. Its limit set is the full circle at infinity.

What about when the genus g is larger than three? This question is widely open. We conjecture that in general the group is (virtually) cyclic...

9. Suspensions and volumes

There is a remarkable connection between the dynamics of pseudo-Anosov homeomorphisms in dimension two and the geometry in dimension three. The relation is given by the (very general) construction of suspension. To each $f: S_g \to S_g$ we associate the 3 dimensional object

$$M_f = S_g \times [0,1]/(1,x) \sim (0,f(x)).$$

Another famous theorem of Thurston states that $f=\psi$ is pseudo-Anosov if and only if M_ψ is an hyperbolic 3-manifold. Thus it has a volume, although it is very hard to express it in terms of ψ. Kojima and McShane have recently established this beautiful inequality relating dynamic and geometric complexities:

$$\log(\lambda(\psi)) \geq \frac{1}{3\pi|\chi(S_g)|}\mathrm{vol}(M_\psi),$$

where $\chi(S_g) = 2-2g$.

10. TO LEARN MORE ABOUT PSEUDO-ANOSOV MAPS

The book by Fathi-Laudenbach-Poenaru [FLP79] is a very good introduction to the topic, containing numerous details. It is based on the work of Thurston [Thu88] on surface homeomorphisms. This book is also available in English.

The book by Farb–Margalit [FM11] is a more modern introduction to the modular group. It contains all prerequisites and details of its study.

If one wants to learn more about pseudo-Anosov maps, the literature is rather vast. The recent works by Agol, Hironaka, Leininger, Margalit provide a nice "state of the art" and propose new approaches to the different problems alluded to above.

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