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RECURSIVENESS IN Π_1^1 PATHS THROUGH $\mathcal{O}$

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ABSTRACT. Kleene's $\mathcal{O}$ is recursive in some Π_1^1 path through $\mathcal{O}$. If every hyp set is recursive in a given Π_1^1 set, then $\mathcal{O}$ is recursive in its triple jump.

Let $\mathcal{O}$ and $<_{\mathcal{O}}$ be defined as in Rogers [7, p. 208]. A path through $\mathcal{O}$ is a subset of $\mathcal{O}$ which is linearly ordered by $<_{\mathcal{O}}$, closed under $<_{\mathcal{O}}$, and has order type ω_1. For $a \in \mathcal{O}$, let $\mathcal{O}(a) = \{b: b <_{\mathcal{O}} a\}$, and $|a|$ be the ordinal of a.

There has been some interest in Π_1^1 paths through $\mathcal{O}$. The existence of such paths was proved in Feferman and Spector [1]. Jockusch [4] has shown that there are Π_1^1 paths P through $\mathcal{O}$ with $\mathcal{O}(e)$ recursive for every $e \in P$, as well as those with $\mathcal{O}(e)$ a complete r.e. set, for the $e \in P$ with ordinal ω^2. Paths with the former property will be called regular.

Questions have arisen concerning the "information coded up" by Π_1^1 paths through $\mathcal{O}$. It is clear that Π_1^1 paths through $\mathcal{O}$ are all nonhyperarithmetic Π_1^1 sets and, hence, all have hyperdegree $\mathcal{O}$. However, from the point of view of enumeration reducibility or truth table reducibility, very little information is coded by Π_1^1 paths through $\mathcal{O}$. Kreisel [5] has noted that any hyperarithmetic set enumeration reducible to a Π_1^1 path P (using the definition in Rogers [7, p. 146]) is enumeration reducible to $\mathcal{O}(e)$, for some $e \in P$. Parikh [6] has shown that any hyperarithmetic set truth table reducible to a Π_1^1 path P is truth table reducible to $\mathcal{O}(e)$, for some $e \in P$.

The above results immediately imply that every hyperarithmetic set enumeration reducible to some Π_1^1 path is r.e., and every hyperarithmetic set truth table reducible to some Π_1^1 path is Δ_2^0. As noted in Jockusch [3], these results also imply that there exists a Π_1^1 path P such that every hyperarithmetic set which is enumeration reducible or truth table reducible to P is recursive.

Theorem 1 below is a strengthening of the Parikh result for truth table reducibility. We originally found the argument in Theorem 1 to give a simplified proof for the truth table reducibility case. We are grateful to C. Jockusch and J. Owings for jointly pointing out to us that our argument works for weak truth table reducibility ($\leqq_W$). See Rogers [7, p. 158] for the definition of $\leqq_W$.

THEOREM 1. *Let P be a Π_1^1 path through $\mathcal{O}$. If $A \leqq_W P$ and A is hyperarithmetic, then $A \leqq_W \mathcal{O}(e)$ for some $e \in P$. In particular, ($A \leqq_W P$ and A is hyp) $\to A$ is Δ_2^0; if P is regular, ($A \leqq_W P$ and A is hyp) $\to A$ is recursive.*

PROOF. The proof proceeds informally, using concepts from metarecursion

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theory. Let f be the unique order preserving map from ω_1 one-one onto P. Then f is metrarecursive. Assume $A \leqq_W P$, and A is hyp.

Let F be the partial recursive operator given by a computation procedure, on $\mathcal{P}(\omega)$ (or 2^ω if characteristic functions are preferred to sets) corresponding to '$A \leqq_W P$'. That is, there is a recursive function h such that $h(n)$ is a bound on the numbers k that are used in F to compute the truth value of '$n \in A$' using the truth values of '$k \in P$'. Without loss of generality, we may assume that $F(C) = D$ implies that $h(n)$ is a bound on the numbers k that are used in F to compute the truth value of '$n \in D$' using the truth values of '$k \in C$'.

Clearly, for each n and sufficiently large $\alpha < \omega_1$, we have $n \in A \leftrightarrow n \in F(\mathcal{O}(f(\alpha)))$. Therefore we may define a sequence of ordinals $\alpha_n < \omega_1$ such that α_n is the least ordinal greater than α_{n-1} with $j \in A \leftrightarrow j \in F(\mathcal{O}(f(\alpha_n)))$, for all $j \leqq n$. Since A, f are metarecursive, it is clear that $\langle \alpha_n \rangle$ is metarecursive. Hence $\lim(\alpha_n) = \lambda < \omega_1$. Since $\lim(\mathcal{O}(f(\alpha_n))) = \mathcal{O}(f(\lambda))$, it is easy to see that $F(\mathcal{O}(f(\lambda))) = A$, and we are done.

We give a brief sketch of our proof of the existence of Π_1^1 paths through $\mathcal{O}$ of Turing degree $\mathcal{O}$. If we take a Π_1^0 predicate of functions that has solutions but no hyperarithmetic solutions (e.g., see Gandy [2]) and pass to the Kleene-Brouwer ordering restricted to the unsecured sequence numbers for the predicate, we obtain a recursive linear ordering with infinite descending sequences but with no hyperarithmetic descending sequences. The well-founded part of such a linear ordering must be Turing equivalent to a Π_1^1 path through $\mathcal{O}$. Now if every solution to the predicate codes up a lot of information, so must the well-founded part (since some infinite descending sequence must be recursive in the well-founded part). For instance, if every hyperarithmetic set is recursive in every solution, then every hyperarithmetic set will be recursive in the well-founded part (thereby answering the question in footnote 2 of [5]). In our proof, the Π_1^0 predicate is chosen so that its solutions code up enough information for us to conclude that the corresponding well-founded part is of Turing degree $\mathcal{O}$.

Let $\mathcal{O}^*$, $\prec$ be as in Harrison [3]. All facts appealed to here about $\mathcal{O}^*$, $\prec$ can be found in Harrison [3].

For sets $B \subset \omega$, let $(B)_n$ be $\{m\colon 2^n 3^m \in B\}$. Define $S(e, B)$ iff (1) $\omega_1^B = \omega_1$, (2) $(B)_0 = \{0\}$, if $0 \prec e$, (3) $(B)_{2^n} = (B)_n \cup \{2^m\colon m \in (B)_n\}$, if $2^n \prec e$, (4) $(B)_{3 \cdot 5^a} = \{3 \cdot 5^m\colon \varphi_m$ is total and $(\forall n)(\varphi_m(n) \prec \varphi_m(n+1))$ and $(\forall n)(\exists k \prec 3 \cdot 5^a)(\varphi_m(n) \in (B)_k)\} \cup \{b\colon (\exists k \prec 3 \cdot 5^a)(b \in (B)_k)\}$, if $3 \cdot 5^a \prec e$.

LEMMA 1. (a) $(\exists e \in \mathcal{O}^* - \mathcal{O})(\exists B)(S(e, B))$; (b) *if* $S(e, B)$, $n \prec e$, $n \in \mathcal{O}$, *then* $(B)_n = \{m\colon |m| \leqq |n|\}$; (c) *if* $S(e, B)$, $e \in \mathcal{O}^* - \mathcal{O}$, *then B is not hyp.*

PROOF. Clearly $(\forall e \in \mathcal{O})(\exists B)(S(e, B))$. Now $\{e\colon e \in \mathcal{O}^* \;\&\; (\exists B)(S(e, x))\}$ is Σ_1^1 and, hence, unequal to $\mathcal{O}$. Since $\mathcal{O}$ is included, clearly $(\exists e)(e \in \mathcal{O}^* - \mathcal{O} \;\&\; (\exists B)(S(e, B)))$.

(b) is shown by straightforward transfinite induction on $|n|$. It is well known that every hyp set is recursive in some $\{n\colon |n| \leqq \alpha\}$. Since $\{a \in \mathcal{O}\colon a \prec e\}$ is a path through $\mathcal{O}$ for $e \in \mathcal{O}^* - \mathcal{O}$, it is clear that $(S(e, B) \;\&\; e \in \mathcal{O}^* - \mathcal{O}) \rightarrow$ every hyp set is recursive in B. Hence (c) holds.

From now on we fix $e \in \mathcal{O}^* - \mathcal{O}$ such that $(\exists B)(S(e, B))$.

LEMMA 2. *There is a recursive tree of finite sequences which has infinite paths, and such that for every infinite path f there is a $B \leqq_T f$ with $S(e, B)$.*

PROOF. Since $S(e, B)$ is Σ_1^1, there is a recursive predicate R such that

$$S(e,B) \leftrightarrow (\exists g)(\forall n)(R(n,\bar{g}(n),\overline{\mathrm{ch}(B)}(n))).$$

Let T be the recursive tree of pairs whose infinite paths correspond to pairs $\langle g, h\rangle$, where $(\forall n)(R(n,\bar{g}(n),\bar{h}(n)))$. Then T has the desired properties.

From now on, we fix the recursive tree T of Lemma 2. Let $<^*$ be the usual Kleene-Brouwer ordering. Then $(T, <^*)$ is a recursive linear ordering. It is easy to verify that $(T, <^*)$ is not a well ordering, yet has no hyperarithmetic descending sequences. Let $A \subset T$ be the well-founded part of $(T, <^*)$.

LEMMA 3. *$(A, <^*)$ has order type ω_1.*

PROOF. If the order type of $(A, <^*)$ was $\alpha < \omega_1$, then A would by hyp, and hence there would be hyperarithmetic descending sequences in $(T, <^*)$. If the order type of $(A, <^*)$ was greater than ω_1, we would have a recursive well ordering of type ω_1.

LEMMA 4. *Some infinite path through T is recursive in A.*

PROOF. Define the function f: $\omega \to \omega$ given by (1) $f(0) = \langle\ \rangle$, (2) $f(n+1) = f(n) * k$, where k is the least number such that $f(n) * k \in T - A$. To see that f is well defined, suppose that $f(m)$ has been defined for all $m \leqq n$, $f(n) = \langle a_0, \ldots, a_{n-1}\rangle \in T - A$. Choose an arbitrary $s <^* f(n)$, $s \in T - A$. If s, $f(n)$ differ somewhere, let $s = \langle a_0, \ldots, a_k, b, \ldots\rangle$, where $b < a_{k+1}$. Then $s \leqq^* \langle a_0, \ldots, a_k, b\rangle \in T - A$, contradicting the choice of $f(k+1)$. Hence s must properly extend $f(n)$. Clearly $s \leqq^* s \restriction n+1 \in T - A$. Choose $f(n+1) = \langle a_0, \ldots, a_{n-1}, k\rangle$, where k is least such that $\langle a_0, \ldots, a_{n-1}, k\rangle \in T - A$. It is immediate that f is an infinite path through T recursive in A.

LEMMA 5. *There is a regular Π_1^1 path P through $\mathcal{O}$ of the same Turing degree as A.*

PROOF. We sketch the proof. The first step is to form the linear ordering (T^*, R), where T^* is the least set such that $3^k \in T^*$ for all $k \in T$, and $2^k \in T^*$ for all $k \in T^*$. Take R to be the linear ordering on T^* defined by $3^k R 3^m \leftrightarrow k <^* m$, $2^k R x \leftrightarrow (kRx \ \&\ x \neq 2^k)$, $xR2^k \leftrightarrow (xRk$ or $x = k)$. Let A^* be the well-founded part of (T^*, R). As in Lemma 3, clearly (A^*, R) has order type ω_1. Note that $A^* \equiv_T A$.

The second step is to apply the recursion theorem to obtain a partial recursive function f with domain T^* such that (1) $f(a) = 0$, where a is the R-least element of T^*, (2) $f(2^b) = 2^{f(b)}$, (3) for $b \neq a$, $f(3^b) > b$, and $f(3^b) = 3 \cdot 5^e$, where e is the Gödel number of some recursive sequence $f(y_0)$, $f(y_1), \ldots$ such that $y_0, y_1, \ldots$ is a strictly R-increasing sequence with limit 3^b in (T^*, R).

It is clear that f maps (T^*, R) isomorphically onto an initial segment of $<_{\mathcal{O}}$. By the order type of (A^*, R), this initial segment of $\mathcal{O}$ must be a path P through $\mathcal{O}$. By a hierarchy computation, P is Π_1^1. Because of the condition $f(3^b) > b$ for $b \neq a$, it is easily seen that $P \equiv_T A^*$, and P is regular. Hence, P is a regular Π_1^1 path through $\mathcal{O}$ of the same Turing degree as A, and we are done.

LEMMA 6. *There is a regular Π_1^1 path P through $\mathcal{O}$ and an x with $S(e, B)$ such that $B \leqq_T P$.*

PROOF. Immediate from Lemmas 2, 4 and 5.

LEMMA 7. *Let P be any Π_1^1 path through $\mathcal{O}$, and $S(e, B)$. Then $\mathcal{O} \leqq_T (P, B)$.*

PROOF. We first claim that

$$n \in \mathcal{O} \leftrightarrow (\exists t \prec e)(\exists k \in P)(k \notin (B)_t \,\&\, n \in (B)_t).$$

Suppose $n \in \mathcal{O}$. Let $t \prec e$, $|t| = |n| < |k|$. Then by Lemma 1, $k \notin (B)_t$, $n \in (B)_t$. Suppose $t \prec e$, $k \in P$, $k \notin (B)_t$, $n \in (B)_t$. By the cumulative way in which S was defined, clearly $|t| < |k|$. Hence, $|n| \leqq |t|$, and so $n \in \mathcal{O}$.

Let $a \in \mathcal{O}^*$ be such that $\{b\colon b \prec a\} \cap \mathcal{O} = P$. We claim

$$n \in \mathcal{O} \rightarrow (\forall t \prec e)(\forall k \prec a)(k \notin P \rightarrow (k \in (B)_t \rightarrow n \in (B)_t)).$$

Suppose $n \in \mathcal{O}$, $t \prec e$, $k \prec a$, $k \notin P$, $n \notin (B)_t$. Then $|t| < |n|$. Hence, $k \notin (B)_t$, since $k \notin \mathcal{O}$.

We claim

$$(\forall t \prec e)(\forall k \prec a)(k \notin P \rightarrow (k \in (B)_t \rightarrow n \in (B)_t)) \rightarrow n \in \mathcal{O}.$$

Suppose

$$(\forall t \prec e)(\forall k \prec a)(k \notin P \rightarrow (k \in (B)_t \rightarrow n \in (B)_t)), \qquad n \notin \mathcal{O}.$$

Now for each $k \in P$ there is a $t \prec e$ such that $n \notin (B)_t$ & $k \in (B)_t$; namely take $|k| = |t|$. Hence, either P is r.e. in B or

$$(\exists k \prec a)(\exists t \prec e)(k \notin P \,\&\, n \notin (B)_t \,\&\, k \in (B)_t).$$

The first contradicts $\omega_1^B = \omega_1$. The second contradicts our assumption.

Summarizing, we see that $\mathcal{O}$ is Δ_1^0 in (P, B). Hence, $\mathcal{O} \leqq_T (P, B)$.

THEOREM 2. *$\mathcal{O}$ is recursive in some regular Π_1^1 path through $\mathcal{O}$.*

PROOF. Immediate from Lemmas 6 and 7.

The proof of Lemma 7 led us to the following theorem and its corollary. Carl Jockusch has recently found a simpler proof of Theorem 3, and we give his proof rather than ours.

THEOREM 3. *Let B be an unbounded subset of $\mathcal{O}$. If every hyperarithmetic set is recursive in B then $\mathcal{O} \leqq_T B'''$.*

PROOF. For each $j \in \mathcal{O}$, $\{k\colon |k| < |j|\} = \mathcal{O}_j$ is recursive in a Π_1^0 singleton in a uniform way. I.e., there are partial recursive functions F, G such that if $j \in \mathcal{O}$, then $F(j)$ is the index of a Π_1^0 predicate $P_{F(j)}$ which has a unique solution g, and $\mathrm{ch}(\mathcal{O}_j)$ is recursive in g with index $G(j)$. Hence

$$a \in \mathcal{O} \leftrightarrow (\exists j)(j \in B \;\&\; (\exists i)(\{i\}^B \text{ is total} \;\&\; P_{F(j)}(\{i\}^B) \;\&\; \{G(j)\}^{\{i\}^B}(a) = 1)).$$

Therefore, $\mathcal{O}$ is Σ_3^0 in B.

COROLLARY. *If every hyperarithmetic set is recursive in a fixed* Π_1^1 *set* B, *then* $\mathcal{O} \leqq_T B'''$.

G. Sacks and S. Simpson have recently and independently constructed a Π_1^1 set B such that every hyperarithmetic set is recursive in B and $\mathcal{O} \equiv_T B'''$.

It is not known whether $\mathcal{O}$ is recursive in every Π_1^1 path through $\mathcal{O}$, or even whether $0'$ is recursive in every Π_1^1 path through $\mathcal{O}$.

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