EXAMPLES OF UNORIENTED
AREA-MINIMIZING SURFACES

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ABSTRACT. A comprehensive study is made of constructions of area-minimizing flat chains modulo two. Many have singularities. For instance, any bounded, area-minimizing submanifold of $\mathbf{R}^n$ occurs as the singular set of some area-minimizing flat chain modulo two in some $\mathbf{R}^N$.

Introduction. Given a $(k-1)$-dimensional boundary in $\mathbf{R}^n$, one can seek to minimize area in some class of surfaces. We want to take a large class that includes nonorientable as well as orientable surfaces, without restrictions on the singularities or topological type. Such a class appears in geometric measure theory as the flat chains modulo two. The existence and regularity of area-minimizing flat chains modulo two are already well established (cf. [7, 4.2.26; 9]). Unfortunately, examples have been rare.

This paper presents several methods for constructing area-minimizing flat chains modulo two. The first five theorems include the following five examples:

1. a Möbius strip,

2. the graph over a convex region of any function satisfying the minimal surface equation,

3. any minimal disc bounded by a curve $C^{2,\alpha}$ close to a circle,

4. any orientable hypersurface bounded by a connected submanifold of the unit sphere and known a priori to minimize area among oriented surfaces, such as the cone over $\mathbf{S}^3 \times \mathbf{S}^3$ in $\mathbf{R}^8$,

5. the union of two area-minimizing flat chains modulo two lying in orthogonal subspaces of $\mathbf{R}^n$.

Examples 4 and 5 sometimes contain singularities. Theorems 4 and 5 seem to produce all known examples of area-minimizing flat chains modulo two with singularities. Corollary 5.4 produces the first examples of nonlinear singular sets.

Many of the results of this paper have been known to workers in the field; others are new.

Comparison with the case of oriented surfaces. Examples of surfaces which minimize area in the smaller class of oriented surfaces (integral currents) have been much more abundant. H. Federer [10 or 7, 5.4.19] proves that bounded portions of complex analytic varieties are area-minimizing. R. Harvey and H. B. Lawson [15]

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give other rich "calibrated geometries" of area-minimizing surfaces. Lawson [18] obtains area-minimizing surfaces invariant under a suitable group of isometries from geodesics in a quotient space. (See also [19].)

Unfortunately these methods seem to break down in the class of flat chains modulo two. In one case [22] symmetry arguments and regularity theory are used to prove the existence of a continuum of area-minimizing flat chains modulo two with a certain boundary, but the proof is not constructive.

Organization of this paper. The statement of each of the six theorems is usually followed by remarks, lemmas and finally the proof.

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Preliminaries. H. Federer's treatise [7] serves as a reference for basic definitions, notations and results.

0.1. *Oriented surfaces.* For the basic class of k-dimensional oriented surfaces in $\mathbf{R}^n$, we take the rectifiable currents $\mathscr{R}_k \mathbf{R}^n$ of geometric measure theory. These surfaces can be built from rectifiable sets. There is no restriction on the singularities or topological type. We also use the subspace $\mathbf{I}_k \mathbf{R}^n$ of integral currents and the larger space $\mathscr{F}_k \mathbf{R}^n$ of integral flat chains:

$$\mathbf{I}_k \mathbf{R}^n \subset \mathscr{R}_k \mathbf{R}^n \subset \mathscr{F}_k \mathbf{R}^n.$$

One can associate to any $S \in \mathscr{F}_k \mathbf{R}^n$ a boundary $\partial S \in \mathscr{F}_{k-1} \mathbf{R}^n$, a mass or area $\mathbf{M}(S) \in [0, \infty]$, a flat norm $\mathscr{F}(S) \in [0, \infty)$, a support spt $S \subset \mathbf{R}^n$, and a positive measure $\|S\|$ on $\mathbf{R}^n$. A rectifiable current R has an oriented tangent plane $\vec{R}$ at $\|R\|$-almost all points in $\mathbf{R}^n$.

The integral currents are just the rectifiable currents whose boundaries are rectifiable currents. The integral flat chains are sums of boundaries of rectifiable currents and rectifiable currents. It turns out that the rectifiable currents are precisely the integral flat chains of finite mass. The usual definitions in Federer refer to a compact set K which will be inconsequential for us, since we will always be able to assume all our currents have supports inside some large closed ball K_0.

Any compact oriented C^1 manifold with boundary or any graph over a compact domain of a Lipschitz function may be viewed as a rectifiable current.

0.2. *Unoriented surfaces.* For our classes of k-dimensional unoriented surfaces in $\mathbf{R}^n$ we take the spaces of flat chains modulo two,

$$\mathbf{I}_k^2 \mathbf{R}^n \subset \mathscr{R}_k^2 \mathbf{R}^n \subset \mathscr{F}_k^2 \mathbf{R}^n,$$

consisting of congruence classes of integral flat chains. For example, two rectifiable currents $S, T \in \mathscr{R}_k \mathbf{R}^n$ are congruent modulo two if $S - T = 2R$ for some $R \in \mathscr{R}_k \mathbf{R}^n$. In particular, reversing the orientation of a rectifiable current does not change its congruence class. The congruence class of $T \in \mathscr{F}_k \mathbf{R}^n$ is denoted by $T^2 \in \mathscr{F}_k^2 \mathbf{R}^n$.

One can associate to any $S \in \mathscr{F}_k^2 \mathbf{R}^n$ a boundary $\partial S \in \mathscr{F}_{k-1}^2 \mathbf{R}^n$, a mass $\mathbf{M}^2(S) \in [0, \infty]$, a flat norm $\mathscr{F}^2(S) \in [0, \infty)$, a support spt^2 $S \subset \mathbf{R}^n$, and, if $S \in \mathscr{R}_k^2 \mathbf{R}^n$, a

positive measure $\|S\|^2$. It turns out that $\mathscr{R}_k^2 \mathbf{R}^n$ consists precisely of the flat chains modulo two of finite mass. An $S \in \mathscr{R}_k^2 \mathbf{R}^n$ has an unoriented tangent plane $\vec{S}$ at $\|S\|^2$-almost all points in $\mathbf{R}^n$. Every $S \in \mathscr{R}_k^2 \mathbf{R}^n$ has a representative modulo two T; that is, a $T \in \mathscr{R}_k \mathbf{R}^n$ of multiplicity one such that $S = T^2$; $\mathbf{M}^2(S) = \mathbf{M}(T)$.

Any compact C^1 manifold with boundary, orientable or nonorientable, or any graph over a compact domain of a Lipschitz function, may be viewed as a flat chain modulo two.

Compactness. The following compactness theorem guarantees the existence of solutions to many variational problems. *For any $C > 0$ and closed ball K_0, the set of all flat chains modulo two S supported in K_0 with*

$$\mathbf{M}^2(S) \leqslant C, \quad \mathbf{M}^2(\partial S) \leqslant C$$

is compact under the flat norm. The analogous theorem holds for integral flat chains.

The *theory* of flat chains modulo two was first developed by W. Ziemer [26] (in 1962) and by W. Fleming [12] (in 1966).

0.3. *Area-minimizing surfaces.*

DEFINITION. A rectifiable current $S \in \mathscr{R}_k \mathbf{R}^n$ is called *area-minimizing* if for all $T \in \mathscr{R}_k \mathbf{R}^n$ with $\partial T = \partial S$, $\mathbf{M}(T) \geqslant \mathbf{M}(S)$. Similarly, a flat chain modulo two $S \in \mathscr{R}_k^2 \mathbf{R}^n$ is *area-minimizing* if for all $T \in \mathscr{R}_k^2 \mathbf{R}^n$ with $\partial T = \partial S$, $\mathbf{M}^2(T) \geqslant \mathbf{M}^2(S)$.

Existence. Given any boundary $B \in \mathscr{F}_{k-1} \mathbf{R}^n$ (resp. $\mathscr{F}_{k-1}^2 \mathbf{R}^n$) with $\partial B = 0$, there is an area-minimizing $S \in \mathscr{R}_k \mathbf{R}^n$ (resp. $\mathscr{R}_k^2 \mathbf{R}^n$) with $\partial S = B$. (Such existence follows from compactness.)

Regularity. If $S \in \mathscr{R}_k \mathbf{R}^n$ and $T \in \mathscr{R}_k^2 \mathbf{R}^n$ are area-minimizing, then spt S − spt ∂S and spt$^2 T$ − spt$^2 \partial T$ are smooth manifolds except for a singular set of Hausdorff dimension at most $k − 2$ [9, Theorem 2; 4].

The theorems. F. J. Almgren and B. Solomon both have showed me a proof of the following theorem by methods to appear in [3].
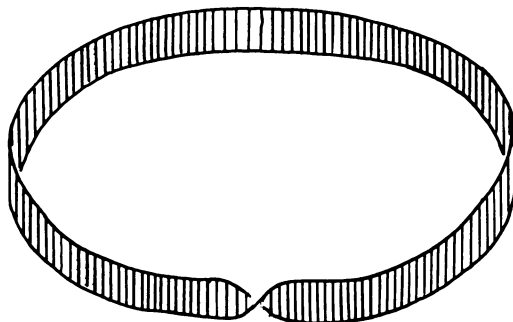

FIGURE 1. A Möbius strip

1. THEOREM. *Let B be the smooth curve (which winds twice around a thin torus) parametrized by*

$$f\colon [0, 2\pi) \to \mathbf{R}^3,$$

$$f(\theta) = (\cos 2\theta(1 + \varepsilon \cos \theta), \sin 2\theta(1 + \varepsilon \cos \theta), \varepsilon \sin \theta).$$

For ε sufficiently small, any area-minimizing flat chain modulo two bounded by B is a Möbius strip, which is an embedded manifold with boundary.

REMARK. The strip can be shown to be close to the strip one would expect.

REMARK. Assuming an appropriate bridge theorem, one now deduces the existence of area-minimizing flat chains modulo two of every finite topological type.
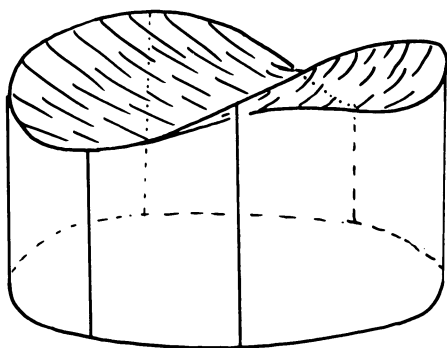

FIGURE 2. The graph of a function satisfying the minimal surface equation

2. THEOREM. *Let K be a compact, convex domain in $\mathbf{R}^n$ and let $f\colon K \to \mathbf{R}$ be a Lipschitz weak solution to the minimal surface equation. Then the graph of f is area-minimizing as a flat chain modulo two.*

REMARK. It follows from [6] that f is real-analytic on the interior of K. The theorem is an easy consequence of a maximum principle of R. Hardt [14, 4.1 and 5].

DEFINITIONS FOR THEOREM 3. For any positive integer q and $0 \leqslant \alpha \leqslant 1$, we use $C^{q,\alpha}$ to indicate α-Hölder-continuous derivatives of order q. For M a compact, $(k + 1)$-dimensional, $C^{q,\alpha}$ submanifold with boundary of $\mathbf{R}^n$, $f \in C^{q,\alpha}(M, \mathbf{R}^N)$, let

$$\|f\|_{C^{q,\alpha}} = \sup\{\|D^j f(x)\|, |x - y|^{-\alpha}\|D^q f(x) - D^q f(y)\|:$$

$$x, y \in M, x \neq y, 0 \leqslant j \leqslant q\}.$$

Let p denote a fixed real-analytic map from M into the space of $(n - k - 1)$-dimensional linear subspaces of $\mathbf{R}^n$ such that at each $x \in M$, $p(x)$ complements the tangent space to M at x. Let

$$C^{q,\alpha}(M, p) = \{f \in C^{q,\alpha}(M, \mathbf{R}^n): f(x) \in p(x) \text{ for all } x \in M\}.$$

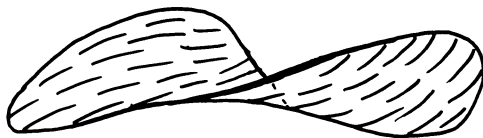Let ι_M denote the inclusion of M in $\mathbf{R}^n$.

FIGURE 3. An embedded area-minimizing manifold without Jacobi fields

3. THEOREM. *For positive integers n and k, $0 < \alpha < 1$, let B be a compact, embedded, k-dimensional, $C^{q,\alpha}$ submanifold of $\mathbf{R}^n$. Suppose that B bounds a unique area-minimizing flat chain modulo two S, that $M = \operatorname{spt}^2 S$ is an embedded manifold with boundary, and that M has no nontangential Jacobi fields which vanish at the boundary. Then given $\varepsilon > 0$, there exists $\delta > 0$, such that if $\|\gamma - \iota_B\|_{C^{2,\alpha}} < \delta$ then $\gamma_{\#} B$ bounds a unique area-minimizing flat chain modulo two T, $\operatorname{spt}^2 T$ is an embedded manifold with boundary, and γ has an extension f on M such that*

$$T = f_{\#} S \quad \text{and} \quad \|f - \iota_M\|_{C^{2,\alpha}} < \varepsilon.$$

3.1. REMARK. The above theorem, an outcome of a conversation with B. White, is a consequence of a theorem of his. White's theorem [24, 3.1], under the same hypotheses, concludes that $\gamma_{\#} B$ bounds an embedded minimal manifold M', unique among embedded minimal manifolds $C^{2,\alpha}$ close to M. Our theorem says that M' uniquely minimizes mass among flat chains modulo two with the same boundary.

3.2. LEMMA. *Let q, α, M, p, $\iota = \iota_M$ be as in the introduction to Theorem 3. Given $x \in M$, $0 < \delta_1 < \delta_2$, there exist ε, $C > 0$ such that if $f \in C^{q,\alpha}(M, \mathbf{R}^n)$,*

$$f|\partial M = \operatorname{id}, \quad \|f - \iota\|_{C^1} < \varepsilon,$$

then there is a unique $g \in C^{2,\alpha}(M, p)$ such that $(\iota + g)(M) = f(M)$. Furthermore,

$$\|g|\mathbf{U}(x, \delta_1)\|_{C^{q,\alpha}} \leqslant C\|(f - \iota)|\mathbf{U}(x, \delta_2)\|_{C^{q,\alpha}}.$$

REMARK. This lemma can be proved by changing coordinates locally to make M an axis $(k + 1)$-plane and $p(x)$ the orthogonal complement to the tangent space to M at x.

SKETCH OF PROOF OF THEOREM 3. Take a sequence of f's such that

$$\|f - \iota_B\|_{C^{2,\alpha}} \to 0.$$

Let T be any area-minimizing flat chain modulo two bounded by $f_{\#} B$. Since S is the unique area-minimizing flat chain modulo two bounded by B, one can infer that $T \to S$ in the flat norm by applying a compactness argument. In fact, the measures $\|T\|^2 \to \|S\|^2$ (cf. [7, 5.4.2]). Now by covering $M = \operatorname{spt}^2 S$ by small balls and applying Allard's regularity theorems [1, §8; 2, §4], one concludes that eventually $\operatorname{spt}^2 T$ is a $C^{1,\alpha}$ manifold with boundary, C^1 close to M.

To state that conclusion more precisely, we adopt the notation of White's paper [24, 3]. Let

$$\phi\colon C^{2,\alpha}(B, \mathbf{R}^n) \to C^{2,\alpha}(M, \mathbf{R}^n)$$

be the bounded linear operator such that for $\gamma \in C^{2,\alpha}(B, \mathbf{R}^n)$, $\phi(\gamma) - \iota_M$ is the unique harmonic map on M with boundary values $\gamma - \iota_B$. It is a standard result that $\|\phi(\gamma) - \iota_M\|_{C^{2,\alpha}} \to 0$. What follows from Allard's theorem and Lemma 3.2 is that $\mathrm{spt}^2\, T$ can be parametrized as $\phi(\gamma) + g$ for some $g \in C_0^{2,\alpha}(M, p)$ with

(1) $\|g\|_{C^1} \to 0$, $\|g\|_{C^{1,\alpha}} \leqslant K_0$.

We claim that $\|g\|_{C^{2,\alpha}} \to 0$. Let $\{C\}$ be a finite collection such that:

(2) $C = D \times E$ is the orthogonal product of a small $(k + 1)$-ball D and a small $(n - k - 1)$-ball E both centered at the same point $a \in M$,

(3) D is tangent to M at a,

(4) $C \cap B = \varnothing$ unless $a \in B$,

(5) $C \cap M$ is the graph over $\Omega \subset D$ of a function with gradient less than say .1 in absolute value,

(6) if for $i = 1, 2, 3, 4$, C_i denotes C shrunken by a factor 2^{-i} by a homothety about a, then $\{C_4\}$ covers M.

Let $\Omega_i = \Omega \cap C_i$. We can assume that $(\iota + g)(M) \cap C$ is the graph of a function $w\colon \Omega \to E$ with $|\nabla w| < .2$, that $\phi(\gamma)(M)$ and $(\phi(\gamma) + g)(M)$ intersect C in graphs over domains in D of functions with gradients less than .2 in absolute value, and that $\phi(M \cap C_1) \subset C$. Define functions ϕ_1, ϕ_2 on Ω such that

$$\phi(\gamma) - \iota\colon M \cap C_1 \to C = D \times E,$$

$$\phi(\gamma) - \iota = (\phi_1, \phi_2) \circ (\text{projection onto } D).$$

Then $\|\phi_2\|_{C^{2,\alpha}} \to 0$, $\|\phi_1\|_{C^{2,\alpha}} \to 0$, $\|\phi_1^{-1}\|_{C^{2,\alpha}} \to 0$. Now since $\mathrm{spt}^2\, T = (\phi(\gamma) + g)(M)$ is minimal, on $\phi_1\Omega_2 \subset D$, $(\phi_2 + w) \circ \phi_1^{-1}$ satisfies the minimal surface system [**23**, Theorem 2.2]

$$\sum_{i,j=1}^{k+1} G^{ij} \frac{\partial^2}{\partial x_i \partial x_j}\left(w \circ \phi_1^{-1}\right) + \sum_{i,j=1}^{k+1} G^{ij} \frac{\partial^2}{\partial x_i \partial x_j}\left(\phi_2 \circ \phi_1^{-1}\right) = 0,$$

where

$$g_{ij} = \delta_{ij} + \left[\frac{\partial}{\partial x_i}\left((\phi_2 + w) \circ \phi_1^{-1}\right)\right] \cdot \left[\frac{\partial}{\partial x_j}\left((\phi_2 + w) \circ \phi_1^{-1}\right)\right],$$

and G^{ij} is the corresponding matrix of cofactors. Hence on Ω_2, w satisfies a similar system of the form

$$\sum_{i,j=1}^{k+1} H^{ij} w_{ij} + h = 0.$$

It follows from (1) that the H^{ij} are uniformly bounded in $C^{0,\alpha}$ norm, that h is uniformly bounded in $C^{0,\alpha}$ norm, and that the system is uniformly elliptic. Along $\Omega_2 \cap \partial\Omega$, $w = 0$. In Ω_2, $\|w\|_{C^1} \to 0$. Therefore by Schauder estimates [**17**, Chapter 3, 1.12 and 1.13], in Ω_3, $\|w\|_{C^{2,\alpha}} \to 0$. By Lemma 3.2, $\|g|M \cap C_4\|_{C^{2,\alpha}} \to 0$. Since finitely many C_4's cover M, $\|g\|_{C^{2,\alpha}} \to 0$. Consequently, taking $f = \phi(\gamma) + g$,

$$T = f_\# S \quad \text{and} \quad \|f - \iota_M\|_{C^{2,\alpha}} \to 0.$$

Finally, since M has no nontangential Jacobi field which vanishes at the boundary, one can apply White's theorem (cf. Remark 3.1) to conclude that T is unique.

3.3. COROLLARY. *A curve in* $\mathbf{R}^n$ *sufficiently* $C^{2,\alpha}$ *close to a circle bounds a unique area-minimizing flat chain modulo two, which is an embedded disc.*

3.4. COROLLARY. *A regular point in a minimal surface has a neighborhood which is area-minimizing as a flat chain modulo two.*

REMARK. Corollary 3.4 improves the older theorem that a regular point in a minimal surface has a neighborhood which is area-minimizing as an integral current (cf. [**27**]).

Results such as Theorem 4 have been known to workers in the field and are implicit in the literature.
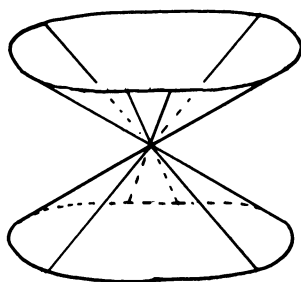

FIGURE 4. The area-minimizing cone over $\mathbf{S}^3 \times \mathbf{S}^3$ in $\mathbf{R}^8$, schematic

4. THEOREM. *Let S be an $(n - 1)$-dimensional area-minimizing integral current in $\mathbf{R}^n$ bounded by a compact, connected submanifold B of the unit sphere. Then S^2 is area-minimizing in the class of flat chains modulo two.*

REMARK. Every flat chain modulo two Y of finite mass has a representative modulo two. The following lemma of B. White gives a simultaneous representation for Y and ∂Y when Y is n-dimensional. It says that every compact hypersurface without boundary is orientable.

4.1. LEMMA [**25**, Corollary 2.3]. *Let Y be an n-dimensional flat chain modulo two in $\mathbf{R}^n$, such that Y and ∂Y have finite mass. Then there is an intgral current S such that S is a representative modulo two for Y and ∂S is a representative modulo two for ∂Y.*

REMARK. The following lemma asserts that two disjoint hypersurfaces with a common boundary are both orientable. It implies, for example, that any surface with the same boundary as a Möbius strip in $\mathbf{R}^3$ must intersect the interior of the Möbius strip.

4.2. LEMMA. *Let X_1, X_2 be $(n - 1)$-dimensional flat chains modulo two in $\mathbf{R}^n$ with $\partial X_1 = \partial X_2$. Suppose that $\mathrm{spt}^2 X_1 \cap \mathrm{spt}^2 X_2 = \mathrm{spt}^2 \partial X_1$ and $\mathcal{H}^{n-1}(\mathrm{spt}^2 \partial X_1) = 0$. Then X_1 has a representative modulo two R with $\mathrm{spt} \partial R = \mathrm{spt}^2 \partial X_1$.*

PROOF. Since $\partial X_1 = \partial X_2$, $X_1 - X_2 = \partial Y$, for some flat chain mod 2 of finite mass Y. By Lemma 4.1, there is an integral current S such that ∂S is a representative

mod 2 for $\partial Y = X_1 - X_2$. Put $R = \partial S \llcorner (\mathbf{R}^n - \mathrm{spt}^2 X_2)$. Then R is a representative mod 2 for X_1 and spt $\partial R \subset \mathrm{spt}^2 \partial X_1$. Hence spt $\partial R = \mathrm{spt}^2 \partial X_1$.

PROOF OF THEOREM 4. Let X_1 be any area-minimizing flat chain mod 2 bounded by B. We will show that $\mathbf{M}(S) \leqslant \mathbf{M}^2(X_1)$. Let X_2 be a flat chain mod 2 in the unit sphere bounded by B. Since X_1 is area-minimizing, $\mathrm{spt}^2 X_1$ intersects the sphere only in B; hence,

$$\mathrm{spt}^2 X_1 \cap \mathrm{spt}^2 X_2 = B.$$

By Lemma 4.2, X_1 has a representative mod 2 R with spt $\partial R = \mathrm{spt}^2 \partial X_1 = B$. Since B is connected, it follows by [7, 4.1.31(2) and 4.2.27] that $\partial R = kB$ for some nonzero integer k. Next it follows from [8, 5.10] that there is a rectifiable current T with $\partial T = B$ and $\mathbf{M}(T) = |1/k|\mathbf{M}(R) = |1/k|M^2(X_1)$. Finally, by the minimizing property of S, $\mathbf{M}(S) \leqslant \mathbf{M}(T) \leqslant \mathbf{M}^2(X_1)$.
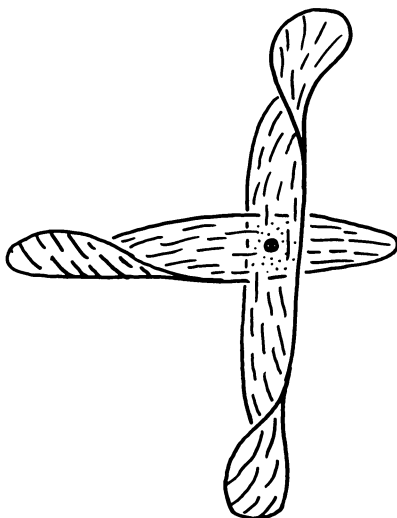

FIGURE 5. The union of two orthogonal area-minimizing surfaces

5. THEOREM. *For integers $2 \leqslant k \leqslant n$ let S_1, S_2 be k-dimensional area-minimizing flat chains modulo two. Suppose that their supports, $\mathrm{spt}^2 S_1$ and $\mathrm{spt}^2 S_2$, lie in subspaces P_1 and P_2 of $\mathbf{R}^n$ with*

$$(1) \qquad\qquad \dim\left(P_1 \cap P_2^{\perp}\right) \geqslant \dim P_1 - k + 2.$$

Then $S = S_1 + S_2$ is area-minimizing. Suppose, further, that

$$(2) \qquad\qquad \dim(P_1 \cap P_2) < k - 2$$

and S' is an area-minimizing flat chain modulo two with $\partial S' = \partial S$. Then $S' = S_1' + S_2'$ with $\mathrm{spt}^2 S_1' \subset P_1$, $\mathrm{spt}^2 S_2' \subset P_2$, $\partial S_1' = \partial S_1$, $\partial S_2' = \partial S_2$.

REMARK. The theorem, proof and Corollary 5.4 hold for rectifiable currents as well. In the special case when $P_1 \perp P_2$, it was proved that S is area-minimizing by H. Federer and W. Fleming in 1960 ([11, 9.15], cf. also [13, §2, Example]). The general case is new.

REMARK. The following lemma also plays a central role in the general study of calibrations. See [**21**, Lemma 4; **5**; **20** and **16**].

5.1. LEMMA. *Consider* $\mathbf{R}^{2m} \cong \mathbf{C}^m$ *with orthonormal basis* $e_1, ie_1, \ldots, e_m, ie_m$. *Let*

$$(3) \qquad \phi \in \bigotimes_{j=1}^{m} \left(\wedge^1 \mathrm{span}\{ e_j, ie_j \} \right) \subset \wedge^m \mathbf{R}^{2m}.$$

Then on the space of unit simple m-vectors, ϕ *has a maximum point of the form*

$$(4) \qquad \exp(i\alpha_1)e_1 \wedge \exp(i\alpha_2)e_2 \wedge \cdots \wedge \exp(i\alpha_m)e_m.$$

(5) *Furthermore, for any j, if there is an* α_j^0 *such that* ϕ *has no maximum of form* (4) *with* $\alpha_j = \alpha_j^0$, *then every maximum point* ξ *has a factor of the form* $\exp(i\alpha_j)e_j$.

(6) *In particular, if* ϕ *has only finitely many maxima of form* (4), *then all of the maxima of* ϕ *are of form* (4).

PROOF. Let $M = \|\phi\| \equiv \max\{\phi(\xi): \xi$ a unit simple m-vector$\}$. It suffices to prove (5), since the other statements are immediate consequences of (5). So assume that there is a $0 \leqslant \alpha_j^0 < 2\pi$ such that ϕ has no maximum of form (4) with $\alpha_j = \alpha_j^0$. For convenience we assume $j = 1$. It is not hard to show (cf. [**15**, Lemma II.7.5]) that there are orthonormal vectors $f_1, f_2, g_1, \ldots, g_m$ with $\mathrm{span}\{ f_1, f_2 \} = \mathrm{span}\{ e_1, ie_1 \}$ and angles $\theta_1, \theta_2 \in [0, \pi/2)$ such that ξ takes the form

$$\xi = (\cos\theta_1 f_1 + \sin\theta_1 g_1) \wedge (\cos\theta_2 f_2 + \sin\theta_2 g_2) \wedge g_3 \wedge \cdots \wedge g_m.$$

By hypothesis (3),

$$\phi(\xi) = a\cos\theta_1\sin\theta_2 + b\sin\theta_1\cos\theta_2$$

(where $a = \langle f_1 \wedge g_2 \wedge \cdots \wedge g_m, \phi \rangle, b = \langle g_1 \wedge f_2 \wedge \cdots \wedge g_m, \phi \rangle$)

$$\leqslant \sqrt{a^2\cos^2\theta_1 + b^2\sin^2\theta_1} \leqslant \max\{ |a|, |b| \} \leqslant M.$$

Hence, equality holds. Unless $a = b = M$, $\{\theta_1, \theta_2\} = \{0, \pi/2\}$ and ξ has as a factor f_1 or f_2, as desired. If $a = b = M$, $f_1 \wedge g_2 \wedge \cdots \wedge g_m$ is a maximum point of ϕ, and hence

$$0 = \frac{d}{dt} \langle f_1 \wedge (\cos tg_2 + \sin tf_2) \wedge \cdots \wedge g_m, \phi \rangle \big|_{t=0}$$

$$= \langle f_1 \wedge f_2 \wedge \cdots \wedge g_m, \phi \rangle.$$

Similarly, $0 = \langle g_1 \wedge g_2 \wedge \cdots \wedge g_m, \phi \rangle$. Therefore, for any θ,

$$\langle (\cos\theta f_1 + \sin\theta f_2) \wedge (-\sin\theta g_1 + \cos\theta g_2) \wedge g_3 \wedge \cdots \wedge g_m, \phi \rangle$$

$$= M\cos^2\theta + M\sin^2\theta + 0 + 0 = M.$$

Choosing θ so that $\exp(i\alpha_1^0)e_1 = \cos\theta f_1 + \sin\theta f_2$ shows that $\|\exp(i\alpha_1^0)e_1 \lrcorner \phi\| = M$. But by induction $\exp(i\alpha_1^0)e_1 \lrcorner \phi$ has a maximum of the form $\exp(i\alpha_2)e_2 \wedge \cdots \wedge \exp(i\alpha_m)e_m$. Therefore ϕ has a maximum point of the form

$$\exp(i\alpha_1^0)e_1 \wedge \exp(i\alpha_2)e_2 \wedge \cdots \wedge \exp(i\alpha_m)e_m,$$

and the hypothesis of (5) fails to hold.

5.2. LEMMA. *Let P_1, P_2 be subspaces of $\mathbf{R}^n$ with*

$$(7) \qquad\qquad \dim\left(P_1 \cap P_2^{\perp} \right) \geqslant \dim P_1 - k + 2.$$

Let ζ be a unit simple vector in $\bigwedge_k \mathbf{R}^n$. Then the projections of ζ satisfy $|P_1\zeta| + |P_2\zeta| \leqslant 1$. If furthermore, $\dim(P_1 \cap P_2) < k - 2$, *then $|P_1\zeta| + |P_2\zeta| = 1$ if and only if ζ lies in P_1 or in P_2.*

REMARK. The following converse holds. Suppose P_1, P_2 are subspaces of $\mathbf{R}^n$ of dimension at least k, such that for all unit simple $\zeta \in \bigwedge_k \mathbf{R}^n$,

$$(8) \qquad\qquad |P_1\zeta| + |P_2\zeta| \leqslant 1.$$

Then $\dim(P_1 \cap P_2^{\perp}) \geqslant \dim P_1 - k + 2$. Suppose, furthermore, that equality in (8) holds only if ζ lies in P_1 or in P_2. Then $\dim(P_1 \cap P_2) < k - 2$.

PROOF OF LEMMA 5.2. We may assume that $P_1\zeta$ and $P_2\zeta$ are nonzero, and for convenience that $n \geqslant 2k$. Let u_1, u_2 be unit k-vectors such that $P_j\zeta$ is a positive multiple of u_j. There are orthonormal vectors

$$e_1, \ldots, e_k, ie_1, \ldots, ie_k$$

such that

$$u_1 = e_1 \wedge \cdots \wedge e_k, \quad u_2 = \pm\exp(i\theta_1)e_1 \wedge \cdots \wedge \exp(i\theta_k)e_k$$

with $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_k \leqslant \pi/2$ (cf. [15, II.7.5]). By (7), $\theta_{k-1} = \theta_k = \pi/2$. By replacing ie_k by $-ie_k$ if necessary, we may assume the ambiguous sign on u_2 is positive. For any vector x, let x^* denote the dual covector, and put $\phi = u_1^* + u_2^*$. By Lemma 5.1, on the space of unit simple k-vectors, ϕ has a maximum point ξ of the form

$$(9) \qquad\qquad \xi = \exp(i\alpha_1)e_1 \wedge \cdots \wedge \exp(i\alpha_k)e_k.$$

We compute that

$$
\begin{aligned}
\phi(\xi) &= \cos\alpha_1 \cdots \cos\alpha_k + \cos(\theta_1 - \alpha_1)\cdots\cos(\theta_k - \alpha_k) \\
(10) \qquad &\leqslant |\cos\alpha_{k-1}\cos\alpha_k| + |\cos(\theta_{k-1} - \alpha_{k-1})\cos(\theta_k - \alpha_k)| \\
&= |\cos\alpha_{k-1}\cos\alpha_k| + |\sin\alpha_{k-1}\sin\alpha_k| \\
&\leqslant \left(\cos^2\alpha_{k-1} + \sin^2\alpha_{k-1}\right)^{1/2}\left(\cos^2\alpha_k + \sin^2\alpha_k\right)^{1/2} = 1.
\end{aligned}
$$

Therefore,

$$|P_1\zeta| + |P_2\zeta| = \phi(\zeta) \leqslant \phi(\xi) \leqslant 1.$$

If, furthermore, $\dim(P_1 \cap P_2) < k - 2$, then $\theta_{k-2} > 0$. We suppose that $|P_1\zeta| + |P_2\zeta| = \phi(\zeta) = 1$ and show that $\zeta = u_1$ or $\zeta = u_2$, which will prove that ζ lies in P_1 or in P_2. By 5.1(6), it suffices to show that u_1 and u_2 are the only maxima of ϕ of the form (4). But for any such maximum ξ, equality must hold in (10). Since $\theta_{k-2} > 0$, either $\alpha_1 = \cdots = \alpha_k = 0$ or $(\theta_1 - \alpha_1) = \cdots = (\theta_k - \alpha_k) = 0$. Hence $\xi = u_1$ or $\xi = u_2$.

5.3. COROLLARY TO LEMMA. *Let P_1, P_2 be subspaces of $\mathbf{R}^n$ with $\dim(P_1 \cap P_2^{\perp}) \geqslant \dim P_1 - k + 2$. Let T be any flat chain modulo two of finite mass. Then*

$$(11) \qquad\qquad \mathbf{M}^2(T) \geqslant \mathbf{M}^2(P_{1\#}T) + \mathbf{M}^2(P_{2\#}T).$$

Furthermore, if $\dim(P_1 \cap P_2) < k - 2$ *and equality holds in* (11), *then for* $\|T\|^2$-*almost all* $x \in \mathbf{R}^n$, $\vec{T}$ *lies in* P_1 *or in* P_2.

PROOF OF THEOREM 5. Let $T \in \mathscr{R}_k^2 \mathbf{R}^n$ with $\partial T = \partial S$. Then by Corollary 5.3,

$$\mathbf{M}^2(T) \geqslant \mathbf{M}^2(P_{1\#}T) + \mathbf{M}^2(P_{2\#}T) \geqslant \mathbf{M}^2(S_1) + \mathbf{M}^2(S_2) = \mathbf{M}^2(S)$$

because S_1 and S_2 are area-minimizing and $\partial P_{j\#}T = \partial P_{j\#}S = \partial S_j$. Hence S is area-minimizing.

Furthermore, suppose that $\dim(P_1 \cap P_2) < k - 2$ and $\mathbf{M}^2(T) = \mathbf{M}^2(S)$. By Corollary 5.3, T has a decomposition $T = T_1 + T_2$ in $\mathscr{R}_k^2 \mathbf{R}^n$ such that for $\|T_j\|$-almost all $x \in \mathbf{R}^n$, $\vec{T_j}$ lies in P_j and $\mathbf{M}^2(T) = \mathbf{M}^2(T_1) + \mathbf{M}^2(T_2)$. Let $X = \partial T_1 - \partial S_1 = \partial S_2 - \partial T_2 \in \mathscr{F}_{k-1}^2 \mathbf{R}^n$. Since $P_{1\#}X = P_{1\#}(\partial S_2 - \partial T_2) = \partial(P_{1\#}S_2 - P_{1\#}T_2) = 0$, $\mathrm{spt}^2 X \subset P_2$. Similarly $\mathrm{spt}^2 X \subset P_1$, so that $\mathrm{spt}^2 X \subset P_1 \cap P_2$. Now $X = \partial Y$ for some $Y \in \mathscr{R}_k^2 \mathbf{R}^n$ with $\mathrm{spt}^2 Y \subset P_1 \cap P_2$. But since $\mathscr{H}^k(P_1 \cap P_2) = 0$, $Y = 0$ (cf. [7, p. 432]). Therefore $X = 0$, $\partial T_1 = \partial S_1$ and $\partial T_2 = \partial S_2$. Since

$$\mathbf{M}^2(T_1) + \mathbf{M}^2(T_2) = \mathbf{M}^2(T) = \mathbf{M}^2(S) = \mathbf{M}^2(S_1) + \mathbf{M}^2(S_2),$$

T_1 and T_2 are area-minimizing. Hence $\mathrm{spt}\, T_1 \subset P_1$ and $\mathrm{spt}\, T_2 \subset P_2$.

5.4. COROLLARY TO THEOREM. *Let* R_1, R_2 *be* l-*dimensional area-minimizing flat chains modulo two in* $\mathbf{R}^n$, *regular on the interior; i.e.,* $M_i = \mathrm{spt}^2 R_i - \mathrm{spt}^2 \partial R_i$ *is a smooth manifold. Then* $M_1 \cap M_2$ *is the interior singular set of some area-minimizing flat chain modulo two. More precisely, with* $m = n - l + 2$, *there is an area-minimizing flat chain modulo two* $T \in \mathscr{R}_{l+m}^2 \mathbf{R}^{n+2m}$ *such that the set of points where* $\mathrm{spt}^2 T - \mathrm{spt}^2 \partial T$ *fails to be a smooth manifold is* $(M_1 \cap M_2) \times \{0\} \subset \mathbf{R}^n \times \mathbf{R}^{2m}$. *In particular, taking* $R_2 = R_1$, *we see that the interior of any regular area-minimizing flat chain modulo two is the interior singular set of some other area-minimizing flat chain modulo two.*

REMARKS. From two-dimensional area-minimizing flat chains modulo two in $\mathbf{R}^3$, the above corollary constructs five-dimensional area-minimizing flat chains modulo two in $\mathbf{R}^9$ with interior singular sets consisting of a smooth nonlinear curve or of an even number of nonlinear curves meeting at a point. These seem to be the first examples of nonlinear singular sets in area-minimizing flat chains modulo two. The corollary also holds for rectifiable currents and provides the first examples of nonlinear curves as singular sets in area-minimizing rectifiable currents.

PROOF. Let $k = l + m$ and define flat chains mod 2, S_1, $S_2 \in \mathscr{R}_k^2(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^m)$, as the following Cartesian products of R_1, R_2 and m-cubes:

$$S_1 = R_1 \times [-1, 1]^m \times \{0\}, \quad S_2 = R_2 \times \{0\} \times [-1, 1]^m.$$

Let $T = S_1 + S_2$. S_1 and S_2 are area-minimizing flat chains mod 2 (cf. Remark on Theorem 6), supported in $P_1 = \mathbf{R}^n \times \mathbf{R}^m \times \{0\}$ and $P_2 = \mathbf{R}^n \times \{0\} \times \mathbf{R}^m$. Since

$$\dim\left(P_1 \cap P_2^{\perp}\right) = m = n - l + 2 = m + n - k + 2 = \dim P_1 - k + 2,$$

Theorem 5 implies that $T = S_1 + S_2$ is area-minimizing. It is easy to check that T has the desired singular set.

REMARK ON THEOREM 6. It is well known that if S is an area-minimizing flat chain modulo two, then the Cartesian product of S with any interval is area-minimizing.

The following theorem, under merely a stability hypothesis on S, concludes that the Cartesian product of S with a sufficiently short interval is area-minimizing.

6. THEOREM. *Suppose S is a k-dimensional flat chain modulo two in $\mathbf{R}^n$, ∂S has finite mass, and S satisfies the following stability condition: there exists $\varepsilon > 0$ such that if T is a flat chain modulo two, $\partial T = \partial S$, and $\mathscr{F}^2(T - S) < \varepsilon$, then $\mathbf{M}^2(T) \geqslant \mathbf{M}^2(S)$.*

Then there is a $\delta > 0$ such that the Cartesian product of S with an interval of length δ is area-minimizing in $\mathbf{R}^{n+1}$.

PROOF. Take any $\delta < \varepsilon/\mathbf{N}^2(S)$ $(\mathbf{N}^2(S) = \mathbf{M}^2(S) + \mathbf{M}^2(\partial S))$. Assume that $S \times [0, \delta]^2$ is not area-minimizing. Then there is a $Q \in \mathbf{I}^2_{k+2}(\mathbf{R}^{n+1})$ with

$$\partial Q = \partial\big(S \times [0, \delta]^2\big) \quad \text{and} \quad \mathbf{M}^2(Q) < \mathbf{M}^2\big(S \times [0, \delta]^2\big).$$

We may assume $\mathrm{spt}^2(Q - S \times [0, \delta]^2) \subset \mathbf{R}^n \times [0, \delta]$. In fact, by increasing δ slightly, we may assume $\mathrm{spt}^2(Q - S \times [0, \delta]^2) \subset \mathbf{R}^n \times (0, \delta]$. For $r \in [0, \delta]$ let

$$Q_r = \langle Q, x_{n+1}, r + \rangle$$

be the slice of Q by the hyperplane $x_{n+1} = r$ (cf. [7, p. 429]). Since

$$\int_{r=0}^{\delta} \mathbf{M}^2(Q_r)\, dr \leqslant \mathbf{M}^2(Q) < \mathbf{M}^2\big(S \times [0, \delta]^2\big) = \int_{r=0}^{\delta} \mathbf{M}^2(S)\, dr,$$

we can choose $r \in (0, \delta)$ such that $\mathbf{M}^2(Q_r) < \mathbf{M}^2(S)$. Since

$$Q_r - Q_0 = \partial\big[Q \llcorner \{0 < x_{n+1} \leqslant r\}\big] - (\partial Q) \llcorner \{0 < x_{n+1} \leqslant r\},$$

we have

$$\mathscr{F}^2(Q_r - Q_0) \leqslant \mathbf{M}^2(Q) + \mathbf{M}^2(\partial Q) \llcorner \{0 < x_{n+1} < \delta\}$$
$$< \mathbf{M}^2\big(S \times [0, \delta]^2\big) + \mathbf{M}^2\big((\partial S) \times [0, \delta]^2\big)$$
$$\leqslant \delta \mathbf{N}^2(S).$$

Now let p denote orthogonal projection of $\mathbf{R}^n \times \mathbf{R}$ into $\mathbf{R}^n$. Since

$$\mathrm{spt}\big(Q - S \times [0, \delta]^2\big) \subset \mathbf{R}^n \times (0, \delta],$$
$$p_{\#} Q_0 = p_{\#}\big(\langle S \times [0, \delta]^2, x_{n+1}, 0 \rangle\big) = S.$$

Put $T = p_{\#} Q_r \in \mathscr{R}^2_k \mathbf{R}^n$. One computes that

$$\partial T = p_{\#}\langle \partial Q, x_{n+1}, r + \rangle = p_{\#}\big\langle \partial\big(S \times [0, \delta]^2\big), x_{n+1}, r + \big\rangle$$
$$= p_{\#}(\partial S \times \delta_r) = \partial S,$$
$$\mathscr{F}^2(T - S) = \mathscr{F}^2\big(p_{\#}(Q_r - Q_0)\big) \leqslant \mathscr{F}^2(Q_r - Q_0) < \delta \mathbf{N}^2(S) < \varepsilon.$$

But $\mathbf{M}^2(T) \leqslant \mathbf{M}^2(Q_r) < \mathbf{M}^2(S)$, a contradiction of the stability condition on S.

6.1. *Question. Is the Cartesian product of two area-minimizing flat chains modulo two area-minimizing?*

(Although, in general, the Cartesian product of rectifiable currents need not be rectifiable, the almost everywhere regularity of area-minimizing flat chains modulo two insures that their product is a flat chain modulo two of finite mass.) The question is open also for normal currents (cf. [20]) but settled negatively for integral currents (cf. [19, 1.2]).

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