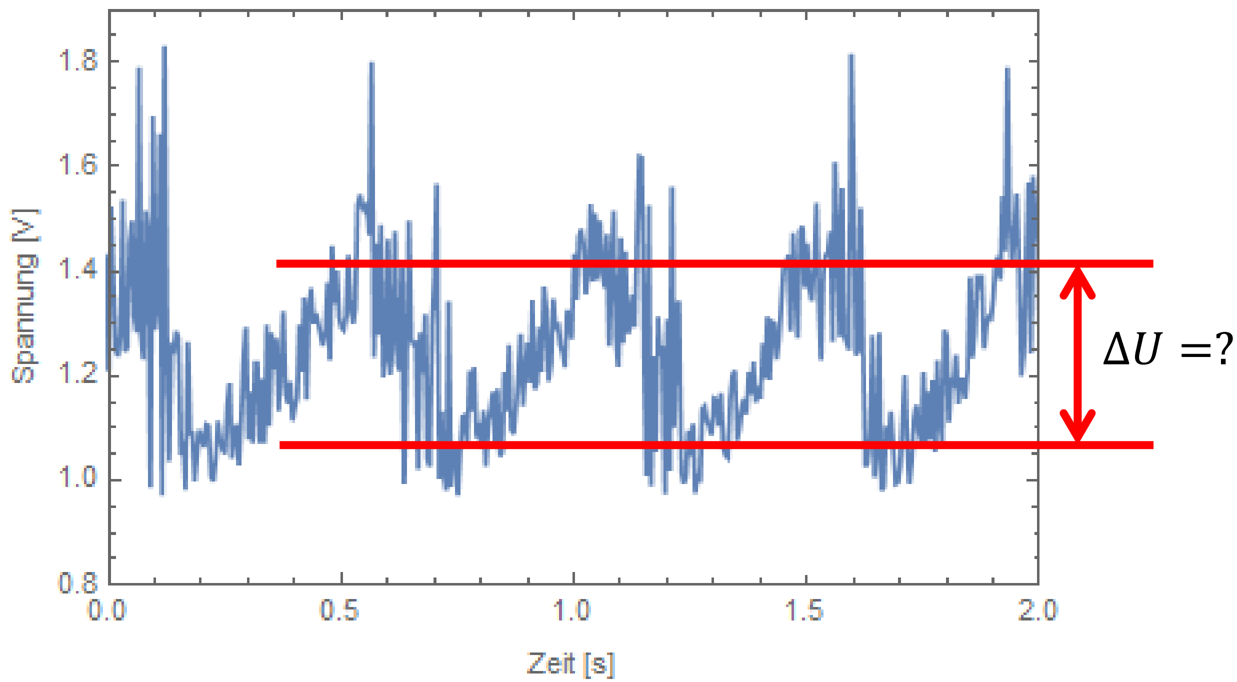
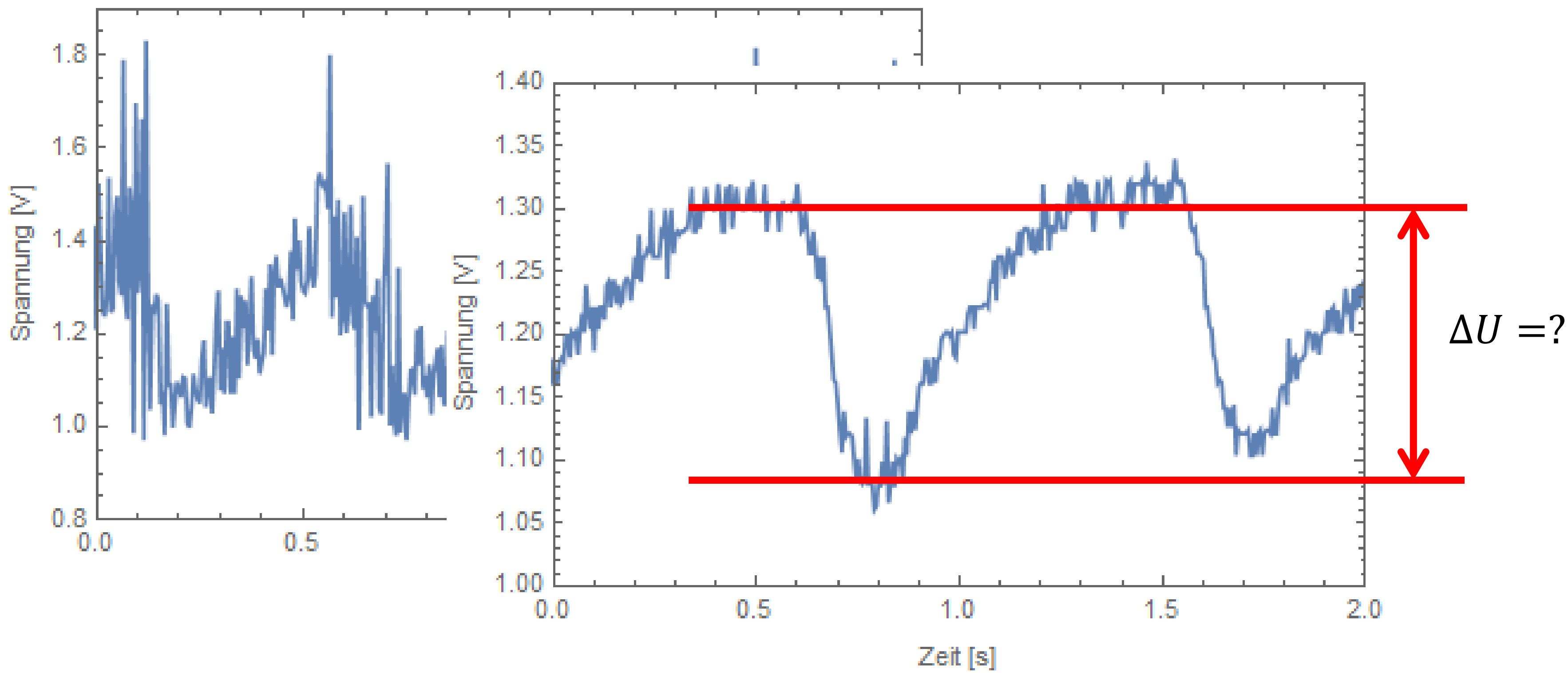
Spektrale Analyse – Fourier Transformation

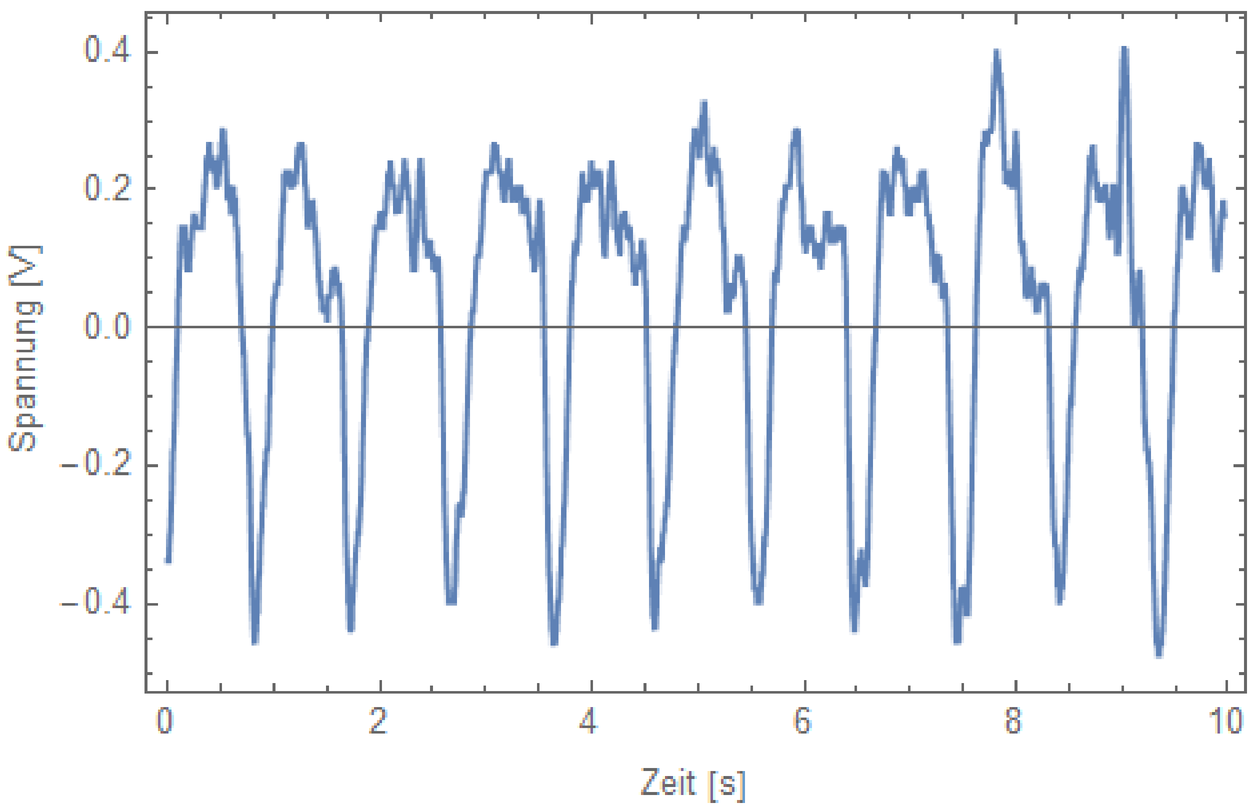
Fragestellung: Bestimmung der Amplitude eines verrauschten Signals



Fragestellung: Bestimmung der Amplitude eines verrauschten Signals



Frequenz-Spektrum eines (Puls-) Signals



Fourier-Analyse

Zeitabhängiges Signal $f(n\,T_a)$ ⇒ Frequenzspektrum $F_d(I\,k\,\Delta\omega)$

Diskrete Fourier-Transformation, DFT: $$F_d(I\,k\,\Delta\omega) = \sum_{n=0}^{N-1} f(n\,T_a)\,e^{-\mathrm{I}\,2\pi\,k\frac{n}{N}}$$

Fourier-Synthese

Frequenzspektrum $F_d(I\,k\,\Delta\omega)$ ⇒ Zeitabhängiges Signal $f(n\,T_a)$

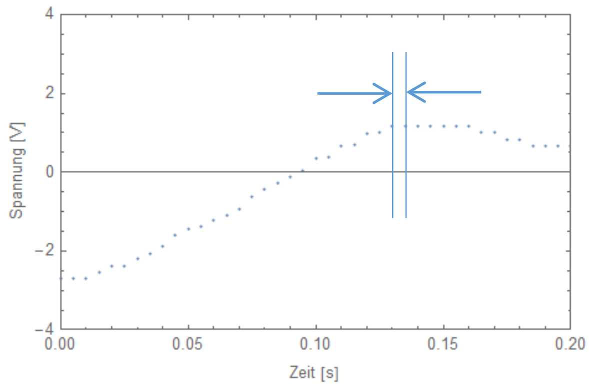
Inverse Fourier-Transformation, IDFT: $$f(n\,T_a) = \frac{1}{N}\sum_{k=0}^{N-1} F_d(I\,k\,\Delta\omega)\,e^{\mathrm{I}\,2\pi\,k\frac{n}{N}}$$

Fourier-Analyse

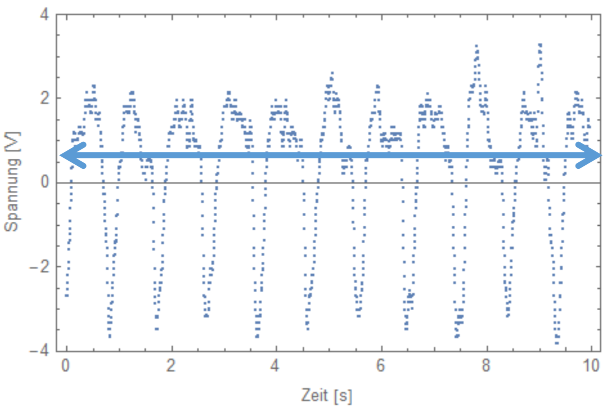
Zeitabhängiges Signal $f(n\,T_a)$ → Frequenzspektrum $F_d(I\,k\,\Delta\omega)$

Diskrete Fourier-Transformation, DFT: $$F_d(I\,k\,\Delta\omega) = \sum_{n=0}^{N-1} f(n\,T_a)\, e^{-I\,2\pi\,k\frac{n}{N}}$$

Abtastintervall T_a



Messzeit $T = N\,T_a$ für N Abtast-Punkte



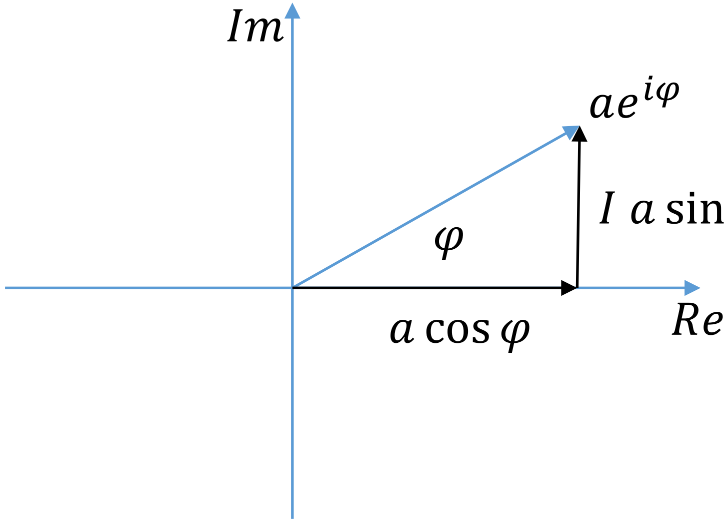
Abstand zwischen zwei Frequenzen: $\Delta f = \frac{1}{N\,T_a}$ oder $\Delta\omega = \frac{2\,\pi}{N\,T_a}$

Fourier-Analyse

Zeitabhängiges Signal $f(n\,T_a)$ → Frequenzspektrum $F_d(I\;k\;\Delta\omega)$

Diskrete Fourier-Transformation, DFT: $$F_d(I\;k\;\Delta\omega) = \sum_{n=0}^{N-1} f(n\,T_a)\,e^{-I\,2\pi\,k\frac{n}{N}}$$

$$= \sum_{n=0}^{N-1} f(n\,T_a)\,\cos 2\pi\,k\,\frac{n}{N} \;-I \sum_{n=0}^{N-1} f(n\,T_a)\,\sin 2\pi\,k\,\frac{n}{N}$$



Fourier-Analyse

Berechnung der reellen Koeffizienten der Sinus- und Cosinus-Funktionen
oder
Berechnung der reellen Koeffizienten der Exponential-Funktionen mit imaginären Argumenten

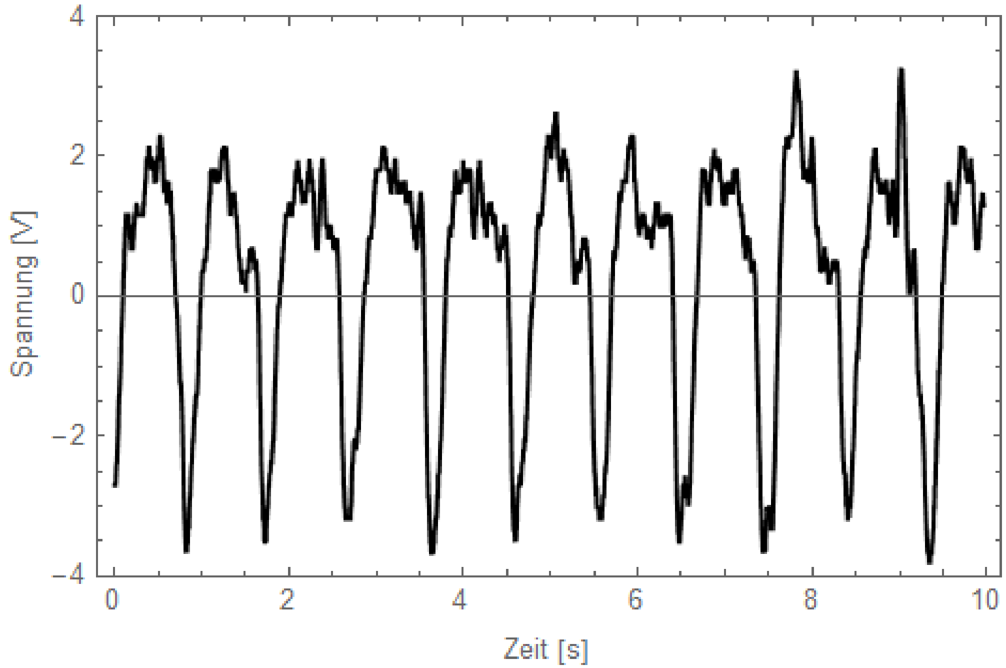
$$\frac{1}{\sqrt{n}}\left(a_0 + a_1\, \mathrm{e}^{-I\omega_1 t} + a_1^*\, \mathrm{e}^{I\omega_1 t} + \cdots\right)$$

$$a_i(\omega_i) = ?$$

Mit Frequenzen $\omega_i = 0, \pm\frac{2\pi}{T}, \pm 2\frac{2\pi}{T}, \pm 3\frac{2\pi}{T}, \ldots$ und Zeitfenster T

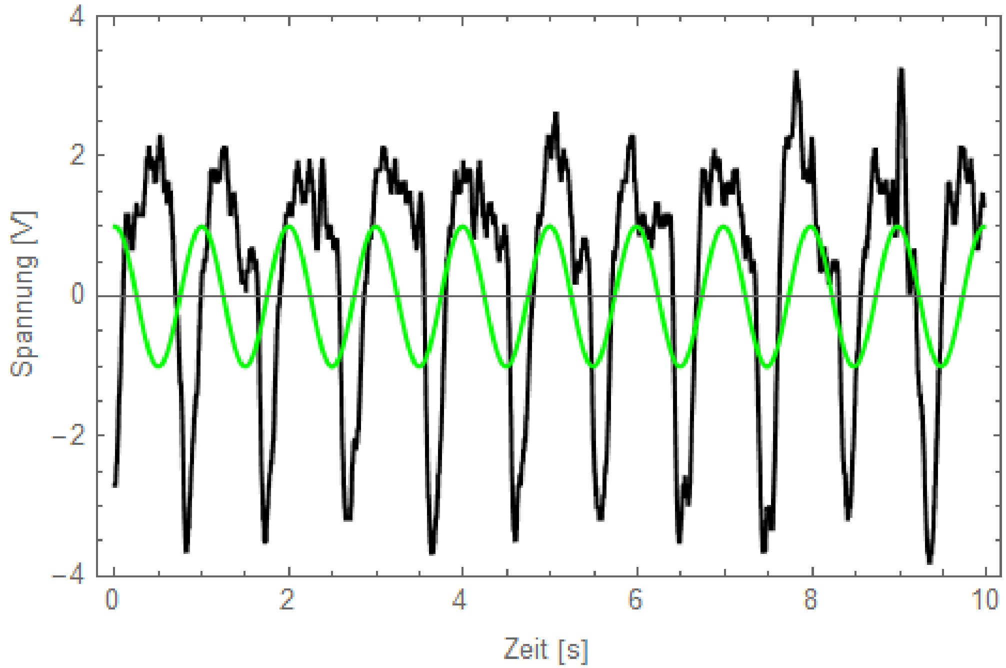
Fourier-Analyse

Signal



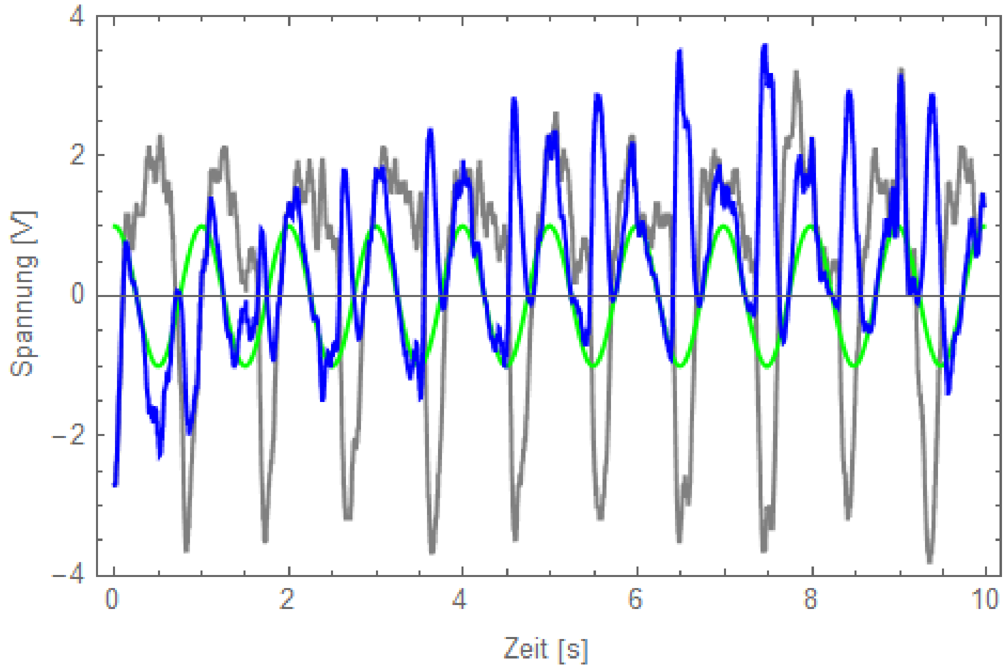
Fourier-Analyse

Signal
Cosinus $f_1 = 1.00\ \mathrm{Hz}$



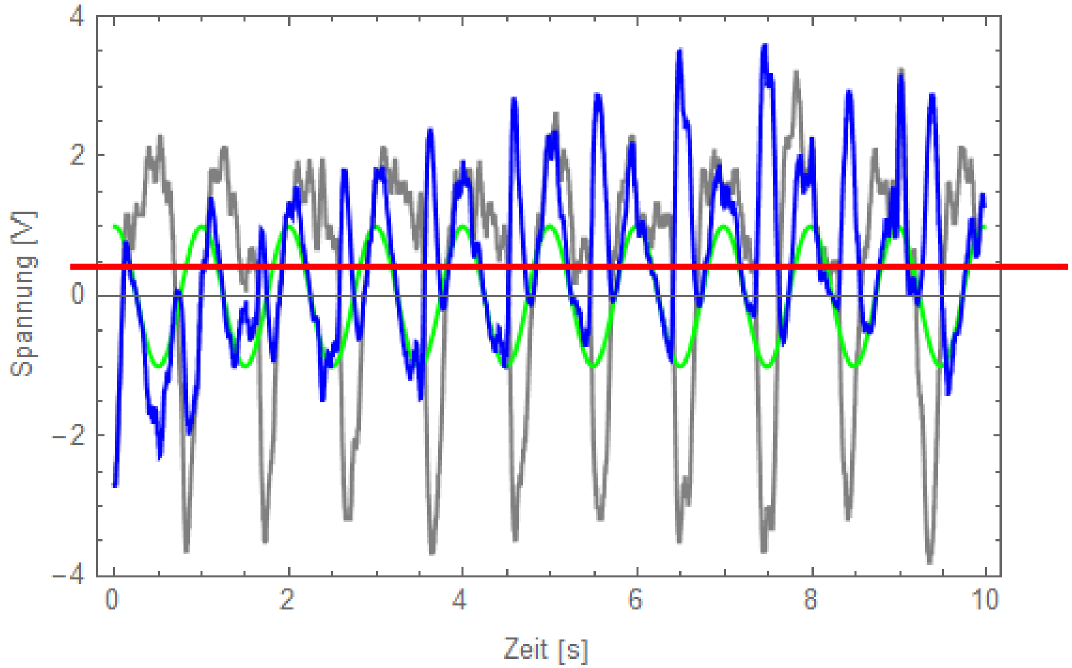
Fourier-Analyse

Signal
Cosinus $f_1 = 1.00\ \mathrm{Hz}$
Signal x Cosinus



Fourier-Analyse

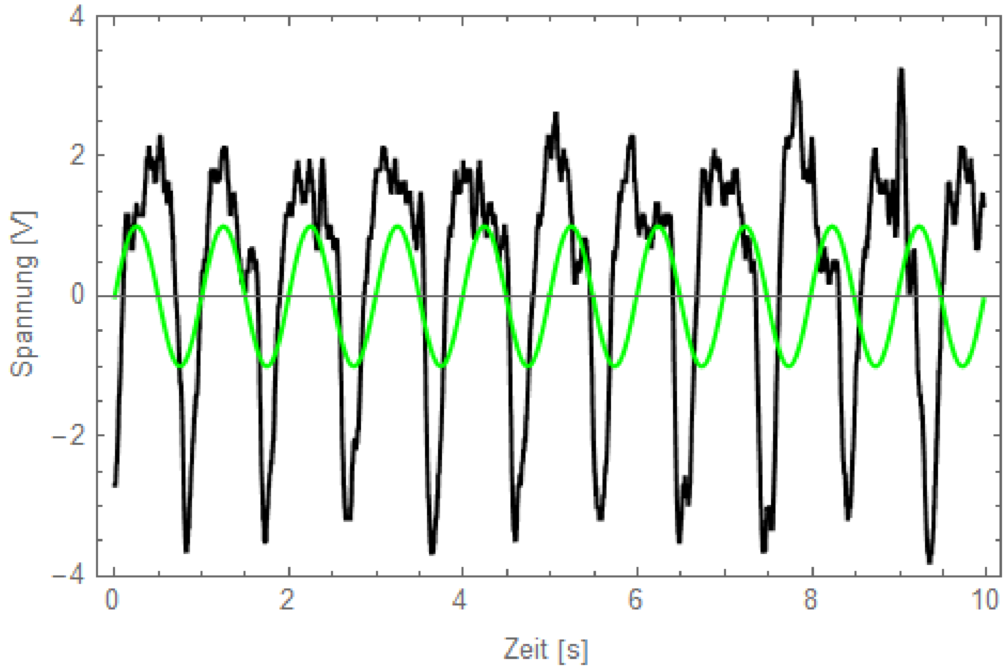
Signal
Cosinus $f_1 = 1.00$ Hz
Signal x Cosinus



Mittelwert = 0.496
= Koeffizient für Cos

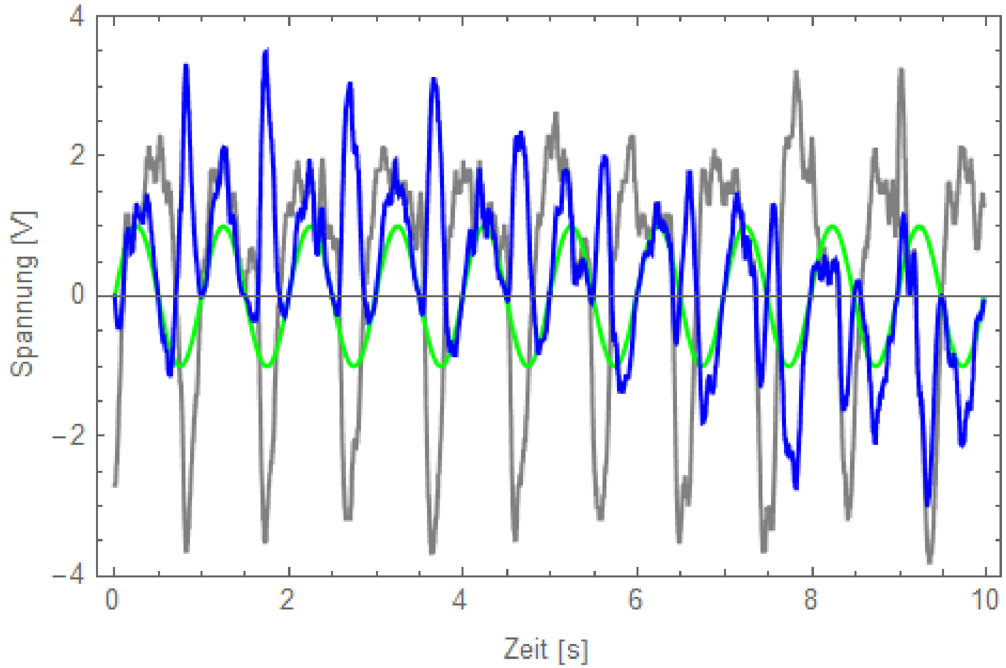
Fourier-Analyse

Signal
Sinus $f_1 = 1.00\ \mathrm{Hz}$



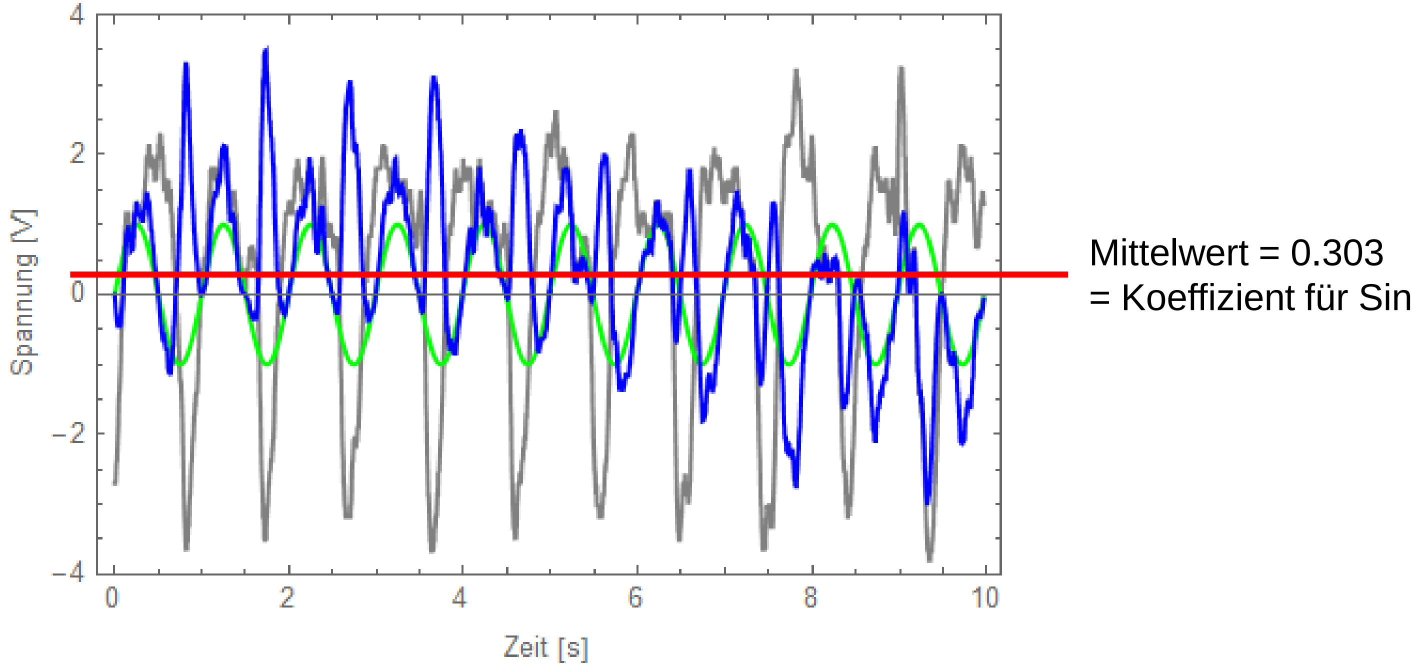
Fourier-Analyse

Signal
Sinus $f_1 = 1.00\ \mathrm{Hz}$
Signal x Sinus



Fourier-Analyse

Signal
Sinus $f_1 = 1.00\ \text{Hz}$
Signal x Sinus



Fourier-Analyse

Spektrale Komponente der Frequenz $f_1 = 1.00\ \mathrm{Hz}$

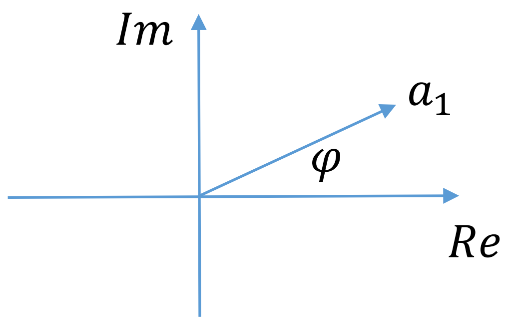
Sinus- und Cosinus-Analyse mit reellen Koeffizienten:

$$0.496\ \cos\omega_1 t + 0.303\ \sin\omega_1 t \qquad \text{mit } \omega_1 = 2\pi f_1 = 2\,\pi \times 1.00\ \mathrm{Hz}$$

Fourier-Analyse mit komplexen Koeffizienten:

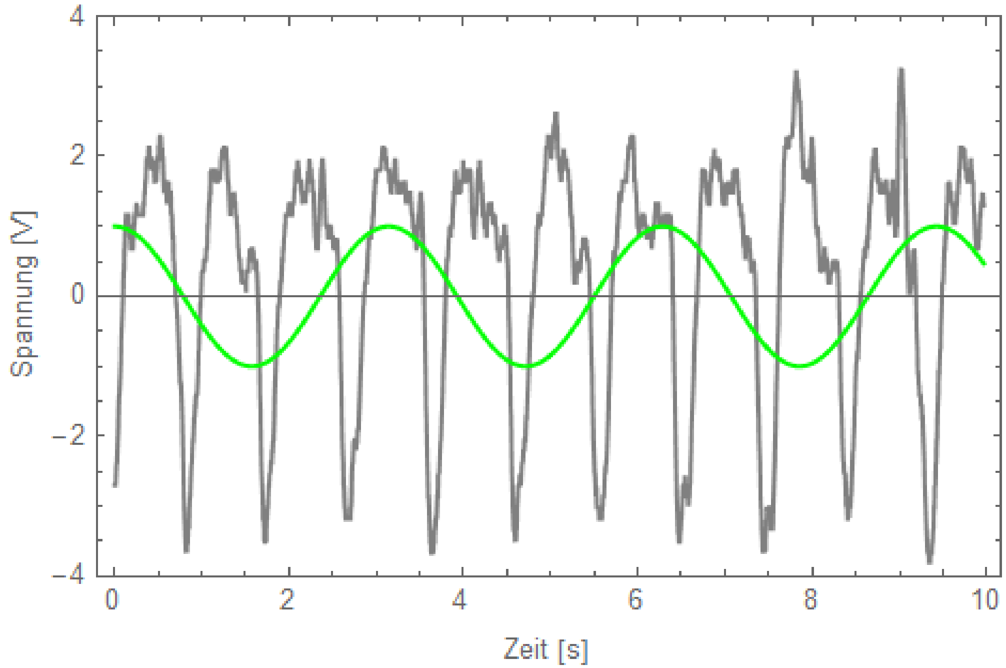
$$a_1\ \mathrm{e}^{-I\omega_1 t} + a_1^*\ \mathrm{e}^{I\omega_1 t}$$

mit $a_1 = 0.496 + I\ 0.303$
und $\omega_1 = 2\pi f_1 = 2\,\pi \times 1.00\ \mathrm{Hz}$



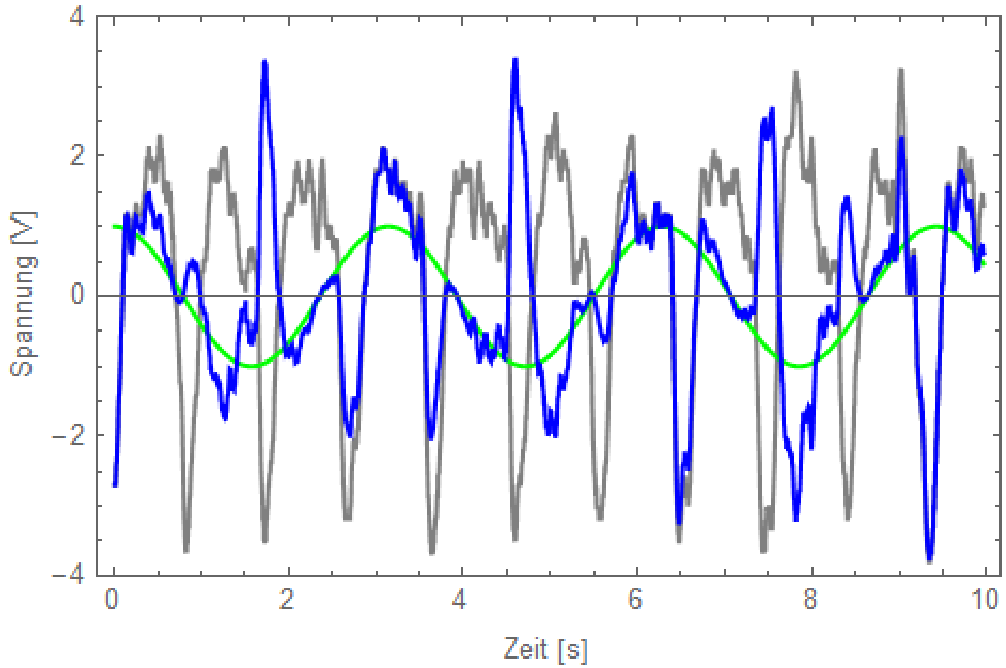
Fourier-Analyse

Signal
Cosinus $f_1 = 0.23\ \mathrm{Hz}$



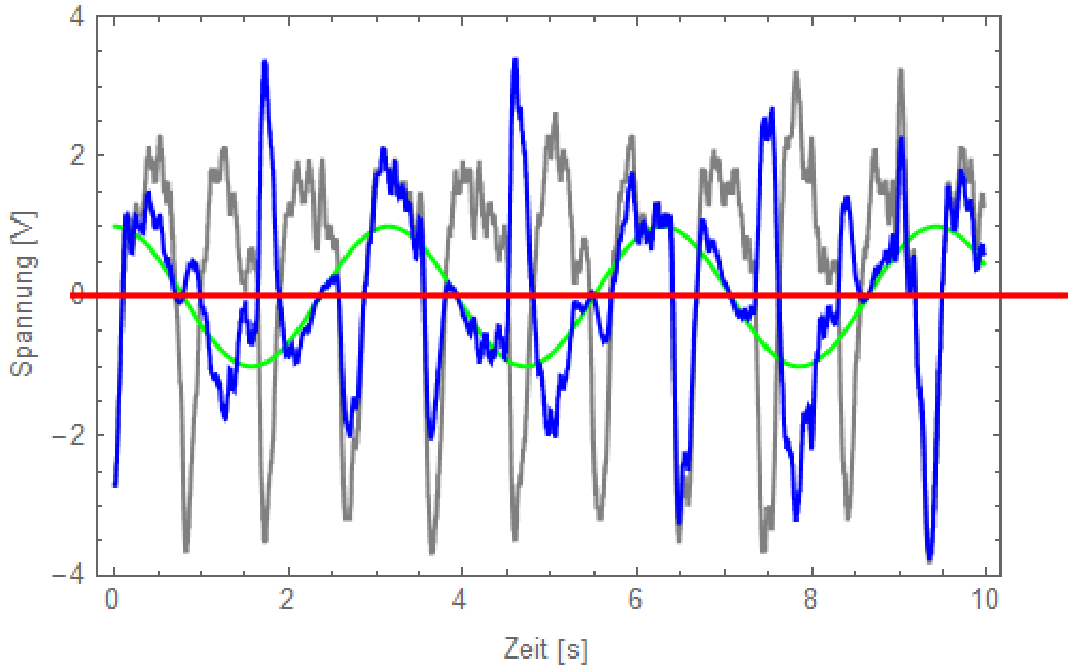
Fourier-Analyse

Signal
Cosinus $f_1 = 0.32\ \mathrm{Hz}$
Signal x Cosinus



Fourier-Analyse

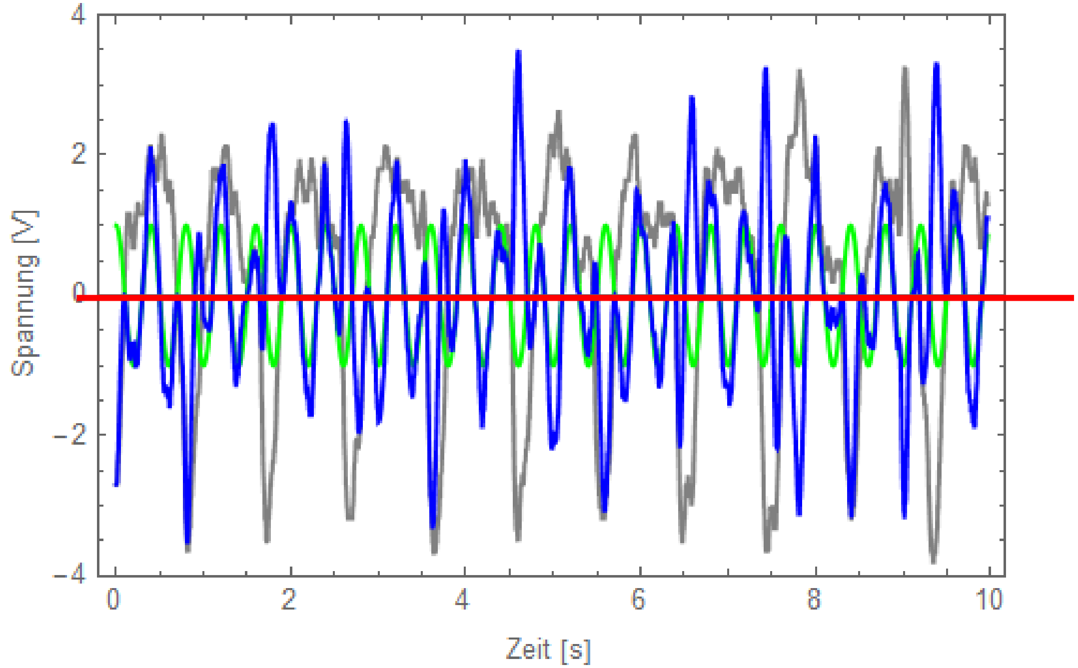
Signal
Cosinus $f_1 = 0.32\ \mathrm{Hz}$
Signal x Cosinus



Mittelwert = 0.023
= Koeffizient für Cos

Fourier-Analyse

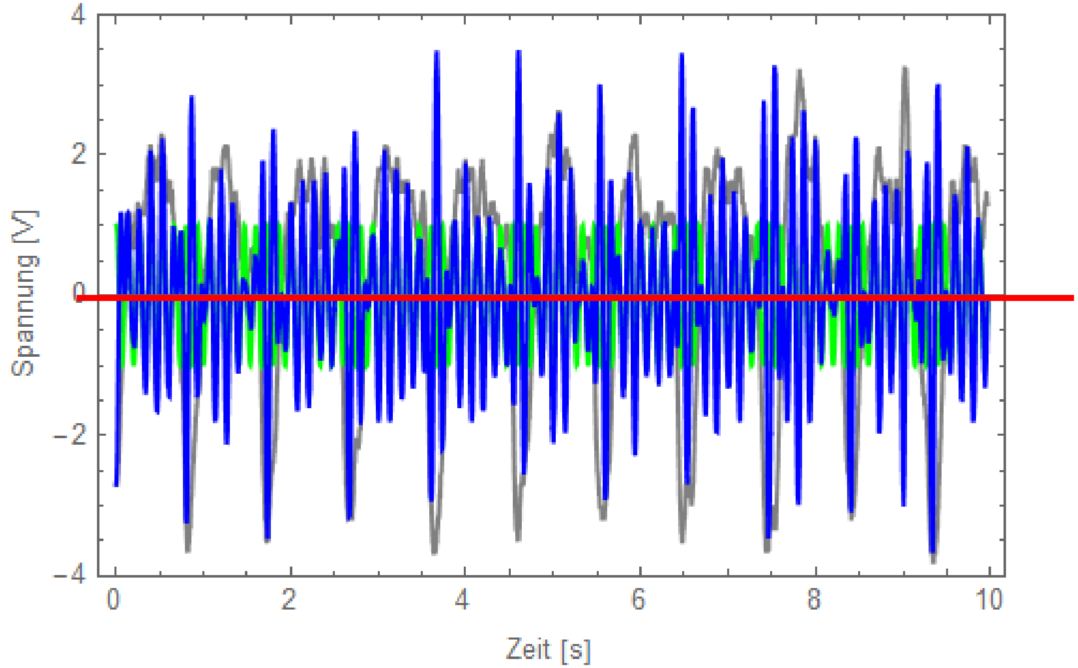
Signal
Cosinus $f_1 = 2.50\ \mathrm{Hz}$
Signal x Cosinus



Mittelwert = -0.042
= Koeffizient für Cos

Fourier-Analyse

Signal
Cosinus $f_1 = 7.50\ \mathrm{Hz}$
Signal x Cosinus



Mittelwert = -0.007
= Koeffizient für Cos

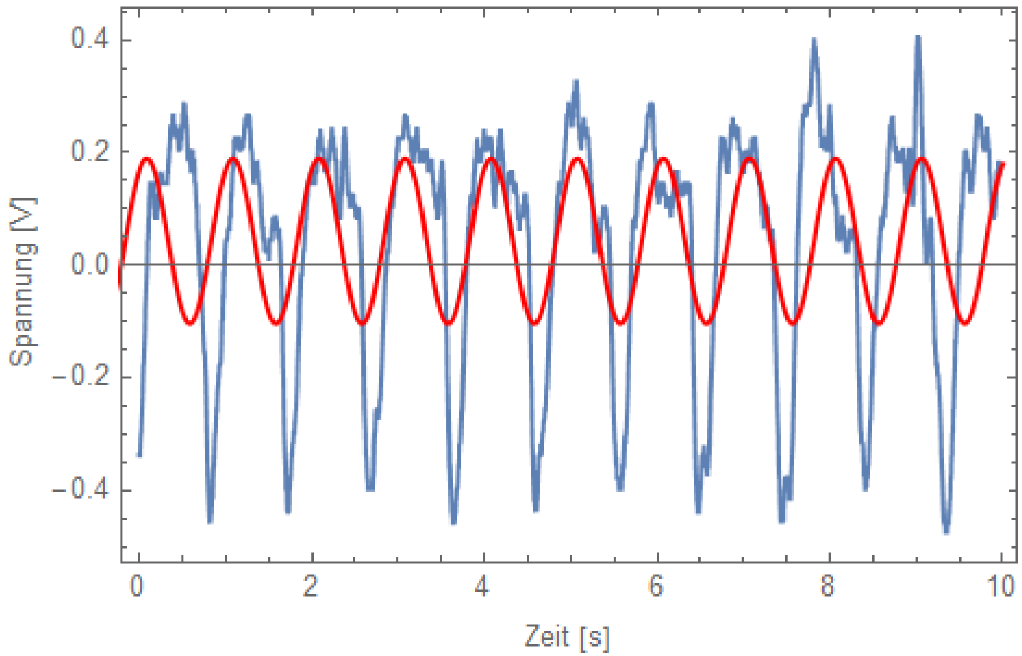
Fourier-Analyse

Fourier Koeffizienten Animated.nb

Fourier-Synthese

Addition von Sinus- und Cosinus-Funktionen
oder
Addition von Exponential-Funktionen mit imaginären Argumenten und komplexen Koeffizienten

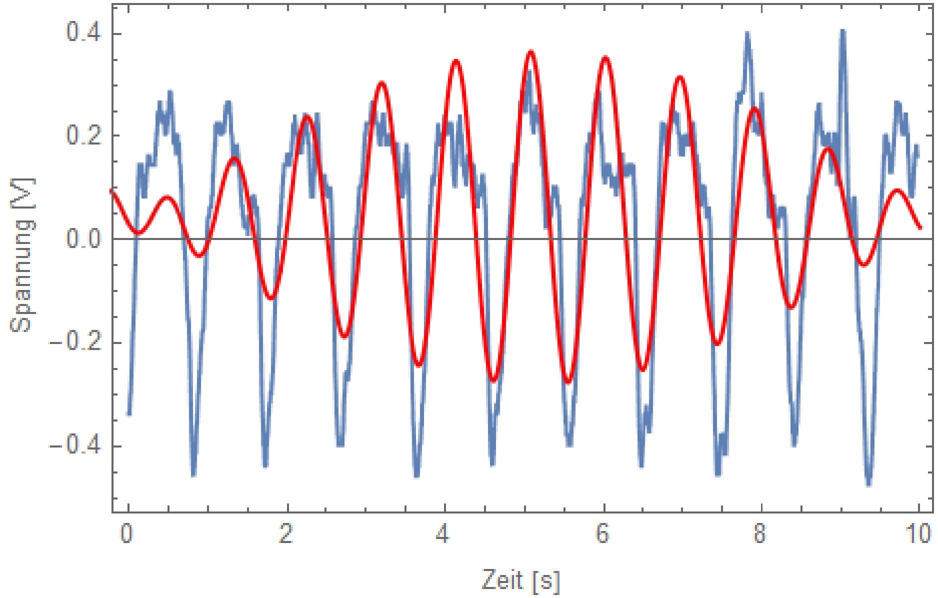
$$\frac{1}{\sqrt{n}}\left(a_0 + a_1\, e^{-I\omega_1 t} + a_1^*\, e^{I\omega_1 t}\right)$$



$$\omega_1 = 1.003\ \text{Hz}$$

Frequenz-Spektrum eines (Puls-) Signals

$$\frac{1}{\sqrt{n}}\left(a_0 + a_1\, e^{-I\omega_1 t} + a_1^*\, e^{I\omega_1 t} + a_2\, e^{-I\omega_2 t} + a_2^*\, e^{I\omega_2 t}\right)$$

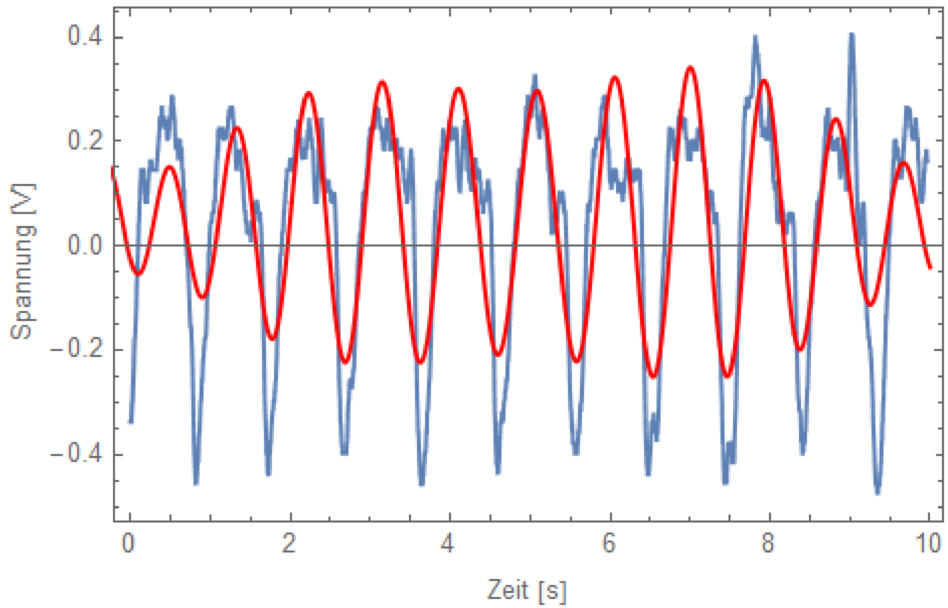


$$\omega_1 = 1.00 \text{ Hz}$$

$$\omega_2 = 1.10 \text{ Hz}$$

Frequenz-Spektrum eines (Puls-) Signals

$$\frac{1}{\sqrt{n}}\left(a_0 + a_1\,\mathrm{e}^{-I\omega_1 t} + a_1^*\,\mathrm{e}^{I\omega_1 t} + a_2\,\mathrm{e}^{-I\omega_2 t} + a_2^*\,\mathrm{e}^{I\omega_2 t} + a_3\,\mathrm{e}^{-I\omega_3 t} + a_3^*\,\mathrm{e}^{I\omega_3 t}\right)$$



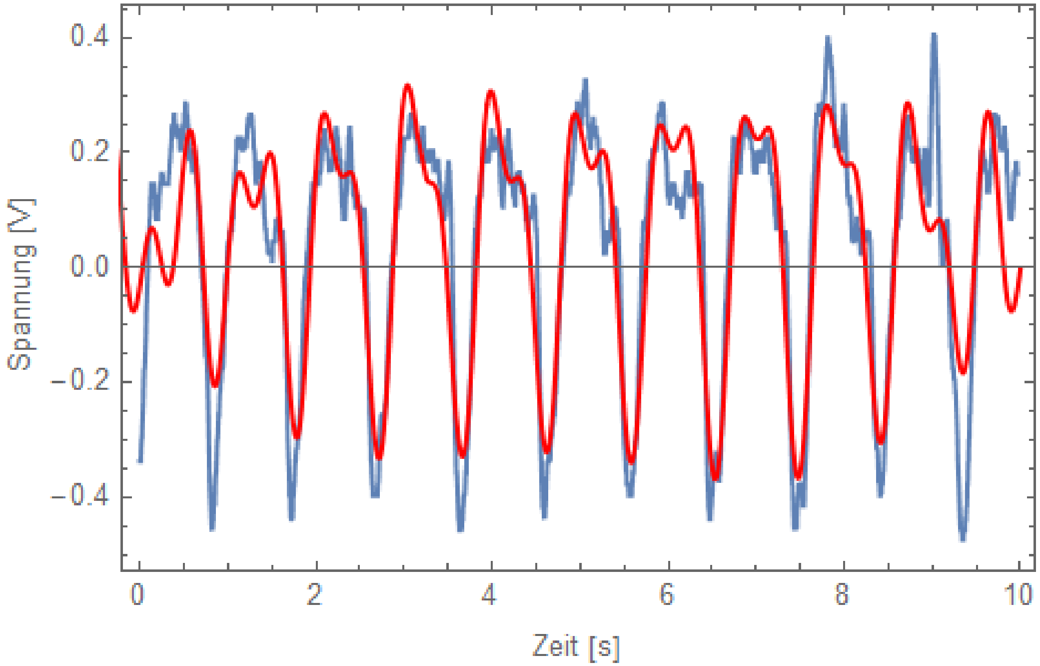
$\omega_1 = 1.00\ \mathrm{Hz}$

$\omega_2 = 1.10\ \mathrm{Hz}$

$\omega_3 = 1.20\ \mathrm{Hz}$

Frequenz-Spektrum eines (Puls-) Signals

$$\frac{1}{\sqrt{n}}\left(a_0 + a_1\, \mathrm{e}^{-I\omega_1 t} + a_1^*\, \mathrm{e}^{I\omega_1 t} + \cdots\right)$$



$\omega_1 = 1.00$ Hz

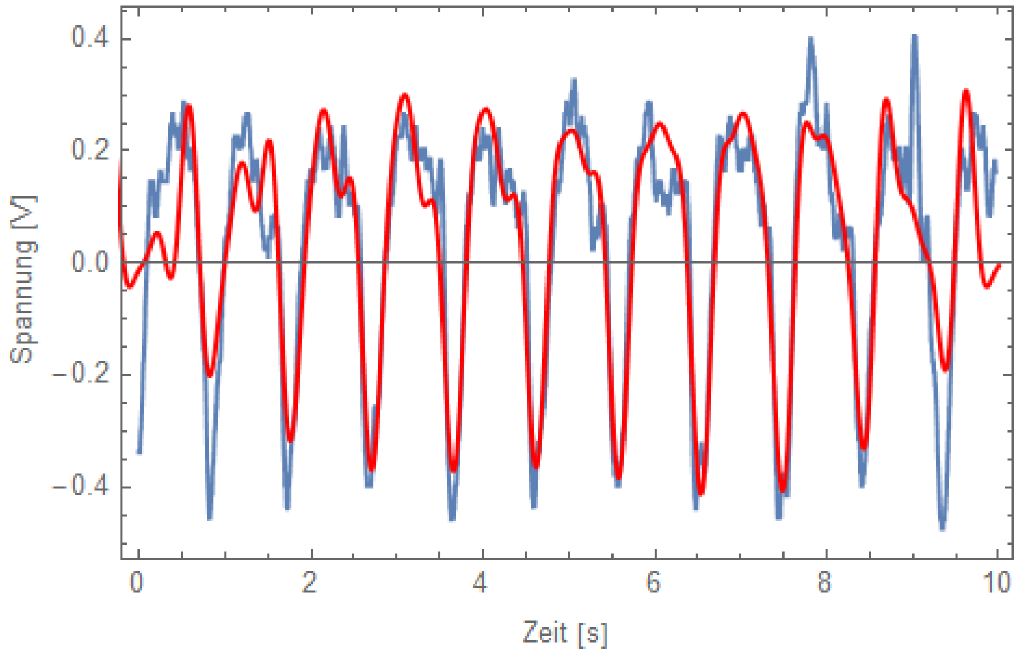
$\omega_2 = 1.10$ Hz

$\omega_3 = 1.20$ Hz

$\omega_4 = 2.10$ Hz

Frequenz-Spektrum eines (Puls-) Signals

$$\frac{1}{\sqrt{n}}\left(a_0 + a_1\, e^{-I\omega_1 t} + a_1^*\, e^{I\omega_1 t} + \cdots\right)$$



$\omega_1 = 1.00\ \mathrm{Hz}$

$\omega_2 = 1.10\ \mathrm{Hz}$

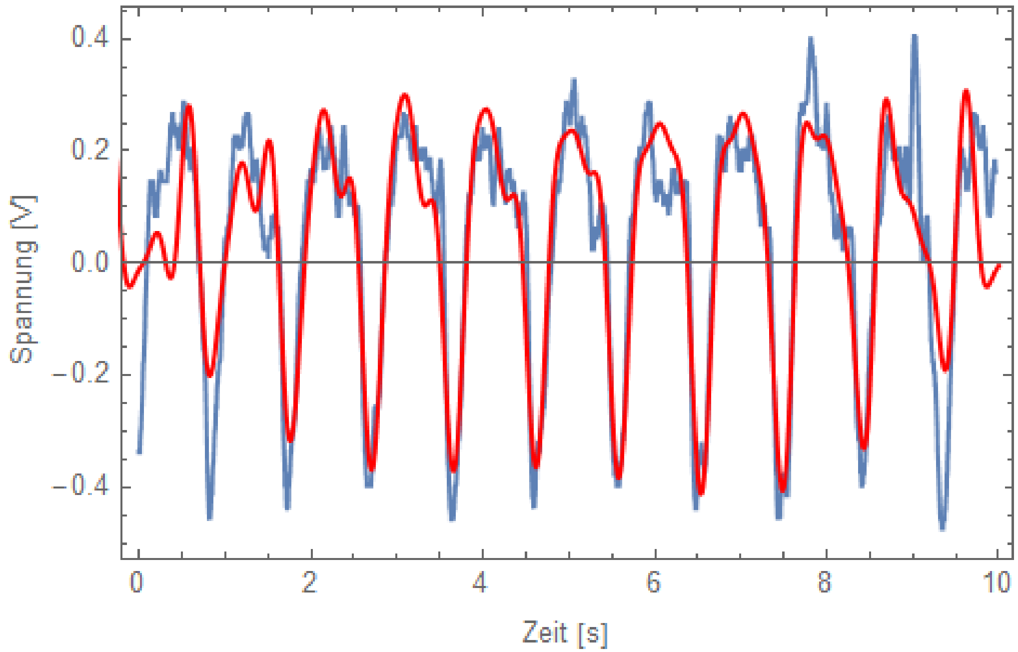
$\omega_3 = 1.20\ \mathrm{Hz}$

$\omega_4 = 2.10\ \mathrm{Hz}$

$\omega_5 = 3.11\ \mathrm{Hz}$

Frequenz-Spektrum eines (Puls-) Signals

$$\frac{1}{\sqrt{n}}\left(a_0 + a_1\, e^{-I\omega_1 t} + a_1^*\, e^{I\omega_1 t} + \cdots\right)$$



$\omega_1 = 1.00\text{ Hz}$

$\omega_2 = 1.10\text{ Hz}$

$\omega_3 = 1.20\text{ Hz}$

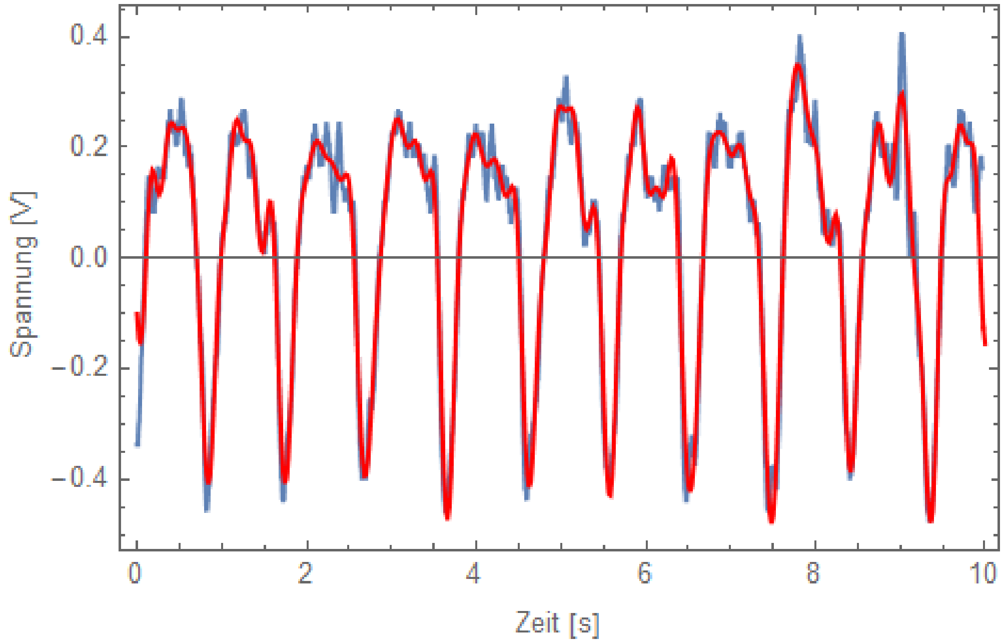
$\omega_4 = 2.10\text{ Hz}$

$\omega_5 = 3.11\text{ Hz}$

Frequenz-Spektrum eines (Puls-) Signals

$$\frac{1}{\sqrt{n}}\left(a_0 + a_1\, e^{-I\omega_1 t} + a_1^*\, e^{I\omega_1 t} + \cdots\right)$$

50 Frequenz-Terme bis $f = 50\ \mathrm{Hz}$



Fourier-Synthese

FourierSeriesOfSimpleFunctions.cdf

Fourier-Analyse

Zeitabhängiges Signal $f(n\,T_a)$ ➡ Frequenzspektrum $F_d(I\,k\,\Delta\omega)$

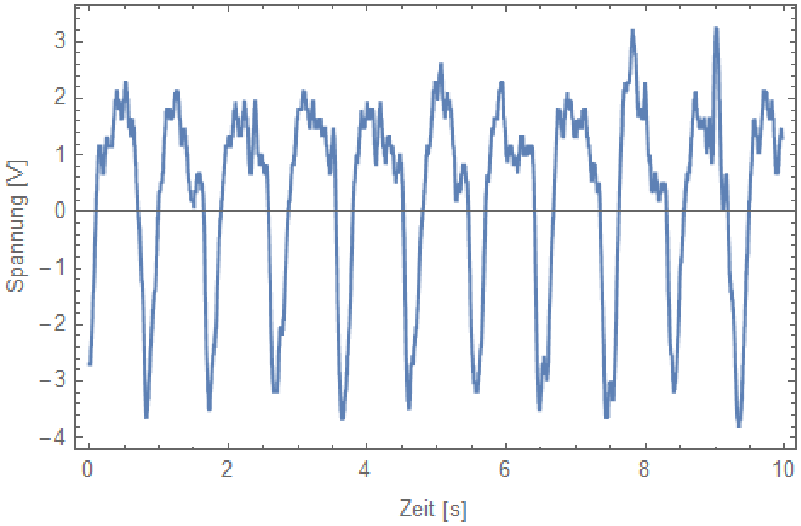
Diskrete Fourier-Transformation, DFT:
$$F_d(I\,k\,\Delta\omega) = \sum_{n=0}^{N-1} f(n\,T_a)\,e^{-I\,2\pi\,k\frac{n}{N}}$$

Fourier-Synthese

Frequenzspektrum $F_d(I\,k\,\Delta\omega)$ ➡ Zeitabhängiges Signal $f(n\,T_a)$

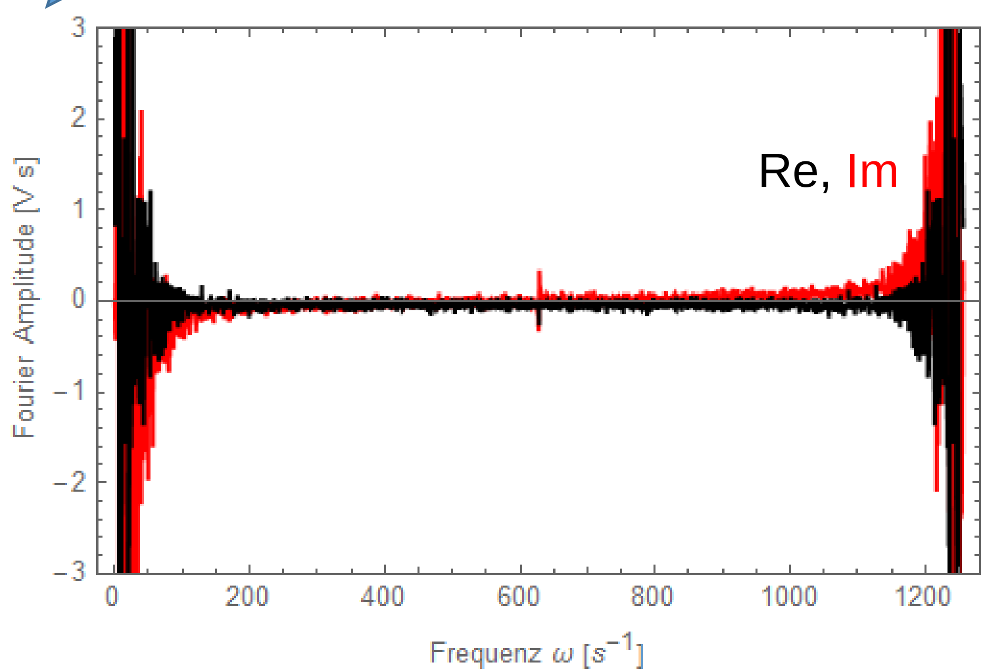
Inverse Fourier-Transformation, IDFT:
$$f(n\,T_a) = \frac{1}{N}\sum_{k=0}^{N-1} F_d(I\,k\,\Delta\omega)\,e^{I\,2\pi\,k\frac{n}{N}}$$

Fourier-Analyse

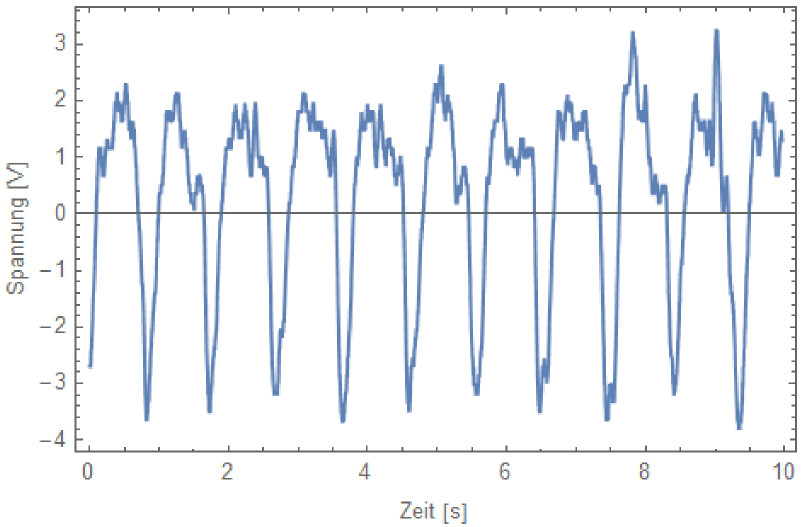


Diskrete Fourier-Transformation, DFT:

$$F_d(I\ k\ \Delta\omega) = \sum_{n=0}^{N-1} f(n\ T_a)\, e^{-I\, 2\pi\, k \frac{n}{N}}$$

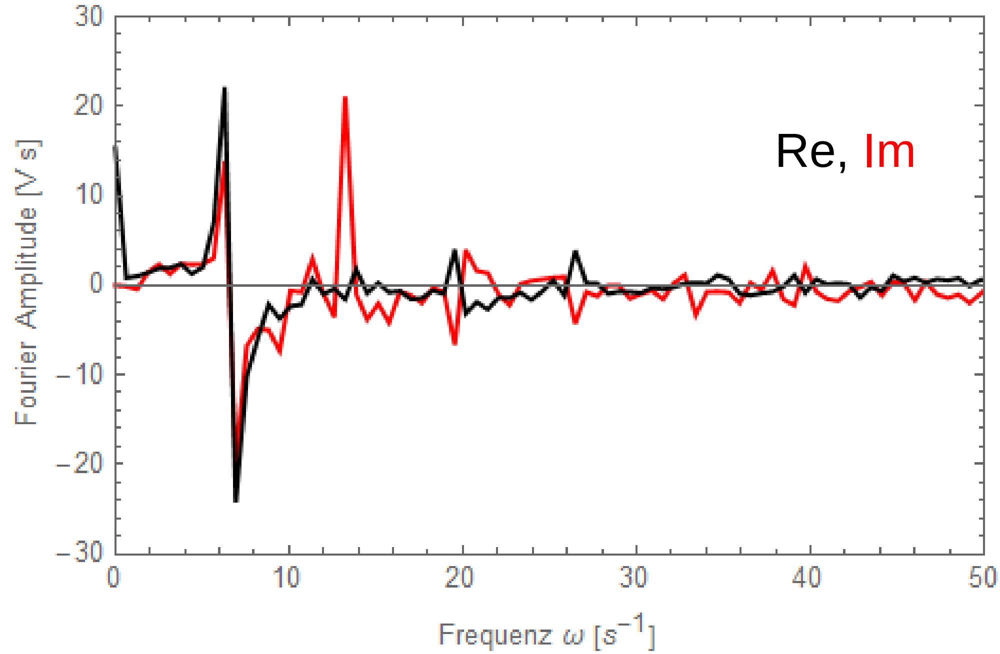


Fourier-Analyse

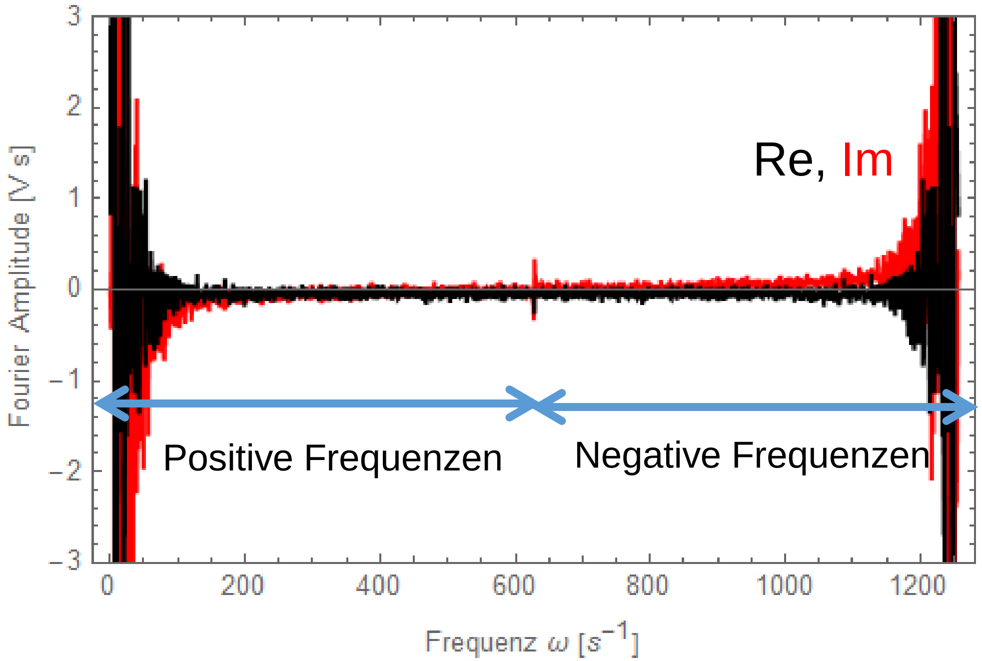


Diskrete Fourier-Transformation, DFT:

$$F_d(I\,k\,\Delta\omega) = \sum_{n=0}^{N-1} f(n\,T_a)\,e^{-I\,2\pi\,k\frac{n}{N}}$$



Fourier-Analyse



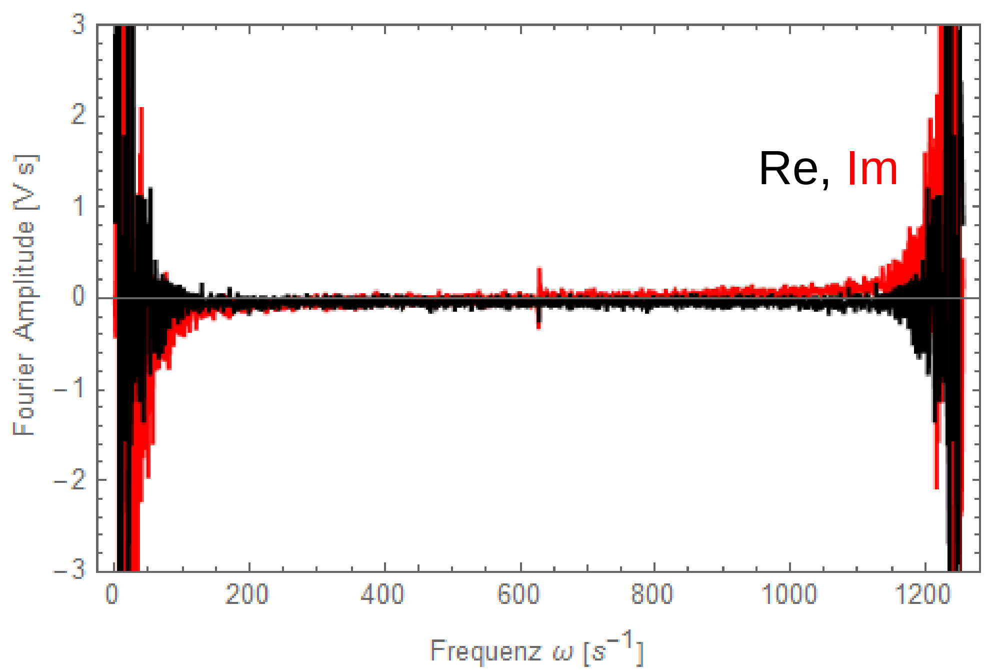
Fourier-Analyse

Wenn die ursprüngliche Funktion reellwertig ist, sind die Fourier-Koeffizienten bei positiven und negativen Frequenzen zueinander komplex konjugiert.

]= $15.3638 + 6.76264 \times 10^{-16}\, i$

ableForm=

0.821074 − 0.12206 i	0.821074 + 0.12206 i
1.01843 − 0.443253 i	1.01843 + 0.443253 i
1.40981 + 1.47039 i	1.40981 − 1.47039 i
1.90044 + 2.33691 i	1.90044 − 2.33691 i
1.93732 + 1.24854 i	1.93732 − 1.24854 i
2.38222 + 2.39835 i	2.38222 − 2.39835 i
1.28257 + 2.31834 i	1.28257 − 2.31834 i
2.03518 + 2.37366 i	2.03518 − 2.37366 i
7.08069 + 3.03615 i	7.08069 − 3.03615 i
22.1715 + 13.8656 i	22.1715 − 13.8656 i
−24.2766 − 19.7931 i	−24.2766 + 19.7931 i
−10.2085 − 6.77119 i	−10.2085 + 6.77119 i
−5.989 − 4.87907 i	−5.989 + 4.87907 i
−2.14631 − 5.00598 i	−2.14631 + 5.00598 i
−3.72947 − 7.32858 i	−3.72947 + 7.32858 i
−2.35733 − 0.606105 i	−2.35733 + 0.606105 i
−2.18119 − 0.763251 i	−2.18119 + 0.763251 i
0.701293 + 2.9373 i	0.701293 − 2.9373 i
−0.965445 − 0.895669 i	−0.965445 + 0.895669 i
−0.398485 − 3.55956 i	−0.398485 + 3.55956 i



Amplituden-Spektrum vs. Leistungs-Spektrum

Leistung ist proportional zum Absolutquadrat der Amplitude.

Leistung eines elektrischen Wechselspannungs- oder Wechselstrom-Signals:

$$P = U \cdot I = U \cdot \frac{U}{R} = \frac{U^2}{R}$$

$$P = U \cdot I = R \cdot I \cdot I = I^2 \cdot R$$

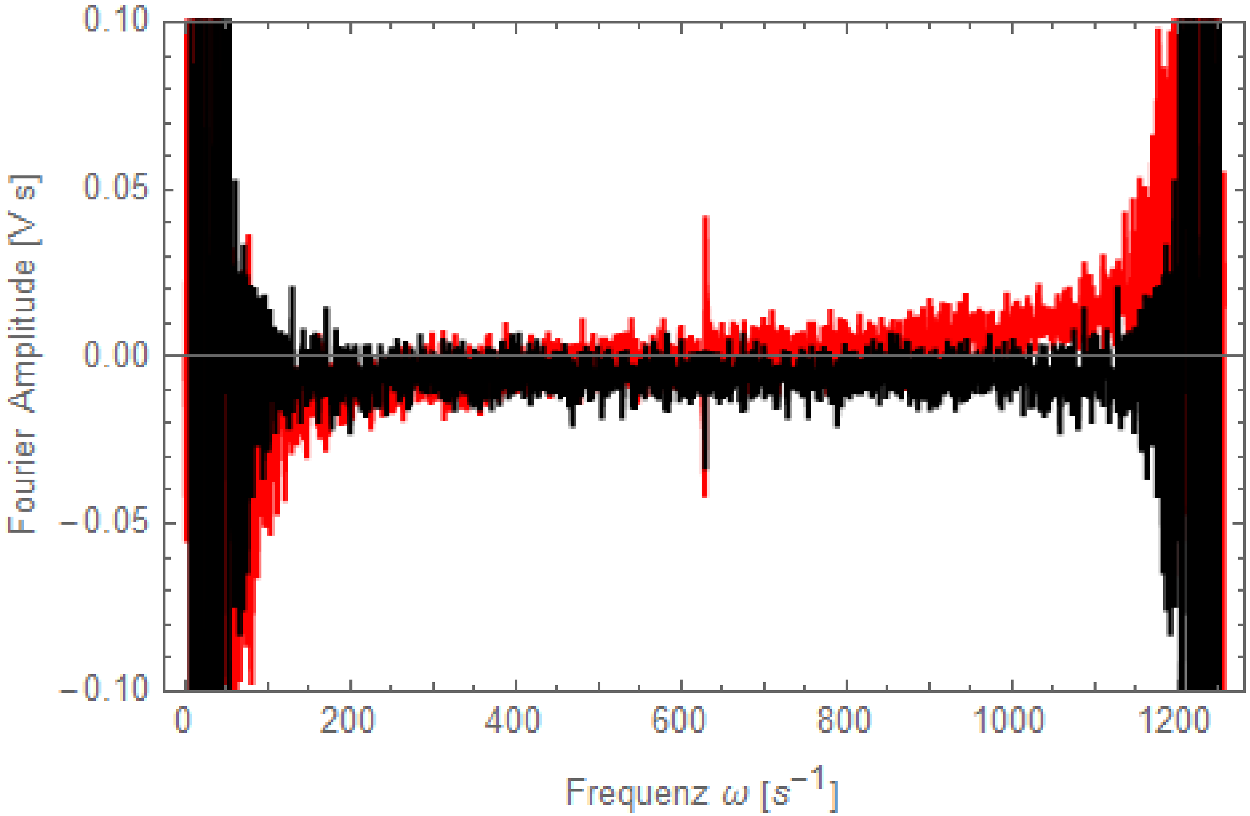
Intensität I einer elektromagnetischen Welle mit Feldstärke $E(t)$:

$$I = \varepsilon_0 \, c \, E^2$$

Dielektrizitätskontante des Vakuums: ε_0
Lichtgeschwindigkeit: c

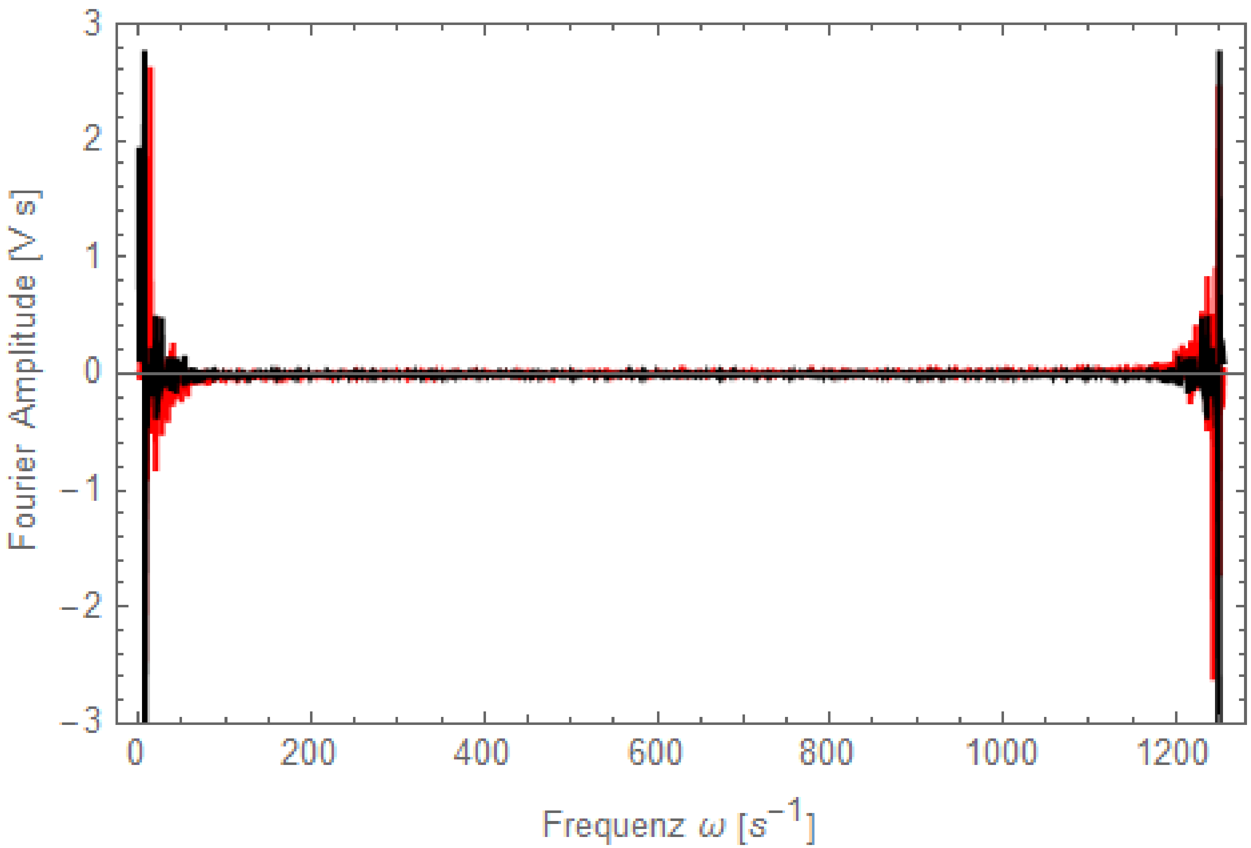
Amplituden-Spektrum

Fourier-Spektrum: Realteil und Imaginärteil der Fourier-Koeffizienten $a(\omega)$



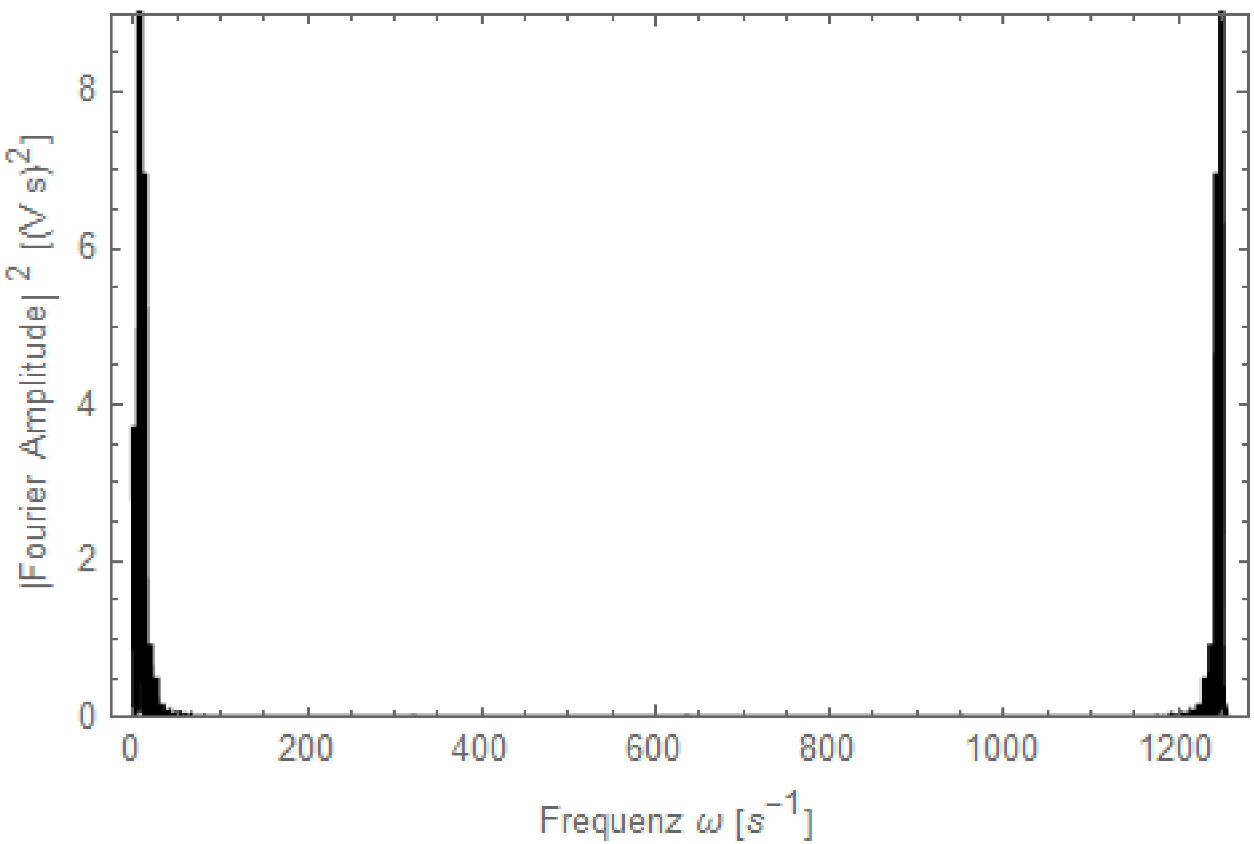
Amplituden-Spektrum

Fourier-Spektrum: Realteil und Imaginärteil der Fourier-Koeffizienten $a(\omega)$



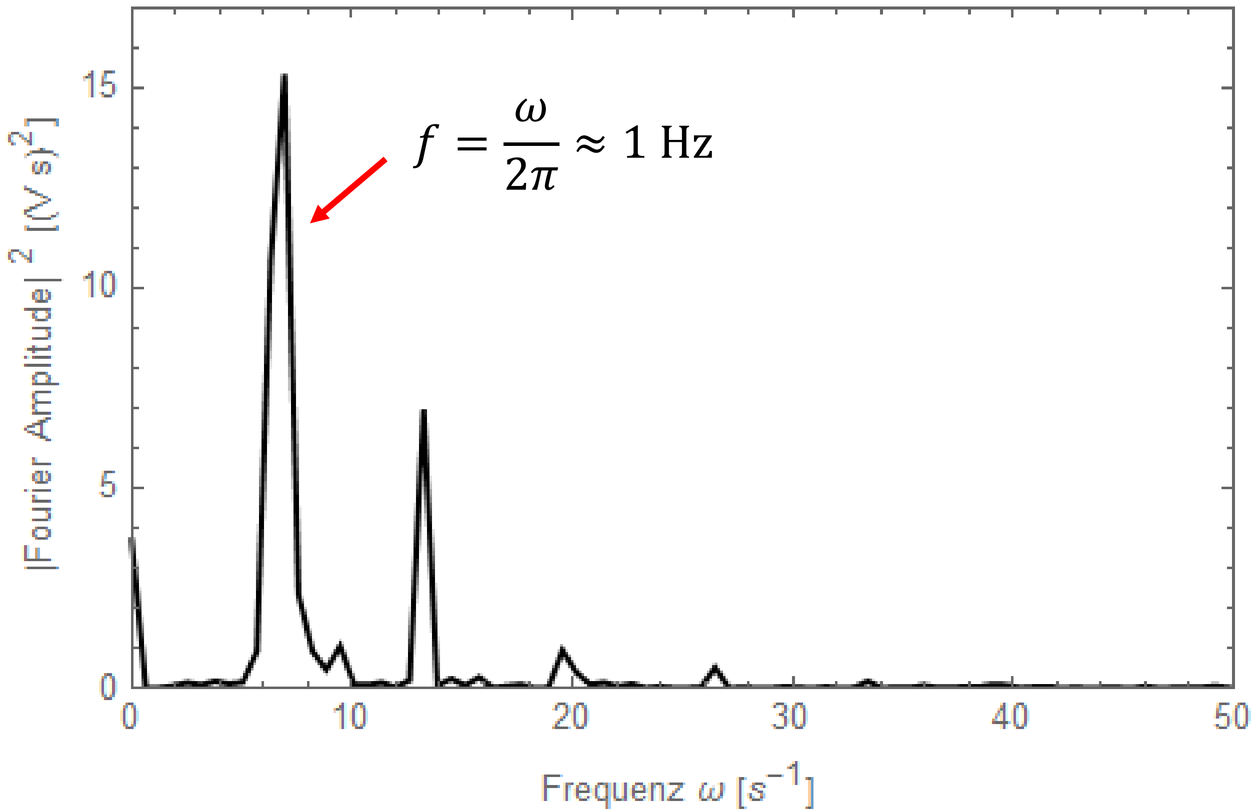
Leistungs-Spektrum (Power-Spektrum)

Absolutquadrat der Fourier-Koeffizienten $|a(\omega)|^2 = a(\omega) \cdot a(\omega)^*$



Leistungs-Spektrum (Power-Spektrum)

Absolutquadrat der Fourier-Koeffizienten $|a(\omega)|^2 = a(\omega) \cdot a(\omega)^*$



Leistungs-Spektrum (Power-Spektrum)

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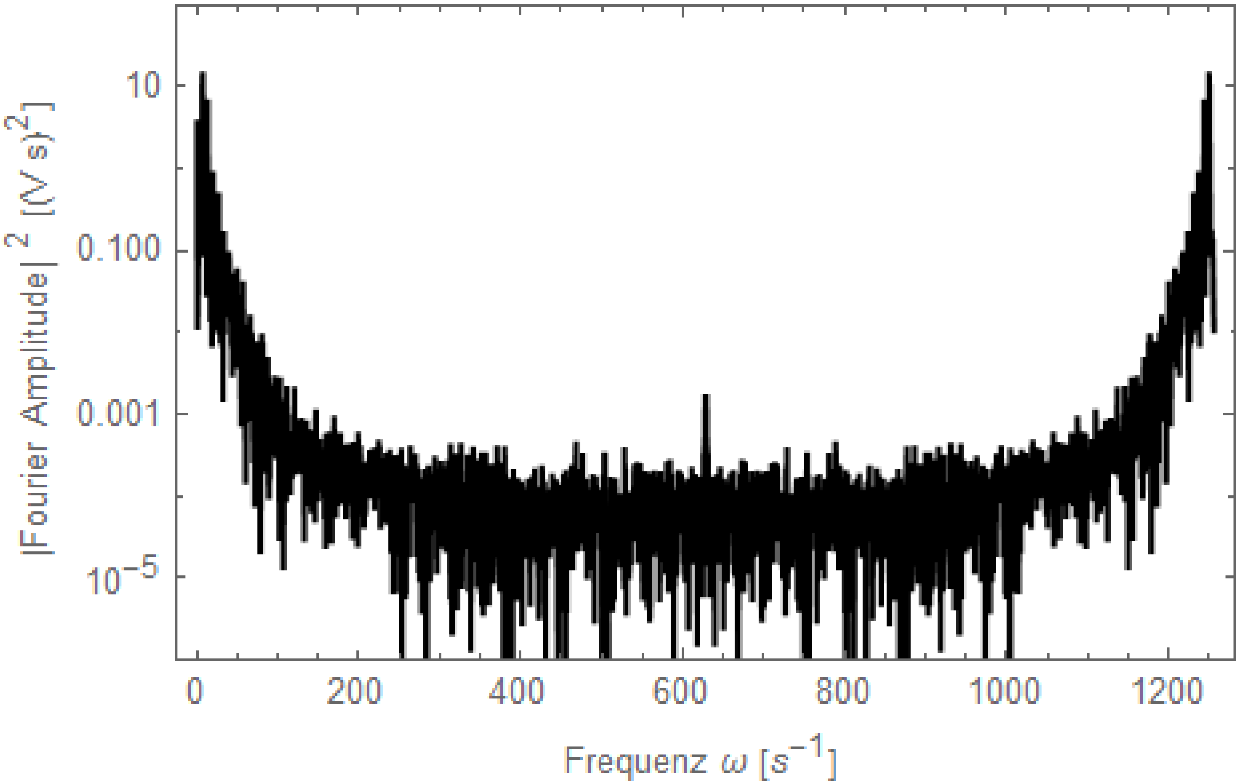
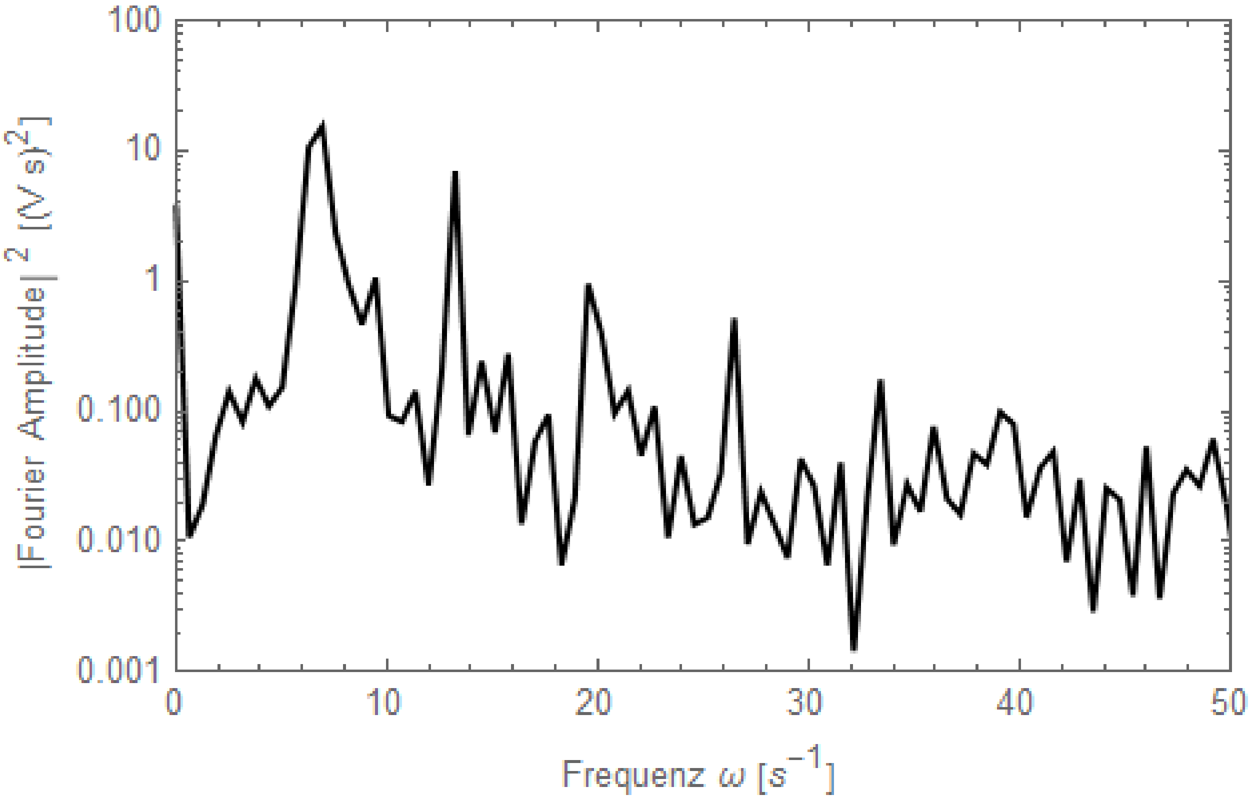


Leistungs-Spektrum (Power-Spektrum)

Absolutquadrat der Fourier-Koeffizienten $|a(\omega)|^2 = a(\omega) \cdot a(\omega)^*$

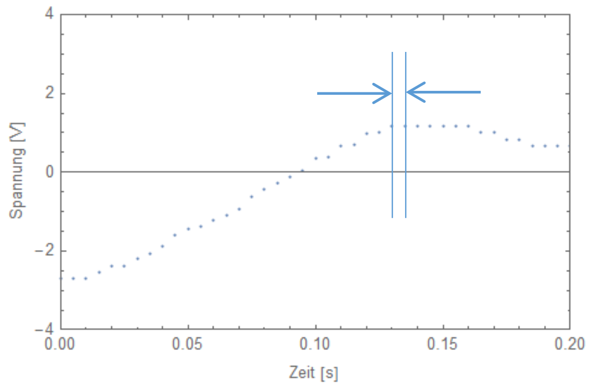


Fourier-Analyse

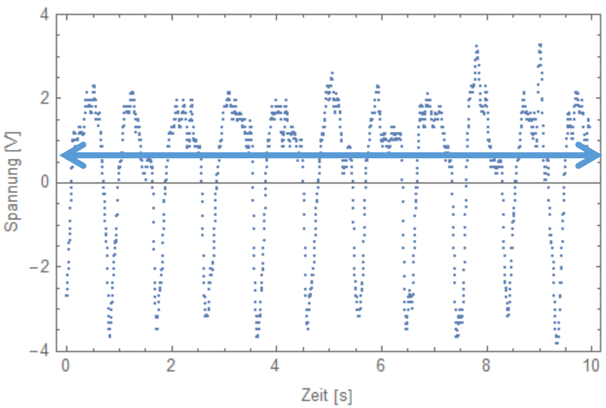
Zeitabhängiges Signal $f(n\,T_a)$ ➔ Frequenzspektrum $F_d(I\,k\,\Delta\omega)$

Diskrete Fourier-Transformation, DFT: $$F_d(I\,k\,\Delta\omega) = \sum_{n=0}^{N-1} f(n\,T_a)\,e^{-I\,2\pi\,k\frac{n}{N}}$$

Abtastintervall T_a



Messzeit $T = N\,T_a$ für N Abtast-Punkte



Abstand zwischen zwei Frequenzen: $\Delta f = \frac{1}{N\,T_a}$ oder $\Delta\omega = \frac{2\,\pi}{N\,T_a}$

Fourier-Analyse