INFINITESIMAL CONFORMAL TRANSFORMATIONS ON TANGENT BUNDLES WITH THE LIFT METRIC I+II

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Abstract. Let M be a complete, simply connected Riemannian manifold with positive constant scalar curvature, and TM its tangent bundle with the lift metric I+II. If TM admits an essential infinitesimal conformal transformation, then M is isometric to the standard sphere. Furthermore if M is compact, then the assumption "essential" is reduced to "non-homothetic".

1. Introduction

Let $(M,\ g)$ be a Riemannian manifold and ∇ its Riemannian connection. A transformation f of M is called a projective transformation if it preserves the geodesics, where each geodesic should be confounded with a subset of M by neglecting its affine parameter. Furthermore, f is called an affine transformation if it preserves the connection ∇. We then remark that an affine transformation may be characterized as a projective transformation which preserves the affine parameter together with geodesics.

Let V be a vector field on M, and let us consider a local one-parameter group $\{f_t\}$ of local transformations of M generated by V. Then V is called an infinitesimal conformal transformation if each f_t is a local conformal transformation. V is called an infinitesimal projective transformation if each f_t is a local projective transformation. Similarly V is called an infinitesimal affine transformation if each f_t is a local affine transformation. Clearly an infinitesimal affine transformation is an infinitesimal projective transformation. The converse is not true in general. Indeed the standard sphere $S^n(c)$ with the the radius $\dfrac{1}{\sqrt{c}}$, which is a space of positive constant curvature c, admits a non-affine infinitesimal projective transformation.

As a converse problem, the following conjecture is known.

Conjecture A. *Let M be a complete, simply connected Riemannian manifold with positive constant scalar curvature. Assume that M admits a non-affine infinitesimal projective transformation. Then is M isometric to the standard sphere*!)

Let TM be the tangent bundle over M. Then we can consider some lift metrics on TM, for example, the complete lift metric, the lift metric I+II, etc.

Recently one of the authors proved the following

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Theorem A ([Y3]). *Let (M, g) be a complete, simply connected Riemannian manifold with positive constant scalar curvature and TM its tangent bundle with the complete lift metric. If TM admits an essential infinitesimal conformal transformation, then M is isometric to the standard sphere.*

In this paper, we prove the following

Theorem 1. *Let (M, g) be a complete, simply connected Riemannian manifold with positive constant scalar curvature and TM its tangent bundle with the lift metric* I+II*. If TM admits an essential infinitesimal conformal transformation, then M is isometric to the standard sphere.*

Theorem 2. *Let (M, g) be a compact, simply connected Riemannian manifold with positive constant scalar curvature and TM its tangent bundle with the lift metric* I+II*. If TM admits a non-homothetic infinitesimal conformal transformation, then M is isometric to the standard sphere.*

Therefore we have the following new conjecture.

Conjecture B. *Let (M, g) be a complete, simply connected Riemannian manifold with positive constant scalar curvature and TM its tangent bundle with the lift metric* I+II*. Assume that TM admits a non-homothetic infinitesimal conformal transformation. Then is M isometric to the standard sphere?*

In the present paper everything will be always discussed in the C^∞-category, and Riemannian manifolds will be assumed to be connected and dimension $n > 1$.

2. Preliminaries

Let (M, g) be a Riemannian manifold and ∇ its Riemannian connection. Let V be a vector field on M. It is well-known that V is an infinitesimal conformal transformation if and only if there exists a function ψ on M satisfying

$$L_V\, g = \psi\, g,$$

where L_V is the Lie derivation with respect to V. In this case, ψ is called the associated function of V. Especially, if ψ is constant, then V is called an infinitesimal homothetic transformation. Furthermore, V is an infinitesimal isometry if and only if ψ is zero constant. A vector field V on M is an infinitesimal projective transformation if and only if there exists a 1-form Ω on M satisfying

$$(L_V \nabla)(X,\, Y) = \Omega(X)Y + \Omega(Y)X$$

for any $X,\ Y \in \mathrm{T}^1_0(M)$. In this case, Ω is called the associated 1-form of V.

We have the following Tanno's Theorem ([O], [T]):

Lemma 1. *Let (M, g) be a complete, simply connected Riemannian manifold. In order that M admits a non-constant scalar function f on M satisfying*

$$\nabla_k \nabla_j \nabla_i f + c(2g_{kj}\nabla_i f + g_{ki}\nabla_j f + g_{ji}\nabla_k f) = 0$$

for some positive constant c, it is necessary and sufficient that M is isometric to the standard sphere of radius $\dfrac{1}{\sqrt{c}}$.

One of the authors proved the following

Lemma 2 ([Y1]). *Let $(M,\ g)$ be a compact, simply connected Riemannian manifold with constant scalar curvature S. If M admits an non-affine infinitesimal projective transformation, then S is positive and M is isometric to the standard sphere.*

Let $\Gamma_{ji}{}^h$ be the coefficients of ∇, i.e., $\nabla_{\partial_j}\partial_i =: \Gamma_{ji}{}^a\partial_a$, where $\partial_h := \dfrac{\partial}{\partial x^h}$ and (x^h) is the local coordinates of M. We define a local frame $\{E_i,\ E_{\bar{i}}\}$ of TM as follows:

$$E_i := \partial_i - y^b\Gamma_{ib}{}^a\partial_{\bar{a}} \quad \text{and} \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $(x^h,\ y^h)$ is the induced coordinates of TM and $\partial_{\bar{i}} := \dfrac{\partial}{\partial y^i}$. $\{E_i,\ E_{\bar{i}}\}$ is called the adapted frame of TM. Then $\{dx^h,\ \delta y^h\}$ is the dual frame of $\{E_i,\ E_{\bar{i}}\}$, where $\delta y^h := dy^h + y^b\Gamma_{ab}{}^h dx^a$.

By straightforward calculations, we have the following

Lemma 3. *The Lie brackets of the adapted frame of TM satisfy the following identities:*

(1) $[E_j,\ E_i] = y^bK_{ijb}{}^aE_{\bar{a}}$,

(2) $[E_j,\ E_{\bar{i}}] = \Gamma_{ji}{}^aE_{\bar{a}}$,

(3) $[E_{\bar{j}},\ E_{\bar{i}}] = 0$,

where $K = (K_{kji}{}^h)$ denotes the Riemannian curvature tensor of M defined by $K_{kji}{}^h := \partial_k\Gamma_{ji}{}^h - \partial_j\Gamma_{ki}{}^h + \Gamma_{ji}{}^a\Gamma_{ka}{}^h - \Gamma_{ki}{}^a\Gamma_{ja}{}^h$.

Lemma 4. *Let $\widetilde{V}$ be a vector field on TM. Then*

(1) $L_{\widetilde{V}}E_i = [\widetilde{V},\ E_i] = -(E_i\widetilde{V}^a)E_a + (\widetilde{V}^cy^bK_{icb}{}^a - \widetilde{V}^b\Gamma_{bi}{}^a - E_i\widetilde{V}^{\bar{a}})E_{\bar{a}}$,

(2) $L_{\widetilde{V}}E_{\bar{i}} = [\widetilde{V},\ E_{\bar{i}}] = -(\partial_{\bar{i}}\widetilde{V}^a)E_a + (\widetilde{V}^b\Gamma_{bi}{}^a - \partial_{\bar{i}}\widetilde{V}^{\bar{a}})E_{\bar{a}}$,

(3) $L_{\widetilde{V}}dx^h = (E_a\widetilde{V}^h)dx^a + (\partial_{\bar{a}}\widetilde{V}^h)\delta y^a$,

(4) $L_{\widetilde{V}}\delta y^h = \{y^c\widetilde{V}^bK_{bac}{}^h + \widetilde{V}^b\Gamma_{ba}{}^h + E_a\widetilde{V}^{\bar{h}}\}dx^a - (\widetilde{V}^b\Gamma_{ba}{}^h - \partial_{\bar{a}}\widetilde{V}^{\bar{h}})\delta y^a$,

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) = \widetilde{V}^aE_a + \widetilde{V}^{\bar{a}}E_{\bar{a}} := \widetilde{V}$.

We denote by $\mathrm{T}^r_s(M)$ the set of all tensor fields of type $(r,\ s)$ on M. Similarly, we denote by $\mathrm{T}^r_s(TM)$ the corresponding set on TM.

3. Infinitesimal conformal transformations on TM

Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle. The lift metric I+II is defined by $\widetilde{g} = g_{ba}dx^bdx^a + 2g_{ba}dx^b\delta y^a$. A vector field $\widetilde{V}$ on TM is an infinitesimal conformal transformation if and only if there exists a function $\widetilde{\rho}$ on TM such that

$$L_{\widetilde{V}}\widetilde{g} = 2\widetilde{\rho}\ \widetilde{g}.$$

The infinitesimal conformal transformation $\widetilde{V}$ is said to be *essential* if $\widetilde{\rho}$ depends on (y^h) essentially ([Y3]). The infinitesimal conformal transformation $\widetilde{V}$ on TM is said to be *inessential* if $\widetilde{\rho}$ depends only on (x^h).

Proposition. *Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle with the lift metric* I+II. *Then $\widetilde{V}$ is an infinitesimal conformal transformation with the associated function $\widetilde{\rho}$ on TM if and only if there exist $\psi \in \mathrm{T}^0_0(M),\ B=(B^h),\ C=(C^h),\ \Phi=(\Phi^h)\in \mathrm{T}^1_0(M)$and $A=(A_i{}^h)\in \mathrm{T}^1_1(M)$ satisfying*

(3.1) $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) = (B^h + y^a A_a{}^h,\ C^h - y^a(A_a{}^h + g^{hb}\nabla_b B_a) + (2\psi + y^a\Phi_a)y^h)$,

(3.2) $\widetilde{\rho} = \psi + y^a\Phi_a$,

(3.3) $A_{ji} + A_{ij} = 0$,

(3.4) $\nabla_j A_i{}^h = \Phi_i\delta^h_j - \Phi^h g_{ji}$,

(3.5) $L_{B+C}\ g_{ji} = \nabla_j C_i + \nabla_i C_j + \nabla_j B_i + \nabla_i B_j = 2\psi g_{ji}$,

(3.6) $L_B \Gamma_{ji}{}^h = \nabla_j\nabla_i B^h + K_{aji}{}^h B^a = \Psi_j\delta^h_i + \Psi_i\delta^h_j - g_{ji}\Phi^h$,

(3.7) $\nabla_j\Phi_i + \nabla_i\Phi_j = 0$,

(3.8) $K_{akji}A_h{}^a = -g_{kj}\nabla_i\Phi_h + g_{ki}\nabla_j\Phi_h$,

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) := \widetilde{V}^a E_a + \widetilde{V}^{\bar{a}}E_{\bar{a}} = \widetilde{V}$, $\Psi_i := \partial_i\psi$, *and* $A_{ji} := A_j{}^a g_{ai}$ *etc.*

Proof. Here we prove only the necessary condition of Proposition because it is easy to prove the sufficient condition.

Let $\widetilde{V}$ be an infinitesimal conformal transformation with the associated function $\widetilde{\rho}$ on TM. Here we have

$$\begin{aligned}(L_{\widetilde{V}}\widetilde{g}) = &\{2g_{aj}(y^c\widetilde{V}^b K_{bic}{}^a + \widetilde{V}^{\bar{b}}\Gamma_{bi}{}^a + E_i\widetilde{V}^{\bar{a}}) + \widetilde{V}^a\partial_a g_{ji} + 2g_{ai}E_j\widetilde{V}^a\}dx^j dx^i\\ &+ 2\{\widetilde{V}^a\partial_a g_{ji} + g_{ai}E_j\widetilde{V}^a - g_{aj}(\widetilde{V}^b\Gamma_{ib}{}^a - \partial_{\bar{i}}\widetilde{V}^{\bar{a}}) + g_{aj}\partial_{\bar{i}}\widetilde{V}^a\}dx^j\delta y^i\\ &+ 2g_{ai}(\partial_{\bar{j}}V^a)\delta y^j\delta y^i.\end{aligned}$$

From $L_{\widetilde{V}}\widetilde{g} = 2\widetilde{\rho}\ \widetilde{g} = 2\widetilde{\rho}\ g_{ji}dx^j dx^i + 4\widetilde{\rho}\ g_{ji}dx^j\delta y^i$, we obtain

$$g_{ai}\partial_{\bar{j}}\widetilde{V}^a + g_{aj}\partial_{\bar{i}}\widetilde{V}^a = 0, \tag{3.9}$$

$$\widetilde{V}^a\partial_a g_{ji} + g_{ai}E_j\widetilde{V}^a - g_{aj}(\widetilde{V}^b\Gamma_{ib}{}^a - \partial_{\bar{i}}\widetilde{V}^a - \partial_{\bar{i}}\widetilde{V}^{\bar{a}}) = 2\widetilde{\rho}\ g_{ji} \tag{3.10}$$

and

$$\begin{aligned}&\widetilde{V}^a\partial_a g_{ji} + g_{aj}(y^c\widetilde{V}^b K_{bic}{}^a + \widetilde{V}^{\bar{b}}\Gamma_{bi}{}^a + E_i\widetilde{V}^a + E_i\widetilde{V}^{\bar{a}})\\ &+ g_{ai}(y^c\widetilde{V}^b K_{bjc}{}^a + \widetilde{V}^{\bar{b}}\Gamma_{bj}{}^a + E_j\widetilde{V}^a + E_j\widetilde{V}^{\bar{a}})\\ &= 2\widetilde{\rho}\ g_{ji}.\end{aligned} \tag{3.11}$$

From (3.9) there exist $B=(B^h)\in \mathrm{T}^1_0(M)$ and $A=(A_i{}^h)\in \mathrm{T}^1_1(M)$ satisfying

$$\widetilde{V}^h = B^h + y^a A_a{}^h \quad \text{and} \quad A_{ji} + A_{ij} = 0. \tag{3.12}$$

Substituting (3.12) into (3.10), we have

$$\nabla_j B_i - A_{ji} + g_{aj}\partial_{\bar{i}}\widetilde{V}^{\bar{a}} + y^a\nabla_j A_{ai} = 2\widetilde{\rho}\ g_{ji}. \tag{3.13}$$

Operating $\partial_{\bar{k}}$ to (3.13), we have

$$\nabla_j A_{ki} + g_{aj}\partial_{\bar{k}}\partial_{\bar{i}}\widetilde{V}^{\bar{a}} = 2g_{ji}\,\partial_{\bar{k}}\widetilde{\rho},$$

from which, changing the roles of k and i and comparing these equations, we get

$$\nabla_j A_{ki} = g_{ji}\partial_{\bar{k}}\widetilde{\rho} - g_{kj}\partial_{\bar{i}}\widetilde{\rho}. \tag{3.14}$$

Here we put $\Phi_i := \dfrac{1}{n-1}\nabla_a A_i{}^a$. Transvecting (3.14) by g^{ji}, we obtain

$$\partial_{\bar{k}}\widetilde{\rho} = \Phi_k, \tag{3.15}$$

from which

$$\nabla_k A_{ji} = \Phi_j g_{ki} - \Phi_i g_{kj} \tag{3.16}$$

and

$$\widetilde{\rho} = \psi + y^a \Phi_a, \tag{3.17}$$

where ψ is a function on M. Substituting (3.16) and (3.17) into (3.13), we have

$$\widetilde{V}^{\bar{h}} = C^h - y^a(A_a{}^h + g^{hb}\nabla_b B_a) + (2\psi + y^a\Phi_a)y^h, \tag{3.18}$$

where $C = (C^h)$ is a vector field on M.

Substituting (3.12), (3.16), (3.17) and (3.18) into (3.11), we have

$$L_{B+C}\; g_{ji} = \nabla_j C_i + \nabla_i C_j + \nabla_j B_i + \nabla_i B_j = 2\psi\; g_{ji}, \tag{3.19}$$

$$L_B \Gamma_{ji}{}^h = \nabla_j \nabla_i B^h + K_{aji}{}^h B^a = \Psi_j \delta_i^h + \Psi_i \delta_j^h - g_{ji}\Phi^h \tag{3.20}$$

and

$$\begin{aligned} &K_{aijk}A_h{}^a + K_{aijh}A_k{}^a + K_{ajik}A_h{}^a + K_{ajih}A_k{}^a \\ &= g_{kj}\nabla_i \Phi_h + g_{hj}\nabla_i\Phi_k + g_{ki}\nabla_j\Phi_h + g_{hi}\nabla_j\Phi_k. \end{aligned} \tag{3.21}$$

Transvecting (3.21) by g^{kh}, we get

$$\nabla_j \Phi_i + \nabla_i \Phi_j = 0. \tag{3.22}$$

Lastly we pove

$$K_{akji}A_h{}^a = -g_{kj}\nabla_i\Phi_h + g_{ki}\nabla_j\Phi_h. \tag{3.23}$$

In fact, we put $P_{kjih} := K_{akji}A_h{}^a + g_{kj}\nabla_i\Phi_h - g_{ki}\nabla_j\Phi_h$. Then (3.21) is rewritten as follows:

$$P_{ikjh} + P_{ihjk} + P_{jkih} + P_{jhik} = 0. \tag{3.24}$$

By virtue of the first Bianchi identity, we have

$$P_{kjih} + P_{jikh} + P_{ikjh} = 0. \tag{3.25}$$

On the other hand, applying the Ricci identity to (3.4), we get

$$P_{kjih} + P_{kijh} = 0. \tag{3.26}$$

Using (3.24), (3.25) and (3.26), we obtain $P_{kjih} = 0$, i.e., (3.23). This completes the proof of the necessary condition.

Q.E.D.

Corollary 1. *Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle with the lift metric* I+II. *Then $\widetilde{V}$ is an inessential infinitesimal conformal transformation with the associated function ψ on TM if and only if there exist $B=(B^h),\ C=(C^h)\in \mathrm{T}^1_0(M)$ and $A=(A_i{}^h)\in \mathrm{T}^1_1(M)$ satisfying*

(1) $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}})=(B^h+y^a A_a{}^h,\ C^h-y^a(A_a{}^h+g^{hb}\nabla_b B_a)+2\psi y^h)$,

(2) $A_{ji}+A_{ij}=0$,

(3) $\nabla_j A_i{}^h=0$,

(4) $L_{B+C}\ g_{ji}=\nabla_j C_i+\nabla_i C_j+\nabla_j B_i+\nabla_i B_j=2\psi\, g_{ji}$,

(5) $L_B \Gamma_{ji}{}^h=\nabla_j\nabla_i B^h+K_{aji}{}^h B^a=\Psi_j\delta_i^h+\Psi_i\delta_j^h$,

(6) $K_{akji}A_h{}^a=0$,

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}):=\widetilde{V}^a E_a+\widetilde{V}^{\bar{a}}E_{\bar{a}}=\widetilde{V}$ *and* $\Psi_i:=\nabla_i\psi=\partial_i\psi$.

4. Proofs of Theorems

Proof of Theorem 1.

Let $(M,\ g)$ be a complete, simply connected Riemannian manifold with positive constant scalar curvature S, and TM its tangent bundle with the lift metric I+II: $\widetilde{g}:=g_{ba}dx^b dx^a+2g_{ba}dx^b\delta y^a$. Assume that TM admits an essential infinitesimal conformal transformation $\widetilde{V}$.

Operating ∇_l to (3.8) and using (3.4), we have

$$(\nabla_l K_{akji})A_h{}^a=-K_{lkji}\Phi_h+g_{lh}K_{akji}\Phi^a-g_{kj}\nabla_l\nabla_i\Phi_h+g_{ki}\nabla_l\nabla_j\Phi_h. \tag{4.1}$$

Transvecting (4.1) by g^{lh} and using the Ricci identity, we get

$$\begin{aligned}(\nabla_b K_{akji})A^{ba}&=(n-1)K_{akji}\Phi^a-g_{kj}\nabla_a\nabla_i\Phi^a+g_{ki}\nabla_a\nabla_j\Phi^a\\&=(n-1)K_{akji}\Phi^a-g_{kj}R_{ai}\Phi^a+g_{ki}R_{aj}\Phi^a,\end{aligned} \tag{4.2}$$

where $R=(R_{ji})$ is the Ricci tensor of M.

On the other hand, using (3.7) and (3.8), we have

$$K_{bjia}A^{ba}=\nabla_j\Phi_i. \tag{4.3}$$

Operating ∇_k to (4.3), and using (3.4) and the first Bianchi identity, we find

$$\begin{aligned}(\nabla_k K_{bjia})A^{ba}&=-K_{bjia}\nabla_k A^{ba}+\nabla_k\nabla_j\Phi_i\\&=\nabla_k\nabla_j\Phi_i+K_{akji}\Phi^a,\end{aligned}$$

from which, we obtain

$$\begin{aligned}&(\nabla_b K_{akji})A^{ba}\\&=(\nabla_i K_{akjb}-\nabla_j K_{akib})A^{ba}\\&=\nabla_i\nabla_j\Phi_k+K_{aijk}\Phi^a-\nabla_j\nabla_i\Phi_k+K_{ajik}\Phi^a\\&=-(K_{akji}+K_{ajik}+K_{aikj})\Phi^a\\&=0.\end{aligned} \tag{4.4}$$

Therefore (4.2) is reduced to

$$(n-1)K_{akji}\Phi^a = (g_{kj}R_{ai} - g_{ki}R_{aj})\Phi^a. \tag{4.5}$$

Transvecting (3.8) by g^{kj}, we have

$$R_{ai}A_h{}^a = (n-1)\nabla_h\Phi_i. \tag{4.6}$$

Operating ∇_k to (4.6) and using (3.4), we have

$$(n-1)\nabla_k\nabla_j\Phi_i = (\nabla_k R_{ai})A_j{}^a + R_{ki}\Phi_j - g_{kj}R_{ai}\Phi^a, \tag{4.7}$$

from which, since the scalar curvature S of M is constant, we obtain

$$nR_{ai}\Phi^a = S\Phi_i. \tag{4.8}$$

From (3.4) and (4.6), we have

$$R_{aj}A_i{}^a + R_{ai}A_j{}^a = 0. \tag{4.9}$$

Operating ∇_k to (4.9) and using (3.4), we get

$$Q_{kji} = -Q_{kij}, \tag{4.10}$$

where $Q_{kji} := (\nabla_k R_{aj})A_i{}^a + R_{kj}\Phi_i - g_{kj}R_{ai}\Phi^a$. Tranvecting (4.1) by g^{lk} and using the second Bianchi identity and the Ricci identity,

$$(\nabla_j R_{ai} - \nabla_i R_{aj})A_h{}^a = 0. \tag{4.11}$$

From the definition of $Q = (Q_{kji})$ and (4.11), we get

$$Q_{kji} = Q_{jki}. \tag{4.12}$$

Therefore, using (4.10) and (4.12), we obtain $Q_{kji} = 0$, i.e.,

$$(\nabla_k R_{aj})A_i{}^a = -R_{kj}\Phi_i + g_{kj}R_{ai}\Phi^a. \tag{4.13}$$

Substituting (4.13) into (4.7) and using (4.5), we obtain

$$L_\Phi \Gamma_{ji}{}^h = \nabla_j\nabla_i\Phi^h + K_{aji}{}^h\Phi^a = 0. \tag{4.14}$$

Here we put $f := \dfrac{n(n-1)}{2S}\Phi_a\Phi^a$. First we assume that f is non-constant. Transvecting (4.5) by Φ^i and using (4.8), we obtain

$$F_i = \nabla_i f = A_i{}^a\Phi_a. \tag{4.15}$$

where $F_i := \nabla_i f = \partial_i f$. Operating ∇_j to (4.15) and using (3.4), we have

$$\begin{aligned} \nabla_j F_i &= A_i{}^a\nabla_j\Phi_a + \Phi_j\Phi_i - (\Phi_a\Phi^a)g_{ji} \\ &= A_i{}^a\nabla_j\Phi_a + \Phi_j\Phi_i - \frac{2S}{n(n-1)}fg_{ji}. \end{aligned} \tag{4.16}$$

Operating ∇_k to (4.16), and using (3.4), (4.5) and (4.14), we obtain

$$
\begin{aligned}
\nabla_k\nabla_j F_i &= A_i{}^a\nabla_k\nabla_j\Phi_a - \frac{S}{n(n-1)}F_j g_{ki} + \Phi_j\nabla_k\Phi_i - \frac{2S}{n(n-1)}F_k g_{ji} \\
&= -A_i{}^a K_{bkja}\Phi^b + +\Phi_j\nabla_k\Phi_i - \frac{S}{n(n-1)}(2F_k g_{ji} + F_j g_{ki}) \qquad (4.17)\\
&= -\frac{S}{n(n-1)}(2F_k g_{ji} + F_j g_{ki} + F_i g_{kj}).
\end{aligned}
$$

Therefore, by virtue of Lemma 1, M is isometric to the standard sphere.

Next, we assume that f is constant. Then f is non-zero because the infinitesimal conformal transformation $\widetilde{V}$ is essential. From (4.15) we have

$$
A_i{}^a\Phi_a = 0. \qquad (4.18)
$$

Transvecting (4.5) by $A_h{}^i$, and using (3.8), (4.6) and (4.8), we have

$$
(R_{ah}A_k{}^a - \frac{S}{n}A_{kh})\Phi_j = 0,
$$

from which, because f is non-zero constant, we get

$$
R_{ah}A_k{}^a = \frac{S}{n}A_{kh}. \qquad (4.19)
$$

Substituting (4.6) and (4.19) into (3.8),

$$
\begin{aligned}
K_{akji}A_h{}^a &= -g_{kj}\nabla_i\Phi_h + g_{ki}\nabla_j\Phi_h \\
&= \frac{1}{n-1}(g_{kj}R_{ai} - g_{ki}R_{aj})A_h{}^a \qquad (4.20)\\
&= \frac{S}{n(n-1)}(g_{ki}A_{jh} - g_{kj}A_{ih}).
\end{aligned}
$$

Operating ∇_l to (4.20), and using (3.4), (4.5) and (4.8), we get

$$
(\nabla_l K_{akji})A_h{}^a + K_{lkji}\Phi_h = \frac{S}{n(n-1)}(g_{li}g_{kj} - g_{lj}g_{ki})\Phi_h. \qquad (4.21)
$$

Because f is non-zero constant, transvecting (4.21) by Φ^h and using (4.18), we obtain

$$
K_{lkji} = \frac{S}{n(n-1)}(g_{li}g_{kj} - g_{lj}g_{ki}). \qquad (4.22)
$$

Thus M is a space of positive constant curvature. Therefore M is isometric to the standard sphere.

Q.E.D.

Proof of Theorem 2.

Let $\widetilde{V}$ be a non-homothetic infinitesimal conformal transformation on TM. If $\widetilde{V}$ is essential, then M is isometric to the standard sphere by virtue of Theorem 1. If $\widetilde{V}$ is inessential, then there exists a non-affine infinitesimal projective transformation B on M by virtue of Corollary 1 (5). In fact, the associated function ψ of $\widetilde{V}$ is non-constant, i.e., the associated 1-form $\Psi = (\Psi_i)$ of B is non-zero, because $\widetilde{V}$ is non-homothetic. Therefore, by virtue of Lemma 2, M is also isometric to the standard sphere.

Q.E.D.

References

[F] S. Fujimura, *On projective transformations of complete Riemannian manifolds with constant scalar curvature*, Memoirs of Institute of Science and Engineering, Ritsumeikan University **59** (2000), 53—62.

[H-Y] I. Hasegawa and K. Yamauchi, *Infinitesimal projective transformations on the tangent bundles with the horizontal lift connection*, Journal of Hokkaido University of Education (Natural Sciences) **52** (2001), 1—5.

[O] M. Obata, Proc. U.S.-Japan Sem. in Differential Geom. (1965), Kyoto Japan, 101—114.

[T] S. Tanno, *Some differential equations on Riemannian manifolds*, J. Math. Soc. Japan **30** (1978), 509—531.

[Y1] K. Yamauchi, *On infinitesimal projective transformations of Riemannian manifolds with constant scalar curvature*, Hokkaido Math. J. **8** (1979), 167—175.

[Y2] ———, *On infinitesimal conformal transformations of tangent bundles with the metric* I+II *over Riemannian manifolds*, Ann. Rep. Asahikawa Med. Coll. **17** (1996), 1—7.

[Y3] ———, *On conformal transformations in tangent bundles*, Hokkaido Math. J. **30** (2001), 359—372.

[Y-I] K.Yano and S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker, 1973.

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