

THE ALGEBRAIC FUNCTIONS AND THEIR  
INTEGRALS.

*Théorie des fonctions algébriques et de leurs intégrales. Étude des fonctions analytiques sur une surface de Riemann.* By PAUL APPELL and ÉDOUARD GOURSAT. With a preface by CH. HERMITE. Paris, Gauthier-Villars et Fils, 1895. 8vo. 22+530 pp.

THE general theories of functions that originated respectively with Cauchy, with Weierstrass and with Riemann have each found in the algebraic functions and their integrals an instructive example, the study of which has turned out to be so fruitful that the results of this research occupy a place of much importance in modern mathematics. Until recently the works of Neumann\* and Clebsch and Gordan† have been the principal introductory treatises in this field. The object of the first of these was to set forth Riemann's theory, that of the second, as Gordan expresses it, "to build the bridge between the theory of the Abelian integrals and the theory of the higher plane curves." In the three decades that have elapsed since the appearance of these works, the activity of mathematicians in this field has been principally in research and not until recently has anyone been found to undertake the task of selecting and arranging the methods and the theorems that lie at the bottom of this subject, and working out a systematic presentation of those things that the beginner ought to learn first. With this purpose in view Messrs. Appell and Goursat applied themselves to the elaboration of their treatise and the result is a work that occupies a place of first importance among the many recent text-books that treat this subject to a greater or less extent.

In the preface Hermite enumerates the principal points of view that have been of importance in the development of the theory of the Abelian integrals, and defines the object of the present work.

The plan of the book is as follows. It is assumed that the reader is acquainted with the rudiments of the theory of functions and a syllabus of the theorems premised forms a nine page introduction. The hyperelliptic equation

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\* *Vorlesungen über Riemann's Theorie der Abelschen Integrale*, Teubner, first ed., 1865. The physical ideas underlying Riemann's theory were treated by Klein: *Ueber Riemann's Theorie der algebraischen Functionen und ihrer Integrale*, Teubner, 1882.

† *Theorie der Abelschen Integrale*, Teubner, 1866.

$$u^2 = A(z-e_1)(z-e_2)\cdots(z-e_n)$$

with its two-leaved Riemann surface presents an example of an algebraic equation of much didactic value; for since the functions that occur in the general case admit here explicit expression in simple form, the algebraic difficulties of the general theory are diminished, and still a degree of generality remains sufficient for purposes of illustration. This hyperelliptic example is discussed at great length and with perfect clearness, so that even the meagre knowledge of the theory of functions demanded of the beginner will suffice. But there is no abrupt transition in the character of the treatment when the general theory of the algebraic functions and their integrals is entered on. Jacobi's problem of inversion, normal curves, the moduli of an algebraic configuration,—these are among the last subjects treated, and thus the line is drawn below the theta functions.

We cannot feel that the omission of all mention of conformal representation is fortunate and we should have been glad to see the study of rational functions and their integrals on an algebraic configuration given a more geometric form by the earlier introduction of higher plane curves. And this for the reason that both of these geometric methods are elementary in their nature and well adapted to give the beginner broader views and a deeper insight into the theories treated.

With regard however to subjects of a more special nature we must acknowledge our indebtedness to Messrs. Appell and Goursat no less for what they have left out of their treatise than for what they have put in. For example, we were gratified to find no homogeneous variables under the sign of integration in so elementary a work. It is a temptation to which pioneer authors are subject to attempt to write at once a text-book and a compendium. Our authors have been content to write a text-book. We congratulate them on the success with which they have accomplished their task; we congratulate the younger students of mathematics on the acquisition of a valuable introduction to an important field of knowledge.

CHAPTER I. The Riemann surface is introduced in the usual way by the example  $u^2=z$ . The construction of the surface is clearly and carefully described and the surface is projected on the sphere. The elliptic and the hyperelliptic surfaces follow, the latter being defined by the equation

$$(1) \quad u^2 = A(z-e_1)(z-e_2)\cdots(z-e_n),$$

and are treated in a similar manner. These first 24 pages give with remarkable clearness the conception of the Riemann surface. The surfaces considered from now on till Chapter IV are the surfaces (1). The residue in an ordinary point or a branch point (finite or infinite) is defined by means of the coefficient of a certain term in the expansion of the function into a power series, the relation between the definitions in the four kinds of points that thus arise being by no means obvious; and it is subsequently shown that the residue is always equal to  $\int f(z) dz$  taken along any closed path about the point in question. In a treatment wedded to the consistent use of power series such a definition is explicable; but one is at a loss here to see why the residue is not defined by the integral and then evaluated in terms of the above coefficients. The behavior of a rational function  $R(u, z)$  on the Riemann surface is studied and it is shown that conversely every single-valued function having no essentially singular points is a rational function of  $u, z$ . An elegant application of the method of residues yields a proof of the theorem that such a function has as many zeros as poles and hence takes on every value the same number of times. § 25 is incorrect, as is shown by the example

$$u^2 = z^4 - 1, \\ v_1 = z^2 + u_1, \quad v_2 = z^2 + u_2, \quad v_1 v_2 = 1$$

Weierstrass's definition of the *deficiency* of an algebraic configuration is given. Rational functions on the hyperelliptic configuration (1) with arbitrarily assigned poles are built up and it is shown by the aid of the explicit expressions obtained that the number of poles must in general be at least equal to  $p+1$ , where  $n = 2p+1$  or  $2p+2$ .

CHAPTER II is entitled: Hyperelliptic Integrals, and deals with the integrals of rational functions of  $u, z$  on the Riemann surface belonging to (1). Such integrals are functions having at most poles and logarithmically singular points. The most general integral of the first class (everywhere finite, *i. e.*, having no singular points) is expressed in terms of the  $p$  linearly independent integrals

$$\int_{(z_0, v_0)}^{(z, v)} \frac{z^k dz}{u}, \quad k=0, 1, \dots, p-1.$$

The integral of the third class  $\bar{\omega}_{a,b}^{a',b'}$  with logarithmic discontinuities at  $(a, b)$ ,  $(a', b')$  is formed for all possible positions of the points  $(a, b)$ ,  $(a', b')$  and for both an odd and

an even  $n$ , and in § 38 a form of such an integral is given (containing however a foreign parameter  $\lambda$ ) that embraces all of these special cases. A reference at this point to Klein's\* normal form of the integral of the second and the third class would not have been out of place, though there are obvious reasons for not discussing these forms in the text. Now come the integrals  $\zeta^{(\nu)}$  of the second class with a single pole of order  $\nu+1$  and the discussion of the relations existing between these integrals and  $\bar{\omega}$ . The treatment here given is careful and exhaustive, but the beginner will need the guidance of a teacher, who will present to him the general case (where the point  $(a, b)$  is neither a branch-point nor infinity) and then give him the special cases to work out for himself. The chapter closes with the theorems regarding the expression of rational functions of  $u, z$  and their integrals in terms of the integrals of the three kinds already considered.

CHAPTER III is entitled: Connection of Two-Leaved Surfaces. The simplification in the analytical work of these first three chapters due to the explicit expressions always obtained for the functions studied finds its geometric counterpart in the avoidance of all general investigations in the field of analysis situs having for their object the proof of the existence of a set of cuts that render the Riemann surface simply connected, such a set of cuts being drawn here by inspection. But while the analytic treatment is always complete, there remain here certain inverse theorems that await their solution in Chapter V; for example, the theorem that  $p$  is the same number for all possible ways of dissecting the surface. The objects that analysis situs in its application to Riemann surfaces has in view are clearly set forth in these pages, which form an excellent introduction to the subject. By the aid of Cauchy's integral theorem (§ 59) extended to regions lying in more than one sheet of the Riemann surface,—in particular, the region  $T'$  obtained by drawing such a set of cuts in the surface  $T$  belonging to (1) as to render it singly connected,—single-valued functions on such regions are defined and thus the integrals of rational functions of  $u, z$  on  $T$  are studied with regard to their moduli of periodicity, the bilinear relations existing between these latter being established. Riemann's normal integrals of the first kind, together with the corresponding normal integrals of the second and third kinds are introduced and the periods of the latter are evaluated.

With CHAPTER IV, p. 165, begins the general treatment of

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\* *Hyperelliptische Sigmafunctionen*, *Math. Ann.* vol. 27, 1886.

algebraic functions, *i. e.*, of the function  $u$  defined by the equation  $F(u, z) = 0$ , where  $F$  denotes any polynomial in  $u, z$ . The chapter opens with Cauchy's proof as given by Briot and Bouquet that such a function is in general continuous; and it is further shown that it has in general a continuous derivative. If  $z$  is confined to a region including no singular point, the  $m$  values  $u$  can be grouped into  $m$  single-valued continuous functions or *branches*. Then comes the establishment of the form of development into power series about any point  $z_0$ , ordinary or singular, finite or infinite, together with the converse theorem (p. 177). Now it seems to us that the presentation would have gained materially at this point by the use of conformal representation and the Riemann surface. Thus the Riemann surface would be constructed in the neighborhood of the branch-point  $z = a$  and transformed on the single-leaved plane by the relation  $z - a = z'^p$ , the function  $u$  becoming thereby a single-valued function of  $z'$ .<sup>\*</sup> Useful symmetry in the form of representation of the algebraic configuration in the neighborhood of the point  $z = a, u = b$  results from the introduction of a parameter  $t$ . We then have

$$z - a = t^p P_1(t), \quad u - b = t^q P_2(t)$$

where  $p, q$  are positive or negative integers and  $P_1(0), P_2(0)$  do not vanish. Geometrically this form means the emphasizing of the fact that the  $u$ -surface in the neighborhood of  $u = b$ , as well as the  $z$ -surface in the neighborhood of  $z = a$ , is transformed conformally on the  $t$ -plane in the neighborhood of  $t = 0$ . This form is so useful in the study of the behavior of a rational function of  $u, z$  (or of its integral) in a point of the surface, for example in the definition of the residue, the number of zeros or poles, the determination of the nature of the singular points of the integral  $\int R(u, z) dz$ , etc., that it seems unfortunate that the authors did not see fit to introduce it. In § 85 it is shown that if  $F(u, z)$  is an irreducible function of  $u, z$  and  $u_1, u_2, \dots, u_m$  are the values of  $u$  corresponding to an ordinary point  $z_0$ , then there must be a path for  $z$  starting at  $z_0$  and ending at  $z_0$ , that will carry  $u_1$  over into any other  $u, u_i$ . The proof is based most simply on the theorem that if  $F(u, z)$  and  $\varphi(u, z)$  are two rational integral functions of the two independent variables  $u, z$  and if  $F$  vanishes whenever  $\varphi$  vanishes, then  $F$  and  $\varphi$  have a common factor. This is indeed a theorem of great importance and frequent application. But can it be assumed

<sup>\*</sup>Cf. Neumann, Chaps. IV, V.

that students for whom it is necessary that the subject of this work be treated in such detail, are so familiar with it that it need not even be quoted? And if this theorem be not assumed, then it is expecting a great deal of the student to require him to supply the proof by the theory of functions. § 86 treats the method of loops. Now follows Puisseaux's method of treating the multiple branch-points of the algebraic configuration. In §§ 92, 93 the Riemann surface for any irreducible algebraic configuration is defined. The treatment begins with well chosen examples and is admirably clear, in fact, one of the clearest that exists. The residue of a function in any point of the surface is defined and many of the theorems of Chapter I are extended to the case of any irreducible surface,—for example, the theorem that a single-valued function on a Riemann surface, having no essentially singular points is a rational function of  $u, z$ . The chapter closes with the proof that branch-points of higher order may be regarded as the limiting case in which several simple branch-points have become coincident.

CHAPTER V begins with a general treatment of analysis situs, principally in two dimensions, modelled after Neumann. The presentation is in general clear, but the intuitional hypotheses might well have been more sharply marked. The theorem already referred to regarding  $p$  is established, Riemann's formula connecting the number of leaves and branch-points with the deficiency:  $w=2(m+p-1)$  is deduced, and Riemann's canonical system of cuts is given. Examples follow. The deficiency of the binomial equation  $u^m=R(z)$ , where  $R$  denotes a rational function, is determined and all such equations for which  $p=0$  or  $1$  are reduced to normal forms. In the course of this work a more general question is suggested—that of determining all "regular" Riemann surfaces of a given deficiency and at this point the surfaces belonging to the regular bodies ( $p=0$ ) appear. The behavior of an Abelian integral, *i. e.*, of the integral of a rational function on the Riemann surface, in the neighborhood of any point is studied, such integrals are shown to have moduli of periodicity as in the hyperelliptic case, the integrals of the three classes are defined, and it is shown that there cannot exist more than  $p$  linearly independent integrals of the first class.

CHAPTER VI is devoted to rational transformations. Let  $f(z, u)=0$  be an irreducible algebraic equation; let

$$Z = \varphi(z, u), \quad U = \psi(z, u)$$

be two rational functions of  $z, u$  having respectively  $\mu$  and  $\nu$

poles on the surface belonging to  $f=0$ . If  $z, u$  be eliminated between these three equations, the result is an algebraic equation  $F(Z, U)=0$  which expresses the necessary and sufficient condition that the above three equations admit a common solution. Thus the chapter begins. Few subjects in elementary mathematics are so deceptive as that of elimination. The student easily mistakes familiarity with the formal operations of combining equations so as to get rid of certain quantities for a knowledge of the theory of elimination and utterly fails to grasp what the resulting equation at which he has arrived really signifies. Moreover, in the present case the quantities that enter are capable of becoming infinite (since homogeneous variables are not used) and thus even the algebraic treatment fails at this point. It therefore seems to us unfortunate that the processes employed should not have been illustrated geometrically by the interpretation of the above equations as plane curves,  $Z, U$  being parameters, and the way thereby pointed out to the analytical treatment by the aid of the theory of functions.—The theorem that the deficiency of a Riemann surface is invariant of a rational transformation is proved both geometrically and analytically and the effect of such transformation on the Abelian integrals is considered. A transformation of Halphen's is given by which any irreducible algebraic curve can be transformed into a curve having only linear cycles and the same transformation is used to prove Noether's theorem that any irreducible curve can be transformed rationally into a curve having only multiple points with distinct tangents, in particular only ordinary double points. The chapter closes with the algebraic representation of the coördinates of curves of deficiency 0, 1 and 2 by means of a parameter, and the transformation of curves of deficiency 0 and 1 into themselves.

CHAPTER VII begins with the subject of Normal Integrals. The form of the integrand of any everywhere finite integral is determined for any irreducible curve  $F(z, u)=0$ , the only restriction being that certain coördinate axes be avoided:

$$\int \frac{Q(z, u)}{F'_u(z, u)} dz$$

where  $Q$  is a polynomial of degree  $m-3$  in  $z, u$ ; and it is shown that there always exist  $p$  such integrals, linearly independent of each other. Picard's proof of this fundamental theorem is also given. An adjoint curve of degree  $m-3$  cuts the ground curve in general in a group of fixed points

and a group of variable points. In § 142 it is shown that this latter group is invariant of one-to-one transformation of the ground curve. Near the end of the section it is assumed that if a curve has only ordinary double-points, the adjoint curve will cut it in  $2p-2$  points distinct from the double-points. This theorem is proved later in § 174; but at present it is not clear that the adjoint curve will not of itself go through some of the double-points oftener than it has been required to do. The integrals of the second and third classes are formed, the algebraic configuration being represented geometrically as a plane curve, while the integrand is built up by the aid of other curves intersecting in certain systems of points. The same geometrical method might well have been introduced into Chapters II and III, either as here, as a means of getting the integrals or merely for the purpose of illustrating the results there obtained.\* Riemann's normal integrals of the first class are introduced and many theorems of Chapters II and III are generalized. A reference for the "well known theorem" of § 154 would not have been out of place.—Any Abelian integral  $I$  can be expressed in the form

$$I = \phi(z, u) + I_1 + I_2,$$

where  $\phi$  denotes a rational function of  $z, u$ ,  $I_1$  an integral having only logarithmic singularities and  $I_2$  an integral having at most poles of the first order in  $p$  points that can be chosen independently of the particular integral  $I$ . The functions  $\phi, I_2$  are discussed to some extent for the case that logarithmic singularities are absent:  $I_1 \equiv 0$  and the conditions that the integrand of  $I$  must satisfy that the integral  $I$  should reduce to a rational function of  $z, u$  are obtained. An interesting example is given. Next the question is proposed: suppose  $\phi \equiv 0, I_2 \equiv 0$ ; when can  $I_1$  be expressed as a sum of logarithms of rational functions and an integral of the first class? and discussed at considerable length. The reduction of the Abelian integrals to integrals on surfaces of lower deficiency is illustrated by examples.

In CHAPTER VIII single-valued functions on a Riemann surface are treated by the aid of the Abelian integrals on the surface. The law of Riemann-Roch and Brill-Noether's law of reciprocity are established. The presentation of this important subject leaves nothing to be desired in regard either to clearness or elegance—at least, this is true if the

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\* For a statement of the reasons for this suggestion see end of the review of Chapter IX.



reader can be trusted to form simple examples for himself, to render concrete the general theorems. *E. g.*, let the ground-curve be a  $C_4$  or  $C_5$  with 0, 1, or 2 double-points; where will the groups of points lie for which  $\sigma > 0$ ? Applications follow. Hyperelliptic configurations are defined and referred to (1) of Chapter I as normal form:  $u^2 = R(\lambda)$ , where  $R$  denotes a polynomial. Two hyperelliptic configurations can be transformed rationally the one into the other when the *binary* forms

$$\lambda_2^{2p+2} R\left(\frac{\lambda_1}{\lambda_2}\right), \quad \mu_2^{2p+2} R_1\left(\frac{\mu_1}{\mu_2}\right)$$

can be transformed linearly into each other. This requires the equality of the  $2p-1$  absolute invariants of  $R, R_1$ ,\* and hence the hyperelliptic configuration has  $2p-1$  *moduli*. The representation of rational functions by the aid of integrals of the third class is given.

CHAPTER IX is entitled: Abel's Theorem. The theorem is first proved by evaluating in two ways a certain integral extended over a closed path. This proof is brief and rigorous, and extremely simple in form; but it is open to the objection to which most proofs by Cauchy's integral are subject, that the thought underlying the theorem is hidden. For this reason the proof as originally given by Abel is to be preferred. This proof is given later, § 189. The proof of § 182 is incomplete; it remains to show that when curves of the sort  $f_1$  are interpolated between  $f$  and  $\varphi$ , the corresponding  $\varepsilon$ 's will never converge toward 0, and hence that one can pass from  $f$  to  $\varphi$  by means of a finite number of curves  $f_1$ . The addition theorem for the Abelian integrals is deduced and a set of Abelian differential equations is integrated. Finally, Abel's theorem is extended to space curves. We should have been grateful to the authors if they had seen fit to take up somewhere in this chapter, perhaps as introductory example, Abel's theorem for the integral of the first kind on the curve

$$u^2 = 4z^3 - g_2z - g_3$$

or

$$u^2 = (1-z^2)(1-k^2z^2),$$

the treatment being put into such form as to yield at once the addition theorem for this elliptic integral. It is true

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\*A reference for this theorem ought certainly to have been given. See, for example, Clebsch: *Binäre Formen*, pp. 365 and 421. Also, for  $p=1$ , Klein: *Modulfunktionen*, I, Chap. I.

that the addition theorem in question is deduced as an example from the general Abelian theorem. But we are interested in seeing an elementary treatment of the curves of deficiency unity and the elliptic integrals, such as might follow or even form a part of a first course in the theory of functions, enriched and to some extent dominated by the general methods set forth in this work; and it seems to us an excellent feature of this treatise that the authors have elsewhere given so much attention to just this subject.

CHAPTER X is devoted to the Problem of Inversion and begins with the question: An irreducible algebraic equation  $F(z, u) = 0$  being given, is it possible to find a rational function  $R(z, u)$  such that, when

$$w = \int_{(z_0, u_0)}^{(z, u)} R(z, u) dz,$$

to each value of  $w$  corresponds one and only one point of the Riemann surface? And the question is answered in the most direct and simplest manner possible. There follows a proof that if the deficiency of  $F$  is 1 and  $w$  is the everywhere finite integral, the points of the  $F$ -surface correspond in a one-to-one manner to the points of a parallelogram in the  $w$ -plane. (Since conformal representation is not considered in this treatise, no reference is made to that property of the transformation.) This theorem is a familiar one, but proofs of it are rare in treatises in which the integral  $w$  (not the doubly periodic function) forms the point of departure. The close analogy that exists between the properties of rational functions on a Riemann surface of deficiency 1 and those of single-valued doubly periodic functions is emphasized, some of the most important corresponding theorems being collected and placed side by side (p. 450). The problem studied by Briot and Bouquet: when will the algebraic differential equation

$$F\left(u, \frac{du}{dz}\right) = 0$$

have a single-valued solution  $u=f(z)$ ? is treated and applications are given, the most important being the ascertainment of all single-valued functions that have an algebraic addition theorem. p. 454, middle, for  $dt/dz$  read  $dz/dt$ . A generalization of Briot and Bouquet's problem follows. After this excellent introduction, which forms in fact a chapter in the theory of the elliptic integrals, the authors proceed to Jacobi's problem of inversion for the Abelian integrals.

CHAPTER XI: Normal curves, Moduli. The chapter opens with the theorem: "There exists no birational transformation, containing an arbitrary parameter, that transforms a curve of deficiency greater than unity into itself." The most general transformation of a curve of deficiency unity into itself is obtained in terms of the parameter  $w$ . Clebsch's and Noether's normal curves are studied and the number of moduli of the algebraic configuration is determined by counting off parameters. Thus a lower limit to the number of the moduli is rigorously established and an elegant application of this result is made to the proof of the theorem that an algebraic plane curve with non-specialized coefficients is not the most general curve of its deficiency.

CHAPTER XII presents some geometric applications of Abel's theorem, principally those to the groups of points of intersection of curves. The points of inflection of a cubic and the points where a conic has contact of the fifth order, the systems of conics tangent to the quartic wherever they meet it, the double-tangents of the quartic, and the discussion of some of these topics when the ground-curve has double-points, are among the subjects treated; but a few examples in areas, angles and lengths of arcs are taken up at the end.

The reviewer has expressed his mind freely on those points where his views differ from those of the authors. He begs leave to say in closing that his admiration for their work is none the less hearty because of these differences of opinion and to add that their treatise seems to him to be a thoroughly desirable book to put into the hands of the student.

W. F. OSGOOD.

HARVARD UNIVERSITY,  
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#### PAPERS OF THE MATHEMATICAL CONGRESS.

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THIS volume, which enjoys the distinction of being the first to be published in the form of a separate book by the