Continuous Groups

- We studied space , time infinitesimal translations

$$x^\mu \to x'^\mu = (1 + \frac{i}{\hbar}\epsilon^\mu.\hat{O}_\mu)x^\mu = x^\mu + \epsilon^\mu$$

- Infinitesimal transformation will give parameters ϵ^μ and generators $\hat{O}_\mu$
- For rotations in space, show that the generators are angular momentum **L** and the rotation angles are parameters
- In spacetime, we have Lorentz transformation- rotations in space and boost transformations giving six parameters and six generators

observation: no of generators equal to # of parameters

Symmetries

- Free particle is described by Hamiltonian $P^2/2m$
- Note that the commutator [H,**p**]**= 0**
- Under space translation, H is invariant which implies the free particle system possesses translation symmetry
- The generators of translation must commute with the H
- Does the free particle system possess rotational symmetry?

Rotations

- Under infinitesimal rotation, the change in position vector will be $\delta\mathbf{r} = \delta\boldsymbol{\vartheta} \times \mathbf{r}$
- Under rotation operation R, the wavefunction obeys $\Psi(\mathbf{r}) = \Psi'(R\mathbf{r}=\mathbf{r} + \delta\mathbf{r})$
- Do similar steps which we did for translation to determine the exponential form denoting rotation operator

Rotations

Infinitesimal rotations $R(\boldsymbol{\delta\theta}) = \mathbb{I} - i\delta\theta\hat{n}.\boldsymbol{L}$

Verify the relation

$$\mathbf{R}(\delta\theta\hat{\mathbf{i}})\mathbf{R}(\delta\theta\hat{\mathbf{j}}) - \mathbf{R}(\delta\theta\hat{\mathbf{j}})\mathbf{R}(\delta\theta\hat{\mathbf{i}}) = \mathbf{R}(\delta\theta^2\hat{\mathbf{k}}) - \mathbb{I}$$

This implies our familiar angular momentum algebra

$$[L_x, L_y] = iL_z$$

Orthogonal Group O(3)

- O(3) consists of set of 3 $\times$3 orthogonal matrices (determinant +1 or -1)

- **Direct product with inversion group**

$$O(3) = SO(3) \otimes C_S$$

- Each element of SO(3) will be specified by three parameters

$$R_{\hat{\mathbf{n}}}(\psi) = R(\psi, \theta, \phi)$$

Special Orthogonal Group SO(3)

Rotation by angle ψ about the direction $\hat{\mathbf{n}} = (\theta, \phi)$:

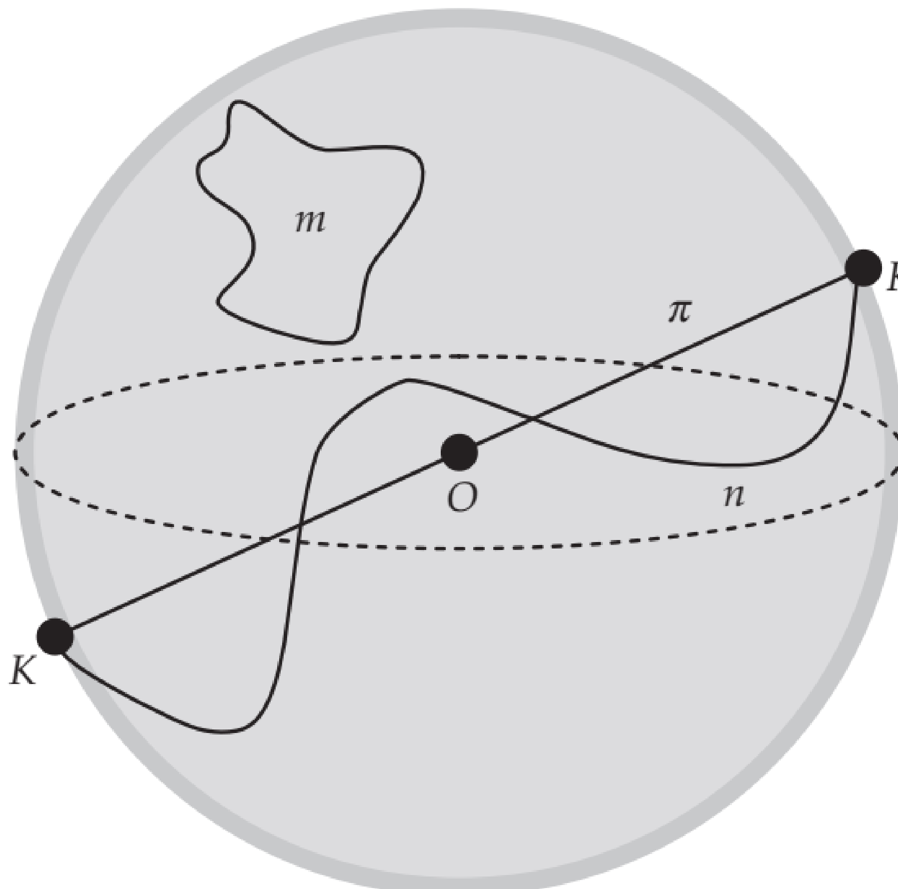
$R_{-\hat{\mathbf{n}}}(\psi) = R_{\hat{\mathbf{n}}}(-\psi)$ which implies $0 \le \psi \le \pi$

$R_{-\hat{\mathbf{n}}}(\pi) = R_{\hat{\mathbf{n}}}(\pi)$

Group manifold is a sphere of radius π.

$\therefore$ SO(3) is a compact group.

SO(3) group manifold



Group manifold is doubly connected

Lorentz Group

Generators are

$$\hat{O}_{\mu\nu} = \hat{L}_{\mu\nu} \text{ where } \hat{L}_{0i} = \hat{K}_i \text{ are boosts}$$
$$\text{and } \hat{L}_{ij} = \epsilon_{ijk}\hat{L}_k \text{ are the rotations in space.}$$

Parameters are $\varepsilon_{\mu\nu}$

Find the algebra of these generators

Special orthogonal group

- Set of n x n orthogonal matrices with det = +1 forms group SO(n)
- These matrices will leave magnitude of position vector in n-dimensional space invariant

$\sum_{i=1}^{n}(x_i x_i)$ = constant

- SO(m,n) refers to (m+n) x (m+n) matrices satisfying $\sum_{i=1}^{n}(x_i x_i) - \sum_{\mathrm{j}=1}^{\mathrm{m}}(y_j y_j) = const$

Lorentz group is SO(3,1)- why?

Lie Algebra

- Lie algebra **g is a vector space on which is defined a binary operation having the following properties**

(1) For all x and y in $\mathfrak{g}$, $[x, y]$ is in $\mathfrak{g}$.

(2) For all x, y and z in $\mathfrak{g}$, and scalars λ and μ, $[\lambda x + \mu y, z] = \lambda[x, z] + \mu[y, z]$.

(3) $[x, y] = -[y, x]$.

(4) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$. [Jacobi identity]

[,] is called Lie Bracket

Lie Bracket [x,y]=0 for all x,y
implies Lie algebra is abelian

Lie algebra continued

- **$[x_s, x_t] = C_{st}^{k} x_k$ where C_{st}^{k} are the structure constants which are antisymmetric is s,t indices**
- **Lie subalgebra $\mathfrak{h}$ is a subset of elements of a Lie algebra g such that the elements of forms a Lie algebra**
- **Further, if [g,h] ε** $\mathfrak{h}$, then $\mathfrak{h}$ is an **invariant subalgebra**

Examples of Lie algebra

- Consider a two dimensional complex vector space. The linear operators acting on such a vector space are 2x2 matrices with complex entries. The Lie algebra is denoted as $\mathfrak{gl}(2, \mathbb{C})$ spanned by

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- It is a 4-dimensional complex vector space of linear operators

Examples Lie algebra

- Subalgebras of $\mathfrak{gl}(2, \mathbb{C})$ - One example is the set of traceless & Hermitian matrices denoted as $\mathfrak{sl}(2, \mathbb{C})$

$$X = \begin{bmatrix} a & z \\ z^* & -a \end{bmatrix}$$

- $\mathfrak{sl}(2, \mathbb{C})$ is a 3-dimensional complex vector space of linear operators
- 3-dimensional real subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ is our familiar su(2) algebra (angular momentum algebra)

Special Unitary group

- SU(2) is obtained by exponential map of su(2) Lie algebra generators (three Pauli matrices)

$$g(\theta\hat{n}) = \cos(\theta/2)\mathbb{I} + i\hat{n}.\sigma\sin(\theta/2),$$

- Unlike SO(3), $g(2\pi) \neq g(0)$ and $g(4\pi) = \mathbf{I}$
- SU(2) group manifold is a solid sphere of radius 2π which is a simply connected manifold
- Two element of SU(2) is mapped to one element of SO(3)– **two to one mapping**

  [double cover of SO(3)]

Lie group

- General linear group of degree n is a set of invertible nxn matrices under matrix multiplication
- Matrices with real entries are GL(n, R) and matrices with complex entries are GL(n,C)
- Subgroups of GL groups are SL(n,R) and SL(n,C)
- Orthogonal groups are subgroups of GL(n,R)
- Symplectic groups Sp(2n,R) are another subgroup of SL(2n,R)