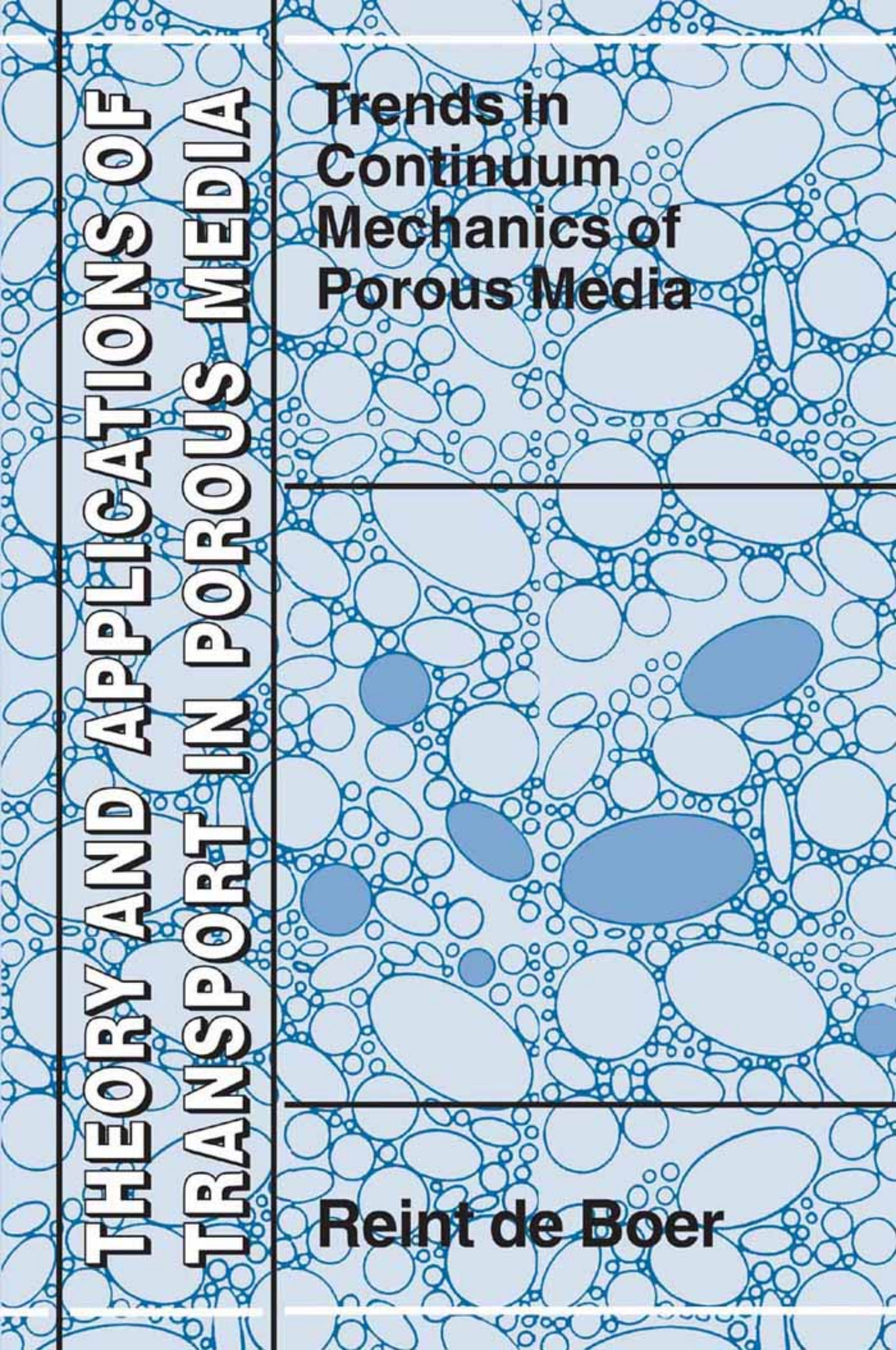

THEORY AND APPLICATIONS OF
TRANSPORT IN POROUS MEDIA

Trends in Continuum Mechanics of Porous Media

Reint de Boer

TRENDS IN CONTINUUM MECHANICS OF POROUS MEDIA

Theory and Applications of Transport in Porous Media

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Trends in Continuum Mechanics of Porous Media

by

REINT DE BOER

University of Duisburg-Essen,
Germany

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in Germany and
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Preface

In the last decade and in particular in recent years, the macroscopic porous media theory has made decisive progress concerning the fundamentals and the development of mathematical models in various fields of engineering and biomechanics. This progress attracted some attention and therefore conferences (colloquia, symposia, etc) devoted almost exclusively to the macroscopic porous media theory have been organized in the last years, e.g., in Cambridge, United Kingdom (1996), Prague, the Czech Republic (1997), Essen, Germany (1997), Metz, France (1999), Stuttgart, Germany (1999), Chicago, USA (2000), and Kerkrade, The Netherlands (2003) in order to collect findings, to present new results, and to discuss new trends. Also in national and international journals a great number of important contributions have been published which has brought the porous media theory, in some parts, namely for binary models, to a close. During the last years ternary models have been the subject of intensive research and characteristic phenomena have been investigated again. This was possible because, today the porous media theory has been developed to a great extent, and the constitutive theory, in particular, has made great progress. The elastic, elasto-plastic and viscous behavior of the constituents have been treated and the constitutive equations have been extended so that they are able to describe the fundamental effects uplift, friction, capillarity, effective stress and phase transition, in liquid and gas filled porous solids in a clearer way. It could be shown that the specific weights of the solid and gas phases are both reduced by uplift. Moreover, the concept of effective stress has been extended to a ternary model and a formula for the effective stress of the solid phase has been derived which is similar to the known relation by Bishop (1960). In recent times the capillarity problem has been revisited and has been successfully investigated; this was a real breakthrough in the continuum mechanical treatment of this complex field.

Moreover, in Biomechanics the transport of multi-electrolytes in charged hydrated biological soft tissues has been incorporated in the Theory of Porous

Media. Even in Botanic, the theory of porous media has been applied to the analysis of plant growth.

Therefore, the time seems to be ripe to review the state of the art and to show the trends in various fields of the Theory of Porous Media.

However, one should not assume that with the above stated contributions, the Theory of Porous Media has come to a close. In this context I would like to remind of the statement of the great poet Hermann Hesse: *"All knowledge and every increase in our knowledge doesn't end with a period, but rather with a question mark. An increase in knowledge means an increase in questions to be posed, and each of them is always replaced in turn by new questions."* In this sense the contents of this book should be understood.

The Introduction is devoted to the historical development up to the end of the 1980s and the beginning of the 1990s (readers interested in an extended description of the historical development of the porous media theory are referred to de Boer, 2000 a). The volume fraction concept is formulated in Chapter 2. An extensive review of the kinematics in porous media theory is presented in Chapter 3. The balance equations and the entropy inequality are discussed in Chapter 4 and 5. Chapter 6 is devoted to the investigation of the constitutive theory with the closure problem and the saturation condition. Moreover, constitutive equations with the description of elastic, elastic-plastic, and viscous states of the porous solid as well as some reflexions on the constitutive behavior of the pore fluids are developed in this Chapter. Fundamental effects in saturated and partly saturated porous solids such as uplift, friction, capillarity, effective stress, and phase transitions are treated im Chapter 7. In Chapter 8 and 9 introductions to Poroelasticity and Poroplasticity are given. Finally, some applications of the porous media theory in various fields (soil mechanics, chemical engineering, biomechanics and building physics as well as in environmental mechanics, soil physics, the petroleum industry, and material science) will demonstrate the usefulness of the macroscopic porous media theory.

I would like to thank Professor Z. Liu, University of Chongqing (PR China) and Professor M. Svanadze, Tbilisi State University (Georgia) for proofreading the Sections 8.5 and 8.4. I would also like to thank Dr.-Ing T. Ricken for many discussions and correcting some sections of the book. Also a big thank-you goes to the editors of *Applied Mechanics Reviews* for the permission to use parts of my review-article (de Boer, 2000 b).

Mrs. G. Bujna, responsible for word processing, has brought the manuscript to its present form. I would like to express my deepest gratitude to her. This is also valid for the work of Mr. J. R. Cambell and Mrs. L. Mensah, who corrected my English. I record here also my heartful thanks to Springer for the careful publishing and the pleasant cooperation.

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Contents

Chapter 1

INTRODUCTION

In many branches of engineering, for example, in chemical engineering, material science, and soil mechanics, as well as in biomechanics, the different reactions of material systems undergoing external and/or internal loadings must be studied and described precisely in order to be able to predict the responses of these systems. Subsequently, the most important point of the investigation is to determine first the composition of the body, because one must know the physically and chemically differing materials that constitute the system under consideration. The material systems (or bodies) in these fields of engineering can be composed in various ways. On the one hand, solids can consist of different solid components, such as dense concrete, without considerable pores. On the other hand, solids can contain closed and open pores, such as ceramics and soils, as well as concrete. The pores can be filled with fluids and, due to the different material properties and the different motions, there may be interaction between the constituents.

Because the exact description of the location of the pores (empty or filled with fluids) and solid material is nearly impossible, the heterogeneous composition can be investigated by using the volume fraction concept (see Chapter 2). This concept results in the effect that "smeared" substitute continua with reduced densities for the solid and fluid phases arise, which can then be treated using elements of the mixture theory.

The combination of the mixture theory with the volume fraction concept touches on the microscopic scale. The question that arises is: at what scale should the mechanical or thermodynamic investigations be performed, on the macro- or microscopic scale? In principle, both strategies are possible. However, the micromechanical approach, with all its averaging processes, is a large field (see, e.g., de Boer and Didwania, 1997, Didwania and de Boer, 1999) and the review of the micromechanical approach would exceed the scope of this book. This is also valid for homogenization processes and other micromechanic approaches. Therefore, only the macroscopic theory, which has recently come to consistent conclusions, will be discussed in the following pages. Of course, the micromechanical effects which are raised by the volume fraction concept and other phenomena will be considered using macromechanical quantities.

Another major problem arising in macroscopic porous media concerns the closure problem. It can easily be shown that, for porous media consisting of α constituents, $\alpha - 1$ field equations are missing taking into account the saturation condition. However, this results only from the macroscopic point of view. Nevertheless, as was mentioned, the porous media theory touches the microscopic scale due to the volume fraction concept. Thus, concerning the closure problem, the microscopic scale should be taken into consideration. Following this idea, the discussion of the closure problem results in some very reasonable conclusions.

Porous media theory has a long tradition (see de Boer, 1996, 2000 a). The purely mechanical theory was founded by Fillunger (1936). Unfortunately, his masterpiece was completely forgotten and ignored. It was rediscovered only fifteen years ago. Also, the valuable contributions of Heinrich (1938) and Heinrich and Desoyer (1955, 1956, 1961) were not noted by the international scientific community.

After the redevelopment of the mixture theory at the end of the 1950s and in the 1960s, it seems that Morland (1972) was the first scientist to use the volume fraction concept with elements of the mixture theory to construct “a simple constitutive theory for a fluid-saturated porous solid”.

Goodman and Cowin (1972) presented a theory for granular materials with interstitial voids making use of formal arguments from continuum mechanics. In order to overcome the lack of their continuum mechanical approach, namely, that one field equation is missing, they defined a balance equation of equilibrated forces and of equilibrated inertia. Such balance equations remain obscure from the physical point of view and should be rejected in light of the modern theory of porous media.

In an extensive paper entitled *Ideal Multiphase Mixtures with Chemical Reactions and Diffusion*, Nunziato and Walsh (1980) gave a review of known theories and extended the multiphase mixture theory to include chemically reacting materials.

It seems that the correct formulation of the amount of constraints in fluid-saturated granular materials with incompressible constituents can be credited to Nunziato and Passman (1981). They consequently introduced Lagrange multipliers to the entropy-inequality, in order to gain restrictions for the constitutive relations. In addition, they considered the volume fraction condition, namely that the sum of the volume fractions had to be equal to one, as a constraint.

The theory of multiphase mixtures was reviewed and extended by Passman, Nunziato, and Walsh (1984). In particular, they clarified the role of the internal constraints.

In two long papers, Bowen (1980, 1982) treated incompressible and compressible porous media by use of the theory of mixtures restricted by the volume fraction concept. He chose another method to describe the thermodynamic behavior of porous media than that of Nunziato and Walsh (1980). In his work on porous media theory with incompressible constituents, Bowen (1980) summarized all the findings of the mixture theory and introduced the volume fraction concept. He defined several new notions and investigated the incompressible porous media model on the basis of second-order materials. This led to a vast formalism, making it difficult to read the paper. In a second paper, Bowen (1982) extended his porous media theory to compressible porous media. In his reflections on the constitutive theory, Bowen (1982) used an idea exploited by Drumheller (1978), "in adopting a rate law to govern the volume fractions". This assumption is hard to understand from the physical point of view and should therefore be avoided.

Mow *et al.* (1980) published a paper *Biphasic Creep and Stress Relaxation of Articular Cartilage in Compression: Theory and Experiments* which is mainly concerned with biomechanical problems. However, in this paper an incompressible binary mixture model was independently developed which is similar to Bowen's (1980) approach.

In the time to follow, in the second half of the 1980s and at the beginning of the 1990s, the research in porous media theory was mainly focused on three directions: first, the implantation of the developed porous media models into numerical algorithms; second, the incorporation of different material behavior into the developed mathematical models; and third, the investigation of special phenomena appearing in saturated and empty porous solids.

Different models have been used for the numerical treatment of initial and boundary-value problems ranging from improved classical models proposed by Biot (1955, 1956) to the model based on the mixture theory restricted by the volume fraction concept. In this connection, one may see, for example, the extended paper by Zienkiewicz *et al.* (1990), which contains an improved Biot model as well as the treatise by Schrefler *et al.* (1993), in which a model based on the mixture theory is treated.

The incorporation of different material behavior has been performed, for example, by de Boer and Kowalski (1983), de Boer and Ehlers (1986 a, b), and de Boer and Lade (1991).

Many efforts were made starting in the early 1980s and the beginning of the 1990s to explain and describe special phenomena occurring in saturated and empty porous solids, which had already been partially recognized a long time ago but which had, however, never been fully founded. Within this framework, the paper by Baer and Nunziato (1986), which was concerned with the deflagration-to-detonation transition in reactive saturated granular materials, is mentioned. In this paper, the authors adopted, in addition to the balance equation of equilibrated force, Bowen's (1982) procedure of assuming an evolution equation for the volume fractions.

Another interesting phenomenon in saturated and partially saturated granular media was investigated by different authors on the basis of the mixture theory (see Passman and McTigue, 1984, and McTigue *et al.*, 1983), namely the concept of effective stresses. In the treatise by de Boer and Ehlers (1990 b), in which Bowen's (1980) model for incompressible constituents was used to prove this important concept, an extended section about the historical development of the concept of effective stresses can be found.

The effects of uplift, friction, and capillarity in liquid saturated porous solids have been extensively discussed by de Boer and Ehlers (1990 a), see also de Boer and Ehlers (1988), in which the historical development of the discovery of these phenomena has also been traced.

In recent times, there have again been new attempts to improve the fundamentals of the porous media theory in such a way that the basic equations be mathematically and/or physically better understood. In particular, the huge formalism of the mixture theory due to the introduction of the barycentric velocity with all its consequences for the kinematics of the mixture body and other average quantities has been abandoned (see, e.g., Kowalski, 1994, de Boer, 1992, 1995 b, and Bluhm, 1997).

Li and Li (1992) constructed a theory on the thermo-elasticity of multicomponent, fluid-saturated, reacting porous media. The authors pointed out that Bowen's (1982) theory had to be expanded through additional constitutive equations. The treatise of Li and Li (1992) remains in some parts unclear. In de Boer (1994) and de Boer and Kowalski (1995), the closure of the theory was obtained by assuming α constitutive equations for the interface pressure of a porous medium consisting of α constituents. However, this model considers only the compressibility due to the interface pressure. New investigations concerning the compressibility of porous solids have been performed by de Boer (1996, 2000 a) and Bluhm (2002).

Recently, some important new findings in the porous media theory have been worked out and partly published, which has led to a widely consistent theory. The main features of this theory are discussed in this book, whereby purely experimental and numerical investigations will not be addressed. Those contributions which repeat older basic approaches which have failed will also not be included.

Chapter 2

VOLUME FRACTION CONCEPT

In the volume fraction concept, it is assumed that the porous solid always models a control space and that only the liquids and/or gases contained in the pores can leave the control space. Furthermore, it is assumed that the pores are statistically distributed and that an arbitrary volume element in the reference and the actual placement is composed of the volume elements of the real constituents.

The basis of the description of porous media, using elements of the theory of mixtures restricted by the volume fraction concept, is the model of a macroscopic body, where neither a geometrical interpretation of the porestructure nor the exact location of the individual components of the body (constituents) are considered.

As mentioned above, the volume fraction concept is a very rigorous condition used in order to create homogenized continua. Therefore, there have been many attempts to improve this concept. A well-known concept is due to Kubik (1979) (see also Kubik and Sawczuk, 1983), who introduced a second-order, symmetric tensor of structural permeability $\mathbf{P}$, which turns in the isotropic case into $\mathbf{P} = \hbar\mathbf{I}$, where $\hbar$ is a scalar representing the surface porosity. It seems that these promising investigations have not been continued.

In order to develop the volume fraction concept we proceed from the fact that a porous medium occupying the control space of the porous solid B_S, with the boundary ∂B_S in the *actual placement*, consists of constituents φ^α, with real volumes v^α, where the index α denotes κ individual constituents. The boundary ∂B_S is a material surface for the solid phase and a non-material surface for the liquid and/or gas phases.

The concept of volume fractions can be formulated as follows:

$$n^\alpha(\mathbf{x}, t) = \frac{dv^\alpha}{dv}\,, \tag{2.1}$$

where $\mathbf{x}$ is the position vector of the actual placement and t the time. Moreover, the volume elements of the real materials and the bulk volume are denoted by dv^α and dv. The volume fractions n^α in (2.1) satisfy the volume fraction condition for κ constituents φ^α,

$$\sum_{\alpha=1}^{\kappa} n^\alpha = 1 \, , \tag{2.2}$$

as can easily be proven. The volume fraction concept leads to the result that the constituents, which are bound against each other, are "smeared" over the control space (partial bodies), which is shaped by the porous solid, i.e., that each substitute constituent occupies the total volume of space simultaneously with the other constituents.

Let the real density and the partial density of the constituent materials be denoted as:

$$\rho^{\alpha R} = \rho^{\alpha R}(\mathbf{x}, t) \; , \;\; \rho^\alpha = \rho^\alpha(\mathbf{x}, t) \; . \tag{2.3}$$

Between the relations $(2.3)_{1,2}$, there exists the relation

$$\rho^\alpha(\mathbf{x}, t) = n^\alpha(\mathbf{x}, t)\rho^{\alpha R}(\mathbf{x}, t) \; . \tag{2.4}$$

Eq. (2.4) is well-known in porous media theory.

The volume fraction concept formulated in the *reference state* at $t = t_0$ can be derived from the corresponding expressions in the actual placement.

The concept of volume fractions, introduced in the preceding forms, is an important part of the theory following in the next sections.

Due to the volume fraction concept, all geometric and physical quantities, such as motion, deformation, and stress, are defined in the total control space, as mentioned above, and thus, they can be interpreted as the statistical average values of the real quantities. Within the framework of the general porous media theory, a saturated porous medium will be treated as an immiscible mixture of all constituents, with particles X_α. This immiscible mixture is, of course, a substitute model; it can be treated with the methods of continuum mechanics, especially with the elements of the mixture theory. For the individual constituents, the kinematics and the balance equations will be discussed extensively; the connection between the balance equations of the partial constituents and those of the mixture body will also be noted. Moreover, the entropy inequality, which yields important restrictions in the constitutive theory, will be addressed.

Chapter 3

KINEMATICS

The kinematics in the porous media theory are based on two fundamental assumptions:

(1) Each spatial point $\mathbf{x}$ of the actual placement is simultaneously occupied by material points X_α of all κ constituents φ^α at the time t. The material points proceed from different reference positions $\mathbf{X}_\alpha$ at time $t = t_0$.

(2) Each constituent is assigned an independent state of motion.

Kinematics, developed on this base, is widely accepted in the literature. In the last years, however, the kinematics of porous media have been specified and extended. We will come back to these contributions later.

3.1 Basic Relations

If the motion of the constituent is understood as a chronological succession of placements $\boldsymbol{\chi}_\alpha$, then for the spatial position vector $\mathbf{x}$ of the material points X_α, which can be identified with the reference position vector $\mathbf{X}_\alpha$ at time $t = t_0$, the following relation holds at time t:

$$\mathbf{x} = \boldsymbol{\chi}_\alpha(\mathbf{X}_\alpha, t) \ . \tag{3.1}$$

The position vector $\mathbf{x}$ is an element of the control space of the porous solid at time t. In general, it is not necessary, e.g., in a ternary model (solid, liquid, and gas), to demand that the reference positions $\mathbf{X}_L$ and $\mathbf{X}_G$ of the liquid and gas particles be elements of the reference placement of the solid phase at time $t = t_0$, i.e., $\mathbf{X}_L \notin B_{0S}$ and $\mathbf{X}_G \notin B_{0S}$. Only for those deformation processes in which the fluid phases leave the control space of the solid phase are the reference positions $\mathbf{X}_L$ and $\mathbf{X}_G$ elements of B_{0S} (see Bluhm, 1997).

A geometrical interpretation of the motion function (3.1), concerning the motion of a solid and a liquid particle, is shown in Fig 3.1.1. Eq. (3.1) represents

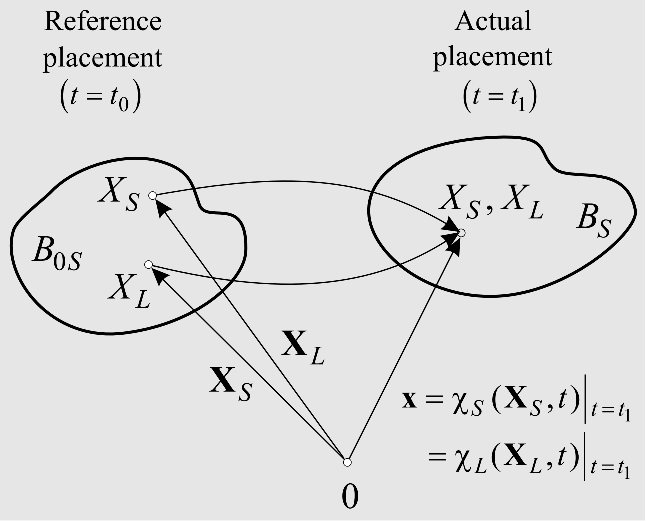

Fig. 3.1.1: Motions of a solid and liquid particle in a liquidsaturated porous solid.

the Lagrange description of motion. The function χ_α is postulated to be unique, and uniquely invertible, at any time t. The existence of a function inverse to (3.1) leads to the Eulerian description of motion, viz

$$\mathbf{X}_\alpha = \chi_\alpha^{-1}(\mathbf{x}, t) \; . \tag{3.2}$$

A mathematically necessary and sufficient condition for the existence of Eq. (3.2) is given, if the Jacobian

$$J_\alpha = \det \mathbf{F}_\alpha \tag{3.3}$$

differs from zero. In (3.3), $\mathbf{F}_\alpha$ is the deformation gradient, which is defined as

$$\mathbf{F}_\alpha = \mathrm{Grad}_\alpha \chi_\alpha \; . \tag{3.4}$$

The differential operator "Grad_α" denotes the partial differentiation with respect to the reference position $\mathbf{X}_\alpha$ of the constituents φ^α. The inverse of (3.4) is given by

$$\mathbf{F}_\alpha^{-1} = \mathrm{grad}\; \mathbf{X}_\alpha \tag{3.5}$$

with the differential operator "grad" referring to the spatial point $\mathbf{x}$. During the deformation process, $\mathbf{F}_\alpha$ is restricted to

$$\det \mathbf{F}_\alpha > 0 \; . \tag{3.6}$$

With the Lagrange description of the motion (3.1), the velocity and the acceleration of a material point of a constituent φ^α are defined by

$$\mathbf{x}'_\alpha = \frac{\partial \boldsymbol{\chi}_\alpha(\mathbf{X}_\alpha, t)}{\partial t} \;, \quad \mathbf{x}''_\alpha = \frac{\partial^2 \boldsymbol{\chi}_\alpha(\mathbf{X}_\alpha, t)}{\partial t^2} \;. \tag{3.7}$$

Using (3.2), the Eulerian description is gained for the velocity $\mathbf{v}_\alpha$ and the acceleration $\mathbf{a}_\alpha$:

$$\mathbf{v}_\alpha = \mathbf{x}'_\alpha = \mathbf{x}'_\alpha(\mathbf{x}, t) \;, \quad \mathbf{a}_\alpha = \mathbf{x}''_\alpha = \mathbf{x}''_\alpha(\mathbf{x}, t) \;. \tag{3.8}$$

As the individual constituents follow, in general, different motions, different material time derivatives must be formulated. This will be shown for an arbitrary scalar function $\Gamma(\mathbf{x}, t)$. Analogous material time derivatives of vector and tensor functions result. If $\Gamma(\mathbf{x}, t)$ is a differentiable function, then its material time derivative, following the motion of the constituent φ^α, is defined by

$$\Gamma'_\alpha = \frac{\partial \Gamma}{\partial t} + \operatorname{grad} \Gamma \cdot \mathbf{x}'_\alpha \;. \tag{3.9}$$

With $(3.7)_1$, the material velocity gradient of the constituent φ_α is obtained:

$$(\mathbf{F}_\alpha)'_\alpha = \operatorname{Grad}_\alpha \mathbf{x}'_\alpha \;. \tag{3.10}$$

The spatial velocity gradient can be calculated from $(3.8)_1$ and results in

$$\mathbf{L}_\alpha = \operatorname{grad} \mathbf{x}'_\alpha = \operatorname{grad} \mathbf{v}_\alpha \;, \tag{3.11}$$

which is connected to the material velocity gradient and the deformation gradient by

$$\mathbf{L}_\alpha = (\mathbf{F}_\alpha)'_\alpha \mathbf{F}_\alpha^{-1} \;. \tag{3.12}$$

Usually, no distinction is made in the literature between $\mathbf{x}'_\alpha$ and $\mathbf{v}_\alpha$, nor between $\mathbf{x}''_\alpha$ and $\mathbf{a}_\alpha$, because it is, in many cases, obvious in connection with the operator as to whether $\mathbf{x}'_\alpha(\mathbf{X}_\alpha, t)$ or $\mathbf{x}'_\alpha(\mathbf{x}, t)$, as well as $\mathbf{x}''_\alpha(\mathbf{X}_\alpha, t)$ or $\mathbf{x}''_\alpha(\mathbf{x}, t)$, is meant; see, for example, (3.10) and (3.11).

The additive decomposition of $\mathbf{L}_\alpha$ yields the symmetrical part $\mathbf{D}_\alpha$ of the spatial velocity gradient and the skew-symmetric spin tensor $\mathbf{W}_\alpha$

$$\mathbf{L}_\alpha = \mathbf{D}_\alpha + \mathbf{W}_\alpha \tag{3.13}$$

with

$$\mathbf{D}_\alpha = \frac{1}{2}(\mathbf{L}_\alpha + \mathbf{L}_\alpha^T) \;, \quad \mathbf{W}_\alpha = \frac{1}{2}(\mathbf{L}_\alpha - \mathbf{L}_\alpha^T) \;. \tag{3.14}$$

Since the local deformations $\mathbf{F}_\alpha$ contain, in general, parts of a rigid body motion, they are less suitable to serve as measurements for the deformations in constitutive equations. For this reason, it is convenient to use the line-elements, in the form of the difference of the squares of the line-elements, in the actual and the reference placements for the measurement of the deformation, in order to avoid irrational operations and to take out the rigid body motions. For the evaluation of the squares of the line-elements, the transport mechanism $d\mathbf{x} = \mathbf{F}_\alpha d\mathbf{X}_\alpha$ gained from (3.4) will be used. After elementary calculations, the following relations are obtained:

$$d\mathbf{x} \cdot d\mathbf{x} \; - \; d\mathbf{X}_\alpha \cdot d\mathbf{X}_\alpha = d\mathbf{X}_\alpha \cdot 2\mathbf{E}_\alpha d\mathbf{X}_\alpha = d\mathbf{x} \cdot 2\mathbf{A}_\alpha d\mathbf{x} \,. \tag{3.15}$$

The introduced symmetric strain tensors $\mathbf{E}_\alpha$ and $\mathbf{A}_\alpha$ are known, respectively, as the Green strain tensor and the Almansi strain tensor. They depend on the deformation gradient $\mathbf{F}_\alpha$ in the following way:

$$\mathbf{E}_\alpha = \frac{1}{2}(\mathbf{C}_\alpha - \mathbf{I}) \,, \quad \mathbf{A}_\alpha = \frac{1}{2}(\mathbf{I} - \mathbf{B}_\alpha^{-1}) \,, \tag{3.16}$$

where

$$\mathbf{C}_\alpha = \mathbf{F}_\alpha^T \mathbf{F}_\alpha \quad \text{and} \quad \mathbf{B}_\alpha = \mathbf{F}_\alpha \mathbf{F}_\alpha^T \tag{3.17}$$

denote the right and left Cauchy-Green deformation tensors, respectively. For further investigations, it is useful to multiplicatively decompose the deformation gradient $\mathbf{F}_\alpha$ into volume-preserving and spherical parts denoted by the symbols $\overset{\smile}{(\ldots)}$ and $\overset{\sim}{(\ldots)}$, see Bluhm (1997):

$$\mathbf{F}_\alpha = \tilde{\mathbf{F}}_\alpha \breve{\mathbf{F}}_\alpha \tag{3.18}$$

with

$$\begin{aligned} \tilde{\mathbf{F}}_\alpha &= (J_\alpha)^{1/3}\mathbf{I} \,, \quad J_\alpha = \det \mathbf{F}_\alpha \,, \\ \breve{J}_\alpha &= \det \breve{\mathbf{F}}_\alpha = 1 \,. \end{aligned} \tag{3.19}$$

With the decomposition (3.18), we obtain for the right Cauchy-Green tensor $(3.17)_1$:

$$\mathbf{C}_\alpha = (J_\alpha)^{2/3} \breve{\mathbf{C}}_\alpha \,, \quad \breve{\mathbf{C}}_\alpha = \breve{\mathbf{F}}_\alpha^T \breve{\mathbf{F}}_\alpha \,. \tag{3.20}$$

Considering (3.3), (3.12), and (3.13), the material time derivative of the Jacobian, as well as of the right Cauchy-Green deformation tensor, the Green strain tensor, and the volume-preserving part of the right Cauchy-Green tensor, yields

$$\begin{aligned}
(J_\alpha)'_\alpha &= J_\alpha(\mathbf{D}_\alpha \cdot \mathbf{I}) \,, \quad (\mathbf{C}_\alpha)'_\alpha = 2\mathbf{F}^T_\alpha \mathbf{D}_\alpha \mathbf{F}_\alpha \,, \\
(\mathbf{E}_\alpha)'_\alpha &= \mathbf{F}^T_\alpha \mathbf{D}_\alpha \mathbf{F}_\alpha \,, \\
(\breve{\mathbf{C}}_\alpha)'_\alpha &= 2\, J_\alpha^{-2/3}\, \mathbf{F}^T_\alpha \mathbf{D}^D_\alpha \mathbf{F}_\alpha \,,
\end{aligned} \tag{3.21}$$

where $\mathbf{D}^D_\alpha$ is the deviatoric part of $\mathbf{D}_\alpha$.

So far, kinematics in the porous media theory has been applied generally. There are, however, as mentioned above, two essential improvements and extensions, namely, the multiplicative decomposition of the deformation gradient into a tensor which describes the deformation of the real material and into a tensor which describes the change of the pores in shape and size. The other extension is based on the classical Cosserat theory.

In what follows, some considerations on the microscopic scale are needed in order to describe the compressibility and incompressibility of the real materials. For this purpose, a macroscopic control space filled with a granular solid phase, and a gas without any physical properties, will be considered. The grains in the control space are represented by small balls. It is assumed that the grains are incompressible, i.e., a hydrostatic stress state in the grains produces no volume change. Although the grains are incompressible, contact forces acting on them cause a volume change of the control space; this results from the change of the porestructure and the volume fraction due to the change of the shapes of the individual grains (see Bluhm, 1997). Therefore, the incompressibility condition cannot be expressed by the deformation gradient $\mathbf{F}_S$ of the partial solid constituent. Rather, the incompressibility condition must be formulated by physical quantities at the microscopic scale. Moreover, statements on the compressibility and other real properties of the constituents must also be expressed by physical quantities at the microscopic scale. In the case of describing compressibility and incompressibility, this means a motion function at the microscopic scale

$$\mathbf{x}_{SR(\text{micro})} = \chi_{SR(\text{micro})}(\mathbf{X}_S + \xi_{SR}, t) \tag{3.22}$$

must be introduced, where $\mathbf{X}_S$ is the center of the volume element dv and ξ_{SR} the vector in the center directed to the material points. From (3.22), the deformation gradient $\mathbf{F}_{SR(\text{micro})}$ can be determined in a way similar to that in (3.4). Then, the incompressibility condition can be reformulated at the microscopic scale:

$$\det \mathbf{F}_{SR(\text{micro})} = J_{SR(\text{micro})} = 1 \,. \tag{3.23}$$

The crucial point of this procedure is however the fact that the motion function $\chi_{SR(\text{micro})}$ in (3.22) is completely unknown and cannot be determined by a balance equation within the framework of the mixture theory (microscopic scale). Therefore, it is advisable in order to describe the phenomena of compressibility and incompressibility, to transfer the microscopic scale deformation behavior of the real solid phase to the macroscopic scale. For this reason, the deformation tensor $\mathbf{F}_{SR}$ is introduced, which is understood to be a part of the deformation Gradient $\mathbf{F}_S$ and is assumed to reflect the microscopic scale deformations of the real solid material at the macroscopic scale. In general, the tensor $\mathbf{F}_{SR}$ is not integrable at the macroscopic scale i.e., the microscopic deformations $\mathbf{F}_{SR(\mathrm{micro})}$ are represented by incompatible deformations at the macroscopic scale. Since

$$\mathbf{F}_S \neq \mathbf{F}_{SR} \tag{3.24}$$

it is necessary to choose a second tensor $\mathbf{F}_{SN}$ to transfer the relation (3.24) into an equation. The part $\mathbf{F}_{SN}$ of the deformation gradient $\mathbf{F}_S$, as well as $\mathbf{F}_{SR}$ is in general not integrable. On the contrary, the deformation tensor $\mathbf{F}_S$ is integrable; thus the deformation gradient at the macroscopic scale must be multiplicatively decomposed into $\mathbf{F}_{SR}$ and $\mathbf{F}_{SN}$.

In the following, some parallels to the theory of the elastic-plastic deformations of metals will be discussed. It is well-known that within the framework of a finite theory a multiplicative decomposition of the deformation gradient into an elastic and a plastic part is widely used. The plastic part of the deformation at the macroscopic scale is caused by dislocations at the microscopic scale. These microscopic dislocations are also in general, represented by incompatible strains at the macroscopic scale. Therefore, the reason for a multiplicative decomposition of the deformation gradient at the macroscopic scale is the same as in the porous media theory, namely, to bring physical phenomena from the microscopic scale to the macroscopic scale.

There are two possibilities of multiplicatively decomposing the deformation gradient $\mathbf{F}_\alpha$ of the constituent φ^α, of which only the following one is suitable (see the extensive discussion of this problem and the consequences concerning the kinematics in Bluhm and de Boer, 1997):

$$\mathbf{F}_\alpha = \mathbf{F}_{\alpha N} \hat{\mathbf{F}}_{\alpha R} \, . \tag{3.25}$$

$\hat{\mathbf{F}}_{\alpha R}$ is the part of $\mathbf{F}_\alpha$ describing the deformation of the real material, whereas $\mathbf{F}_{\alpha N}$ describes the remaining part of the deformation of the control space, namely the change of the pores in size and shape. The parts $\mathbf{F}_{\alpha N}$ and $\hat{\mathbf{F}}_{\alpha R}$ are to be understood as local mappings of tangent (vector) spaces in each material point of the body. In the case of homogeneous deformations, the multiplicative decomposition (3.25) leads to an intermediate state $\widehat{(\ldots)}$ (see Fig 3.1.2). The proof of the multiplicative decomposition (3.25) of $\mathbf{F}_\alpha$ into quantities describing properties of the microscopic scale is still awaiting research. However, the

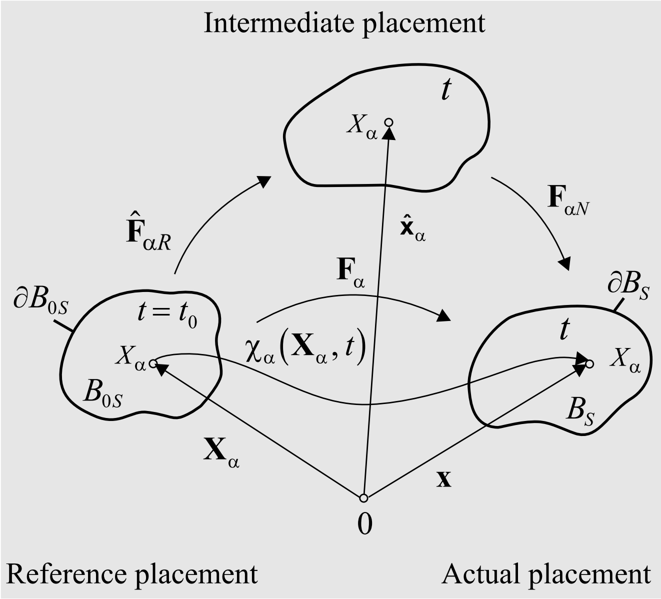

Fig. 3.1.2: Geometrical interpretation of the decomposition
$\mathbf{F}_\alpha = \mathbf{F}_{\alpha N}\hat{\mathbf{F}}_{\alpha R}$

multiplicative decomposition is a useful concept to describe the effects on the microscopic scale. In analogy to (3.12) through (3.14), material time derivatives of the deformation tensor $\hat{\mathbf{F}}_{\alpha R}$ can be introduced:

$$\begin{aligned}
\hat{\mathbf{L}}_{\alpha R} &= (\hat{\mathbf{F}}_{\alpha R})'_\alpha(\hat{\mathbf{F}}_{\alpha R})^{-1}\ , \quad \hat{\mathbf{L}}_{\alpha R} = \hat{\mathbf{D}}_{\alpha R} + \hat{\mathbf{W}}_{\alpha R}\ ,\\
\hat{\mathbf{D}}_{\alpha R} &= \frac{1}{2}(\hat{\mathbf{L}}_{\alpha R} + \hat{\mathbf{L}}^T_{\alpha R})\ , \quad \hat{\mathbf{W}}_{\alpha R} = \frac{1}{2}(\hat{\mathbf{L}}_{\alpha R} - \hat{\mathbf{L}}^T_{\alpha R})\ ,\\
\hat{\mathbf{L}}_{\alpha R} &= \frac{\partial(\hat{\mathbf{x}}_\alpha)'_\alpha}{\partial \mathbf{X}_\alpha}\frac{\partial \mathbf{X}_\alpha}{\partial \hat{\mathbf{x}}_\alpha} = \frac{\partial(\hat{\mathbf{x}}_\alpha)'_\alpha}{\partial \hat{\mathbf{x}}_\alpha}\\
&\quad \text{(homogeneous deformations)}.
\end{aligned} \tag{3.26}$$

The introduction of the material time derivative of $\mathbf{F}_{\alpha N}$, namely $\mathbf{L}_{\alpha N}$, is less useful due to the fact that $\mathbf{L}_{\alpha N}$ is not a spatial velocity Gradient (see Bluhm and de Boer, 1997). Now, the volume fraction concept and the incompressibility condition under consideration of the multiplicative decomposition of the deformation gradient $\mathbf{F}_\alpha$ of the constituent φ^α will be discussed. We proceed from (3.25) and split the two parts, $\mathbf{F}_{\alpha N}$ and $\hat{\mathbf{F}}_{\alpha R}$, of the deformation Gradient into volume-preserving and spherical parts denoted by the symbols $(\overset{\smile}{\ldots})$ and $(\tilde{\ldots})$. Thus,

$$\mathbf{F}_{\alpha N} = \tilde{\mathbf{F}}_{\alpha N}\breve{\mathbf{F}}_{\alpha N}\ , \quad \hat{\mathbf{F}}_{\alpha R} = \tilde{\mathbf{F}}_{\alpha R}\breve{\mathbf{F}}_{\alpha R} \tag{3.27}$$

with

$$\begin{aligned} &\tilde{\mathbf{F}}_{\alpha N} = (J_{\alpha N})^{1/3}\mathbf{I}\ , \quad J_{\alpha N} = \det \mathbf{F}_{\alpha N}\ , \\ &\breve{J}_{\alpha N} = \det \breve{\mathbf{F}}_{\alpha N} = 1\ , \\ &\tilde{\mathbf{F}}_{\alpha R} = (\hat{J}_{\alpha R})^{1/3}\mathbf{I}\ , \quad \hat{J}_{\alpha R} = \det \hat{\mathbf{F}}_{\alpha R}\ , \\ &\breve{J}_{\alpha R} = \det \breve{\mathbf{F}}_{\alpha R} = 1\ . \end{aligned} \tag{3.28}$$

With these quantities, kinematic expressions corresponding to those of $\tilde{\mathbf{F}}_\alpha$ and $\breve{\mathbf{F}}_\alpha$ can be formulated. The following derivatives in particular, are valid:

$$\begin{aligned} (\hat{J}_{\alpha R})'_\alpha &= \hat{J}_{\alpha R}(\hat{\mathbf{D}}_{\alpha R} \cdot \mathbf{I})\ , \\ (J_{\alpha N})'_\alpha &= J_{\alpha N}(\mathbf{D}_{\alpha N} \cdot \mathbf{I}) \end{aligned} \tag{3.29}$$

with

$$\mathbf{D}_{\alpha N} \cdot \mathbf{I} = \mathbf{D}_\alpha \cdot \mathbf{I} - \hat{\mathbf{D}}_{\alpha R} \cdot \mathbf{I}\ . \tag{3.30}$$

For further investigations, the volume elements in the reference and actual placements have to be considered. The relations between the volume elements in the two placements and the intermediate configuration give a deeper insight into the volumetric strains in the various placements.

In continuum mechanics, it is well-known that the following transport theorem concerning the volume elements is valid:

$$dv = J_\alpha dv_{0\alpha}\ , \tag{3.31}$$

where

$$dv_{0\alpha} = dv_{0\alpha}(\mathbf{X}_\alpha, t = t_0)\ , \quad dv = dv(\mathbf{x}, t) \tag{3.32}$$

are the volume elements in the reference placement at the position $\mathbf{X}_\alpha$, denoted by the subscript index α, and in the actual placement at the position $\mathbf{x}$. In consideration of (3.25) through (3.28),

$$dv = J_{\alpha N}\hat{J}_{\alpha R}dv_{0\alpha} \tag{3.33}$$

is gained. By using (3.25), a differential volume element $d\hat{v}_\alpha$, at a material point $\mathbf{X}_\alpha$ of a local intermediate placement in the tangent space is related to the differential volume elements in the reference placement and the actual placements by (see de Boer, 1996, and Bluhm and de Boer, 1997):

$$d\hat{v}_\alpha = \hat{J}_{\alpha R} dv_{0\alpha} \; , \quad dv = J_{\alpha N} d\hat{v}_\alpha \; . \tag{3.34}$$

With the relations (3.31) through (3.34), it is easy to formulate various kinds of volume strains which will be important for the investigations in the following paragraphs. In analogy to the volume strain of the partial material of the constituent φ^α,

$$e_\alpha = \frac{dv - dv_{0\alpha}}{dv_{0\alpha}} = \frac{dv}{dv_{0\alpha}} - 1 = J_\alpha - 1 \; , \tag{3.35}$$

where (3.31) has been used, the volume strain of the real material of φ^α is defined as

$$\begin{aligned} e_{\alpha R} &= \frac{dv^\alpha - dv^\alpha_{0\alpha}}{dv^\alpha_{0\alpha}} = \frac{n^\alpha dv - n^\alpha_{0\alpha} dv_{0\alpha}}{n^\alpha_{0\alpha} dv_{0\alpha}} \\ &= \frac{n^\alpha}{n^\alpha_{0\alpha}} \frac{dv}{dv_{0\alpha}} - 1 \\ &= \frac{n^\alpha}{n^\alpha_{0\alpha}} J_\alpha - 1 = \frac{n^\alpha}{n^\alpha_{0\alpha}} J_{\alpha N} \hat{J}_{\alpha R} - 1 \; , \end{aligned} \tag{3.36}$$

where $dv^\alpha_{0\alpha} = n^\alpha_{0\alpha} dv_{0\alpha}$ as well as (2.1), (3.31), and (3.33) have been used. In consideration of the transport theorems (3.34), further real volume strains can be formulated:

$$\begin{aligned} \hat{e}_{\alpha R} &= \frac{d\hat{v}^\alpha_\alpha - dv^\alpha_{0\alpha}}{dv^\alpha_{0\alpha}} = \frac{\hat{n}^\alpha_\alpha d\hat{v}_\alpha - n^\alpha_{0\alpha} dv_{0\alpha}}{n^\alpha_{0\alpha} dv_{0\alpha}} \\ &= \frac{\hat{n}^\alpha_\alpha}{n^\alpha_{0\alpha}} \frac{d\hat{v}_\alpha}{dv_{0\alpha}} - 1 = \frac{\hat{n}^\alpha_\alpha}{n^\alpha_{0\alpha}} \hat{J}_{\alpha R} - 1 \; , \\ \tilde{e}_{\alpha R} &= \frac{dv^\alpha - d\hat{v}^\alpha_\alpha}{d\hat{v}^\alpha_\alpha} = \frac{n^\alpha dv - \hat{n}^\alpha_\alpha d\hat{v}_\alpha}{\hat{n}^\alpha_\alpha d\hat{v}_\alpha} \\ &= \frac{n^\alpha}{\hat{n}^\alpha_\alpha} \frac{dv}{d\hat{v}_\alpha} - 1 = \frac{n^\alpha}{\hat{n}^\alpha_\alpha} J_{\alpha N} - 1 \; , \end{aligned} \tag{3.37}$$

where the part $d\hat{v}^\alpha_\alpha$ of the differential volume element $d\hat{v}_\alpha$, in the local intermediate placement in the tangent space, is defined via the volume fraction $\hat{n}^\alpha_\alpha$ as

$$d\hat{v}^\alpha_\alpha = \hat{n}^\alpha_\alpha d\hat{v}_\alpha \; . \tag{3.38}$$

The determinants of $\hat{\mathbf{F}}_{\alpha R}$ and $\mathbf{F}_{\alpha N}$ can be expressed depending on the real volume strains $\hat{e}^{\alpha R}$ and $\tilde{e}^{\alpha R}$, namely

$$\hat{J}_{\alpha R} = \frac{n^\alpha_{0\alpha}}{\hat{n}^\alpha_\alpha}(\hat{e}_{\alpha R} + 1) \; , \quad J_{\alpha N} = \frac{\hat{n}^\alpha_\alpha}{n^\alpha}(\tilde{e}_{\alpha R} + 1) \; . \tag{3.39}$$

With (3.39), the real volume strain (3.36) can be reformulated as

$$e_{\alpha R} = \hat{e}_{\alpha R} + \tilde{e}_{\alpha R} + \hat{e}_{\alpha R}\tilde{e}_{\alpha R} \; . \tag{3.40}$$

The tensor $\hat{\mathbf{F}}_{\alpha R}$ is interpreted as that part of the deformation gradient which includes the whole deformation of the real material of the constituent φ^α. Thus, the determinant $J_{\alpha R}$ must reflect the volume strain of the real material of φ^α, i.e., the real volume strain is the difference between the part $d\hat{v}^\alpha_\alpha = \hat{n}^\alpha_\alpha \, d\hat{v}_\alpha$ of the differential volume $d\hat{v}_\alpha$ in the local intermediate placement in the vector space and the part $dv^\alpha_{0\alpha} = n^\alpha_{0\alpha} \, dv_{0S}$ of the volume element dv_{0S} in the reference placement at the position $\mathbf{X}_S$. Therefore, the following relations concerning the real volume strains hold:

$$e_{\alpha R} = \hat{e}_{\alpha R} \; , \quad \tilde{e}_{\alpha R} = 0 \; . \tag{3.41}$$

Furthermore, the transport theorem $(3.34)_1$ excludes the change of the volume fraction by mapping $dv_{0\alpha}$ from the reference to the local intermediate placement, i.e.,

$$\hat{n}^\alpha_\alpha = n^\alpha_{0\alpha} \; . \tag{3.42}$$

With (3.41) and (3.42), the determinants $\hat{J}_{\alpha R}$ and $J_{\alpha N}$, see (3.39), read as follows:

$$\hat{J}_{\alpha R} = \hat{e}_{\alpha R} + 1 = e_{\alpha R} + 1 \; , \quad J_{\alpha N} = \frac{n^\alpha_{0\alpha}}{n^\alpha} \; . \tag{3.43}$$

In the case of incompressibility,

$$e_{\alpha R} = 0 \; , \quad \hat{J}_{\alpha R} = 1 \; , \quad \text{and} \quad \hat{\mathbf{D}}_{\alpha R} \cdot \mathbf{I} = 0 \tag{3.44}$$

are valid, whereby $(3.44)_3$ is the rate formulation of the incompressibility condition (see Bluhm and de Boer, 1997) and where $\hat{\mathbf{D}}_{\alpha R}$ can be interpreted as the Lie derivative of the strain tensor $\mathbf{E}_{\alpha R} = \frac{1}{2}(\mathbf{F}^T_{\alpha R}\mathbf{F}_{\alpha R} - \mathbf{I})$.

With the help of the multiplicative decomposition of the deformation gradient $\mathbf{F}_\alpha$ (3.25), it can be shown that the statement of Mills (1966, 1967) and Bowen (1980) concerning the incompressibility of the real material which was described by setting the real densities of the constituents constant, is only identical with the kinematic constraints (3.44) in the case of a thermodynamic process without any mass exchange (see Bluhm and de Boer, 1997).

As there is no difference in the volume changes of the real material in the actual and intermediate placements, we will omit the superscript sign $(\hat{\ldots})$ in the following sections.

3.2 Kinematics of Microscopic Polar Constituents

Diebels and Ehlers (1996) and Diebels (1999) extended the kinematics of the porous media theory to porous microscopic polar continua. In this case, the material points are assumed to be rigid particles on the microscopic scale. Each particle in the reference placement possesses an attached director $\mathbf{d}_\alpha^R$ which is rotated into the director $\mathbf{d}_\alpha$ of the actual placement. The microscopic motion is described by an orthogonal tensor $\bar{\mathbf{R}}_\alpha$:

$$\begin{aligned} \mathbf{d}_\alpha = \bar{\mathbf{R}}_\alpha \mathbf{d}_\alpha^R\ , \quad \bar{\mathbf{R}}_\alpha \bar{\mathbf{R}}_\alpha^T &= \bar{\mathbf{R}}_\alpha^T \bar{\mathbf{R}}_\alpha = \mathbf{I}\ , \\ \det \bar{\mathbf{R}}_\alpha &= 1\ . \end{aligned} \tag{3.45}$$

In order to describe the deformation of microscopic polar continua two additional deformation tensors, the first Cosserat deformation tensor $\mathbf{U}_\alpha$ and the curvature tensor $\mathbf{K}_\alpha$ are required. They are defined by (see Diebels, 1999):

$$\begin{aligned} \bar{\mathbf{U}}_\alpha &= \bar{\mathbf{R}}_\alpha^T \mathbf{F}_\alpha\ , \\ \bar{\mathbf{K}}_\alpha &= -\frac{1}{2}[\overset{3}{\mathbf{E}}\ (\bar{\mathbf{R}}_\alpha^T \mathrm{Grad}_\alpha \bar{\mathbf{R}}_\alpha)^{\underline{3}}]^{\underline{2}}\ . \end{aligned} \tag{3.46}$$

In the relation for the curvature tensor $\bar{\mathbf{K}}_\alpha$, the third-order tensor $\overset{3}{\mathbf{E}}$ is a fundamental tensor and the underlined superscript denotes the order of the resulting tensor (de Boer, 1982). Diebels and Ehlers (1996) showed that the first Cosserat deformation tensor $\bar{\mathbf{U}}_\alpha$ is related to the scalar product of the direction vector and line elements of the actual and reference placements:

$$\mathbf{d}_\alpha \cdot d\mathbf{x}_\alpha = \mathbf{d}_\alpha^R \cdot \bar{\mathbf{U}}_\alpha \mathbf{X}_\alpha\ . \tag{3.47}$$

Moreover, Diebels and Ehlers (1996) revealed that $\bar{\mathbf{K}}_\alpha$ is a generalized Christoffel symbol.

The material time derivative of $\mathbf{d}_\alpha$ can be described by the gyration tensor $\bar{\mathbf{W}}_\alpha$:

$$(\mathbf{d}_\alpha)'_\alpha = \bar{\mathbf{W}}_\alpha \mathbf{d}_\alpha\ , \quad \bar{\mathbf{W}}_\alpha = (\bar{\mathbf{R}}_\alpha)'_\alpha \bar{\mathbf{R}}_\alpha^T\ . \tag{3.48}$$

From $(3.45)_2$ it follows that $\bar{\mathbf{W}}_\alpha$ is a skew-symmetric tensor. In this case, the angular velocity is given by the axial vector of the gyration tensor:

$$\bar{\omega}_\alpha = \frac{1}{2}\overset{3}{\mathbf{E}}\,\bar{\mathbf{W}}_\alpha^T\ . \tag{3.49}$$

The deformation rates $(\bar{\mathbf{U}})'_\alpha$ and $(\bar{\mathbf{K}}_\alpha)'_\alpha$ are determined by

$$\begin{aligned}(\bar{\mathbf{U}}'_\alpha)'_\alpha &= \bar{\mathbf{R}}_\alpha^T(\mathbf{L}_\alpha - \bar{\mathbf{W}}_\alpha)\mathbf{F}_\alpha := \bar{\mathbf{R}}_\alpha^T\bar{\mathbf{L}}_\alpha\mathbf{F}_\alpha\ ,\\ (\bar{\mathbf{K}}_\alpha)'_\alpha &= \bar{\mathbf{R}}_\alpha^T \operatorname{grad}\bar{\omega}_\alpha\mathbf{F}_\alpha\ .\end{aligned} \tag{3.50}$$

Ehlers and Volk (1997 a,b, 1998, 1999) developed the corresponding Cosserat kinematics for porous media within the framework of the geometrically linear theory. They introduced the Gradient of the displacement $\mathbf{u}_S$ of the solid skeleton:

$$\mathbf{H}_S = \operatorname{Grad}_S\,\mathbf{u}_S\ . \tag{3.51}$$

With the displacement gradient $\mathbf{H}_S$, the classical deformation measures, namely the Lagrange strain tensor $\mathbf{E}_S = \boldsymbol{\varepsilon}_S$ and the continuum rotation vector $\boldsymbol{\varphi}_s$, can be reformulated:

$$\begin{aligned}\mathbf{E}_S = \quad \mathbf{H}_{S\mathrm{sym}} &= \frac{1}{2}\,(\mathbf{H}_S + \mathbf{H}_S^T) =: \boldsymbol{\varepsilon}_s\ ,\\ \mathbf{H}_{S\mathrm{skw}} &= \frac{1}{2}\,(\mathbf{H}_S - \mathbf{H}_S^T) =: \boldsymbol{\varphi}_S \times \mathbf{I}\ .\end{aligned} \tag{3.52}$$

The cross tensor product between the vector $\boldsymbol{\varphi}_S$ and the identity $\mathbf{I}$ is defined by

$$\boldsymbol{\varphi}_S \times \mathbf{I} = -\overset{3}{\mathbf{E}}\,\boldsymbol{\varphi}_S\ , \tag{3.53}$$

see de Boer, 1982.

From $(3.52)_2$,

$$\boldsymbol{\varphi}_S = -\frac{1}{2}\ \overset{3}{\mathbf{E}}\,(\mathbf{H}_{S\mathrm{skw}}) \tag{3.54}$$

is obtained.

For microscopic polar continua, the continuum mechanical description must be extended and additional microscopic polar degrees of freedom have to be introduced via the independent rotation $\overset{*}{\boldsymbol{\varphi}}_S$.

Then, the total average rotation $\bar{\boldsymbol{\varphi}}_S$ (within the framework of geometrically linear theories) is given as the sum of the continuum rotation and the additional microscopic polar rotation:

$$\bar{\boldsymbol{\varphi}}_S = \boldsymbol{\varphi}_S + \overset{*}{\boldsymbol{\varphi}}_S\ \ . \tag{3.55}$$

The linear Cosserat strain tensor $\boldsymbol{\varepsilon}_{SC}$ and the linear curvature tensor $\bar{\boldsymbol{\kappa}}_S$ are determined by Ehlers and Volk (1999) as:

$$\boldsymbol{\varepsilon}_{SC} = \mathbf{H}_S + \overset{3}{\mathbf{E}}\,\bar{\boldsymbol{\varphi}}_S\ , \quad \bar{\boldsymbol{\kappa}}_S = \mathrm{Grad}_S\,\bar{\boldsymbol{\varphi}}_S\ . \tag{3.56}$$

The symmetric and skew-symmetric parts of the Cosserat strain tensor can easily be obtained:

$$\begin{aligned} \boldsymbol{\varepsilon}_{sc\;sym} &= \frac{1}{2}(\mathbf{H}_S + \mathbf{H}_S^T)\ , \\ \boldsymbol{\varepsilon}_{sc\;skw} &= \frac{1}{2}(\mathbf{H}_S - \mathbf{H}_S^T) + \overset{3}{\mathbf{E}}\,\bar{\boldsymbol{\varphi}}_S\ . \end{aligned} \tag{3.57}$$

It is recognized that $\boldsymbol{\varepsilon}_{SC\;sym}$ is equal to the linearized Lagrange strain tensor $\boldsymbol{\varepsilon}_S$ of non-polar materials, whereas

$$\boldsymbol{\varepsilon}_{sc\;skw} = \overset{3}{\mathbf{E}}\,(\bar{\boldsymbol{\varphi}}_S - \boldsymbol{\varphi}_S) = \overset{3}{\mathbf{E}}\,\overset{*}{\boldsymbol{\varphi}}_S \tag{3.58}$$

is a tensorial measure for the additional microscopic polar rotation $\overset{*}{\boldsymbol{\varphi}}_S$.

Ehlers and Volk (1999) rounded off their investigations on Cosserat kinematics in the theory of porous media by stating the microscopic polar compatibility condition,

$$\mathrm{Grad}_S\,\boldsymbol{\varepsilon}_{SC} - \mathrm{Grad}_S^{\overset{23}{T}}\,\boldsymbol{\varepsilon}_{SC} = (\overset{3}{\mathbf{E}}\,\bar{\boldsymbol{\kappa}}_S)^{\underline{3}} - (\overset{3}{\mathbf{E}}\,\bar{\boldsymbol{\kappa}}_S)^{\underline{3}\overset{23}{T}} \tag{3.59}$$

and by reformulating and solving (3.59) with respect to $\bar{\boldsymbol{\kappa}}_s$:

$$\begin{aligned} \bar{\boldsymbol{\kappa}}_S = \frac{1}{2}\,\overset{3}{\mathbf{E}}\,(\mathrm{Grad}_S\,\boldsymbol{\varepsilon}_{SC} \quad &+ \quad \mathrm{Grad}_S^{\overset{13}{T}}\,\boldsymbol{\varepsilon}_{SC} - \\ &- \quad \mathrm{Grad}_S^{\overset{23}{T}}\,\boldsymbol{\varepsilon}_{SC})^{\underline{2}}\ . \end{aligned} \tag{3.60}$$

In the papers mentioned above, the Cosserat kinematics for porous solids is clearly developed and can easily be duplicated. However, the question remains (raised in the scientific community since the first zenith of Cosserat mechanics for one-component bodies in the 1960s) as to how the Cosserat kinematics can be founded physically. It seems that the additional kinematic quantities introduced in the Cosserat mechanics are, in many cases in the present-day, not yet accessible and measurable. This is valid, in particular, for saturated porous solids. Thus, much research and experience is needed in order to determine whether the Cosserat kinematics is as valuable in describing the rotation of granules as claimed by the authors above.

We will come back to the problem of including Cosserat kinematics into the porous media theory in the sections on balance equations, constitutive theory and numerical investigations.

Now, we will leave the development of kinematics and will turn to the formulation of the balance equations.

Chapter 4

BALANCE PRINCIPLES

In the mixture theory and porous media theory, balance principles – balance of mass, balance of momentum and moment of momentum, as well as balance of energy – have to be established for each constituent φ^α in consideration of all interaction and external agencies. There have been no substantial changes in these equations in the recent past. This means that all quantities resulting from long and short range effects which influence the individual constituents, as well as the interaction effects between the constituents, have to be considered in the balance principles.

The interaction effects (supply terms) have to be in the sum equal to zero. These conditions are founded in the fact that – at least in the case of a common velocity $\mathbf{v}$ and acceleration $\mathbf{a}$, a common external acceleration $\mathbf{b}$, as well as a common internal energy ε, and external heat supply r – the sum of the balance principles must formally become the corresponding balance principles of a one-component body.

The following balance principles will be formulated in global and local forms for the individual constituents in consideration of all interaction effects, and, finally, some conclusions in view of the comparison with the balance principles of the mixture body will be drawn.

4.1 Balance of Mass

As has already been mentioned, two possibilities exist concerning the formulation of the *balance principle of mass*. On the one hand, this equation can be given for the bulk mixture body; on the other hand, the mass balance principles can be formulated for each individual constituent in such a way that the superposition of the mass balance principles for the individual constituents turn, for special cases, into the balance principle of the mixture body as a one-component body.

The balance of mass for the individual constituents φ^α requires that the rate of the mass M^α equal a mass term $\int\limits_{B_\alpha} \hat{\rho}^\alpha dv$ caused by the other constituents, where $\hat{\rho}^\alpha$ is the mass supply per volume element:

$$(M^\alpha)'_\alpha = (\int\limits_{B_\alpha} \rho^\alpha dv)'_\alpha = \int\limits_{B_\alpha} \hat{\rho}^\alpha dv \ . \tag{4.1}$$

The integration in (4.1) covers the domain B_α of each individual constituent. With the help of the transport theorem

$$(dv)'_\alpha = \text{div } \mathbf{v}_\alpha dv \ , \tag{4.2}$$

from (4.1) the local statement

$$(\rho^\alpha)'_\alpha + \rho^\alpha \text{div } \mathbf{v}_\alpha = \hat{\rho}^\alpha \tag{4.3}$$

or

$$\frac{\partial \rho^\alpha}{\partial t} + \text{div } (\rho^\alpha \mathbf{v}_\alpha) = \hat{\rho}^\alpha \tag{4.4}$$

is derived. Assuming a common velocity $\mathbf{v}$ for all phases φ^α, the summation of (4.4) over all κ constituents φ^α leads to

$$\sum_{\alpha=1}^{\kappa} \left[\frac{\partial \rho^\alpha}{\partial t} + \text{div } (\rho^\alpha \mathbf{v}) \right] = \sum_{\alpha=1}^{\kappa} \hat{\rho}^\alpha \ . \tag{4.5}$$

With the statement that the sum of the densities of the individual constituents is equal to the density ρ of the mixture body,

$$\rho = \sum_{\alpha=1}^{\kappa} \rho^\alpha \ , \tag{4.6}$$

one obtains

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho \mathbf{v}) = \sum_{\alpha=1}^{\kappa} \hat{\rho}^\alpha \ . \tag{4.7}$$

For deriving Eq. (4.7), we have made use of the fact that the summation and the derivative are interchangeable. Equation (4.7) only results in the form valid for the mixture body if the constraint

$$\sum_{\alpha=1}^{\kappa} \hat{\rho}^\alpha = 0 \tag{4.8}$$

is introduced, i.e., if the sum of the local mass supplies of all κ constituents φ^α is equal to zero.

If all mass exchange is excluded, the relation (4.3) can be integrated, and it follows that:

$$
\begin{aligned}
\rho^\alpha = &\ \rho^\alpha_{0\alpha}(\det \mathbf{F}_\alpha)^{-1} \\
\text{with} & \qquad\qquad\qquad\qquad (4.9) \\
\rho^\alpha_{0\alpha} = &\ \rho^\alpha_{0\alpha}(\mathbf{X}_\alpha, t = t_0)\ .
\end{aligned}
$$

The quantity $\rho^\alpha_{0\alpha}$ denotes the partial density of the constituent φ^α (superscript index) in the reference placement at the position $\mathbf{X}_\alpha$ (subscript index).

4.2 Balance of Momentum and Moment of Momentum

In this section, the consequences of applying the axioms of the *balance of momentum* to each individual constituent will be discussed first.

The balance principle of momentum states that the material time derivative of the momentum is equal to the sum of external forces. Thus,

$$
(\mathbf{l}^\alpha)'_\alpha = \mathbf{k}^\alpha\ . \tag{4.10}
$$

Herein, the momentum $\mathbf{l}^\alpha$ for the constituent φ^α is defined by

$$
\mathbf{l}^\alpha = \int\limits_{B_\alpha} \rho^\alpha \mathbf{v}_\alpha dv\ . \tag{4.11}
$$

The external forces $\mathbf{k}^\alpha$ are given by the sum of the forces $\mathbf{f}^\alpha$, which are caused by long- and short-range effects and are acting on the constituents as volume forces $\rho^\alpha \mathbf{b}^\alpha$ and surface forces $\mathbf{t}^\alpha$, as well as of the interaction forces $\hat{\mathbf{p}}^\alpha$ which belong to the volume forces. The resulting force vector $\mathbf{k}^\alpha$ is thus given by

$$
\mathbf{k}^\alpha = \mathbf{f}^\alpha + \int\limits_{B_\alpha} \hat{\mathbf{p}}^\alpha dv\ , \tag{4.12}
$$

where $\mathbf{f}^\alpha$ is composed of

$$
\mathbf{f}^\alpha = \int\limits_{B_\alpha} \rho^\alpha \mathbf{b}^\alpha dv + \int\limits_{\partial B_\alpha} \mathbf{t}^\alpha da\ . \tag{4.13}
$$

Now, all the terms which are contained in the balance principle of momentum have been listed. With Cauchy's theorem,

$$
\mathbf{t}^\alpha = \mathbf{T}^\alpha \mathbf{n}\ , \tag{4.14}
$$

where $\mathbf{T}^\alpha$ is Cauchy's stress tensor of the constituent φ^α and $\mathbf{n}$ is the unit normal at the surface of the individual constituent body. With the divergence theorem, as well as with either mass balance principle, (4.3) or (4.4), Cauchy's first equation of motion (balance of momentum) for φ^α is obtained from (4.10):

$$\operatorname{div} \mathbf{T}^\alpha + \rho^\alpha \mathbf{b}^\alpha + \hat{\mathbf{p}}^\alpha = \rho^\alpha \mathbf{a}_\alpha + \hat{\rho}^\alpha \mathbf{v}_\alpha \ . \tag{4.15}$$

In this equation, the expression $\hat{\rho}^\alpha \mathbf{v}_\alpha$ represents the exchange of linear momentum through the density supply $\hat{\rho}^\alpha$.

The balance of momentum for the mixture body can be gained by superimposition of the momenta of all κ constituents φ^α, assuming a common velocity $\mathbf{v}$ and acceleration $\mathbf{a}$, as well as a common external acceleration $\mathbf{b}$ for all constituents φ^α:

$$\sum_{\alpha=1}^{\kappa} (\operatorname{div} \mathbf{T}^\alpha + \rho^\alpha \mathbf{b} + \hat{\mathbf{p}}^\alpha) = \sum_{\alpha=1}^{\kappa} (\rho^\alpha \mathbf{a} + \hat{\rho}^\alpha \mathbf{v}) \ . \tag{4.16}$$

Introducing the requirements

$$\begin{aligned} \mathbf{T} &= \sum_{\alpha=1}^{\kappa} \mathbf{T}^\alpha \ , \quad \rho\mathbf{b} = \sum_{\alpha=1}^{\kappa} \rho^\alpha \mathbf{b} \ , \\ \dot{\mathbf{l}} &= \sum_{\alpha=1}^{\kappa} [\rho^\alpha \mathbf{a} + \hat{\rho}^\alpha \mathbf{v}] = \rho\mathbf{a} \ , \quad \sum_{\alpha=1}^{\kappa} \hat{\mathbf{p}}^\alpha = \mathbf{0} \ , \end{aligned} \tag{4.17}$$

where $\mathbf{T}$, $\rho\mathbf{b}$, and $\dot{\mathbf{l}}$ are, respectively, Cauchy's stress tensor, the volume force, and the time rate of the momentum of the mixture body, the balance principle of the mixture body is gained:

$$\operatorname{div} \mathbf{T} + \rho\mathbf{b} = \dot{\mathbf{l}} \ . \tag{4.18}$$

The material time derivative $(\dot{\ldots})$ has to be formed with the velocity $\mathbf{v}$. The requirement of $(4.17)_4$ is a constraint for the momentum supplies. Note that the second summation in $(4.17)_3$ disappears due to (4.8).

The *balance of moment of momentum* for non-polar materials states that the material time derivative of the moment of momentum is equal to the moments of all external forces, where the moments are referred to a fixed point 0:

$$(\mathbf{h}^\alpha_{(0)})'_\alpha = \mathbf{m}^\alpha_{(0)} \ . \tag{4.19}$$

We do not consider local moment of momentum supply vectors $\hat{\mathbf{m}}^\alpha$ or the corresponding tensors $\hat{\mathbf{M}}^\alpha$. For the formulation of the moment of momentum balance principle for polar materials, the reader is referred to Ehlers and Volk (1997 a,b) or Diebels (1999).

The moment of momentum $\mathbf{h}^{\alpha}_{(0)}$ for the constituent φ^{α} is given in consideration of (4.11) by:

$$\mathbf{h}^{\alpha}_{(0)} = \int\limits_{B_\alpha} \mathbf{x} \times \rho^{\alpha} \mathbf{v}_{\alpha} dv \; . \tag{4.20}$$

The moment of the external forces $\mathbf{m}^{\alpha}_{(0)}$ can be calculated considering (4.12) and (4.13) from

$$\mathbf{m}^{\alpha}_{(0)} \quad = \quad \int\limits_{B_\alpha} \mathbf{x} \times (\rho^{\alpha}\mathbf{b}^{\alpha} + \hat{\mathbf{p}}^{\alpha}) dv + \int\limits_{\partial B_\alpha} \mathbf{x} \times \mathbf{t}^{\alpha} da \; . \tag{4.21}$$

From (4.20), considering the mass balance (4.3), the material time derivative of the moment of momentum leads to

$$(\mathbf{h}^{\alpha}_{(0)})'_{\alpha} = \int\limits_{B_\alpha} \mathbf{x} \times (\rho^{\alpha}\mathbf{a}_{\alpha} + \hat{\rho}^{\alpha}\mathbf{v}_{\alpha}) dv \; . \tag{4.22}$$

Moreover, the evaluation of the expression for the moment $\mathbf{m}^{\alpha}_{(0)}$, considering the balance principle of momentum (4.15), yields:

$$\mathbf{m}^{\alpha}_{(0)} \; = \int\limits_{B_\alpha} \mathbf{x} \times (\rho^{\alpha}\mathbf{a}_{\alpha} + \hat{\rho}^{\alpha}\mathbf{v}_{\alpha}) dv + \int\limits_{B_\alpha} \mathbf{I} \times \mathbf{T}^{\alpha} dv \; , \tag{4.23}$$

from which the local statement

$$\mathbf{I} \times \mathbf{T}^{\alpha} = \mathbf{0} \tag{4.24}$$

is obtained.

The above statement is fulfilled, if

$$\mathbf{T}^{\alpha} = (\mathbf{T}^{\alpha})^{T} \; , \tag{4.25}$$

i.e., if Cauchy's stress tensor is symmetric.

The requirement that the mixture, as the sum of all κ constituents, should behave as a one-component material contains the condition:

$$\sum_{\alpha=1}^{\kappa} \mathbf{T}^{\alpha} = \sum_{\alpha=1}^{\kappa} (\mathbf{T}^{\alpha})^{T} \; , \quad \mathbf{T} = \mathbf{T}^{T} \; , \tag{4.26}$$

whereby $(4.17)_1$ has been considered. The result of the balance of moment of momentum is the evaluation of the statement that the stress tensor of the mixture body is also symmetric.

For a binary model consisting of a micropolar skeleton and a non-polar fluid the balance principle of moment of momentum for the skeleton is obtained as (see Ehlers and Volk, 1999):

$$\mathbf{I} \times \mathbf{T}^S + \operatorname{div} \mathbf{M}^S + \rho^S \mathbf{c}^S = \mathbf{0} \ , \tag{4.27}$$

where $\mathbf{T}^S$ and $\mathbf{M}^S$ are the non-symmetric Cauchy stress tensor and the couple stress tensor, respectively, and $\mathbf{c}^S$ the external body couple per unit mass.

4.3 Balance of Energy

The first law of thermodynamics (balance of energy), which has been repeatedly proven in the past, is the most fundamental relation in the thermodynamics of one-component materials. It states that the sum of the material time derivatives of the internal and kinetic energies equals the rates of the mechanical work and the heat. This balance principle is transferred to the individual constituents. Applying the above statement to the constituents, the following balance principle is obtained:

$$(E^\alpha)'_\alpha + (K^\alpha)'_\alpha = W^\alpha + Q^\alpha + \int\limits_{B_\alpha} \hat{e}^\alpha dv \ , \tag{4.28}$$

where $E^\alpha, K^\alpha, W^\alpha, Q^\alpha$, and $\hat{e}^\alpha$ are, respectively, the internal energy, the kinetic energy, the rate of the mechanical energy, the rate of the heat of the constituent φ^α, and the energy supply to φ^α caused by all other constituents. The internal energy, kinetic energy, and the rate of the mechanical work, as well as the rate of the heat, are given by

$$E^\alpha = \int\limits_{B_\alpha} \rho^\alpha \varepsilon^\alpha dv \ , \quad K^\alpha = \int\limits_{B_\alpha} \frac{1}{2} \rho^\alpha \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha dv \ , \tag{4.29}$$

$$W^\alpha = \int\limits_{B_\alpha} \mathbf{v}_\alpha \cdot \rho^\alpha \mathbf{b}^\alpha dv + \int\limits_{\partial B_\alpha} \mathbf{v}_\alpha \cdot \mathbf{t}^\alpha da \ , \tag{4.30}$$

and

$$Q^\alpha = \int\limits_{B_\alpha} \rho^\alpha r^\alpha dv - \int\limits_{\partial B_\alpha} \mathbf{q}^\alpha \cdot d\mathbf{a} \ . \tag{4.31}$$

Here, $\varepsilon^\alpha = \varepsilon^\alpha(\mathbf{x}, t)$ is the specific internal energy, $r^\alpha = r^\alpha(\mathbf{x}, t)$ the partial energy source, and $\mathbf{q}^\alpha = \mathbf{q}^\alpha(\mathbf{x}, t)$ the partial heat flux vector, which is positive when entering the body.

The balance principle of energy (4.28) yields, in connection with (4.29), (4.30), and (4.31), the local statement:

$$\begin{aligned}&\rho^\alpha(\varepsilon^\alpha)'_\alpha \ + \ \hat{\rho}^\alpha\varepsilon^\alpha + \rho^\alpha\mathbf{v}_\alpha\cdot\mathbf{a}_\alpha + \ \frac{1}{2}\hat{\rho}^\alpha\mathbf{v}_\alpha\cdot\mathbf{v}_\alpha\\ &= \rho^\alpha r^\alpha \ - \ \mathrm{div}\ \ \mathbf{q}^\alpha + \mathbf{v}_\alpha\cdot(\mathrm{div}\ \ \mathbf{T}^\alpha + \rho^\alpha\mathbf{b}^\alpha) + \\ &+ \ \mathbf{T}^\alpha\cdot\mathbf{L}_\alpha + \hat{e}^\alpha\ .\end{aligned} \tag{4.32}$$

With Cauchy's first equation of motion (4.15), the relation (4.32) can be reformulated as:

$$\begin{aligned}&\rho^\alpha(\varepsilon^\alpha)'_\alpha \ - \ \mathbf{T}^\alpha\cdot\mathbf{L}_\alpha - \rho^\alpha r^\alpha + \mathrm{div}\,\mathbf{q}^\alpha\\ &= \hat{e}^\alpha - \hat{\mathbf{p}}^\alpha\cdot\mathbf{v}_\alpha - \hat{\rho}^\alpha(\varepsilon^\alpha - \frac{1}{2}\mathbf{v}_\alpha\cdot\mathbf{v}_\alpha)\ .\end{aligned} \tag{4.33}$$

Excluding all mass and moment of momentum exchanges, the relation (4.33) simplifies to

$$\begin{aligned}&\rho^\alpha(\varepsilon^\alpha)'_\alpha \ - \ \mathbf{T}^\alpha\cdot\mathbf{D}_\alpha - \rho^\alpha r^\alpha + \mathrm{div}\,\mathbf{q}^\alpha\\ &= \hat{e}^\alpha - \hat{\mathbf{p}}^\alpha\cdot\mathbf{v}_\alpha\ .\end{aligned} \tag{4.34}$$

It should be mentioned that at times it is useful to replace the specific internal ε^α by the enthalpy or, in particular, by the free Helmholtz energy and the entropy. In the latter case it is advisable to choose the free Helmholtz energy function after the introduction of the notion of entropy, i.e. after the formulation and discussion of the second law of thermodynamics. Thus, in this place we keep the notion of specific internal energy.

In order to be able to compare the results stated above with those of a one-component material and to gain restrictions for the energy supply terms it is advisable to add the balances of energy of all individual constituents. Then it is possible to introduce adequate abbreviations which can be identified as appropiate terms of a one-component material.

The summation of (4.32) over all κ constituents results in

$$\begin{aligned}&\sum_{\alpha=1}^{\kappa}[\rho^\alpha(\varepsilon^\alpha)'_\alpha + \hat{\rho}^\alpha\varepsilon^\alpha] + \sum_{\alpha=1}^{\kappa}(\rho^\alpha\mathbf{v}_\alpha\cdot\mathbf{a}_\alpha + \frac{1}{2}\hat{\rho}^\alpha\mathbf{v}_\alpha\cdot\mathbf{v}_\alpha)\\ &= \sum_{\alpha=1}^{\kappa}\rho^\alpha r^\alpha - \sum_{\alpha=1}^{\kappa}\mathrm{div}\ \ \mathbf{q}^\alpha + \\ &+ \sum_{\alpha=1}^{\kappa}[\mathbf{v}_\alpha\cdot(\mathrm{div}\ \ \mathbf{T}^\alpha + \rho^\alpha\mathbf{b}^\alpha) + \mathbf{T}^\alpha\cdot\mathbf{L}_\alpha] + \sum_{\alpha=1}^{\kappa}\hat{e}^\alpha\ .\end{aligned} \tag{4.35}$$

With the assumptions stated at the beginning of this section, namely $\mathbf{v}_\alpha = \mathbf{v},\ \mathbf{a}_\alpha = \mathbf{a},\ \mathbf{b}^\alpha = \mathbf{b},\ \varepsilon^\alpha = \varepsilon$, and $r^\alpha = r$, and the appropriate statements

$$\begin{aligned}
&\rho\dot{\varepsilon} = \sum_{\alpha=1}^{\kappa}[\rho^\alpha(\dot{\varepsilon}) + \hat{\rho}^\alpha\varepsilon] \quad , \rho\dot{k} = \sum_{\alpha=1}^{\kappa}[\rho^\alpha\mathbf{v}\cdot\mathbf{a} + \frac{1}{2}\hat{\rho}^\alpha\mathbf{v}\cdot\mathbf{v}] \ , \\
&\rho r = \sum_{\alpha=1}^{\kappa}\rho^\alpha r \ , \quad \mathbf{q} = \sum_{\alpha=1}^{\kappa}\mathbf{q}^\alpha \ , \\
&w = \sum_{\alpha=1}^{\kappa}[\mathbf{v}\cdot(\mathrm{div}\mathbf{T}^\alpha + \rho^\alpha\mathbf{b}) + \mathbf{T}^\alpha\cdot\mathbf{L}] \ , \quad \sum_{\alpha=1}^{\kappa}\hat{e}^\alpha = 0 \ ,
\end{aligned} \tag{4.36}$$

we have

$$\rho\dot{\varepsilon} + \rho\dot{k} = w + \rho r - \mathrm{div}\ \mathbf{q} \tag{4.37}$$

for the mixture body, an equation which is formally equivalent to the energy balance principle of a one-component material. Eq $(4.36)_6$ represents a constraint on the energy supplies.

In this section, the introduction of the so-called barycentric velocity – a mass weighted average velocity – and its derivations have been avoided due to the fact that the introduction of this velocity would lead to some obscure statements in the mixture theory (see the extensive discussion of this problem in de Boer, 1995 b).

It should again be pointed out that, in the global form of the balance principle, the integration process covers the body B_α of the constituent φ^α. Only in this case do the local statements of the balance principles given in this section come out right. If the global forms of the balance principles are referred to the partial solid body φ^S, an additional term connected with the difference velocity between φ^S and $\varphi^\alpha(\alpha \neq S)$ must be added. This is also valid for the entropy inequality in the next section.

Chapter 5

BASIC INEQUALITY (ENTROPY PRINCIPLE)

5.1 Preliminaries

In order to gain restrictions for constitutive equations, the second law of thermodynamics (entropy principle) has been usefully applied in continuum mechanics, in the mixture theory and in the theory of porous media, in particular. This procedure was created by Coleman and Noll (1963) and modified by Müller and Liu (see Müller, 1985). Coleman and Noll's method has been repeatedly examined in many fields of continuum mechanics and has yielded excellent results. Its contribution to porous media theory has been in the form of restrictions for the constitutive response functions which helped to formulate a consistent theory for saturated elastic porous solids, where both constituents (solid and fluid) can either be compressible or incompressible (see, e.g., de Boer, 1993, de Boer and Kowalski, 1995, and de Boer, 1996, 1997 b). Moreover, in a recent paper de Boer and Didwania (2000, 2002, 2004) have exploited the entropy inequality for capillary porous solids and have gained new constitutive equations for the volume forces $\hat{\mathbf{p}}^{\alpha}$.

The second law of thermodynamics follows from the balance of energy, after some manipulations with the absolute temperature. It can be shown (see, e.g., Planck, 1897) that the absolute temperature *can* serve as an integrating factor for the sum of the rate of internal energy and that of the stress power (see also Müller, 1979). This expression, the above stated sum (or the corresponding heat) divided by the absolute temperature, is denoted as entropy; thermodynamics is based on this notion (introduced by Clausius, 1865). However, this conception seems to be clear only for reversible processes. For irreversible processes, the entropy notion and the whole procedure of constructing a fundamental inequality remains in some parts obscure and mysterious (see Balian, 1991/92). For this reason, it is not surprising that it is extremely difficult to impart the entropy notion to students. Baierlein (1992) characterized this situation with the words: "Students find entropy a mysterious concept – and not surprisingly so, for it is a difficult notion." There have been of course, many attempts to support the entropy notion for irreversible processes by results obtained in other fields of science such as statistical mechanics and information theory (see Balian, 1991/92). However, these efforts are not very convincing and degenerate at times to purely philosophical discussions. There have however been attempts to avoid the notion of entropy, which is in general neither accessible nor measurable (see Balian, 1991/92).

This critique of the notion of entropy seems also valid in part for other concepts in thermodynamics. Balian (1991/92) stated:

"Still worse, thermodynamics is based upon many, more or less intuitive, concepts which cannot be readily formulated mathematically and whose nature is far from clear on a microscopic scale: temperature, pressure, work, heat, entropy, ..."

5.2 Basic Inequality for Non-Polar Constituents and the Mixture Body

As it is not the aim of this section to work on the entropy principle and other concepts of thermodynamics, we will use the entropy principle in the classical form in the porous media theory for gaining restrictions for constitutive equations (see, e.g., de Boer and Ehlers, 1986 b), namely as the sum of all entropy inequalities for the individual constituents. The assumption that the entropy inequality has to be fulfilled for every individual constituent φ^α is indeed a sufficient – though too restrictive – condition. At the same time, the postulate of a common entropy inequality for all constituents is both a necessary and a sufficient condition for the existence of dissipation mechanisms within the mixture, and will therefore be preferred:

$$\sum_{\alpha=1}^{\kappa}(H^\alpha)'_\alpha \;\geq\; \sum_{\alpha=1}^{\kappa}\int_{B_\alpha}\frac{1}{\Theta^\alpha}\rho^\alpha r^\alpha dv \;-\; \sum_{\alpha=1}^{\kappa}\int_{\partial B_\alpha}\frac{1}{\Theta^\alpha}\mathbf{q}^\alpha\cdot d\mathbf{a}\,. \tag{5.1}$$

The quantity

$$H^\alpha = \int_{B_\alpha}\rho^\alpha\eta^\alpha dv \tag{5.2}$$

denotes the entropy of the constituent φ^α, whereby η^α is the specific entropy. Moreover, Θ^α is the absolute temperature of φ^α. In consideration of the transport theorem (3.31) in connection with $(3.21)_1$ and the mass balance equation (4.3), from (5.2) the material time derivative of the entropy is obtained:

$$\begin{aligned}\sum_{\alpha=1}^{\kappa}(H^\alpha)'_\alpha &= \sum_{\alpha=1}^{\kappa}\int\limits_{B_\alpha}\Big[\rho^\alpha(\eta^\alpha)'_\alpha+\hat{\rho}^\alpha\eta^\alpha\Big]dv \geq \\ &\geq \sum_{\alpha=1}^{k}\int\limits_{B_\alpha}\Big[\frac{1}{\Theta^\alpha}\rho^\alpha r^\alpha - \operatorname{div}\Big(\frac{1}{\Theta^\alpha}\,\mathbf{q}^\alpha\Big)\Big]dv\ .\end{aligned} \tag{5.3}$$

In (5.3), the divergence theorem has been used to transform the surface integral into a volume integral. From (5.3), the local form of the entropy inequality is gained:

$$\sum_{\alpha=1}^{\kappa}\Big[\rho^\alpha(\eta^\alpha)'_\alpha+\hat{\rho}^\alpha\eta^\alpha-\frac{1}{\Theta^\alpha}\rho^\alpha r^\alpha+\operatorname{div}\Big(\frac{1}{\Theta^\alpha}\mathbf{q}^\alpha\Big)\Big]\geq 0\ . \tag{5.4}$$

Considering the balance equation of energy (4.32) and the free Helmholtz energy

$$\psi^\alpha=\varepsilon^\alpha-\Theta^\alpha\eta^\alpha\ , \tag{5.5}$$

the entropy inequality (5.4) can be rewritten as

$$\begin{aligned}\sum_{\alpha=1}^{\kappa}\frac{1}{\Theta^\alpha}\{&-\rho^\alpha[(\psi^\alpha)'_\alpha+(\Theta^\alpha)'_\alpha\eta^\alpha]- \\ &-\hat{\rho}^\alpha(\psi^\alpha-\frac{1}{2}\mathbf{v}_\alpha\cdot\mathbf{v}_\alpha)+\ \mathbf{T}^\alpha\cdot\mathbf{L}_\alpha- \\ &-\hat{\mathbf{p}}^\alpha\cdot\mathbf{x}'_\alpha-\ \frac{1}{\Theta^\alpha}\mathbf{q}^\alpha\cdot\operatorname{grad}\,\Theta^\alpha+\hat{e}^\alpha\}\geq 0\ .\end{aligned} \tag{5.6}$$

If all constituents have the same temperature Θ, i.e., $\Theta^\alpha=\Theta$, Inequality (5.6) simplifies, using $(4.36)_6$, to:

$$\begin{aligned}\sum_{\alpha=1}^{\kappa}\{&-\rho^\alpha[(\psi^\alpha)'_\alpha+(\Theta)'_\alpha\eta^\alpha]-\hat{\rho}^\alpha(\psi^\alpha-\frac{1}{2}\mathbf{v}_\alpha\cdot\mathbf{v}_\alpha)\ +\ \mathbf{T}^\alpha\cdot\mathbf{L}_\alpha- \\ &-\hat{\mathbf{p}}^\alpha\cdot\mathbf{v}_\alpha-\frac{1}{\Theta}\mathbf{q}^\alpha\cdot\operatorname{grad}\,\Theta\}\geq 0\ .\end{aligned} \tag{5.7}$$

Apart from the forms of the entropy inequality of the mixture body represented here, there are further alternative forms possible which however, essentially contain only other transformations in energies. In order to gain restrictions for constitutive relations in the constitutive theory the forms of the entropy inequality (5.6) and (5.7) are both sufficient and convenient. Thus, we will not mention other forms of the second law of thermodynamics.

Chapter 6

CONSTITUTIVE THEORY

6.1 Preliminaries

As has already been mentioned, in order to close the system of field equations, it is necessary to introduce constitutive equations. These equations connect certain mechanical or thermodynamic quantities via material-dependent constants which are determined by test observations. Thus, it is ensured that the constitutive relations introduced are able to describe the test results.

In the past, a great number of constitutive equations for empty and saturated porous media had been derived. However, many of them are very complicated due to the use of inadequate mechanical or thermodynamic concepts. Without a doubt, these constitutive equations may closely describe the stress-strain (rate) relations of the special mechanical behavior of materials. However, in many cases this can only be achieved by introducing many parameters and neglecting requirements due to mechanical and thermodynamic “principles”. These constitutive equations are meaningless in view of the calculation of general boundary and initial value problems within the framework of geometrically linear and non-linear theories. The goal should be to formulate relatively simple constitutive equations. This statement becomes even more evident in the field of saturated porous media. In this field, not only the thermodynamic behavior of the skeleton and the content of the pores but also various interaction phenomena have to be described. Therefore, the idea of formulating relatively simple constitutive equations is of relevance. In order to derive consistent constitutive equations that are relatively simple, some strong assumptions have to be introduced at times; for example, the compressibility of the solid matrix material can be neglected in many cases in comparison with the compressibility of the matrix. Thus, the mathematical model reduces to an incompressible model. Another example is the elastic-plastic model, where some simplifying assumptions also have to be introduced in order to predict the essential properties of the porous solids under study by using relatively simple constitutive relations.

However, it is not sufficient to only fulfill requirements due to test observations; rather more general "principles", which were developed in continuum mechanics in the 1950s and 1960s, should be fulfilled; these are: *determinism, local action, material objectivity,* and *dissipation*. Some of the above-stated "principles" should however, not be understood as axioms, but rather be considered as convenient work hypotheses because the real mechanical or thermodynamic behavior of solids and fluids is in most cases, too complex to be described by relatively simple "principles".

From the above-stated "principles" the material objectivity and dissipation principles are the most important. The material objectivity principle states that the constitutive equations have to be formulated in such a way that they are not influenced by superimposed rigid body motions. The dissipation principle results from the second law of thermodynamics. Both principles have a big impact on the development of consistent constitutive equations.

In the section on the constitutive theory, some known papers in this field will be reviewed considering the above-stated requirements concerning test observations and general "principles".

In Section 3, only the dissipation principle will be discussed. Concerning the objectivity principle, the reader is referred to Ehlers (1989 a,b) or de Boer (2000 a).

6.2 Closure Problem and Constraints

Mixture theory – the basis of porous media theory – is closed, i.e., the number of unknown fields is equal to the sum of the balance principles and the constitutive relations. This can easily be proven. However, by the introduction of the volume fractions n^S, n^L and n^G for the real constituents φ^S, φ^L and φ^G in the porous media theory (in order to obtain homogenized (smeared) continua which can be treated by continuum mechanical methods), a problem arises in which two field equations are missing if (2.2) is considered. This causes a considerable difference between the porous media theory and the mixture theory as well as the continuum mechanics of one-component materials. Also, other existing theories in continuum mechanics are closed and every new condition leads to an equation in excess. This condition must be provided with a Lagrange multiplier for the evaluation process of the entropy inequality. If the equation in excess is a constraint of motion, then the Lagrange multiplier will become an unknown reaction force.

In the porous media theory, on the other hand, one has to look for additional equations in order to close the fields. It is however difficult to gain additional fields since the volume fractions contain quantities of the microscale for which balance or constitutive equations are not contained in the macroscopic mixture theory. Therefore, much effort has been made to overcome this crucial problem. This effort starts by introducing an additional balance principle to the formulation of an evolution equation for the volume fraction. This procedure solves – from the mathematical point of view – the closure problem. However – from the physical point of view – this method is completely insufficient because one must be aware of an important constraint namely, the saturation condition (2.2). This constraint restricts, in the rate formulation, the rates of the volumetric changes and must therefore, be considered in the evaluation of the entropy inequality. By differentiating the saturation condition (2.2) with respect to the solid phase (the same result can be obtained by differentiating with respect to the fluid or the gas phase), we have

$$(n^S)'_S + (n^L)'_S + (n^G)'_S = 0 \tag{6.1}$$

or

$$\begin{aligned} &- (n^S)'_S - (n^L)'_L - (n^G)'_G + \\ &+ \operatorname{grad} n^L \cdot (\mathbf{v}_L - \mathbf{v}_S) + \operatorname{grad} n^G \cdot (\mathbf{v}_G - \mathbf{v}_S) = 0\ . \end{aligned} \tag{6.2}$$

Considering (2.4), we obtain from (6.2):

$$\begin{aligned} &- n^S \frac{(\rho^S)'_S}{\rho^S} + n^S \frac{(\rho^{SR})'_S}{\rho^{SR}} - n^L \frac{(\rho^L)'_L}{\rho^L} + n^L \frac{(\rho^{LR})'_L}{\rho^{LR}} - \\ &- n^G \frac{(\rho^G)'_G}{\rho^G} + n^G \frac{(\rho^{GR})'_G}{\rho^{GR}} + \operatorname{grad} n^L \cdot (\mathbf{v}_L - \mathbf{v}_S) + \\ &+ \operatorname{grad} n^G \cdot (\mathbf{v}_G - \mathbf{v}_S) = 0 \end{aligned} \tag{6.3}$$

or considering the mass balance principles (4.3), in connection with (2.4), $(3.21)_1$, (3.27), (3.28), (3.29), and (3.43), as well as neglecting the mass supplies,

$$\begin{aligned} &n^S(\mathbf{D}_{SN} \cdot \mathbf{I}) + n^L(\mathbf{D}_{LN} \cdot \mathbf{I}) + n^G(\mathbf{D}_{GN} \cdot \mathbf{I}) + \\ &+ \operatorname{grad} n^L \cdot (\mathbf{v}_L - \mathbf{v}_S) + \operatorname{grad} n^G \cdot (\mathbf{v}_G - \mathbf{v}_S) = 0 \end{aligned} \tag{6.4}$$

or

$$\begin{aligned} &n^S(\mathbf{D}_S \cdot \mathbf{I}) - n^S(\mathbf{D}_{SR} \cdot \mathbf{I}) + n^L(\mathbf{D}_L \cdot \mathbf{I}) - \\ &-n^L(\mathbf{D}_{LR} \cdot \mathbf{I}) + n^G(\mathbf{D}_G \cdot \mathbf{I}) - n^G(\mathbf{D}_{GR} \cdot \mathbf{I}) + \\ &+ \operatorname{grad} n^L \cdot (\mathbf{v}_L - \mathbf{v}_S) + \operatorname{grad} n^G \cdot (\mathbf{v}_G - \mathbf{v}_S) = 0 \end{aligned} \tag{6.5}$$

is obtained. In (6.4) and (6.5), use is made of the relations

$$\frac{(n^\alpha)'_\alpha}{n^\alpha} + \mathbf{D}_{\alpha N} \cdot \mathbf{I} = 0 \; , \quad \mathbf{D}_{\alpha N} \cdot \mathbf{I} = \mathbf{D}_\alpha \cdot \mathbf{I} - (\mathbf{D}_{\alpha R} \cdot \mathbf{I}) \tag{6.6}$$

and

$$\frac{(\rho^{\alpha R})'_\alpha}{\rho^{\alpha R}} + \mathbf{D}_{\alpha R} \cdot \mathbf{I} = 0 \; , \tag{6.7}$$

which results from the balance principles of mass (4.3) or (4.4), excluding all mass exchanges, considering (2.4) and (3.29) (see Bluhm, 1997).

The relation (6.5) clearly reveals that the rates of the volumetric strains of the partial bodies $\mathbf{D}_\alpha \cdot \mathbf{I}$ and of the real compressible materials $\mathbf{D}_{\alpha R} \cdot \mathbf{I}$ are dependent. Please note in passing that, we have omitted the signs of $\mathbf{D}_{SR}$ and $\mathbf{D}_{LR}$ as well as of $\mathbf{D}_{GR}$ which characterize the intermediate state at this point and in the following section, for the sake of simplicity.

The problem to be solved depends as to whether the constraint in the forms (6.2), (6.3), or (6.4), (6.5) should be used. If all mass exchanges are neglected, then the constraints in the forms (6.4) and (6.5) are convenient. If, however, mass exchange occurs, then the forms (6.2) and (6.3) have to be used in the evaluation of the entropy inequality.

As has already been mentioned, the saturation constraint (2.2) in the versions (6.2) and (6.3), or (6.4) and (6.5) has to be considered in the evaluation of the entropy inequality because the rates of either the densities or of the volumetric strains of the solid and the fluid as well as the gas phases are dependent. In order to obtain a stress-power-like expression, the constraints (6.2) and (6.3), or (6.4) and (6.5), which contain the rates of the volumetric strains, will be multiplied by a hydrostatic interface pressure λ. It is true that the saturation condition (2.2) is an equation to further reduce the number of unknown volume fractions, but the grade of indetermination does not change by the introduction of the interface pressure λ. Therefore, it will be postulated that three constitutive equations for λ must be introduced (or three constitutive relations for the hydrostatic pressure in the solid material) which contain properties of both the constituents of the partial solid and of partial fluid phases, in order to achieve closure. This is a reasonable demand from the mechanical point of view, because the interface pressure acts in the solid as well as the fluid and gas phases. It will be seen that the introduced requirement leads to excellent physical results.

If the materials of the three individual constituents behave as incompressible phases, then additional constraints have to be considered in the evaluation of the entropy inequality.

6.3 Reformulation of the Entropy Inequality

As has already been pointed out in Section 5.1, the entropy inequality has yielded excellent results in the porous media theory (see the review article of de Boer, 1996). However, in this field, one should be careful when evaluating the entropy inequality purely mathematically, because the mechanical and thermodynamic behaviors of saturated porous solids are very complex. If one evaluates the entropy inequality in a purely stereotyped way, without considering the special physical properties of the complex material under study, one can arrive at results which fulfill the entropy inequality but fail to predict physical phenomena arrived at by experiment. The second law of thermodynamics is an inequality and there are many possibilities to satisfy this inequality. Some evaluations may be less restrictive than others, without violating the inequality.

Therefore, the entropy inequality has to be manipulated in order to include fundamental physical phenomena known from experience, test observations, and theories, which appear independent of the special constitutive behavior of the individual partial constituents, such as elastic, plastic, or viscous behavior. These phenomena mainly concern the concept of effective stresses in different versions (see, e.g., de Boer, 1996, Lade and de Boer, 1997). The concept of effective stresses is caused by the saturation condition and is valid for all kinds of material of the individual partial constituents. However, in order to describe the different versions of the effective stress concept, the closure problem must also be considered, whereby one must distinguish between the compressible and incompressible behavior of the real material of the constituents.

As the second law of thermodynamics is an inequality where the entropy is always greater than zero, this inequality can also be considered as a minimum problem which reaches its minimum at zero. All additional constraints, like the saturation or incompressibility conditions, provided with multipliers, can be taken into consideration by adding to the entropy inequality. It depends on the closure problem as to whether the multipliers are constitutively determined or not.

The ternary model under study consists of an elastic porous solid filled with a mixture of an inviscid, incompressible liquid and a compressible gas.

The goal of the following investigations is to gain from the evaluation of the entropy inequality proper constitutive equations which meet the thermodynamic restrictions and accurately describe physical observations.

The entropy inequality for the sum of all partial constituents states in local form:

$$\begin{aligned}\sum_{\alpha=1}^{\kappa} \Big\{ - \rho^\alpha [{\psi^\alpha}'_\alpha + (\Theta^\alpha)'_\alpha \, \eta^\alpha] + \mathbf{T}^\alpha \cdot \mathbf{D}_\alpha - \hat{\mathbf{p}}^\alpha \cdot \mathbf{v}_\alpha - \\ - \frac{1}{\Theta} \mathbf{q}^\alpha \cdot \operatorname{grad} \Theta^\alpha - \hat{\rho}^\alpha (\psi^\alpha - \frac{1}{2} \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha) \Big\} \geq 0 \, .\end{aligned} \tag{6.8}$$

In (6.8) it is assumed that no moment of momentum and no energy exchanges occur. The consequence of this assumption is that Cauchy's stress tensor is symmetric. Moreover, it is assumed that the thermodynamic process is governed by the same temperature for all constituents φ^α. In (6.8) ρ^α, ψ^α, and η^α denote the partial density, the partial free Helmholtz energy function, and the specific entropy, respectively, and $\mathbf{D}_\alpha, \hat{\mathbf{p}}^\alpha, \mathbf{v}_\alpha, \mathbf{q}^\alpha$ and $\hat{\rho}^\alpha$ stand for the symmetric part of the velocity gradient, the interaction force due to the influence of all other constituents on φ^α, the velocity, the heat flux vector, and the mass supply, respectively.

Before we can start with the evaluation of the entropy inequality (6.8), some important constraints, which restrict the free motions of the phases, must be considered in the entropy inequality. These constraints concern the saturation condition, and the incompressibility of the liquid.

The rate version of the saturation condition is stated in (6.2).

$$-(n^S)'_S - (n^L)'_L - (n^G)'_G + \operatorname{grad} n^L \cdot \mathbf{v}_{LS} + \operatorname{grad} n^G \cdot \mathbf{v}_{GS} = 0 \qquad (6.9)$$

with

$$\mathbf{v}_{LS} = \mathbf{v}_L - \mathbf{v}_S \qquad \mathbf{v}_{GS} = \mathbf{v}_G - \mathbf{v}_S \ . \qquad (6.10)$$

The rate version of the saturation constraint (6.9) provided with the scalar multiplier λ will be added to the entropy inequality (6.8) in order to consider the influence of the restricted motion, due to the saturation constraint, on the constitutive relations.

Finally, the incompressibility of the liquid phase will be considered. The incompressibility condition can be stated with the real density of the liquid:

$$\rho^{LR} = \text{const.} \qquad (6.11)$$

From (6.11) we have, by differentiating with respect to time and performing some manipulations:

$$\frac{n^L(\rho^{LR})'_L}{\rho^{LR}} = 0 \ . \qquad (6.12)$$

This constraint (6.12) connected with a scalar multiplier κ^{LR} will also be considered in the entropy inequality.

Returning to the entropy inequality (6.8) and considering the constraints discussed above as well as the mass balance (4.3) and relation (2.4), we obtain:

$$-\rho^S(\psi^S)'_S - \rho^L(\psi^L)'_L - \rho^G(\psi^G)'_G - \rho^S\eta^S(\Theta)'_S -$$

$$-\rho^L\eta^L(\Theta)'_L - \rho^G\eta^G(\Theta)'_G + \mathbf{T}^{SD} \cdot \mathbf{D}_{SD} + \mathbf{T}^{LD} \cdot \mathbf{D}_L^D + \mathbf{T}^{GD} \cdot \mathbf{D}_G^D -$$

$$
\begin{aligned}
&-\frac{p^S(\rho^{SR})'_S}{\rho^{SR}} - (p^S + \lambda n^S)\frac{(n^S)'_S}{n^S} - (p^L + \kappa^{LR} n^L)\frac{(\rho^{LR})'_L}{\rho^{LR}} - \\
&-(p^L + \lambda n^L)\frac{(\rho^L)'_L}{\rho^L} - p^G\frac{(\rho^{GR})'_G}{\rho^{GR}} - (p^G + \lambda n^G)\frac{(n^G)'_G}{n^G} - \\
&-(\hat{\mathbf{p}}^L - \lambda \operatorname{grad} n^L)\cdot \mathbf{v}_{LS} - (\hat{\mathbf{p}}^G - \lambda \operatorname{grad} n^G)\cdot \mathbf{v}_{GS} - \\
&-\frac{1}{\Theta}(\mathbf{q}^S + \mathbf{q}^L + \mathbf{q}^G)\cdot \operatorname{grad}\Theta - \hat{\rho}^L(\mu^L - \mu^S) - \qquad (6.13)\\
&-\hat{\rho}^G(\mu^G - \mu^G) \geq 0\,, \\
&p^\alpha = \frac{1}{3}(\mathbf{T}^\alpha \cdot \mathbf{I})\,, \\
&\mu^\alpha = \psi^\alpha - \frac{p^\alpha}{\rho^\alpha} - \frac{1}{2}\mathbf{v}_\alpha \cdot \mathbf{v}_\alpha \;\; \text{(chemical potential)}\,.
\end{aligned}
$$

Use is made of the additive decomposition (3.30) and the balance equation of mass (4.3) as well as of the fact that

$$
\frac{(n^L)'_L}{n^L} = \frac{(\rho^L)'_L}{\rho^L} \qquad (6.14)
$$

due to the incompressibility of the liquid phase.

6.4 Exploitation of the Inequality for Ternary and Binary Capillary Porous Models

The goal of this section is the evaluation of the entropy inequality for a ternary model within the framework of the geometrically non-linear theory in order to gain restrictions for the constitutive equations involved. The ternary model consists of an elastic skeleton filled with an incompressible, inviscid liquid and a compressible gas. It is assumed that the liquid is the carrier for the gas bubbles in the pores. This means that in the hydrostatic case the stress state of the gas is influenced by the liquid pressure and by the surface tension of the bubble. On the other hand, if the gas acts as carrier for liquid droplets, the stress state of the liquid is composed of a part of the gas pressure and the surface tension of the droplet. In the exploitation of the entropy principle use will be made of the Principle of Phase Separation by Passman *et al.* (1984):

“The dependent variables of the α-th constituent that are material-specific depend only on the independent variables of the α-th constituent. The growth dependent variables depend on all of the independent variables.”

They justified the Principle of Phase Separation as follows:

"In mulitphase mixtures, the individual constituents are clearly separated physically, and it is plausible to think of the mixture as being ideal, or phase separated."

The Principle of Phase Separation is clearly physically founded by the authors. However, one must bear in mind that the individual phases are described by partial quanties involving the volume fractions and that volume fractions are constraint by the saturation condition (2.2). Thus, one has to be careful in exploiting the entropy inequality and must keep an eye on those problems where the volume fractions are involved.

It has already been indicated that the model under study consists of an elastic porous solid, an inviscid, incompressible liquid and a compressible gas. Furthermore, it is the first goal of the subsequent investigations to include such porous solids which possesses pores with a characteristic diameter in the range between 1 mm and 50 nm (1 nm $\widehat{=}$ 10^{-6}mm). That means that in these pores capillary forces due to intermolecular effects occur. These capillary forces cause motion of the liquid and gas in the pores. The constitutive equations for the capillary forces should contain the interaction between the solid, liquid and gas phases and the density change of the liquid. The rigorous thermodynamical treatment of this model reveals that the capillary forces, as volume forces, gained by the exploitation of the entropy inequality shows exactly these properties (see also Chapter 7, Section 5).

The second goal is to make the elastic behavior of the porous solid more transparent. It is principally possible to introduce the right Cauchy-Green deformation tensor $\mathbf{C}_S$ as a process variable and to derive with this variable a constitutive equation for Cauchy's stress tensor $\mathbf{T}^S$. However, such procedure suppresses several information of the behavior of the real elastic material in the hydrostatic stress and deformation state such as compressibility and configuration pressure. Moreover, no statement concerning the influence of the hydrostatic stress upon phase transition is possible. Therefore, the following procedure seems to be useful. First, the deformation tensor $\mathbf{C}_S$ will be multiplicatively decomposed (see (3.17) through (3.20)) in a spherical tensor $\tilde{\mathbf{C}}_S$ permitting a volume change and a volume-preserving part $\breve{\mathbf{C}}_S$. The volume preserving part $\breve{\mathbf{C}}_S$ will be kept as a process variable because there is no way, until now, to separate properties which are concerned with the microscopic scale from this tensor. This is completely different to the spherical tensor $\tilde{\mathbf{C}}_S$. This tensor contains the determinant of the deformation gradient $\mathbf{F}_S$. From the mass balance (4.3) in the rate formulation it is recognized that the rate of the determinant of the deformation gradient is proportional to the rate of density and the density supply. Thus it is advisable to include the balance equation of mass into the entropy inequality in order to bring the mass supply in consideration for the description of phase transitions.

Moreover, the mass of the partial solid can be multiplicatively decomposed, according to (2.4), into the volume fraction n^S and the real density of the solid material. Due to the fact that n^S and ρ^{SR} are completely independent it is advisable to choose n^S and ρ^{SR} as independent process variables hoping that both variables bring us more information, than the process variable ρ^S, the density of the partial porous solid in the exploitation of the entropy inequality. We will see that this is the case.

Constitutive equations are necessary for the free Helmholtz energy functions ψ^α:

$$\psi^\alpha = \psi^\alpha(s) \; . \tag{6.15}$$

These functions may depend on the following set s of process variables or on parts of it:

$$s = \{n^S,\ \rho^{SR},\ \breve{\mathbf{C}}_S,\ \rho^L, \rho^{GR}, \mathbf{v}_{LS}, \mathbf{v}_{GS}, \Theta,\ \text{grad}\ \Theta\} \; . \tag{6.16}$$

The choice of the set s is motivated by the following statements. The volume fraction n^S will describe the configuration pressure (intergranular forces), ρ^{SR} the real hydrostatic deformation of the solid material and $\breve{\mathbf{C}}_S$ the volume preserving part of the right Cauchy-Green tensor, see (3.20). The density of the liquid governs the partial hydrostatic deformations of the liquid body and plays an important role in the description of the capillarity phenomenon as we will see later. The real density ρ^{GR} of the gas phase enter the constitutive equation for the description of the hydrostatic deformations of the gas. The different velocities $\mathbf{v}_{LS}$ and $\mathbf{v}_{GS}$ are responsible for the frictions forces amongst the individual constituents and finally, Θ and grad Θ describe thermal effects.

In order to develop thermodynamic restrictions from the entropy inequality (6.13), it is first necessary to calculate the material time derivatives of the free Helmholtz energy functions of all three constituents. With

$$\psi^\alpha = \hat{\psi}^\alpha\{n^S,\ \rho^{SR},\ \breve{\mathbf{C}}_S,\ \rho^L, \rho^{GR}, \mathbf{v}_{LS}, \mathbf{v}_{GS}, \Theta,\ \text{grad}\ \Theta\} \; , \tag{6.17}$$

the material time derivatives of ψ^α depending on the above process variables yield:

$$\begin{aligned} (\psi^\alpha)'_\alpha = {} & \frac{\partial \hat{\psi}^\alpha}{\partial n^S}[(n^S)'_S + \ \text{grad}\ n^S \cdot \mathbf{v}_{\alpha S}] + \\ & + \frac{\partial \hat{\psi}^\alpha}{\partial \rho^{SR}}[(\rho^{SR})'_S + \ \text{grad}\ \rho^{SR} \cdot \mathbf{v}_{\alpha S}] + \\ & + \frac{\partial \hat{\psi}^\alpha}{\partial \breve{\mathbf{C}}_S} \cdot [(\breve{\mathbf{C}}_S)'_S + \ \text{grad}\ \breve{\mathbf{C}}_S \mathbf{v}_{\alpha S}] + \end{aligned} \tag{6.18}$$

$$
\begin{aligned}
&+\frac{\partial\hat{\psi}^{\alpha}}{\partial\rho^{L}}[(\rho^{L})'_{L}+\ \text{grad}\ \rho^{L}\cdot\mathbf{v}_{\alpha L}]+\\
&+\ \frac{\partial\hat{\psi}^{\alpha}}{\partial\rho^{GR}}[(\rho^{GR})'_{G}+\ \text{grad}\ \rho^{GR}\cdot\mathbf{v}_{\alpha G}]+\\
&+\ \frac{\partial\hat{\psi}^{\alpha}}{\partial\mathbf{v}_{LS}}\cdot[(\mathbf{v}_{L})'_{L}-(\mathbf{v}_{S})'_{S}-(\text{grad}\ \mathbf{v}_{S})\mathbf{v}_{\alpha S}+(\text{grad}\ \mathbf{v}_{L})\mathbf{v}_{\alpha L}]+\\
&+\ \frac{\partial\hat{\psi}^{\alpha}}{\partial\mathbf{v}_{GS}}\cdot[(\mathbf{v}_{G})'_{G}-(\mathbf{v}_{S})'_{S}-(\text{grad}\ \mathbf{v}_{S})\mathbf{v}_{\alpha S}+(\text{grad}\ \mathbf{v}_{G})\mathbf{v}_{\alpha G}]+\\
&+\ \frac{\partial\hat{\psi}^{\alpha}}{\partial\Theta}\Theta'_{\alpha}+\frac{\partial\hat{\psi}^{\alpha}}{\partial\text{grad}\ \Theta}\ (\text{grad}\ \Theta)'_{\alpha}\ .
\end{aligned}
$$

The insertion of (6.18) into (6.13) yields the entropy inequality in the following form:

$$
\begin{aligned}
-&\quad \rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial n^{S}}(n^{S})'_{S}-\rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\rho^{SR}}(\rho^{SR})'_{S}-\rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\breve{\mathbf{C}}_{S}}\cdot(\breve{\mathbf{C}}_{S})'_{S}-\\
-&\quad \rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\rho^{L}}(\rho^{L})'_{L}+\rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\rho^{L}}\,\text{grad}\,\rho^{L}\cdot\mathbf{v}_{LS}-\rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\rho^{GR}}(\rho^{GR})'_{G}+\\
+&\quad \rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\rho^{GR}}\,\text{grad}\,\rho^{GR}\cdot\mathbf{v}_{GS}-\rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\mathbf{v}_{LS}}\cdot[(\mathbf{v}_{L})'_{L}-(\mathbf{v}_{S})'_{S}]+\\
+&\quad \rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\mathbf{v}_{LS}}\cdot(\text{grad}\ \mathbf{v}_{L})\mathbf{v}_{LS}-\rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\mathbf{v}_{GS}}\cdot[(\mathbf{v}_{G})'_{G}-(\mathbf{v}_{S})'_{S}]+\\
+&\quad \rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\mathbf{v}_{GS}}\cdot(\text{grad}\ \mathbf{v}_{G})\mathbf{v}_{GS}-\rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial\Theta}\Theta'_{S}\ -\\
-&\quad \rho^{S}\frac{\partial\hat{\psi}^{S}}{\partial(\text{grad}\,\Theta)}(\text{grad}\,\Theta)'_{S}-\rho^{L}\frac{\partial\hat{\psi}^{L}}{\partial n^{S}}(n^{S})'_{S}-\\
-&\quad \rho^{L}\frac{\partial\hat{\psi}^{L}}{\partial n^{S}}\,\text{grad}\,n^{S}\cdot\mathbf{v}_{LS}-\rho^{L}\frac{\partial\hat{\psi}^{L}}{\partial\rho^{SR}}(\rho^{SR})'_{S}-\\
-&\quad \rho^{L}\frac{\partial\hat{\psi}^{L}}{\partial\rho^{SR}}\,\text{grad}\,\rho^{SR}\cdot\mathbf{v}_{LS}-\rho^{L}\frac{\partial\hat{\psi}^{L}}{\partial\breve{\mathbf{C}}_{S}}(\breve{\mathbf{C}})'_{S}-\\
-&\quad \rho^{L}\frac{\partial\hat{\psi}^{L}}{\partial\breve{\mathbf{C}}_{S}}\cdot\text{grad}\,\breve{\mathbf{C}}_{S}\mathbf{v}_{LS}-\rho^{L}\frac{\partial\hat{\psi}^{L}}{\partial\rho^{L}}(\rho^{L})'_{L}-\\
-&\quad \rho^{L}\frac{\partial\hat{\psi}^{L}}{\partial\rho^{GR}}(\rho^{GR})'_{G}-\rho^{L}\frac{\partial\hat{\psi}^{L}}{\partial\rho^{GR}}\,\text{grad}\,\rho^{GR}\cdot\mathbf{v}_{LS}+
\end{aligned}
\qquad (6.19)
$$

$$
\begin{aligned}
+ \quad & \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}} \operatorname{grad} \rho^{GR} \cdot \mathbf{v}_{GS} - \ \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{LS}} \cdot [(\mathbf{v}_L)'_L - (\mathbf{v}_S)'_S] + \\
+ \quad & \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{LS}} \cdot (\operatorname{grad} \mathbf{v}_S) \mathbf{v}_{LS} - \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \cdot [(\mathbf{v}_G)'_G - (\mathbf{v}_S)'_S] + \\
+ \quad & \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \cdot (\operatorname{grad} \mathbf{v}_S) \mathbf{v}_{LS} - \ \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \cdot (\operatorname{grad} \mathbf{v}_G) \mathbf{v}_{LS} + \\
+ \quad & \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} (\operatorname{grad} \mathbf{v}_G) \mathbf{v}_{GS} - \rho^L \frac{\partial \hat{\psi}^L}{\partial \Theta} \Theta'_L \ - \\
- \quad & \rho^L \frac{\partial \hat{\psi}^L}{\partial (\operatorname{grad} \Theta)} (\operatorname{grad} \Theta)'_L - \rho^G \frac{\partial \hat{\psi}^G}{\partial n^S} (n^S)'_S - \\
- \quad & \rho^G \frac{\partial \hat{\psi}^G}{\partial n^S} \operatorname{grad} n^S \cdot \mathbf{v}_{GS} - \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^{SR}} (\rho^{SR})'_S - \\
- \quad & \rho^G \frac{\partial \hat{\psi}^G}{\partial \breve{\mathbf{C}}_S} \cdot (\breve{\mathbf{C}}^S)'_S - \hat{\rho}^G \frac{\partial \hat{\psi}^G}{\partial \breve{\mathbf{C}}_S} \cdot \operatorname{grad} \breve{\mathbf{C}}^S \mathbf{v}_{GS} - \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} (\rho^L)'_L + \\
+ \quad & \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \operatorname{grad} \rho^L \cdot \mathbf{v}_{LS} - \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \operatorname{grad} \rho^L \cdot \mathbf{v}_{GS} \ - \\
- \quad & \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^{GR}} (\rho^{GR})'_G - \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} \cdot [(\mathbf{v}_L)'_L - (\mathbf{v}_S)'_S] + \\
+ \quad & \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} \cdot (\operatorname{grad} \mathbf{v}_S) \mathbf{v}_{GS} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} \cdot (\operatorname{grad} \mathbf{v}_L) \mathbf{v}_{LS} - \\
- \quad & \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} (\operatorname{grad} \mathbf{v}_L) \mathbf{v}_{GS} - \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{GS}} \cdot [(\mathbf{v}_G)'_G - (\mathbf{v}_S)'_S] + \\
+ \quad & \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{GS}} \cdot (\operatorname{grad} \mathbf{v}_S) \mathbf{v}_{GS} - \rho^G \frac{\partial \hat{\psi}^G}{\partial \Theta} \Theta'_G - \rho^G \frac{\partial \hat{\psi}^G}{\partial (\operatorname{grad} \Theta)} (\operatorname{grad} \Theta)'_G - \\
- \quad & \rho^S \eta^S \Theta'_S - \rho^L \eta^L \Theta'_L - \rho^G \eta^G \Theta'_G + \\
+ \quad & \mathbf{T}^{SD} \cdot \mathbf{D}^D_S + \mathbf{T}^{LD} \cdot \mathbf{D}^D_L + \mathbf{T}^{GD} \cdot \mathbf{D}^D_G - \\
- \quad & (p^S + \lambda n^S) \frac{(n^S)'_S}{n^S} - p^S \frac{(\rho^{SR})'_S}{\rho^{SR}} - (p^L + \kappa^{LR} n^L) \frac{(\rho^{LR})'_L}{\rho^{LR}} \ - \\
- \quad & (p^L + \lambda n^L) \frac{(\rho^L)'_L}{\rho^L} - p^G \frac{(\rho^{GR})'_G}{\rho^{GR}} - (p^G + \lambda n^G) \frac{(n^G)'_G}{n^G} -
\end{aligned}
$$

$$
\begin{aligned}
- \quad & (\hat{\mathbf{p}}^L - \lambda \,\mathrm{grad}\, n^L) \cdot \mathbf{v}_{LS} - (\hat{\mathbf{p}}^G - \lambda \,\mathrm{grad}\, n^G) \cdot \mathbf{v}_{GS} - \\
- \quad & \frac{1}{\Theta}\mathbf{q}^S \cdot \mathrm{grad}\,\Theta - \frac{1}{\Theta}\mathbf{q}^L \cdot \mathrm{grad}\,\Theta - \frac{1}{\Theta}\mathbf{q}^G \cdot \mathrm{grad}\,\Theta - \\
- \quad & \hat{\rho}^L(\mu^L - \mu^S) - \hat{\rho}^G(\mu^G - \mu^S) \geq 0 \; .
\end{aligned}
$$

With

$$
\begin{aligned}
& \frac{\partial \hat{\psi}^G}{\partial \breve{\mathbf{C}}_S} \cdot \ (\mathrm{grad}\ \breve{\mathbf{C}}_S)\mathbf{v}_{LS} = \mathbf{v}_{LS} \cdot \ (\mathrm{grad}^{\overset{13}{T}} \breve{\mathbf{C}}_S) \frac{\partial \hat{\psi}^L}{\partial \breve{\mathbf{C}}_S} \ , \\
& \frac{\partial \hat{\psi}^G}{\partial \breve{\mathbf{C}}_S} \cdot \ (\mathrm{grad}\ \breve{\mathbf{C}}_S)\mathbf{v}_{GS} = \mathbf{v}_{GS} \cdot \ (\mathrm{grad}^{\overset{13}{T}} \breve{\mathbf{C}}_S) \frac{\partial \hat{\psi}^G}{\partial \breve{\mathbf{C}}_S} \ , \\
& \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{LS}} \cdot \ (\mathrm{grad}\ \mathbf{v}_L)\mathbf{v}_{LS} = \mathbf{v}_{LS} \cdot \ (\mathrm{grad}^T \mathbf{v}_L) \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{LS}} \\
& = \mathbf{v}_{LS} \cdot \mathbf{D}_L \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{LS}} + \mathbf{v}_{LS} \cdot \mathbf{W}_L^T \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{LS}} \ , \\
& \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{GS}} \cdot \ (\mathrm{grad}\ \mathbf{v}_G)\mathbf{v}_{GS} = \mathbf{v}_{GS} \cdot \ (\mathrm{grad}^T \mathbf{v}_G) \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{GS}} \\
& = \mathbf{v}_{GS} \cdot \mathbf{D}_G \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{GS}} + \mathbf{v}_{GS} \cdot \mathbf{W}_G^T \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{GS}} \ , \\
& \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{LS}} \cdot \ (\mathrm{grad}\ \mathbf{v}_S)\mathbf{v}_{LS} = \mathbf{v}_{LS} \cdot \ (\mathrm{grad}^T \mathbf{v}_S) \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{LS}} \ , \\
& \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \cdot \ (\mathrm{grad}\ \mathbf{v}_G)\mathbf{v}_{GS} = \mathbf{v}_{GS} \cdot \ (\mathrm{grad}^T \mathbf{v}_G) \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \ , \\
& \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \cdot \ (\mathrm{grad}\ \mathbf{v}_G)\mathbf{v}_{LS} = \mathbf{v}_{LS} \cdot \ (\mathrm{grad}^T \mathbf{v}_G) \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \ , \\
& \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \cdot \ (\mathrm{grad}\ \mathbf{v}_S)\mathbf{v}_{LS} = \mathbf{v}_{LS} \cdot \ (\mathrm{grad}^T \mathbf{v}_S) \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \ , \\
& \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \cdot \ (\mathrm{grad}\ \mathbf{v}_G)\mathbf{v}_{LG} = \mathbf{v}_{LG} \cdot \ (\mathrm{grad}^T \mathbf{v}_G) \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \\
& = \mathbf{v}_{LG} \cdot \mathbf{D}_G \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} + \mathbf{v}_{LG} \cdot \mathbf{W}_G^T \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} \ ,
\end{aligned}
\tag{6.20}
$$

$$\frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} \cdot \ (\text{grad } \mathbf{v}_S)\mathbf{v}_{GS} = \mathbf{v}_{GS} \cdot \ (\text{grad}^T \mathbf{v}_S)\frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} \ ,$$

$$\frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} \cdot \ (\text{grad } \mathbf{v}_L)\mathbf{v}_{LS} = \mathbf{v}_{LS} \cdot \ (\text{grad}^T \mathbf{v}_L)\frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}}$$

$$= \mathbf{v}_{LS} \cdot \mathbf{D}_L \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} + \mathbf{v}_{LS} \cdot \mathbf{W}_L^T \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} \ ,$$

$$\frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{GS}} \cdot \ (\text{grad } \mathbf{v}_S)\mathbf{v}_{GS} = \mathbf{v}_{GS} \cdot \ (\text{grad}^T \mathbf{v}_S)\frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{GS}} \ ,$$

$$\frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} \cdot (\text{grad}\, \mathbf{v}_L)\mathbf{v}_{GS} = \mathbf{v}_{GS} \cdot (\text{grad}^T \, \mathbf{v}_L)\frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}}$$

$$= \mathbf{v}_{GS} \cdot \mathbf{D}_L \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} + \mathbf{v}_{GS} \cdot \mathbf{W}_L^T \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}}$$

it follows that:

$$- \frac{(n^S)'_S}{n^S}[p^S + \lambda\, n^S + n^S \rho^S \frac{\partial \hat{\psi}^S}{\partial n^S} \ + n^S \rho^L \frac{\partial \hat{\psi}^L}{\partial n^S} + n^S \rho^G \frac{\partial \hat{\psi}^G}{\partial n^S}] -$$

$$- \frac{(\rho^{SR})'_S}{\rho^{SR}}[p^S + \rho^{SR} \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^{SR}} + \rho^{SR} \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{SR}} + \rho^{SR} \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^{SR}}] -$$

$$- \frac{(\rho^L)'_L}{\rho^L}[p^L + \lambda\, n^L + \rho^L \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} + (\rho^L)^2 \frac{\partial \hat{\psi}^L}{\partial \rho^L} + \rho^L \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L}] -$$

$$- \frac{(\rho^{LR})'_L}{\rho^{LR}}[p^L + \kappa^{LR}\, n^L] + \mathbf{D}_S^D \cdot [\mathbf{T}^{SD} -$$

$$- 2\rho^S J_S^{-2/3}(\mathbf{F}_S \frac{\partial \hat{\psi}^S}{\partial \breve{\mathbf{C}}_S} \mathbf{F}_S^T)^D - 2\rho^S J_S^{-2/3}(\mathbf{F}_S \frac{\partial \hat{\psi}^L}{\partial \breve{\mathbf{C}}_S} \mathbf{F}_S^T)^D -$$

$$-2\rho^S J_S^{-2/3}(\mathbf{F}_S \frac{\partial \hat{\psi}^G}{\partial \breve{\mathbf{C}}_S} \mathbf{F}_S^T)^D] + \ \mathbf{D}_L^D \cdot \mathbf{T}^{LD} + \ \mathbf{D}_G^D \cdot \mathbf{T}^{GD} -$$

$$- \frac{(n^G)'_G}{n^G}[p^G + \lambda\, n^G] - \frac{(\rho^{GR})'_G}{\rho^{GR}}[p^G + \rho^{GR} \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^{GR}} +$$

$$+\rho^{GR} \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}} + \rho^{GR} \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^{GR}}] -$$

$$- \mathbf{v}_{LS} \cdot [\hat{\mathbf{p}}^L - \lambda \ \text{ grad } n^L - \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} \text{ grad } \rho^L -$$

$$\begin{aligned}
&-\rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \text{ grad } \rho^L - \rho^S \ (\text{grad}^T \mathbf{v}_L) \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{LS}} + \\
&+ \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}} \text{ grad } \rho^{GR} - \rho^L \ (\text{grad}^T \mathbf{v}_S) \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{LS}} - \\
&-\rho^L \ (\text{grad}^T \mathbf{v}_S) \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} + \rho^L \ (\text{grad}^T \mathbf{v}_G) \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} - \\
&-\rho^G \ (\text{grad}^T \mathbf{v}_L) \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}}] - \mathbf{v}_{GS} \cdot [\hat{\mathbf{p}}^G \ - \ \lambda \text{ grad } n^G - \\
&-\rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^{GR}} \text{ grad } \rho^{GR} - \rho^S (\text{grad}^T \mathbf{v}_G) \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{GS}} - \\
&-\rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}} \text{ grad } \rho^{GR} \ - \rho^L (\text{grad}^T \mathbf{v}_G) \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} + \\
&+\rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \text{ grad} \rho^L \ - \ \rho^G \ (\text{grad}^T \mathbf{v}_S) \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} + \\
&+\rho^G \ (\text{grad}^T \mathbf{v}_L) \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} - \rho^G \ (\text{grad}^T \mathbf{v}_S) \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{GS}}] - \\
&- \ \Theta'_S [\rho^S \eta^S + \rho^S \frac{\partial \hat{\psi}^S}{\partial \Theta}] - \Theta'_L [\rho^S \eta^L + \rho^L \frac{\partial \hat{\psi}^L}{\partial \Theta}] - \\
&-\Theta'_G [\rho^G \eta^G + \rho^G \frac{\partial \hat{\psi}^G}{\partial \Theta}] - \ \text{grad} \Theta \cdot \frac{1}{\Theta} \mathbf{q}^S - \\
&-\rho^S \frac{\partial \hat{\psi}^S}{\partial (\text{grad}\, \Theta)} \cdot (\text{grad}\, \Theta)'_S - \text{grad} \Theta \cdot \frac{1}{\Theta} \mathbf{q}^L - \\
&-\rho^L \frac{\partial \hat{\psi}^L}{\partial (\text{grad}\, \Theta)} \cdot (\text{grad}\, \Theta)'_L - \ \text{grad} \Theta \cdot \frac{1}{\Theta} \mathbf{q}^G - \\
&-\rho^G \frac{\partial \hat{\psi}^G}{\partial (\text{grad}\, \Theta)} \cdot (\text{grad}\, \Theta)'_G - \ \{ (\mathbf{v}_L)'_L - (\mathbf{v}_S)'_S \} \cdot [\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{LS}} + \\
&+\rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{LS}} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}}] - \ \{ (\mathbf{v}_G)'_G - (\mathbf{v}_S)'_S \} \cdot [\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{GS}} + \\
&+\rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{GS}}] - \ (\mathbf{D}_L)'_L \cdot [\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{D}_L} + \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{D}_L} + \\
&+\rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{D}_L}] - \hat{\rho}^L (\mu^L - \mu^S) - \hat{\rho}^G (\mu^G - \mu^S) \geq 0 \ .
\end{aligned} \tag{6.21}$$

We assume that the terms connected with the time rates of the process variables, see (6.16) in (6.21) do not depend on these time rates. The evaluation of the entropy inequality then yields

$$p^S + \lambda n^S + n^S \rho^S \frac{\partial \hat{\psi}^S}{\partial n^S} + n^S \rho^L \frac{\partial \hat{\psi}^L}{\partial n^S} + n^S \rho^G \frac{\partial \hat{\psi}^G}{\partial n^S} = 0 \, , \tag{6.22}$$

$$p^S + \rho^{SR} \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^{SR}} + \rho^{SR} \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{SR}} + \rho^{SR} \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^{SR}} = 0 \, , \tag{6.23}$$

$$\begin{gathered} \mathbf{T}^{SD} - 2\rho^S J_S^{-2/3} (\mathbf{F}_S \frac{\partial \hat{\psi}^S}{\partial \breve{\mathbf{C}}_S} \mathbf{F}_S^T)^D - 2\rho^S J_S^{-2/3} (\mathbf{F}_S \frac{\partial \hat{\psi}^L}{\partial \breve{\mathbf{C}}_S} \mathbf{F}_S^T)^D - \\ -2\rho^S J_S^{-2/3} (\mathbf{F}_S \frac{\partial \hat{\psi}^G}{\partial \breve{\mathbf{C}}_S} \mathbf{F}_S^T)^D = \mathbf{O} \, , \end{gathered} \tag{6.24}$$

$$p^L + \kappa^{LR}\, n^L = 0 \, , \quad \mathbf{T}^{LD} = \mathbf{O} \, , \quad \mathbf{T}^{GD} = \mathbf{O} \, , \tag{6.25}$$

$$p^L + \lambda n^L + \rho^L \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} + (\rho^L)^2 \frac{\partial \hat{\psi}^L}{\partial \rho^L} + \rho^L \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} = 0 \, , \tag{6.26}$$

$$p^G + \lambda n^G = 0 \, , \tag{6.27}$$

$$p^G + \rho^{GR} \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^{GR}} + \rho^{GR} \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}} + \rho^{GR} \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^{GR}} = 0 \, , \tag{6.28}$$

$$\eta^S + \frac{\partial \hat{\psi}^S}{\partial \Theta} = 0 \, , \quad \eta^L + \frac{\partial \hat{\psi}^L}{\partial \Theta} = 0 \, , \; \eta^G + \frac{\partial \hat{\psi}^G}{\partial \Theta} = 0 \, , \tag{6.29}$$

$$\frac{\partial \hat{\psi}^S}{\partial \operatorname{grad} \Theta} = \mathbf{0} \, , \quad \frac{\partial \hat{\psi}^L}{\partial \operatorname{grad} \Theta} = \mathbf{0} \, , \quad \frac{\partial \hat{\psi}^G}{\partial \operatorname{grad} \Theta} = \mathbf{0} \, , \tag{6.30}$$

$$\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{LS}} + \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{LS}} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}} = \mathbf{0} \, , \tag{6.31}$$

$$\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{D}_L} + \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{D}_L} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{D}_L} = \mathbf{O} \, , \tag{6.32}$$

$$\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{GS}} + \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{GS}} = \mathbf{0} \, . \tag{6.33}$$

Moreover, an important dissipation inequality remains:

$$\begin{aligned}\mathcal{D} \ = \ & -\mathbf{v}_{LS}\cdot[\hat{\mathbf{p}}^L-\lambda\,\mathrm{grad}\,n^L-(\rho^S\frac{\partial\hat{\psi}^S}{\partial\rho^L}+\rho^G\frac{\partial\hat{\psi}^G}{\partial\rho^L})\,\mathrm{grad}\,\rho^L \\ & -(\mathrm{grad}^T\,\mathbf{v}_L)\,(\rho^S\frac{\partial\hat{\psi}^S}{\partial\mathbf{v}_{LS}}+\rho^G\frac{\partial\hat{\psi}^G}{\partial\mathbf{v}_{LS}})+ \\ & +\ (\mathrm{grad}\,\rho^{GR})\,(\rho^L\frac{\partial\hat{\psi}^L}{\partial\rho^{GR}})-\ (\mathrm{grad}^T\,\mathbf{v}_S)\,(\rho^L\frac{\partial\hat{\psi}^L}{\partial\mathbf{v}_{LS}}+ \\ & +\rho^L\frac{\partial\hat{\psi}^L}{\partial\mathbf{v}_{GS}})+\ (\mathrm{grad}^T\,\mathbf{v}_G)\,(\rho^L\frac{\partial\hat{\psi}^L}{\partial\mathbf{v}_{GS}})]-\mathbf{v}_{GS}\cdot[\hat{\mathbf{p}}^G- \\ & -\lambda\,\mathrm{grad}\,n^G-\ (\rho^S\frac{\partial\hat{\psi}^S}{\partial\rho^{GR}}+\rho^L\frac{\partial\hat{\psi}^L}{\partial\rho^{GR}})\,\mathrm{grad}\,\rho^{GR}- \qquad (6.34) \\ & -(\mathrm{grad}^T\,\mathbf{v}_G)\,(\rho^S\frac{\partial\hat{\psi}^S}{\partial\mathbf{v}_{GS}}+\rho^L\frac{\partial\hat{\psi}^L}{\partial\mathbf{v}_{GS}})- \\ & -(\mathrm{grad}^T\,\mathbf{v}_S)(\rho^G\frac{\partial\hat{\psi}^G}{\partial\mathbf{v}_{LS}}+\rho^G\frac{\partial\hat{\psi}^G}{\partial\mathbf{v}_{GS}})+\ \rho^G\frac{\partial\hat{\psi}^G}{\partial\rho^L}\,\mathrm{grad}\,\rho^L+ \\ & +(\mathrm{grad}^T\,\mathbf{v}_L)\,(\rho^G\frac{\partial\hat{\psi}^G}{\partial\mathbf{v}_{LS}})]-\mathrm{grad}\,\Theta\cdot\frac{1}{\Theta}[\mathbf{q}^S+\mathbf{q}^L+\mathbf{q}^G]- \\ & -\hat{\rho}^L[\mu^L-\mu^S]-\hat{\rho}^G[\mu^G-\mu^S]\geq 0\ .\end{aligned}$$

Inequality (6.34) can be simplified considering (6.31) through (6.33):

$$\begin{aligned}\mathcal{D} \ = \ & -[\hat{\mathbf{p}}^L-\lambda\,\mathrm{grad}\,n^L-(\rho^S\frac{\partial\hat{\psi}^S}{\partial\rho^L}+\rho^G\frac{\partial\hat{\psi}^G}{\partial\rho^L})\,\mathrm{grad}\,\rho^L+ \\ + \ & (\mathrm{grad}^T\,\mathbf{v}_L)(\rho^L\frac{\partial\hat{\psi}^L}{\partial\mathbf{v}_{LS}})-(\mathrm{grad}^T\,\mathbf{v}_S)(\rho^L\frac{\partial\hat{\psi}^L}{\partial\mathbf{v}_{LS}}+\rho^L\frac{\partial\hat{\psi}^L}{\partial\mathbf{v}_{GS}})+ \\ + \ & \mathrm{grad}\,\rho^{GR}(\rho^L\frac{\partial\hat{\psi}^L}{\partial\rho^{GR}})+(\mathrm{grad}^T\,\mathbf{v}_G)\rho^L\frac{\partial\hat{\psi}^L}{\partial\mathbf{v}_{GS}}]\cdot\mathbf{v}_{LS}- \qquad (6.35) \\ - \ & [\hat{\mathbf{p}}^G-\lambda\,\mathrm{grad}\,n^G-(\rho^S\frac{\partial\hat{\psi}^S}{\partial\rho^{GR}}+\rho^L\frac{\partial\hat{\psi}^L}{\partial\rho^{GR}})\,\mathrm{grad}\,\rho^{GR}+ \\ + \ & (\mathrm{grad}^T\,\mathbf{v}_G)(\rho^G\frac{\partial\hat{\psi}^G}{\partial\mathbf{v}_{GS}})-(\mathrm{grad}^T\,\mathbf{v}_S)(\rho^G\frac{\partial\hat{\psi}^G}{\partial\mathbf{v}_{LS}}+\rho^G\frac{\partial\hat{\psi}^G}{\partial\mathbf{v}_{GS}})+ \\ + \ & (\mathrm{grad}^T\,\mathbf{v}_L)(\rho^G\frac{\partial\hat{\psi}^G}{\partial\mathbf{v}_{LS}})+\rho^G\frac{\partial\hat{\psi}^G}{\partial\rho^L}\,\mathrm{grad}\,\rho^L]\cdot\mathbf{v}_{GS}-\end{aligned}$$

$$
\begin{aligned}
&- \quad \operatorname{grad}\Theta \cdot \frac{1}{\Theta}[\mathbf{q}^S + \mathbf{q}^L + \mathbf{q}^G] - \\
&- \quad \hat{\rho}^L[\mu^L - \mu^S] - \hat{\rho}^G[\mu^G - \mu^S] \geq 0 \; .
\end{aligned}
$$

Moreover, one can show that, in the case of the mixture equilibrium, the free Helmholtz energy functions ψ^L and ψ^G do not depend on the relative velocities $\mathbf{v}_{LS}$ and $\mathbf{v}_{GS}$ and of $\mathbf{D}_L$ in (6.35) (see de Boer, 2000 a). Then,

$$
\begin{aligned}
\mathcal{D} \;=\; & -[\hat{\mathbf{p}}^L - \lambda \operatorname{grad} n^L - \Big(\rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L}\Big) \operatorname{grad} \rho^L + \\
+ \; & \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}} \operatorname{grad} \rho^{GR}] \cdot \mathbf{v}_{LS} - \\
- \; & [\hat{\mathbf{p}}^G - \lambda \operatorname{grad} n^G + \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \operatorname{grad} \rho^L - \\
- \; & \Big(\rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^{GR}} + \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}}\Big) \operatorname{grad} \rho^{GR}] \; \cdot \mathbf{v}_{GS} - \\
- \; & \operatorname{grad} \Theta \cdot \frac{1}{\Theta}[\mathbf{q}^S + \mathbf{q}^L + \mathbf{q}^G] - \\
- \; & \hat{\rho}^L(\mu^L - \mu^G) - \hat{\rho}^G(\mu^G - \mu^S) \geq 0 \; .
\end{aligned}
\tag{6.36}
$$

We assume that the expressions in the brackets depend on the set s (6.16) of process variables.

Close to the mixture equilibrium state, we develop the first terms in a Taylor series. In this connection the mixture equilibrium is defined by

$$
\mathbf{v}_{LS} = \mathbf{0} \; , \; \mathbf{v}_{GS} = \mathbf{0} \; , \;\; \operatorname{grad} \Theta = \mathbf{0} \; . \tag{6.37}
$$

Next we investigate the interaction forces $\hat{\mathbf{p}}^L$ and $\hat{\mathbf{p}}^G$, which are obtained by a Taylor series expansion:

$$
\begin{aligned}
\hat{\mathbf{p}}^L - \lambda \operatorname{grad} n^L \;&-\; \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} \operatorname{grad} \rho^L - \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \operatorname{grad} \rho^L \\
&= \; - \; \beta_L \operatorname{grad} \Theta - S_L \mathbf{v}_L - \bar{S}_G \mathbf{v}_G \; , \\
\hat{\mathbf{p}}^G - \lambda \operatorname{grad} n^G \;&+\; \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \operatorname{grad} \rho^L \\
&= \; - \; \beta_G \operatorname{grad} \Theta - \bar{S}_L \mathbf{v}_L - S_G \mathbf{v}_G \; ,
\end{aligned}
\tag{6.38}
$$

where β_L to S_G are response function which depend on $s_0 = \{\rho^L,\ \rho^{GR},\ 0\}$. In (6.38) we have neglected the influence of grad ρ^{GR}. The operation grad ρ^{GR} as well as the derivatives of the free Helmholtz energy functions with respect to ρ^{GR} are supposed to describe diffusion processes. However, this phenomenon has not been discussed in this context yet.

Moreover, the heat flux vectors $\mathbf{q}^\alpha$ are determined by:

$$\begin{aligned}
\frac{1}{\Theta}\mathbf{q}^S &= -\ \alpha_{S\Theta}\ \mathrm{grad}\ \Theta - \bar{\alpha}_L \mathbf{v}_L - \bar{\alpha}_G \mathbf{v}_G\ , \\
\frac{1}{\Theta}\mathbf{q}^L &= -\ \alpha_{L\Theta}\ \mathrm{grad}\ \Theta - \alpha_L \mathbf{v}_L - \bar{\bar{\alpha}}_G\ \mathbf{v}_G\ , \\
\frac{1}{\Theta}\mathbf{q}^G &= -\ \alpha_{G\Theta}\ \mathrm{grad}\ \Theta - \bar{\bar{\alpha}}_L\ \mathbf{v}_L - \alpha_G \mathbf{v}_G
\end{aligned} \tag{6.39}$$

with $\alpha_{S\Theta}$ through α_G as response parameter.

It should be mentioned that, in order to gain the constitutive equations (6.38), and (6.39), the principle of material objectivity has been used, which leads to some restrictions for the response parameters introduced above (see de Boer, 2000 a).

The constitutive equations (6.25) and (6.26) as well as (6.27) and (6.28) describe the mechanical behavior of the liquid and gas phases, where the gas phase is the carrier for the droplets. In this model the interface "pressure" λ can be identified according to (6.27) by the hydrostatic pressure p^{GR}. If one is, however, interested in a model in which the liquid phase is the carrier and the gas is contained in the liquid phase as, e.g., bubbles, then one has to reformulate a part of the entropy inequality and to replace the process variable ρ^L with n^G. This happens when a lot of droplets exist in the gas phase caused by water supply from a reservoir and when the droplets unify. Then, gradually the liquid becomes the carrier and the gas phase takes the form of bubbles. The onset of the change of the carrier is still an open question. A first attempt to solve such an "instable" behavior can be found in the book by Müller (1994).

That part A of the entropy inequality, which has to be reformulated, concerns the following expressions in the third line of Inequality (6.21):

$$A = -\ \frac{(\rho^L)'_L}{\rho^L}\ [\rho^L \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} + (\rho^L)^2 \frac{\partial \hat{\psi}^L}{\partial \rho^L} + \rho^L \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L}]\ . \tag{6.40}$$

In the constitutive ansätze for the free Helmholtz energy functions ψ^S, ψ^L, and ψ^G we replace ρ^L with n^G, viz:

$$\psi^S = \hat{\psi}^S = \bar{\psi}^S\ ,\ \psi^L = \hat{\psi}^L = \bar{\psi}^L\ ,\ \psi^G = \hat{\psi}^G = \bar{\psi}^G\ , \tag{6.41}$$

where

$$\psi^\alpha = \bar{\psi}^\alpha(n^S\ ,\ \rho^{SR}\ ,\ \breve{\mathbf{C}}_S\ ,\ n^G\ ,\ \rho^{GR}\ ,\ \mathbf{v}_{LS}\ ,\ \mathbf{v}_{GS}\ ,\ \Theta\ ,\ \operatorname{grad}\Theta)\ . \qquad (6.42)$$

In order to reformulate A in (6.40) we have to find a relation between ρ^L and n^G. With (2.4) and the saturation condition (2.2) we have:

$$\rho^L = \rho^{LR} n^L\ ,\ n^L = 1 - n^S - n^G\ . \qquad (6.43)$$

Bearing in mind that ρ^{LR} is constant due to the incompressiblity of the real liquid we can easily express ρ^L by n^G. From (6.43), after some manipulations we obtain

$$n^G = 1 - n^S - \frac{\rho^L}{\rho^{LR}}. \qquad (6.44)$$

With (6.44) and (6.43) we first calculate

$$\begin{aligned} \frac{(\rho^L)'_L}{\rho^L} = \frac{(n^L)'_L}{n^L} &= \frac{1}{n^L}[-\,(n^S)'_S - \,(n^G)'_G + \\ &+ \operatorname{grad} n^G \cdot \mathbf{v}_{GS} - (\operatorname{grad} n^S + \operatorname{grad} n^G)\cdot \mathbf{v}_{LS}]\ . \end{aligned} \qquad (6.45)$$

Then, we manipulate the energy terms. We obtain with (6.44) the relation

$$\frac{\partial \psi^\alpha}{\partial \rho^L} = \frac{\partial \bar{\psi}^\alpha}{\partial n^G}\frac{\partial n^G}{\partial \rho^L} = -\,\frac{1}{\rho^{LR}}\frac{\partial \bar{\psi}^\alpha}{\partial n^G}\ . \qquad (6.46)$$

Thus, with (6.45) and (6.46), we obtain for A in (6.40)

$$\begin{aligned} A = -\,\frac{1}{n^L}\ & [(n^S)'_S\ + (n^G)'_G\ -\ \operatorname{grad} n^G \cdot \mathbf{v}_{GS} + \\ & +(\operatorname{grad} n^G + \operatorname{grad} n^S)\cdot \mathbf{v}_{LS}]\times \\ & \times\left[n^L \rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + n^L \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} + n^L \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G}\right] \end{aligned} \qquad (6.47)$$

or

$$\begin{aligned} A = \ & -\left(n^S \rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + n^S \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} + n^S \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G}\right)\frac{(n^S)'_S}{n^S} - \\ & -\left(n^G \rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + n^G \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} + n^G \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G}\right)\frac{(n^G)'_G}{n^G} + \\ & +\left(\rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} + \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G}\right)\operatorname{grad} n^G \cdot \mathbf{v}_{GS} + \\ & +\left(\rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} + \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G}\right)(\operatorname{grad} n^G + \operatorname{grad} n^S)\cdot \mathbf{v}_{LS}\ . \end{aligned} \qquad (6.48)$$

From (6.21) in connection with (6.48) we obtain

$$
\begin{aligned}
-\frac{(n^S)'_S}{n^S} \quad & [p^S + \lambda\, n^S + n^S \rho^S \frac{\partial \hat{\psi}^S}{\partial n^S} + n^S \rho^L \frac{\partial \hat{\psi}^L}{\partial n^S} + n^S \rho^G \frac{\partial \hat{\psi}^G}{\partial n^S} - \\
& - n^S \rho^S \frac{\partial \hat{\psi}^S}{\partial n^G} - \ n^S \rho^L \frac{\partial \hat{\psi}^L}{\partial n^G} - n^S \rho^S \frac{\partial \hat{\psi}^G}{\partial n^G}] - \\
-\frac{(\rho^{SR})'_S}{\rho^{SR}} \quad & [p^S + \rho^{SR} \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^{SR}} + \rho^{SR} \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{SR}} + \rho^{SR} \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^{SR}}] - \\
-\frac{(\rho^L)'_L}{\rho^L} \quad & [p^L + \lambda\, n^L] - \frac{(\rho^{LR})'_L}{\rho^{LR}} [p^L + \kappa^{LR}\, n^L] + \\
+\mathbf{D}_S^D \ \cdot \quad & [\mathbf{T}^{SD} - \ 2 \rho^S J_S^{-2/3} (\mathbf{F}_S \frac{\partial \hat{\psi}^S}{\partial \breve{\mathbf{C}}_S} \mathbf{F}_S^T)^D - \\
& -2 \rho^S J_S^{-2/3} (\mathbf{F}_S \frac{\partial \hat{\psi}^L}{\partial \breve{\mathbf{C}}_S} \mathbf{F}_S^T)^D - 2 \rho^S J_S^{-2/3} (\mathbf{F}_S \frac{\partial \hat{\psi}^S}{\partial \breve{\mathbf{C}}_S} \mathbf{F}_S^T)^D] - \\
-\frac{(n^G)'_G}{n^G} \quad & [p^G + \lambda\, n^G - n^G (\rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} + \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G})] - \\
-\mathbf{v}_{LS} \ \cdot \quad & [\hat{\mathbf{p}}^L - \lambda \operatorname{grad} n^L - \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} \operatorname{grad} \rho^L - \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \operatorname{grad} \rho^L + \\
& + (\rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} + \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G}) (\operatorname{grad} n^G + \operatorname{grad} n^S) - \qquad (6.49) \\
& - (\operatorname{grad}^T \mathbf{v}_L)(\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{LS}}) + \ \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}} \operatorname{grad} \rho^{GR} - \\
& - (\operatorname{grad}^T \mathbf{v}_S)(\rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{LS}}) - (\operatorname{grad}^T \mathbf{v}_S)(\rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}}) + \\
& + (\operatorname{grad}^T \mathbf{v}_G)\ (\rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}}) - (\operatorname{grad}^T \mathbf{v}_L)(\rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}})] - \\
-\mathbf{v}_{GS} \ \cdot \quad & [\hat{\mathbf{p}}^G \ - \ \lambda \operatorname{grad} n^G - \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \operatorname{grad} \rho^L - \\
& - \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^{GR}} \operatorname{grad} \rho^{GR} \ - \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}} \operatorname{grad} \rho^{GR} + \\
& + (\rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} + \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G}) \operatorname{grad} n^G \ - \\
& - (\operatorname{grad}^T \mathbf{v}_G)(\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{GS}}) - (\operatorname{grad}^T \mathbf{v}_G)(\rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}}) - \\
& - (\operatorname{grad}^T \mathbf{v}_S)(\rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}}) + (\operatorname{grad}^T \mathbf{v}_L)(\rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}}) -
\end{aligned}
$$

$$
\begin{aligned}
&- \quad (\operatorname{grad}^T \mathbf{v}_S)(\rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{GS}})] \; - \Theta'_S[\rho^S \eta^S + \rho^S \frac{\partial \hat{\psi}^S}{\partial \Theta}] - \\
&- \quad \Theta'_L[\rho^L \eta^L + \rho^L \frac{\partial \hat{\psi}^L}{\partial \Theta}] - \; \Theta'_G[\rho^G \eta^G + \rho^G \frac{\partial \hat{\psi}^G}{\partial \Theta}] - \\
&- \quad \operatorname{grad}\Theta \cdot \frac{1}{\Theta}\mathbf{q}^S - \operatorname{grad}\Theta \cdot \frac{1}{\Theta}\mathbf{q}^L - \operatorname{grad}\Theta \cdot \frac{1}{\Theta}\mathbf{q}^G - \\
&- \quad [(\mathbf{v}_L)'_L - (\mathbf{v}_S)'_S] \cdot [\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{LS}} + \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{LS}} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{LS}}] - \\
&- \quad [(\mathbf{v}_G)'_G - (\mathbf{v}_S)'_S] \cdot [\rho^S \frac{\partial \hat{\psi}^S}{\partial \mathbf{v}_{GS}} + \rho^L \frac{\partial \hat{\psi}^L}{\partial \mathbf{v}_{GS}} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \mathbf{v}_{GS}}] \; .
\end{aligned}
$$

From Inequality (6.49) we get the new constitutive restrictions:

$$
\begin{aligned}
& p^G + \lambda n^G - \rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} - \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} - \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G} = 0 \; , \\
& p^L + \kappa^{LR} n^L = 0 \; , \\
& p^L + \lambda n^L = 0
\end{aligned}
\tag{6.50}
$$

or

$$
p^L + p \, n^L = 0 \; , \tag{6.51}
$$

where we have identified the indeterminate quantity κ^{LR}, which is related to the incompressibility of the real liquid, as the real liquid pressure p.

Moreover, we obtain a useful inequality $\mathcal{G}$:

$$
\begin{aligned}
\mathcal{G} \;\; = \;\; & - [\hat{\mathbf{p}}^L - \lambda \operatorname{grad} n^L - (\rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L}) \operatorname{grad} \rho^L + \\
& + \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}} \operatorname{grad} \rho^{GR} - (\rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} - \\
& - \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G})(\operatorname{grad} n^G + \operatorname{grad} n^S] \cdot \mathbf{v}_{LS} - [\hat{\mathbf{p}}^G - \lambda \operatorname{grad} n^G - \\
& - \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \operatorname{grad} \rho^L - (\rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} + \rho^L \frac{\partial \hat{\psi}^L}{\partial \rho^{GR}}) \operatorname{grad} \rho^{GR} + \\
& + (\rho^S \frac{\partial \bar{\psi}^S}{\partial n^G} + \rho^L \frac{\partial \bar{\psi}^L}{\partial n^G} + \rho^G \frac{\partial \bar{\psi}^G}{\partial n^G}) \operatorname{grad} n^G] \cdot \mathbf{v}_{GS} - \\
& - \operatorname{grad} \Theta \cdot \frac{1}{\Theta}[\mathbf{q}^S + \mathbf{q}^L + \mathbf{q}^G] - \\
& - \hat{\rho}^L(\mu^L - \mu^S) - \hat{\rho}^G(\mu^G - \mu^S) \geq 0 \; .
\end{aligned}
\tag{6.52}
$$

In obtaining (6.52) we have introduced the same assumptions as in (6.36).

6.5 Elastic Behavior of the Solid Skeleton

From the evaluation of the entropy inequality, it follows that, due to the saturation constraint, constitutive equations must be developed for the effective stresses, which are a part of the total stresses in the partial solid body. It was quite natural that the first constitutive equations for the effective stresses in the porous media theory were developed for the elastic behavior of the solid skeleton, since the elasticity theory was the best developed theory for one-component continua, in particular, within the geometrically linear theory.

a) Finite Theories

The forerunner of finite elasticity laws, Hooke's law, which describes linear-elastic behavior, is a creation of the 19th century. Later, within the framework of the geometrically linear theory, the generalized Hooke's law was extended in such a way that non-linear deformations could also be included. A well-known, non-linear elasticity law for small deformations has been introduced by Kauderer (1958).

General non-linear elasticity laws to describe finite distortions have only been developed in this century. The treatise of Murnaghan (1937) on finite elastic deformations seems to be incomplete (see Truesdell and Noll, 1965). The efforts of Mooney (1940) and Rivlin (1948), as well as Rivlin and Saunders (1951) were, on the other hand, much more successful. A review of the development of finite elasticity theory can be found in Truesdell and Noll (1965).

In extension of the Neo-Hooke model (Rivlin, 1948), Simo and Pister (1984) developed a new elasticity law which contains two response parameters and permits the description of large volumetric strains. All discussed elasticity laws are related to the elastic behavior of non-porous solids. The conversion of these laws to porous solids was performed by, among others, Morland (1972) and Ehlers (1989 a). However, their contributions are based on simplified models. This is also valid for the investigations of Bluhm and de Boer (1994). Therefore, these papers will not be discussed.

In the meantime new contributions have clarified some issues in poroelasticity with incompressible constituents. For example, Ehlers and Eipper (1997, 1999) developed within the framework of finite deformations the following hyperelastic equation of the Kirchhoff effective stress tensor, which is a weighted stress tensor ($\boldsymbol{\tau}_E^S = J_S \mathbf{T}_E^S$):

$$\begin{aligned} \boldsymbol{\tau}_E^S &= \mu^S(\mathbf{B}_S - \mathbf{I}) + \\ &+ \lambda^S(1-n_{0S}^S)^2\Big(\frac{J_S}{1-n_{0S}^S} - \frac{J_S}{J_S - n_{0S}^S}\Big)\mathbf{I}\,, \end{aligned} \tag{6.53}$$

where μ^S and λ^S are the Lamé constants.

It should be mentioned that if all pores are closed the compaction point is reached due to the assumed incompressibility of the porous solid. Thus, with (4.9), (2.4), (3.3), and $n^S_{0S} \le n^S \le 1$, the Jacobian is restricted by $n^S_{0S} < J_S < \infty$.

Ehlers and Eipper (1999) have shown that the Oldroyd stress rate is given by

$$(\boldsymbol{\tau}^S_E)^{\nabla} =: \overset{4}{\mathbf{C}}\, \mathbf{D}_S \ , \tag{6.54}$$

where the fourth-order tensor $\overset{4}{\mathbf{C}}$ for the model under study is represented by

$$\overset{4}{\mathbf{C}} = 2\alpha(\mathbf{I}\otimes\mathbf{I})^{\overset{23}{\mathrm{T}}} + \beta(\mathbf{I}\otimes\mathbf{I}) \ , \tag{6.55}$$

with

$$\begin{aligned} \alpha(J_S) &= [\mu^S - \lambda^S(1-n^S_{0S})J_S\frac{J_S-1}{J_S-n^S_{0S}}] \ , \\ \beta(J_S) &= \lambda^S[(1-n^S_{0S})J_S\frac{J_S^2-2n^S_{0S}J_S+n^S_{0S}}{(J_S-n^S_{0S})^2}] \ . \end{aligned} \tag{6.56}$$

Recall that in the case of incompressibility J_S can be replaced by J_{SN}, see Eqs. (3.25) through (3.28), because J_{SR} is equal to unity.

In the meantime Bluhm (1999, 2002) independently developed a non-linear elasticity law for porous solids and extended this law to compressible solid material behavior and thermal effects. For the solid stress due to temperature changes, he chose a linear relation. In the incompressible case he arrived for the Kirchhoff effective stress tensor at:

$$\begin{aligned} \boldsymbol{\tau}^S_E &= 2\mu^S J_S^{-2/3}\mathbf{K}^D_S + \\ &+ K^{SN}\Big[\log J_{SN} + 1 - \frac{1}{J_{SN} - \frac{n^S_{0S}}{1-n^S_{0S}}(\frac{n^S_{0S}}{J_{SN}} - n^S_{0S})}\Big]\mathbf{I} - \\ &- 3\alpha^S K^S(\Theta - \Theta_{0S})\mathrm{I} \ , \end{aligned} \tag{6.57}$$

where $\mathbf{K}^D_S$ is the deviator of the Karni-Reiner tensor $\mathbf{K}_S = \frac{1}{2}(\mathbf{B}_S - \mathbf{I})$, μ^S and K^{SN} as well as K^S are the shear modulus and compression moduli of the skeleton and the mixture body, α^S is the thermal expansion coefficient, and Θ the absolute temperature.

b) Linear Theory

In order to show the relationship to Hooke's law, Bluhm (1999, 2002) linearized the constitutive equation (6.57),

$$\begin{aligned} \boldsymbol{\tau}^S_{E\ lin} = & \ -\mu^S \triangle \mathbf{B}_S^{-1} - \frac{1}{2}\Big[K^{SN} \frac{2(1-n^S_{0S}) + (n^S_{0S})^2}{1-n^S_{0S}} - \\ & -\frac{2}{3}\mu^S \Big] (\triangle \mathbf{B}_S^{-1} \cdot \mathbf{I})\mathbf{I} - 3\alpha^S K^S \triangle \Theta \mathbf{I}\ , \end{aligned} \tag{6.58}$$

where

$$\begin{aligned} &\triangle \mathbf{B}_S^{-1} = \mathbf{B}_S^{-1} - \mathbf{B}_S^{-1}|_{\hat{P}_0} = \mathbf{B}_S^{-1} - \mathbf{I} = -\ 2\mathbf{A}_S\ , \\ &\triangle \Theta = \Theta - \Theta|_{\hat{P}_0\ =\ \Theta - \Theta_{0S}}\ . \end{aligned} \tag{6.59}$$

In (6.59) the symbol

$$(\ldots)\ \Big|_{\hat{\mathcal{P}}_0} = (\ldots)\ \Big|_{\mathbf{B}_S^{-1}\ =\ \mathbf{I},\ \Theta\ =\ \Theta_{0S}} \tag{6.60}$$

implies that the quantity $(\ldots)$ must be evaluated for $\mathbf{B}_S^{-1} = \mathbf{I}$ and $\Theta = \Theta_{0S}$, whereby

$$\boldsymbol{\tau}^S_E|_{\hat{P}_0} = \mathbf{O}\ . \tag{6.61}$$

With (6.59) and (6.60), the linearized law of Hooke-type with respect to the actual placement (6.58) can be reformulated as

$$\begin{aligned} \boldsymbol{\tau}^S_{E\ lin} \quad = \quad & 2\mu^S \mathbf{A}_S + [K^{SN} \frac{2(1-n^S_{0S}) + (n^S_{0S})^2}{1-n^S_{0S}} - \\ & -\ \frac{2}{3}\ \mu^S](\mathbf{A}_S \cdot \mathbf{I})\mathbf{I} - 3\alpha^S K^S (\Theta - \Theta_{0S})\mathbf{I}\ . \end{aligned} \tag{6.62}$$

In accordance with the linear theory of one component elastic material, the response parameters μ^S,

$$\lambda^S = K^{SN} \frac{2(1-n^S_{0S}) + (n^S_{0S})^2}{1-n^S_{0S}} - \frac{2}{3}\ \mu^S \tag{6.63}$$

and

$$K^S = K^{SN} \frac{2(1-n^S_{0S}) + (n^S_{0S})^2}{1-n^S_{0S}} = \frac{2}{3}\ \mu^S + \lambda^S \tag{6.64}$$

will be interpreted as the macroscopic Lamé constants and macroscopic compression modulus of the constituents φ^S. With this interpretation of the response parameters μ^S, λ^S, and K^S, the linearized constitutive relation (6.62) simplifies to:

$$\begin{aligned} \boldsymbol{\tau}^S_{E\,lin} &= 2\mu^S \mathbf{A}_S + \lambda^S(\mathbf{A}_S \cdot \mathbf{I})\mathbf{I} - \ 3\alpha^S K^S(\Theta - \Theta_{0S})\mathbf{I} \\ &= 2\mu^S \mathbf{A}^D_S + K^S(\mathbf{A}_S \cdot \mathbf{I})\mathbf{I} - \ 3\alpha^S K^S(\Theta - \Theta_{0S})\mathbf{I}\,. \end{aligned} \tag{6.65}$$

Furthermore, Bluhm (1999, 2002) developed a corresponding constitutive equation for the symmetric Piola-Kirchhoff stress tensor

$$\begin{aligned} \mathbf{S}^S_{E\,lin} &= 2\mu^S \mathbf{E}_S + \lambda^S(\mathbf{E}_S \cdot \mathbf{I})\mathbf{I} - 3\alpha^S K^S(\Theta - \Theta_{0S})\mathbf{I} \\ &= 2\mu^S \mathbf{E}^D_S + K^S(\mathbf{E}_S \cdot \mathbf{I})\mathbf{I} - 3\alpha^S K^S(\Theta - \Theta_{0S})\mathbf{I}\,. \end{aligned} \tag{6.66}$$

It should be mentioned that the linearization of $\boldsymbol{\tau}^S_E$ in the form (6.65) only makes sense for moderate deformations where the elastic tangent does not vary too much; in other words, it is a useful approximation.

Finally, for two elastic porous solids with the porosities in the reference placement $n^S_{0S} = 0.8$ and $n^S_{0S} = 0.2$, the free Helmholtz energy function ψ^S and the effective hydrostatic stress state p^S_E in dependence of $J_{SN} = \dfrac{n^S_{0S}}{n^S}$ for $\breve{\mathbf{C}}_S = \mathbf{I}$ and $\Theta = \Theta_{0S}$ are depicted (see Fig. 6.5.1).

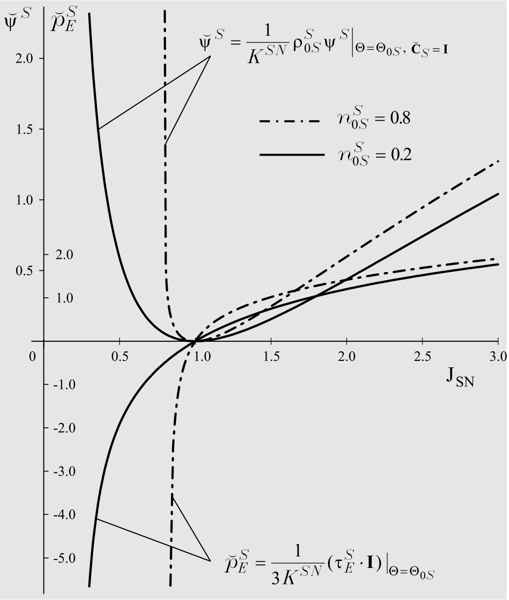

Fig. 6.5.1: Representation of the free Helmholtz energy and the effective hydrostatic stress

c) Other Approaches

Cieszko and Kubik (1997) developed non-linear and linear constitutive relations for fluid-saturated porous elastic solids undergoing purely mechanical deformations. Based on thermodynamic considerations, they arrived at the constitutive equation

$$\overset{*}{\mathbf{T}}^S = -\,p\mathbf{I} + 2\rho^{SR}\mathbf{F}\frac{\partial \hat{e}^S}{\partial \mathbf{C}}\mathbf{F}^T \, , \tag{6.67}$$

where

$$\overset{*}{\mathbf{T}}^S = \frac{1}{n^S}\mathbf{T}^S \, , \quad p = (\rho^{LR})^2\frac{\partial \hat{e}^L}{\partial \rho^{LR}} \tag{6.68}$$

with ρ^{SR} and ρ^{FR} as the real densities, $\mathbf{T}^S$ as Cauchy's stress tensor as well as $\hat{e}^S$ and $\hat{e}^F$ as the internal energies for the fluid and the porous skeleton (The statement by Ciezko and Kubik, that $\overset{*}{\mathbf{T}}{}^S$ in (6.68) is defined as the effective stress tensor in the skeleton, which is due to Morland, cannot be held in light of the modern porous media theory, see de Boer, 1996). Ciezko and Kubik (1997) specialized (6.67) and $(6.68)_2$ to *linear* constitutive relations for fluid-filled elastic porous media with anisotropic and isotropic properties of the skeleton. Finally, they compared their results with the Biot relations.

However, this procedure is questionable; as stated earlier, Biot's poroelasticity is an ad-hoc approach, which does not consider the main principles of continuum mechanics.

Ehlers and Volk (1999) developed, in their treatise on localization phenomena in liquid-saturated and empty porous solids, constitutive equations for the effective stress $\mathbf{T}^S_E$ and the couple stress tensor $\mathbf{M}^S$:

$$\mathbf{T}^S_E = [2\mu^S \overset{4}{\mathbf{I}}_{sym} + 2\mu^S_C \overset{4}{\mathbf{I}}_{skw} + \lambda^S \, (\mathbf{I}\otimes\mathbf{I})]\varepsilon_{sce} \, , \tag{6.69}$$

where

$$\begin{aligned} \overset{4}{\mathbf{I}}_{sym} &= \frac{1}{2}[(\mathbf{I}\otimes\mathbf{I})^{\overset{23}{T}} + (\mathbf{I}\otimes\mathbf{I})^{\overset{13}{T}}] \, , \\ \overset{4}{\mathbf{I}}_{skw} &= \frac{1}{2}[(\mathbf{I}\otimes\mathbf{I})^{\overset{23}{T}} - (\mathbf{I}\otimes\mathbf{I})^{\overset{13}{T}}] \, . \end{aligned} \tag{6.70}$$

In (6.69) and (6.70), $\overset{4}{\mathbf{I}}_{sym}$ and $\overset{4}{\mathbf{I}}_{skw}$ are fundamental tensors of the fourth-order. The response parameters μ^S and λ^S are the Lamé constants of the solid phase, and μ^S_C is an additional response parameter describing the influence of the skew-symmetric part of the elastic Cosserat strain ε_{sc} on the effective stress $\mathbf{T}^S_E$.

For the couple stress tensor $\mathbf{M}^S$, Ehlers and Volk (1999) used the simplified constitutive equation

$$\mathbf{M}^S = 2\mu^S \, (l_C^S)^2 \bar{\boldsymbol{\kappa}}_{Se} \tag{6.71}$$

after de Borst (1991). Therein, l_C^S can be interpreted as an internal length scale parameter.

Giovine (1999) presented a linear theory of porous elastic solids. He described the goal of his contribution as follows: "... we propose the suitable constitutive properties for a porous elastic material which does not conduct heat, including a rate effect in the microstructural response due to inelastic surface effects associated with pore compaction and distention, and establish the restrictions imposed on these equations by thermodynamical considerations." In these considerations, Giovine (1999) assumed the following set of process variables $s := \{\mathbf{F}_S, \mathbf{U}_S, \mathrm{Grad}_S \;\, \mathbf{U}_S, \;\Theta\}$ (where $\mathbf{U}_S = \;\mathbf{U}_S(\mathbf{X}_\alpha, t)$ is the microstructural tensor field describing the changes in the porestructure) for the set $\mathcal{R}$ of response functions: $\mathcal{R}(s, \dot{\mathbf{U}}_S) := \{\psi^S, \eta^S, \mathbf{T}^S, \mathbf{Z}^S, \mathbf{\Sigma}^S\}$ (where $\mathbf{Z}^S$ is the conjugate variable associated with $\mathbf{U}_S$, and $\mathbf{\Sigma}^S$ the microstress third-order tensor associated with $\mathrm{Grad}_S \; \mathbf{U}_S$). From his evaluation of the entropy inequality, Giovine (1999) obtained certain restrictions for the response functions stated above. Then, he presented his constitutive relations in the linear case: "The constitutive equation for the linear theory of a porous elastic material with empty large voids which do not diffuse through the matrix are the following:

$$\mathbf{T}^S \;=\; (\lambda^S \mathrm{tr}\, \mathbf{E}_S \;+\; \omega_5^S \mathrm{tr} \mathbf{V}_S)\mathbf{I} \;+\; 2\mu^S \mathbf{E}_S \;+\; \omega_6^S \mathbf{V}_S \text{ .''} \tag{6.72}$$

Corresponding constitutive equations (extended versions with numerous response parameters) were added by Giovine (1999) for the response functions $\mathbf{\Sigma}^S$and $\mathbf{Z}^S$. In (6.72), $\mathbf{E}_S$ is the linearized Green strain tensor and $\mathbf{V}_S := \mathbf{U}_S - \mathbf{I}$ a symmetric tensor, describing microscopic deformation characteristics. The response parameters are denoted by λ^S, μ^S (Lamé constants), and ω_5^S as well as ω_6^S.

From the balance equations of mass, linear momentum, and micromomentum, Giovine (1999) derived the equations of motion.

He concluded his paper with two sections on quasi-static homogeneous deformations and small-amplitude acoustic waves using the constitutive equations developed by him.

The whole paper remains in many parts obscure. Giovine (1999) does not explain the statement in the beginning of his paper cited above, namely that he considered "rate effect in the microstructural response due to inelastic surface effects associated with pore compaction and distension." How can a porous elastic solid possesses inelastic properties? Also, the introduction of the balance equation for microscopic momentum seems to be artificial and the solution of this balance equation for well-defined initial and boundary value problems will become difficult. Moreover, the introduction of many response parameters are not well physically founded.

New contributions to the different elastic models, namely incompressible, compressible, and hybrid models, are contained in the book of de Boer (2000 a). For a binary model, consisting of an elastic incompressible or compressible skeleton and an incompressible or compressible fluid, the following results are obtained in the case of a purely hydrostatic stress state $p^{\alpha R}$ in the real material.

The main results will be summarized in the following paragraphs: For the incompressible and compressible solid materials

$$p^{SR} = -\,\kappa^{SR}\ ,\ \ p^{SR} = -\,(\rho^{SR})^2 \frac{\partial \psi^S}{\partial \rho^{SR}}\ , \tag{6.73}$$

and for the incompressible and compressible fluid materials

$$\begin{aligned} p^{LR} &= -\,\kappa^{LR} = -\,p\ , \\ p^{LR} &= -\,(\rho^{LR})^2 \frac{\partial \psi^L}{\partial \rho^{LR}} \end{aligned} \tag{6.74}$$

are obtained where κ^{SR} as well as κ^{LR} and p, respectively, are unknown Lagrange multipliers and where

$$p^{\alpha R} = \frac{1}{n^\alpha} p^\alpha\ . \tag{6.75}$$

Moreover, the hydrostatic pressure p^{SR} can also be expressed by the fluid pressure p^{LR} and the configuration pressure p^{con}.

$$p^{SR} = p^{LR} - p^{\mathrm{con}}\ , \quad p^{\mathrm{con}} = \rho^S \frac{\partial \psi^S}{\partial n^S}\ . \tag{6.76}$$

With the constitutive Eqs. (6.73) through (6.76), the mechanical behavior of the aforementioned different models for hydrostatic stress states can be described.

The corresponding constitutive equations for empty porous elastic solids can be gained from the above stated constitutive relation by setting p^{LR} equal to zero.

A few remarks will be made concerning the configuration pressure p^{con}. For a binary model, consisting of solid and fluid phases, the configuration pressure is caused by the “intergranular” forces (see Baer and Nunziato, 1986). In a binary model, consisting of a liquid and a gas (bubbles), the corresponding “configuration pressure” can be interpreted as the surface tension of the bubble. In a similar way, in the binary model with gas (air) and droplets (water), the “configuration pressure” of the water can be seen as the surface tension of the droplet and the surface tension of the bubbles.

6.6 Elastic-Plastic Behavior of the Solid Skeleton

The consideration of plastic material behavior in the theory of porous media happened relatively late, namely when already some elasticity theories had been developed. The first investigations in the plastic region of porous solids go back to de Boer and Kowalski (1983). However, these were based on simplified kinematics and elements of Biot's theory. Further investigations with this model by de Boer and Kowalski (1985, 1986) as well as de Boer and Ehlers (1986 a) are referred to metallic porous solids. In 1986, de Boer and Ehlers (1986 b) abandoned the described model and made consequent use of the mixture theory restricted by the concept of volume fractions. Ehlers (1989) developed with this model and de Boer's (1988) failure condition for brittle and metallic porous solids a simplified plasticity theory. Finally, de Boer and Brauns (1990) completed the first developments in the creation of a consistent plasticity for porous solids.

Today, based on several articles by the Essen school of mechanics a complete elasto-plasticity is available (see Bluhm *et al.*, 1996).

a) General Theory

The development of an elastic-plasticity theory for porous materials with saturated pores has made great progress in recent time as indicated in the preceding paragraph. From thermodynamic considerations and the saturation constraint, it follows (as in the elasticity theory) that the constitutive equations must be formulated with effective stresses.

For convenience we follow the notations in de Boer (1988, 2000 a, b,) as well as de Boer and Brauns (1990) and Jägering (1998) and replace Cauchy's stress tensor with the symmetric Piola-Kirchhoff stress tensor in the sections on elasto-plasticity. This is only approximately valid within the frame-work of the geometrically linear theory, see Eq. (6.66). Otherwise one has to consider certain transformation rules.

It is quite natural that the first elastic-plastic theories were derived within the geometrically linear framework. In this framework, the development of constitutive equations for elastic-plastic behavior is based on the additive decomposition of the linearized Green strain $\mathbf{E}_S$:

$$\mathbf{E}_S = \mathbf{E}_{Se} + \mathbf{E}_{Sp} \ , \tag{6.77}$$

in which, $\mathbf{E}_{Se}$ and $\mathbf{E}_{Sp}$ are the elastic and plastic parts of $\mathbf{E}_S$. In plasticity theories the plastic part $\mathbf{E}_{Sp}$ is determined by constitutive relations. Then, the elastic part $\mathbf{E}_{Se}$ can be obtained from (6.77). From thermodynamic investigations (see e.g. Ehlers 1989 a,b; de Boer and Lade, 1991) it follows that the stress tensor $\mathbf{S}^S$ can be additively split into

$$\mathbf{S}^S = \ - n^S p \mathbf{I} + \mathbf{S}^S_E \ , \tag{6.78}$$

where p is the hydrostatic pressure of the incompressible and inviscid fluid, and $\mathbf{S}^S_E$ the effective stress which can be derived from the following constitutive equation:

$$\mathbf{S}^S_E = \rho^S_{0S} \frac{\partial \psi^S}{\partial \mathbf{E}_{Se}} \,, \tag{6.79}$$

whereby ρ^S_{0S} is the partial density of the solid phase in the reference placement of ρ^S. Moreover, the entropy inequality yields an important dissipation inequality which restricts the constitutive relation for the rate of the plastic strain tensor:

$$(\mathbf{E}_{Sp})'_S \cdot (\mathbf{S}^S_E - \mathbf{Z}^S) \geq 0 \,. \tag{6.80}$$

In (6.80) the second-order tensor $\mathbf{Z}^S$ denotes the so-called backstress tensor, which strongly depends on the deformation history.

Whereas the description of the elastic behavior of porous media, within the framework of the constitutive theory, is characterized by the consideration of the deformation gradient (or due to invariant requirements, by other kinematic quantities), the description of plastic response is governed by the total deformation process. Thus, the total deformation process has to be known, since the response of porous media to different loading processes is different.

It is well-known that the classical plasticity theory is based, first, upon a yield condition that indicates the onset of plastic deformations and describes the hardening range and the failure state; second, upon the consistency condition; third, upon loading criteria that are gained from the consistency condition and the assumption of stable material behavior; and fourth, upon the flow rule that connects strain increments with stresses and stress increments. The constitutive relations must reflect test observations and thermodynamic restrictions. The main feature of the test results and the thermodynamic restrictions is the indication of kinematic hardening, which has to be described by a tensor containing all the information on the deformation history of the skeleton. However, test results also reveal that isotropic hardening is involved. Thus, the main aim of the following investigations is to elaborate on this behavior.

The yield condition marks the onset of plastic deformations when a determined stress state is achieved. This condition is material-dependent of course. Therefore, a constitutive relation is necessary and is understood to separate the elastic and plastic ranges at a stage where no plastic deformations have occurred (initial yield condition), as well as in the hardening and failure states. As pointed out, thermodynamic investigations reveal that for the saturated porous solid, the constitutive equations have to be formulated with the effective stresses, and therefore also the yield condition.

The plastic behavior of porous bodies depends strongly on the structure of the solid material. In contrast to metals, granular and brittle materials show a completely different behavior in extension and compression tests. Due to the complex behavior of granular and brittle materials (empty porous solids) in the plastic range, several different yield conditions have been developed (see, e.g., the review in de Boer, 2000 a).

Yield condition

The yield function has to be constructed in such a way that it reflects all relevant test observations and invariant requirements. In the hardening range in particular, it has to depend on the deformation history, which will be denoted by the symbol $\mathcal{H}^S$ (see Fromm, 1933). At the onset of plastic deformations and in the failure state the yield function does not depend on the deformation history, and, therefore, $\mathcal{H}^S$ becomes zero and a constant value. The yield function, generally assumed to depend on the stress tensor $\mathbf{S}_E^S$, is a scalar function, and is described by

$$Y = Y(\mathbf{S}_E^S, \mathcal{H}^S) \ . \tag{6.81}$$

If the deformation history $\mathcal{H}^S$ is known, Y depends only on $\mathbf{S}_E^S$:

$$Y = Y_{(\mathcal{H}^S)}(\mathbf{S}_E^S) \ . \tag{6.82}$$

$\mathcal{H}^S$ is equal to zero, especially for the state at the onset of plastic deformation which is free from the deformation history, and

$$Y = Y_{(0)}(\mathbf{S}_E^S) = F(\mathbf{S}_E^S) \ . \tag{6.83}$$

The failure state also does not depend on the deformation history. In this case, $\mathcal{H}^S$ is to be considered as a constant C:

$$Y = Y_{(C)}(\mathbf{S}_E^S) = F(\mathbf{S}_E^S) \ . \tag{6.84}$$

At the onset of plastic deformations, the yield function $F(\mathbf{S}_E^S)$ (6.83) depends on the invariants of $\mathbf{S}_E^S$, if the material under discussion is isotropic. This is not always valid for the other forms of (6.81). However, it is possible to express $Y = Y(\mathbf{S}_E^S, \mathcal{H}^S)$ with a scalar valued function H, which is invariant against rotations, that is

$$Y = Y(\mathbf{S}_E^S, \mathcal{H}^S) = H(\mathbf{S}_E^S - \mathbf{Z}^S) \ , \tag{6.85}$$

where $\mathbf{Z}^S$ is the so-called backstress or translation tensor depending on the stress $\mathbf{S}_E^S$ and the deformation history $\mathcal{H}^S$. It is possible to replace H with F, and in this case $\mathbf{S}_E^S$ is returned through $\mathbf{Z}^S$ to its initial state. Thus, in the following, yield functions of the kind

$$Y = Y(\mathbf{S}_E^S, \mathbf{Z}^S) = F(\mathbf{S}_E^S - \mathbf{Z}^S) \tag{6.86}$$

are considered, especially those that are represented by the invariants of the stress difference

$$\bar{\mathbf{S}} = \mathbf{S}_E^S - \mathbf{Z}^S \ . \tag{6.87}$$

Then

$$Y = F(\mathrm{I}_{\bar{\mathbf{S}}}, \mathrm{II}_{\bar{\mathbf{S}}}, \mathrm{III}_{\bar{\mathbf{S}}}) \ , \tag{6.88}$$

or

$$Y = F(\mathrm{I}_{\bar{\mathbf{S}}}, \mathrm{II}_{\bar{\mathbf{S}}^D}, \mathrm{III}_{\bar{\mathbf{S}}^D}) \ , \tag{6.89}$$

is assumed.

The above yield functions are limited if the plastic state is attained, i.e.,

$$Y = F(\mathrm{I}_{\bar{\mathbf{S}}}, \mathrm{II}_{\bar{\mathbf{S}}}, \mathrm{III}_{\bar{\mathbf{S}}}) = 0 \tag{6.90}$$

or

$$Y = F(\mathrm{I}_{\bar{\mathbf{S}}}, \mathrm{II}_{\bar{\mathbf{S}}^D}, \mathrm{III}_{\bar{\mathbf{S}}^D}) = 0 \ . \tag{6.91}$$

For the discussion of proposed yield functions the geometric representation is very useful (see Fig. 6.6.1). The representation of these functions in the hydrostatic and octahedral planes gives useful information for the parameters involved, see de Boer (2000 a). Moreover, the geometric representation sometimes gives many hints for improving the yield function in so far as the yield function can be better adjusted to test results.

It can be seen that, in a geometric representation, yield functions that are located inside the Mohr-Coulomb lines and are adjusted to the plane deviatoric stresses can describe the test results. In this case, the convexity of the yield function is ensured. In geometric representation, the backstress tensor causes a translation of the center of the yield condition, with all the other properties remaining unchanged. Thus, all known yield conditions for granular and brittle materials can be used for kinematic-hardening, if the stress tensor $\mathbf{S}_E^S$ is replaced by the difference tensor $\bar{\mathbf{S}}$, according to (6.87). In this sense, the yield function

$$\begin{aligned} F &= \sqrt{\Phi}(1+\gamma\vartheta)^{1/m} + \beta \mathrm{I}_{\mathbf{S}_E^S} + \varepsilon \mathrm{I}^2_{\mathbf{S}_E^S} - \kappa = 0 \ , \\ \Phi &= \mathrm{II}_{\mathbf{S}_E^{SD}} + \frac{1}{2}\alpha^2 \mathrm{I}^2_{\mathbf{S}_E^S} \ , \quad \vartheta = \frac{\mathrm{III}_{\mathbf{S}_E^{SD}}}{\sqrt{\Phi^m}} \sqrt{\mathrm{II}^{m\text{-}3}_{\mathbf{S}_E^{SD}}} \ , \end{aligned} \tag{6.92}$$

developed by de Boer (1988), extended by de Boer and Dresenkamp (1989), as well as by Bluhm (1994), can also be formulated in the above defined stress subspace $\bar{\mathbf{S}} = \mathbf{S}_E^S - \mathbf{Z}^S$:

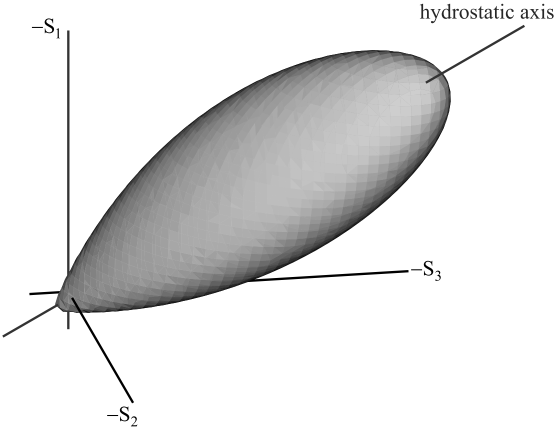

Fig. 6.6.1: Yield function with $\alpha = 0.0775, \beta = 0.274,$
$\varepsilon = 0.0003\ \frac{mm^2}{N},\ \kappa = 10.27\ \frac{N}{mm^2},\ \gamma = 1.299,$
$m = 3$ ($\mathrm{S}_1, \mathrm{S}_2$ and S_3 are principal stresses)

$$\begin{aligned} F &= \sqrt{\bar{\Phi}}(1+\gamma\bar{\vartheta})^{1/m} + \beta \mathrm{I}_{\bar{\mathbf{S}}} + \varepsilon \mathrm{I}^2_{\bar{\mathbf{S}}} - \kappa = 0 \ , \\ \bar{\Phi} &= \mathrm{II}_{\bar{\mathbf{S}}^D} + \frac{1}{2}\alpha^2 \mathrm{I}^2_{\bar{\mathbf{S}}} \ , \quad \bar{\vartheta} = \frac{\mathrm{III}_{\bar{\mathbf{S}}^D}}{\sqrt{\bar{\Phi}^m}} \sqrt{\mathrm{II}^{m\text{-}3}_{\bar{\mathbf{S}}^D}} \ . \end{aligned} \tag{6.93}$$

The six quantities $\alpha, \beta, \gamma, \varepsilon, \kappa$, and m, in (6.92) and (6.93), are response parameters which have to be adjusted to test observations. In the special case of $m = 3$, the yield condition results in the approach of de Boer and Dresenkamp (1989).

It is well-known, in the classical plasticity theory for metallic materials, that the second invariant of the stress state $\mathrm{II}_{\mathbf{S}^D}$ or $\mathrm{II}_{\bar{\mathbf{S}}^D}$ is a circle in the graphic representation in the deviatoric plane. The term $(1+\gamma\vartheta)^m$, or $(1+\gamma\bar{\vartheta})^m$, governs the typical shape of the yield surface of frictional materials in the deviatoric plane. In particular, the right choice of the parameter m allows the correct approach to the Mohr-Coulomb failure condition. Moreover, the incorporation of $\frac{1}{2}\alpha^2 \mathrm{I}^2_{\mathbf{S}}$ or $\frac{1}{2}\alpha^2 \mathrm{I}^2_{\bar{\mathbf{S}}}$ into Φ or $\bar{\Phi}$ guarantees the influence of the first invariant of the stress tensor on the deviatoric shape of the yield surface. This is an essential feature in the plasticity theory of frictional materials. It is easily recognized that a hydrostatic pressure has to influence the deviatoric shape of the yield surface. For example, loose sand under low hydrostatic pressure will exhibit another shape in the deviatoric plane than sand under high hydrostatic pressure. Furthermore, the parameters β and ε, as well as the parameter α^2, govern the form of the yield surface in the hydrostatic plane.

The response parameters may depend on the plastic deformation history. If this is the case, then the yield condition (6.93) represents a combination of kinematic and isotropic hardening. The above-stated yield condition is therefore very general. It is suitable for describing the onset of plastic deformations as well as the plastic hardening range, or the failure state of kinematically and isotropically hardening brittle and granular skeletons, including the effects of compression and extension.

The great advantage of the yield condition (6.92) is that this condition represents a “single-surface” condition. Thus, no cap-model is needed to limit the yield curve in the hydrostatic plane. Further discussions of the yield condition introduced here are contained in the papers of de Boer (1988), de Boer and Dresenkamp (1989), de Boer and Lade (1991), as well as Bluhm (1994) and Bluhm *et al.* (1996). In these papers, the convexity of the yield condition (6.92), in particular, is investigated.

Finally, it should be mentioned that the general yield condition (6.92) contains some special cases of yield conditions. If the response parameters γ and ε are equal to zero, we obtain a yield condition proposed earlier by de Boer (1986). With α, γ, and ε being equal to zero, the condition (6.92) converts to the failure condition of Drucker and Prager (1952). Moreover, if γ, β, and ε are neglected, the general yield condition turns into the yield condition of Green (1972). Finally, with α, γ, β, and ε equal to zero, the famous yield condition of von Mises (1913) is realized.

Let us now turn again to the yield function (6.93). In order to include *isotropic hardening*, we assume that the response parameters $\alpha, \beta, \gamma, \varepsilon, \kappa$, and m are functions of the plastic work W. We choose

$$\begin{aligned} &\alpha = \alpha(W)\ , \quad \beta = \beta(W)\ , \quad \gamma = \gamma(W)\ , \\ &\varepsilon = \varepsilon(W)\ , \quad \kappa = \kappa(W)\ , \quad \text{and} \quad m = m(W)\ , \end{aligned} \tag{6.94}$$

where the rate of the plastic work is given by

$$DW = \mathbf{S}_E^S \cdot D\mathbf{E}_{Sp}\ . \tag{6.95}$$

In classical plasticity theory, the constitutive equations are, in part, formulated in the time rates. However, the constitutive relations do not explicitly depend on time. Therefore, in (6.94) and in the sequel, we use only the rates, for example, DW and $D\mathbf{E}_{Sp}$.

Kinematic hardening is described by the backstress tensor $\mathbf{Z}^S$. According to Fromm (1933), we propose a general ansatz:

$$\mathbf{Z}^S = C_1(\mathbf{S}_E^S, \mathbf{H}_S^D)\mathbf{H}_S^D + \ C_2(\mathbf{S}_E^S, \mathbf{H}_S)\mathrm{I}_{H_S}\mathbf{I}\ . \tag{6.96}$$

The tensor of deformation history $\mathbf{H}_S$ is strongly influenced by the plastic deformations due to experience. The following relation for the rate of $\mathbf{H}_S$ may be appropriate:

$$D\mathbf{H}_S = f(\varphi, D\mathbf{E}_{Sp})D\mathbf{E}_{Sp} \ , \tag{6.97}$$

where $f(\varphi, D\mathbf{E}_{Sp})$ has to be a positive scalar value function, depending on the plastic strain rates and on the angle of internal friction φ. Relation (6.97) is similar to Fromm's (1933) general approach. However, Fromm's relation depends only on the deviatoric strain rates, while the function f depends on the stresses and the strain rates. It has already been proven by de Boer and Brauns (1990) that the angle of internal friction of the Mohr-Coulomb theory can serve as a parameter for describing the hardening process. Although the Mohr-Coulomb theory deals only with the failure state, the above-mentioned procedure is promising.

In the general formulas (6.96) and (6.97), the approaches of Melan (1938), Eisenberg and Phillips (1968), and de Boer and Brauns (1990) – representing an extended version of the Eisenberg and Phillips formula – are contained as special cases. It has been shown by de Boer and Brauns (1990) in a long paper, that the extended version efficiently describes (for the backstress tensor) the test results gained by Lade (1979), which are the basis for many theoretical investigations.

Bluhm (1994) transferred the ansatz for the backstress tensor $\mathbf{Z}^S$ (6.96) to the geometrically non-linear case, where the backstress tensor is represented as a function of the plastic part of the Karni-Reiner strain tensor. In an example, he showed the path of the backstress tensor for a fictitious loading and unloading process. The results fit very well to the test observations of many experiments described in the literature. Thus, the proposed yield condition seems to be suited to serve as a basis for a plasticity theory of frictional materials.

Consistency condition

As has already been mentioned, in order to describe the mechanical behavior of a plastically deformed solid, the consistency condition has to be formulated. If the consistency condition is fulfilled, the plastic flow continues.

The condition of consistency states that, if the yield condition is fulfilled, it must also be satisfied after a step. Thus, the condition of consistency guarantees plastic deformations:

$$\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot D\bar{\mathbf{S}} + \frac{\partial F}{\partial W} DW = 0 \ , \tag{6.98}$$

or, using (6.86),

$$\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathrm{D}\mathbf{S}_E^S = \frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathrm{D}\mathbf{Z}^S - \frac{\partial F}{\partial W} \cdot \mathrm{D}W \ . \tag{6.99}$$

If the stress state satisfies (6.93), plastic deformations can occur. If, however,

$$F < 0 \,, \tag{6.100}$$

there will be no plastic deformations.

We use (6.95), (6.96), and (6.97) to explicitly determine the consistency condition (6.99). After elementary – albeit lengthy and laborious – calculations, we arrive at

$$\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot D\mathbf{S}_E^E = (\mathbf{M} - \mathbf{N}) \cdot D\mathbf{E}_{Sp} \,, \tag{6.101}$$

where

$$\begin{aligned} \mathbf{N} &= \frac{\partial F}{\partial W} \mathbf{S}_E^S \,, \\ \mathbf{M} &= f(\varphi, D\mathbf{E}_{Sp}) \Big(\frac{\partial \mathbf{Z}^S}{\partial \mathbf{H}_S} \Big)^T \frac{\partial F}{\partial \bar{\mathbf{S}}} \,, \end{aligned} \tag{6.102}$$

$$\begin{aligned} \frac{\partial F}{\partial \bar{\mathbf{S}}} \quad = &\quad \frac{1}{2} \, (\bar{\Phi})^{-1/2} (1 + \gamma\bar{\vartheta})^{1/m} \, (\bar{\mathbf{S}}^D + \alpha^2 \mathrm{I}_{\bar{\mathbf{S}}} \mathbf{I}) \, + \\ + &\quad \frac{1}{m} (1 + \gamma\bar{\vartheta})^{\frac{1-m}{m}} \gamma \, \{ [(\bar{\Phi})^{-\frac{m-1}{2}} (\bar{\mathbf{S}}^D \bar{\mathbf{S}}^D - \frac{2}{3} \, \mathrm{II}_{\bar{\mathbf{S}}^D} \mathbf{I}) \, - \\ - &\quad \frac{m}{2} \bar{\Phi}^{-\frac{m+1}{2}} (\bar{\mathbf{S}}^D + \alpha^2 \mathrm{I}_{\bar{\mathbf{S}}}) \, \mathrm{III}_{\bar{\mathbf{S}}^D}] (\mathrm{II}_{\bar{\mathbf{S}}^D})^{\frac{m-3}{2}} \, + \\ + &\quad \frac{m-3}{2} \bar{\Phi}^{-\frac{m-1}{2}} \, \mathrm{III}_{\bar{\mathbf{S}}^D} (\mathrm{II}_{\bar{\mathbf{S}}^D})^{\frac{m-5}{2}} \bar{\mathbf{S}}^D \} + \; (2\varepsilon \mathrm{I}_{\bar{\mathbf{S}}} + \beta) \mathbf{I} \,, \end{aligned} \tag{6.103}$$

and

$$\begin{aligned} \frac{\partial F}{\partial W} \quad = &\quad \frac{1}{2} (\bar{\Phi})^{-1/2} \, \mathrm{I}_{\bar{\mathbf{S}}}^2 \, \alpha (1 + \gamma\bar{\vartheta})^{1/m} \alpha,_W \, + \\ + &\quad \frac{1}{m} (1 + \gamma\bar{\vartheta})^{\frac{1-m}{m}} \, [\bar{\vartheta}\gamma,_W \, + \\ + &\quad \bar{\Phi}^{-\frac{m+2}{2}} \sqrt{\mathrm{II}_{\bar{\mathbf{S}}^{SD}}^{m\text{-}3}} \;\; \mathrm{III}_{\bar{\mathbf{S}}^D} \, \mathrm{I}_{\bar{\mathbf{S}}}^2 \, m\alpha \; \alpha,_W \,] + \\ + &\quad \mathrm{I}_{\bar{\mathbf{S}}} \, \beta,_W + \mathrm{I}_{\bar{\mathbf{S}}}^2 \, \varepsilon,_W + \kappa,_W \end{aligned} \tag{6.104}$$

with $(\ldots),_W$ marking the partial derivative with respect to W. The tensor $\mathbf{N}$ results from the assumption of isotropic hardening, see (6.94) and (6.95), and the tensor $\mathbf{M}$ is calculated from the ansätze for kinematic hardening, see (6.96) and (6.97).

A modified yield function was introduced by de Boer, Ehlers and Bluhm (1991) for finite deformations (see also de Boer and Lade, 1991, and Ehlers, 1995). The modification consists of an additional term $\delta^2 \mathrm{I}^4_{\mathbf{S}_E}$ and the simplification of the expression ϑ, namely $\vartheta = \dfrac{\mathrm{III}_D}{\mathrm{II}_D^{\;3/2}}$, where the invariants III_D and II_D are formulated with the deviator of the effective Kirchhoff stress tensor. This modification has some disadvantages. The modified version of (6.92) contains seven response parameters instead of six. Moreover, in ϑ the first invariant of the stress tensor is not contained. As previously mentioned the influence of the hydrostatic pressure on the deviatoric shape of the yield surface for granular media cannot be described. Thus, we will not further discuss the modification of (6.92).

Bluhm (1994) transferred (6.92) to the case of finite deformations and formulated the invariants of the stress state with the Kirchhoff stress tensor. He showed that the number of response parameters in (6.92) can be reduced to five assuming that m = 4,5. In this case, the shape of the yield curve in the deviatoric plane for low hydrostatic stress states becomes more triangular, and one obtains a better adaption to test observations, see also Bücker and Bluhm (1994).

Ehlers and Volk (1997 a,b, 1999) extended the modified version of the yield condition (6.92) to micropolar cohesive frictional materials (see Ehlers and Volk, 1999):

$$
\begin{aligned}
F^c &= \phi^{1/2} + \beta \mathrm{I}_{\mathbf{T}^S_E} + \varepsilon \mathrm{I}^2_{\mathbf{T}^S_E} - \kappa = 0\,, \\
\phi &= \mathrm{II}_{\mathbf{T}^{SD}_{E\,sym}} (1+\gamma\vartheta)^m + \frac{1}{2}\alpha \mathrm{I}^2_{\mathbf{T}^S_E} + \delta^2 \mathrm{I}^4_{\mathbf{T}^{SD}_{E\,sym}} + \\
&\quad + k_1 \mathbf{M}^S \cdot \mathbf{M}^S + k_2 \mathrm{II}_{\mathbf{T}^S_{E\,skw}}\,, \\
\vartheta &= \frac{\mathrm{III}_{\mathbf{T}^{SD}_{E\,sym}}}{(\mathrm{II}_{\mathbf{T}^{SD}_{E\,sym}})^{3/2}}\,.
\end{aligned}
\tag{6.105}
$$

"In this representation, $\{\alpha, \beta, \gamma, \delta, \varepsilon, m, \kappa, k_1, k_2\}$ is a set of nine material parameters, $\mathrm{I}_{\mathbf{T}^S_E}$, $\mathrm{II}_{\mathbf{T}^{SD}_{E\,sym}}$ and $\mathrm{III}_{\mathbf{T}^{SD}_E}$ are the first and deviatoric (negative) second and third principal invariants of the symmetric part of the effective stress $\mathbf{T}^S_E$, whereas $\mathrm{II}_{\mathbf{T}^S_{E\,skw}}$ defines the second principal invariant of the skew-symmetric part of $\mathbf{T}^S_E$ (We use the original notation of Ehlers and Volk, 1999; a confusion with the notations chosen by the author seems to be improbable):

$$
\begin{aligned}
\mathrm{I}_{\mathbf{T}^S_E} &= \mathbf{T}^S_E \cdot \mathbf{I}\,, \quad \mathrm{II}_{\mathbf{T}^{SD}_{E\,sym}} = \frac{1}{2}\mathbf{T}^{SD}_{E\,sym} \cdot \mathbf{T}^{SD}_{E\,sym}\,, \\
\mathrm{II}_{\mathbf{T}^S_{E\,skw}} &= \frac{1}{2}\mathbf{T}^S_{E\,skw} \cdot \mathbf{T}^S_{E\,skw}\,, \\
\mathrm{III}_{\mathbf{T}^{SD}_{E\,sym}} &= \frac{1}{3}\mathbf{T}^{SD}_{E\,sym} \cdot \mathbf{T}^{SD}_{E\,sym}\,\mathbf{T}^{SD}_{E\,sym}\,.\text{"}
\end{aligned}
\tag{6.106}
$$

It should be mentioned that the two terms in (6.105) connected with the material parameters k_1 and k_2 (describing micropolar effects) are also contained in Ehlers and Volk (1997 a,b). However, in this paper they are not related to ϕ but to κ. The reason for this change is discussed by Ehlers and Volk (1998), who justified it by saying that "the influence of micropolarity has been included into the function ϕ in order to additionally govern the deviatoric shape of F." Again, the shape of the yield surface is not dependent on the hydrostatic pressure.

Loading criteria

Now, the *loading criteria* will be formulated:

$$F = 0 \text{ and}$$

$$\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot D\mathbf{S}_E^S \begin{cases} > 0 \text{ loading} & D\mathbf{E}_{Sp} \neq \mathbf{O} \, , \\ = 0 \text{ neutral state} & D\mathbf{E}_{Sp} = \mathbf{O} \, , \\ < 0 \text{ unloading} & D\mathbf{E}_{Sp} = \mathbf{O} \, . \end{cases} \tag{6.107}$$

In the case of *ideal-plastic behavior* (critical state), the loading criteria take the forms:

$$F = 0 \text{ and}$$

$$\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot D\mathbf{S}_E^S \begin{cases} = 0 \text{ neutral state} & D\mathbf{E}_{Sp} \neq \mathbf{O} \, , \\ < 0 \text{ unloading} & D\mathbf{E}_{Sp} = \mathbf{O} \, . \end{cases} \tag{6.108}$$

As has already been mentioned, the yield condition (6.93) is very general, with six response parameters, including the description of kinematic and isotropic hardening. It depends on the problem as to which response parameters have to be considered in order to correctly describe the plastic behavior. Moreover, for kinematic hardening, some simplifications can be introduced if the characteristic stress-strain curve is nearly linear, and, for isotropic hardening, some response parameters may be constant, or may depend on the problem under study.

The set of constitutive relations will be completed with the development of the flow rule.

Flow rule

In order to describe the complete motion of an initial and boundary value problem, a constitutive equation for the rates $D\mathbf{E}_{Sp}$ of the plastic strains of the partial solid constituent (flow rule) is needed. This constitutive equation must also reflect the deformation history according to test observations. This will be done in such a way that the flow rule will be formulated in the special stress space created with the stress difference $\bar{\mathbf{S}}$, see (6.87). In continuum mechanics of isotropic materials in general it is stated that the change of the strain deviator and the volume change are completely independent of each other. Here, the same statement is introduced and a relatively simple flow rule is formulated:

$$D\mathbf{E}_{Sp} = D\lambda[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] \ , \tag{6.109}$$

where $D\lambda$ is a spatial-dependent scalar and

$$\mu = \mu(\mathrm{I}_{\bar{\mathbf{S}}}, \mathrm{II}_{\bar{\mathbf{S}}^D}) \tag{6.110}$$

is a function governing the volume change. There is no reason for the flow rule to have another form. Neither the test results nor the theoretical investigations, indicate this as necessary. The formulation of the flow rule with a plastic potential widely used for geomechanical materials can only be understood from the practical point of view. On the one hand, it is easy to construct such a potential because, in the geometric representation the potential lines are always perpendicular to the strain rate directions observed in tests results. On the other hand, a lot of software based on the plastic potential concept already exists.

The quantities $D\lambda$ and μ must always be positive due to the restriction (6.80) in connection with the relation (6.109). The scalar value function $D\lambda$ in (6.109) can be determined from the condition of consistency (6.101), in connection with (6.102), (6.103), and (6.104). For the function μ, a backstress-dependent equation is required, since μ has to reflect the deformation history. The following relation seems to be reasonable:

$$\mu = \delta(\varphi)\mathrm{I}_{\bar{\mathbf{S}}} + \xi(\varphi)\sqrt{\mathrm{II}_{\bar{\mathbf{S}}^D}} \ , \tag{6.111}$$

where the material-dependent parameters $\delta(\varphi)$ and $\xi(\varphi)$ are assumed to depend on the angle of internal friction φ. As has been proven by de Boer and Brauns (1990), it seems reasonable to use the angle of internal friction which, although introduced for the failure state as a constant, also exists in the hardening range as a parameter. The following approaches for determining δ and ξ yield excellent results (see de Boer and Brauns, 1990):

$$\delta = -\ m_1[\sin(\varphi_B - \varphi)]^k \ , \tag{6.112}$$

$$\xi = -\ m_2[\sin(\varphi_B - \varphi)]^k \qquad (k = 1, 2, 3, \ldots) \ . \tag{6.113}$$

Here, the factors m_1 and m_2 denote parameters which are a result of the adjustment of the flow rule to special test results, and φ_B represents the angle of internal friction in the failure state; the exponent k influences the shape of the curve of the volume change in the hardening range. For the special case of exclusive contracted volume changes due to hydrostatic pressure, the parameter ξ has to be dropped. In general however, the sum of the terms of the right-hand side of (6.111) causes a sign change of the function μ with increasing hardening due to $\mathrm{I}_{\bar{\mathbf{S}}} < 0$, so that through the appropriate choice of the parameters m_1, m_2, and k, the size of the region of the contracted volume changes can be adjusted to the test results (see the extensive discussions in de Boer and Brauns, 1990, as well as in de Boer and Lade, 1991).

The flow rule (6.109) was transferred to finite deformations by Ehlers (1991) and Bluhm (1994). Moreover, Ehlers and Volk (1999) extended the considerations concerning the formulation of the flow rule to frictional micropolar skeleton materials (see also Ehlers and Volk, 1997 a,b, 1998). They proceeded from the assumption that the Cosserat strain ε_{SC} (3.56)$_1$ can be additively split into elastic and plastic parts. In this case, it follows from (3.60) that $\bar{\kappa}_S$ can also be additively decomposed:

$$\begin{aligned} \varepsilon_{SC} &= \varepsilon_{SCe} + \varepsilon_{SCp} \, , \\ \bar{\kappa}_S &= \bar{\kappa}_{Se} + \bar{\kappa}_{Sp} \, . \end{aligned} \tag{6.114}$$

For the plastic part of the Cosserat strain, Ehlers and Volk (1999) introduced an evolution equation in the form

$$(\varepsilon_{SCp})'_S = \Lambda \frac{\partial G^c}{\partial \mathbf{T}_E^S} \, , \tag{6.115}$$

where Λ is the usual plastic multiplier and G^c an additional plastic potential function:

$$\begin{aligned} G^c &= \Gamma^{1/2} + \beta \mathrm{I}_{\mathbf{T}_E^S} + \varepsilon \mathrm{I}^2_{\mathbf{T}^{SD}_{E\,sym}} - g(\mathrm{I}_{\mathbf{T}_E^S}) = 0 \, , \\ \Gamma &= \mathrm{II}_{\mathbf{T}^{SD}_{E\,sym}} + \frac{1}{2}\alpha \mathrm{I}_{\mathbf{T}_E^S} + \delta^2 \mathrm{I}^4_{\mathbf{T}^{SD}_{E\,sym}} + k_2 \mathrm{II}_{\mathbf{T}^S_{E\,skw}} \, . \end{aligned} \tag{6.116}$$

In view of the flow rule (6.115), Ehlers and Volk (1999) stated: “In good accordance with experimental data, this relation predicts coaxial behaviour of the deviatoric part of $(\varepsilon_{SCp})'_S$. Concerning the volumetric part, the function $g(\mathrm{I}_{\mathbf{T}_E^S})$ is chosen in such a way that the direction of $(\varepsilon_{SCp})'_S$, in the brittle range, approximately lies in the middle between the associated and the deviatoric direction. In the ductile range and at the top of the yield surface, $g(\mathrm{I}_{\mathbf{T}_E^S})$ predicts associated behaviour...”

From (3.60) and (6.115) the rate of the plastic part of the curvature tensor $\bar{\kappa}_S$ is gained:

$$\begin{aligned} (\bar{\kappa}_{Sp})'_S &= \frac{1}{2} \overset{3}{\mathbf{E}} \, [\mathrm{Grad}_S (\varepsilon_{SCp})'_S + \\ &\quad + \overset{13}{\mathrm{Grad}^T_S} (\varepsilon_{SCp})'_S - \overset{23}{\mathrm{Grad}^T_S} (\varepsilon_{SCp})'_S]^{\underline{2}} \, . \end{aligned} \tag{6.117}$$

Thus, with (6.69), (6.70), and (6.71), as well as constitutive equations for ε_{SCp} and $\bar{\kappa}_{Sp}$ with (6.115), (6.116), and (6.117), the total Cosserat strain and curvature tensor can be formulated considering (6.114).

The flow rule (6.109) was reformulated and extended by Bluhm (1994) within the framework of kinematic hardening in order to describe finite deformations. He consequently proceeded from the intermediate placement and formulated the flow rule with the symmetric part of the plastic velocity gradient, the Kirchhoff stress tensor, and the corresponding backstress tensor. To what extent the extension of the flow rule (6.109) with the deviator of the tensor product of the stress difference (6.87) is an improvement of (6.109) was not discussed.

b) Special Stress-Strain Relations

With the development of the constitutive relations for the elastic and plastic strain rates $D\mathbf{E}_{Se}$ and $D\mathbf{E}_{Sp}$ of the solid skeleton (see Chapter 6.5), we are able to construct a corresponding relation for $D\mathbf{E}_S$ according to (6.66) for non-polar materials. For the elastic and plastic states, we have the constitutive equations (6.66) (without thermal effects) and (6.109)

$$\begin{aligned} D\mathbf{E}_{Se} &= (\overset{4}{\mathbf{K}}{}^S)^{-1} D\mathbf{S}_E^S\,, \\ D\mathbf{E}_{Sp} &= D\lambda[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}}\cdot\mathbf{I})\mathbf{I}]\,. \end{aligned} \tag{6.118}$$

In (6.118), the fourth-order symmetric tensors $\overset{4}{\mathbf{K}}{}^S$ and $(\overset{4}{\mathbf{K}}{}^S)^{-1}$ are given by:

$$\begin{aligned} \overset{4}{\mathbf{K}}{}^S &= 2\mu^S(\overset{4}{\mathbf{I}} + \frac{\nu^S}{1-2\nu^S}\,\overset{4}{\bar{\mathbf{I}}})\,, \\ 2\mu^S &= \frac{E^S}{1+\nu^S}\,, \\ (\overset{4}{\mathbf{K}}{}^S)^{-1} &= \frac{1}{E^S}[(1+\nu^S)\,\overset{4}{\mathbf{I}} - \nu^S\,\overset{4}{\bar{\mathbf{I}}}]\,, \end{aligned} \tag{6.119}$$

where, in (6.66), the Lamé constants have been replaced by Young's modulus E^S and Poison's ratio ν^S. Moreover, $\overset{4}{\mathbf{I}}$ and $\overset{4}{\bar{\mathbf{I}}}$ denote fourth-order identity tensors (see de Boer, 1982).

The multiplier $D\lambda$ is obtained from the consistency condition. After some calculations, we have

$$D\lambda = \frac{\partial F}{\partial \bar{\mathbf{S}}}\cdot D\mathbf{S}_E^S\{[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}}\cdot\mathbf{I})\mathbf{I}]\cdot(\mathbf{M}-\mathbf{N})\}^{-1}\,. \tag{6.120}$$

Thus, considering (6.118) through (6.120), we arrive with (6.77) at

$$\begin{aligned} D\mathbf{E}_S = &\{(\overset{4}{\mathbf{K}}{}^S)^{-1} + \\ &+ \frac{[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}}\cdot\mathbf{I})\mathbf{I}]\otimes\dfrac{\partial F}{\partial\bar{\mathbf{S}}}}{[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}}\cdot\mathbf{I})\mathbf{I}]\cdot(\mathbf{M}-\mathbf{N})}\}D\mathbf{S}_E^S\,. \end{aligned} \tag{6.121}$$

With the hardening parameter

$$h = [\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] \cdot (\mathbf{M} - \mathbf{N}) \tag{6.122}$$

and with

$$(\overset{4}{\mathbf{P}}{}^S)^{-1} = (\overset{4}{\mathbf{K}}{}^S)^{-1} + \frac{1}{h}[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] \otimes \frac{\partial F}{\partial \bar{\mathbf{S}}} \ , \tag{6.123}$$

we obtain, instead of (6.121),

$$D\mathbf{E}_S = (\overset{4}{\mathbf{P}}{}^S)^{-1} D\mathbf{S}^S_E \ . \tag{6.124}$$

In the next step, we are concerned with the inversion of (6.124). We commence with the constitutive relation for the rates of $\mathbf{E}_S$ (6.121). With the hardening parameter h (6.122), we can write

$$D\mathbf{E}_S = \{(\overset{4}{\mathbf{K}}{}^S)^{-1} + \frac{1}{h}[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] \otimes \frac{\partial F}{\partial \bar{\mathbf{S}}}\} D\mathbf{S}^S_E \tag{6.125}$$

or

$$\begin{aligned} D\mathbf{E}_S &= (\overset{4}{\mathbf{K}}{}^S)^{-1} D\mathbf{S}^S_E + \\ &+ \frac{1}{h}[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}](\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot D\mathbf{S}^S_E) \ . \end{aligned} \tag{6.126}$$

With $(6.118)_2$ and (6.122), we can express the loading condition (6.107) by the following relation:

$$\frac{1}{h}\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot D\mathbf{S}^S_E = D\lambda \ . \tag{6.127}$$

Now, the stress-strain relation (6.126) will be multiplied by $\overset{4}{\mathbf{K}}{}^S \dfrac{\partial F}{\partial \bar{\mathbf{S}}}$ considering (6.127):

$$\begin{aligned} D\mathbf{E}_S \cdot \overset{4}{\mathbf{K}}{}^S \frac{\partial F}{\partial \bar{\mathbf{S}}} &= (\overset{4}{\mathbf{K}}{}^S)^{-1} D\mathbf{S}^S_E \cdot \overset{4}{\mathbf{K}}{}^S \frac{\partial F}{\partial \bar{\mathbf{S}}} + \\ &+ D\lambda[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] \cdot \overset{4}{\mathbf{K}}{}^S \frac{\partial F}{\partial \bar{\mathbf{S}}} \ . \end{aligned} \tag{6.128}$$

The first term on the right-hand side of (6.128) reduces, due to the symmetry of $\overset{4}{\mathbf{K}}{}^S$, to:

$$\begin{aligned} (\overset{4}{\mathbf{K}}{}^S)^{-1} D\mathbf{S}^S_E \cdot \overset{4}{\mathbf{K}}{}^S \frac{\partial F}{\partial \bar{\mathbf{S}}} &= (\overset{4}{\mathbf{K}}{}^S)^T (\overset{4}{\mathbf{K}}{}^S)^{-1} D\mathbf{S}^S_E \cdot \frac{\partial F}{\partial \bar{\mathbf{S}}} \\ &= \frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot D\mathbf{S}^S_E \ . \end{aligned} \tag{6.129}$$

Thus, from (6.128), considering (6.127) and (6.129), we obtain another version of the multiplier $D\lambda$:

$$D\lambda = \frac{D\mathbf{E}_S \cdot \overset{4}{\mathbf{K}}{}^S \, \dfrac{\partial F}{\partial \bar{\mathbf{S}}}}{h + \overset{4}{\mathbf{K}}{}^S \cdot [\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] \otimes \dfrac{\partial F}{\partial \bar{\mathbf{S}}}} \, . \tag{6.130}$$

Moreover, we again return to Eq. (6.126), considering (6.127):

$$D\mathbf{E}_S = (\overset{4}{\mathbf{K}}{}^S)^{-1} D\mathbf{S}^S_E + D\lambda[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] \, , \tag{6.131}$$

and applying the fourth-order elasticity tensor $\overset{4}{\mathbf{K}}{}^S$, we get:

$$\begin{aligned} \overset{4}{\mathbf{K}}{}^S \, D\mathbf{E}_S \quad = \quad & \overset{4}{\mathbf{K}}{}^S \, (\overset{4}{\mathbf{K}}{}^S)^{-1} D\mathbf{S}^S_E \, + \\ & + \overset{4}{\mathbf{K}}{}^S \, [\bar{\mathbf{S}}^D + \, \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] D\lambda \, . \end{aligned} \tag{6.132}$$

Equation (6.132) will be rearranged and, with (6.130), leads to

$$D\mathbf{S}^S_E = \overset{4}{\mathbf{P}}{}^S \, D\mathbf{E}_S \tag{6.133}$$

with

$$\overset{4}{\mathbf{P}}{}^S = \overset{4}{\mathbf{K}}{}^S - \frac{\overset{4}{\mathbf{K}}{}^S \, [\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] \otimes \overset{4}{\mathbf{K}}{}^S \, \dfrac{\partial F}{\partial \bar{\mathbf{S}}}}{h + \overset{4}{\mathbf{K}}{}^S \cdot \{[\bar{\mathbf{S}}^D + \mu(\bar{\mathbf{S}} \cdot \mathbf{I})\mathbf{I}] \otimes \dfrac{\partial F}{\partial \bar{\mathbf{S}}}\}} \, . \tag{6.134}$$

The "elastic-plastic tangent" (6.134) is important for numerical calculations of initial and boundary value problems.

Note in passing that for ideal-plastic behavior, e.g., in the critical state of soils, the hardening parameter h disappears. Since, in this case, the loading criterion is also equal to zero, see (6.108), an undetermined term arises in the constitutive equation for $D\mathbf{E}_S$ (6.126). In the constitutive equations for $D\mathbf{S}^S_E$ (6.133), along with (6.134), however, this indefiniteness disappears.

Finally, it can be stated that the fourth-order tensor $(\overset{4}{\mathbf{P}}{}^S)^{-1}$ in (6.123) is indeed the inverse tensor to $\overset{4}{\mathbf{P}}{}^S$ in (6.134) and that

$$\overset{4}{\mathbf{P}}{}^S \, (\overset{4}{\mathbf{P}}{}^S)^{-1} \, = \overset{4}{\mathbf{I}} \tag{6.135}$$

is valid.

Finite elastic-plastic deformations have been addressed by many authors, e.g. by Bluhm (1994) and Ehlers (1989 a, b, 1991). Basis of the finite elastoplasticity is Lee's multiplicative decomposition of the deformation gradient into an elastic and plastic part. This procedure and the complex elastic and plastic constitutive equations for the strain measurements containing the elastic properties and the plastic effects with hardening and the loading history lead to a vast formalism which makes a closed theory very difficult. Therefore, the usual way to calculate elastic-plastic behavior in the finite range of boundary and initial value problems is to incorporate the constitutive equations and the balance equations into numerical simulations.

6.7 Viscous Behavior of the Solid Skeleton

A great variety of organic and living solid matrices show time-dependent properties, such as increasing or decreasing stresses and deformations. Living tissues in particular, exhibit a distinct viscous behavior (see, e.g., Mow *et al.*, 1990). However, plastic or other artificially created porous media like concrete, can also show viscous properties (see Lenk, 1971).

It seems that, in the porous media theory, the *viscoelastic or viscoplastic* behavior of porous solids have only been treated in a few contributions. One can state that the viscoelasticity and viscoplasticity theories have not reached the high standard that the elasticity and plasticity theories within the theory of porous media have, although in classical continuum mechanics, for one-component materials, general schemes for such mechanical behavior exist (see Truesdell and Noll, 1965, Noll, 1974, Haupt, 1977). However, the constitutive theory within the porous media theory is rather complex (see the preceeding sections), so it is understandable that a general constitutive theory for saturated porous solids with time dependent properties does not yet exist.

The greatest progress in describing viscous behavior has been made within the range of linear viscoelasticity in the field of biomechanics. Mow *et al.* (1990) stated concerning the flow independent (intrinsic) viscoelastic behavior of the solid matrix: "The previous discussions on hydrated soft tissues have focused on how interstitial fluid flow influences the viscoelastic creep and stress-relaxation behavior of cartilage, meniscus and intervertebral disc materials. Interstitial fluid flow always occurs when these tissues are compressed or stretched since a volume change always occurs, or when a pressure gradient is applied. However, when these tissues are subjected to the action of pure shear, no volume change occurs and no pressure gradients exist; thus no interstitial fluid flow occurs. When tissue specimens are subjected to pure shear, the response is that offered by the intrinsic viscoelasticity of the collagen-proteoglycan solid matrix... For pure shear experiments on cartilage, we have found that the quasi-linear viscoelastic (QLV) model proposed by Fung (1972,

1981) provides an excellent theory to describe the response of the collagen-proteoglycan solid matrix."

We refer in this treatise to Fung (1993), who proceeded from the Boltzmann formulation of viscoelasticity. After discussing the one-dimensional case, he developed a linear viscoelastic law (convolution integral) for the three-dimension stress and strain state within the framework of the geometrically linear theory "with infinitesimal displacements, strains, and velocities":

$$\mathbf{T}_E^S(\mathbf{x}, t) = \int\limits_{-\infty}^{t} \overset{4}{\mathbf{G}^S} \, \frac{\partial \mathbf{E}_S}{\partial \boldsymbol{\tau}} d\boldsymbol{\tau} \tag{6.136}$$

or its inverse

$$\mathbf{E}_S(\mathbf{x}, t) = \int\limits_{-\infty}^{t} \overset{4}{\mathbf{J}^S} \, \frac{\partial \mathbf{T}_E^S}{\partial \boldsymbol{\tau}} d\boldsymbol{\tau} \ , \tag{6.137}$$

where the fourth-order tensor $\overset{4}{\mathbf{G}^S} \ (\mathbf{x}, t - \boldsymbol{\tau})$ is called the relaxation function and the tensor $\overset{4}{\mathbf{J}^S} \ (\mathbf{x}, t - \boldsymbol{\tau})$ is called the creep function.

"Note that the lower limit of integration is taken as $-\infty$, which into means that the integration is to be taken before the very beginning of motion. If the motion starts at time $t = 0$, and $\mathbf{T}_E^S = \mathbf{O}, \mathbf{E}_S = \mathbf{O}$ for $t < 0$, then Eq. (6.136) reduces to" (the notations of Fung (1993) have been changed by the author in order to conform to modern standard):

$$\begin{aligned} \mathbf{T}_E^S(\mathbf{x}.t) &= \overset{4}{\mathbf{G}^S} \ (\mathbf{x}, t)\mathbf{E}_S(\mathbf{x}, 0+) + \\ &+ \int\limits_0^t \overset{4}{\mathbf{G}^S} \ (\mathbf{x}, t - \boldsymbol{\tau}) \frac{\partial \mathbf{E}_S}{\partial \boldsymbol{\tau}}(\mathbf{x}, \boldsymbol{\tau}) d\boldsymbol{\tau} \ . \end{aligned} \tag{6.138}$$

In this equation, $\mathbf{E}_S(\mathbf{x}, 0+)$ "is the limiting value of $\mathbf{E}_S(\mathbf{x}, t)$ when $t \to 0$ from the positive side. The first term in Eq. (6.138) gives the effect of initial disturbance: it arises from the jump of $\mathbf{E}_S(\mathbf{x}, t)$ at $t = 0$. If the strain history contains other jumps at other instants of time, then each jump calls for an additional term similar to the first term in Eq. (6.138)."

After stating the fundamental equations of viscoelasticity (in the Bolzmann version), Fung discussed in greater detail the case where biological tissues are subjected to periodic oscillations. This case is important in experimentally determining the viscoelastic properties. Finally, he discussed the use of viscoelastic models.

In Setton *et al.* (1993) the extended viscoelasticity theory of Mak (1986) (which is capable of describing both the flow-dependent and flow-independent mechanism of viscoelasticity in articular cartilage) was analyzed experimentally. Readers who are interested in further details of the viscoelasticity theory are referred to Houben (1996), in which several references are listed.

Recently Ehlers and Markert (2001) discussed viscoelastic deformations in fluid-saturated solids.

The transfer of various kinds of viscoplasticity theories to saturated and empty porous solids seems not to have taken place, in general. There are only a few known papers which are concerned with the description of viscoplastic behavior in the porous media theory, e.g., Diebels *et al.* (1996) discussed a special case in viscoplasticity theories, namely the rate dependent plastic behavior of geomaterials under dynamic loading, extending the classical approach for metalic materials by Perzyna (1966).

Rubin *et al.* (2000) developed a model to describe the elastic-viscoplastic deformations of porous materials and constructed restrictions from thermodynamics. Moreover, Ekh *et al.* (2000) included thermal effects in their investigations of viscoplastic porous solids.

6.8 Thermomechanical Behavior of Porefluids

We will consider only simple fluids which are isotropic by definition. Within the framework of simple fluids, there are many ansätze to describe the thermomechanical behavior of the fluids. In this section, however, we will discuss only a few of them.

The difficulty in formulating consistent constitutive equations for porefluids arises from the fact that one has to possibly combine quantities on the microscopic level namely the real hydrostatic pressure in the porefluid, with those on the macroscopic level, namely, the symmetrical part of the partial velocity gradient (for non-Newtonian and Newtonian fluids).

a) Inviscid Porefluids

Let us consider *inviscid porefluids* first. From thermodynamic restrictions (see, e.g., de Boer, 2000 a), it turns out that the partial stress tensor of the fluid $\mathbf{T}^L$ can be represented by the simple formula

$$\mathbf{T}^L = -\, n^L p \mathbf{I} \,, \tag{6.139}$$

where p is the porefluid pressure. From a historical point of view, it is interesting to note that Fillunger (1936) had already correctly determined the partial stress tensor of the fluid by applying the cut principle and the Dellessian law; this law states that the surface porosity is equal to volume porosity if the pores are statistically distributed.

For *incompressible* porefluids, the hydrostatic pressure cannot be determined by a constitutive equation, but it can be obtained with the help of the balance equation of momentum for the fluid phase and the incompressibility constraint.

For *compressible* porefluids the hydrostatic porefluid pressure can be expressed by constitutive equations which relate the porefluid pressure to the density and the temperature. For example, an *ideal gas* is characterized by the constitutive relation for p (linear in ρ^{GR} and Θ^G):

$$p = \bar{R}\rho^{GR}\Theta^G \ , \quad \bar{R} = \frac{R}{M} \ , \tag{6.140}$$

where M is the molecular weight of the gas, R is the so-called gas constant (the same for all ideal gases), ρ^{GR} is the density of the ideal gas and Θ^G is the absolute temperature of the fluid. The *general constitutive equation* for the porefluid pressure p results from thermodynamic considerations (see, e.g., de Boer, 2000 a) with the free Helmholtz energy function ψ^G:

$$p = (\rho^{GR})^2 \frac{\partial \hat{\psi}^G}{\partial \rho^{GR}} \ , \quad \psi^G = \hat{\psi}^G(\rho^{GR}, \Theta^G) \ . \tag{6.141}$$

With an appropriate ansatz for $\hat{\psi}^G$, considering test observations, the porefluid pressure can be determined from $(6.141)_1$.

b) Viscous Porefluids

We may begin our discussion with a constitutive assumption of a non-linear viscous fluid in which the response functions for the free Helmholtz energy ψ^L and the partial stress tensor $\mathbf{T}^L$ (we will not mention other response functions; for a complete thermodynamic treatment, see, e.g., de Boer, 2000 a) depend on the variables

$$s = \{\rho^{LR}, \Theta^L, \mathbf{D}_L\} \ . \tag{6.142}$$

Thus,

$$\begin{aligned} \psi^L &= \hat{\psi}^L(\rho^{LR}, \Theta^L, \mathbf{D}_L) \ , \\ \mathbf{T}^L &= \hat{\mathbf{T}}^L(\rho^{LR}, \Theta^L, \mathbf{D}_L) \ . \end{aligned} \tag{6.143}$$

We must note here that the corresponding mechanical theory of a one-component fluid characterized by $\mathbf{T} = \hat{\mathbf{T}}(\rho, \mathbf{D})$ is known as the *Reiner-Rivlin fluid.*

The general representation of the response function for $\mathbf{T}^L$ is, according to Truesdell and Noll (1965),

$$\mathbf{T}^L = -\, n^L p \mathbf{I} + \kappa_0 \mathbf{I} + \kappa_1 \mathbf{D}_L + \kappa_2 (\mathbf{D}_L)^2 \ , \tag{6.144}$$

where p is a function of the real density ρ^{LR} and where the coefficients κ_0, κ_1, and κ_2 are functions of the temperature Θ^L and of the partial density $\rho^L = n^L \rho^{LR}$, as well as of the three principal invariants of $\mathbf{D}_L$. In the linear case, Eq. (6.144) simplifies to

$$\mathbf{T}^L = -\, n^L p\mathbf{I} + \kappa_0\mathbf{I} + \kappa_1\mathbf{D}_L \ . \tag{6.145}$$

For

$$\kappa_0 = \lambda^L \mathrm{tr}\mathbf{D}_L \ , \quad \kappa_1 = 2\mu^L \ , \tag{6.146}$$

where λ^L and μ^L are constants, Eq. (6.145) turns into the corresponding version of the Navier-Stokes equation.

It is worth mentioning that, in the case of an incompressible porefluid, the hydrostatic pressure p is not determined by a constitutive equation. Moreover, $\mathrm{tr}\mathbf{D}_L$ does not vanish (as in the Navier-Stokes equation for a one-component fluid) due to the change in pores.

A special form of (6.145) is used by Fung (1993) for the description of the flow of blood (as a one-component material):

$$\mathbf{T} = -\, p\mathbf{I} + 2\,\overset{\circ}{\mu}\,(J_2)\mathbf{D} \ , \tag{6.147}$$

where

$$J_2 \ = \ \frac{1}{2}\mathbf{D}\cdot\mathbf{D} \ = \ \frac{1}{2}\mathrm{I}_1^2 - \mathrm{II}_2 \ , \tag{6.148}$$

with $\mathbf{T}$, J_2, and $\overset{\circ}{\mu}$ denoting the stress tensor, the difference of half of the square of the first invariant I_1 and the second invariant I_2 of symmetrical part of the velocity gradient $\mathbf{D}$, and a response function of the real blood body, respectively. This constitutive equation can be transferred to the partial fluid body by an extended version of (6.147)

$$\mathbf{T}^L = -\, n^L p\mathbf{I} + \lambda^L \mathrm{tr}\mathbf{D}_L\mathbf{I} + 2\overset{\circ}{\mu}{}^L\mathbf{D}_L \ , \tag{6.149}$$

where λ^L and $\overset{\circ}{\mu}{}^L$ can depend on the first and second invariants of $\mathbf{D}_L$, possibly in the version (6.148). However, in order to verify (6.149), experimental and numerical investigations are necessary.

Chapter 7

FUNDAMENTAL EFFECTS IN GAS- AND LIQUID-FILLED POROUS SOLIDS

7.1 Introduction

There is no field in continuum mechanics which shows such a variety of typical effects as porous materials filled with liquid and/or gas. Due to the enormous influence of the liquid and gas in the pores on the porous solid phase and quite the contrary there arises such effects as uplift, friction, capillarity, effective stresses and phase transitions.

The fundamental effects in saturated porous solids – uplift, friction, capillarity, and effective stress – were the main subjects of investigation into saturated porous media in the first half of this century (see the extensive discussion by de Boer, 2000 a). However, the theoretical foundation of these effects was established more or less intuitively and was in no way based on profound mechanical or thermodynamic principles. This was impossible at the time because the porous media theory and the constitutive theory within continuum mechanics, with all the restrictions gained from the objectivity principle and thermodynamic considerations, were all still awaiting further research. With the development of the first porous media theories, on the basis of the mixture theory and the volume fraction concept in the late 1970s and the beginning of the 1980s, a tool was provided to examine the aforementioned effects theoretically. This was done by de Boer and Ehlers (1988, 1990a), who were able to confirm some results on the uplift and friction problems as well as on the effective stress concept which had been intuitively stated in the early stages of the development of the porous media theory, and thereby were able to correct some statements in older literature. However, the investigations of de Boer and Ehlers were based on a binary model by Bowen (1980) with incompressible constituents. Even though such a model can describe the basic effects quite accurately, it is advisable to investigate the above stated physical phenomena on the basis of a more general model (developed in the last five years). This ternary model contains not only incompressible, but also compressible constituents.

It is the goal of the following investigations to derive main features of the fundamental effects uplift, friction, capillarity, effective stresses and phase transitions.

7.2 Basic Equations

For the following investigations the balance equations of momentum for the individual constituents are necessary. They read as:

$$\operatorname{div} \mathbf{T}^\alpha + \rho^\alpha \mathbf{b}^\alpha + \hat{\mathbf{p}}^\alpha = \rho^\alpha \mathbf{a}_\alpha + \hat{\rho}^\alpha \mathbf{v}_\alpha \ . \tag{7.1}$$

In (7.1) Cauchy's stress tensor is denoted by $\mathbf{T}^\alpha$ for the constituent φ^α, where $\alpha = S$ (solid), L (liquid), and G (gas), and the external acceleration is indicated by $\mathbf{b}^\alpha$. The quantitites $\mathbf{a}_\alpha$ and $\mathbf{v}_\alpha$ contain the acceleration and velocity of a material point of the individual constituents φ^α. Moreover, ρ^α is the partial density, which is related to the real density by

$$\rho^\alpha = n^\alpha \rho^{\alpha R} \ , \tag{7.2}$$

with n^α being volume fractions, which depend on the position vector $\mathbf{x}$ marking the place x and the time t. The volume fractions have to satisfy the volume fraction condition

$$n^S + n^L + n^G = 1 \ . \tag{7.3}$$

The interaction volume forces between the constituents are marked by $\hat{\mathbf{p}}^\alpha$. They are restricted by

$$\hat{\mathbf{p}}^S + \hat{\mathbf{p}}^L + \hat{\mathbf{p}}^G = \mathbf{0} \ . \tag{7.4}$$

The mass supply terms $\hat{\rho}^\alpha$, which come from the mutual exchange of mass between the constituents are constrained, namely

$$\hat{\rho}^S + \hat{\rho}^L + \hat{\rho}^G = 0 \ . \tag{7.5}$$

Furthermore, constitutive equations are required for Cauchy's stress tensor of the individual constituents and for the interaction forces between the solid and liquid phases and between the gas and liquid phases.

For the *elastic porous solid* phase Cauchy's stress tensor is determined by

$$\mathbf{T}^S = -\, n^S p \mathbf{I} + \mathbf{T}^S_E \ , \tag{7.6}$$

where p is the real hydrostatic pressure of the liquid and where the effective stress $\mathbf{T}^S_E$ is given by the motion of the capillary-porous solid:

$$\mathbf{T}_E^S = 2\rho^S \mathbf{F}_S \frac{\partial \psi^S}{\partial \mathbf{C}_S} \mathbf{F}_S^T \ . \tag{7.7}$$

In (7.7), $\mathbf{F}_S$ denotes the deformation gradient, $\mathbf{C}_S$ the right Cauchy-Green deformation tensor, and ψ^S the free Helmholtz energy function of the solid phase. The stress states of the *inviscid liquid* and the *partial gas phases* turn out to be simple:

$$\mathbf{T}^L = -\, n^L p \mathbf{I} \ , \quad \mathbf{T}^G = -\, n^G p \mathbf{I} + \mathbf{T}_E^G \ \text{ or } \ \mathbf{T}^G = -\, n^G p^{GR} \mathbf{I} \tag{7.8}$$

with the effective stress

$$\mathbf{T}_E^G = p_E^G \mathbf{I} = \rho^G \frac{\partial \psi^G}{\partial n^G} \mathbf{I} \ , \ \ p^{GR} = (\rho^{GR})^2 \frac{\partial \psi^G}{\partial \rho^{GR}} \ . \tag{7.9}$$

In (7.9), ψ^G is the free Helmholtz energy functions of the gas phase.

Finally, the interaction forces should be addressed.

$$\hat{\mathbf{p}}^L \ = \ p \,\mathrm{grad}\, n^L + \hat{\mathbf{p}}_E^L, \quad \hat{\mathbf{p}}^G \ = \ p \,\mathrm{grad}\, n^G + \hat{\mathbf{p}}_E^G \ , \tag{7.10}$$

where

$$\hat{\mathbf{p}}_E^L = -\, \hat{\mathbf{t}}^S - \hat{\mathbf{t}}^G - \beta_\Theta \,\mathrm{grad}\,\Theta - \beta_V^L \mathbf{v}_{LS} - \beta_V^G \mathbf{v}_{GS} \ , \tag{7.11}$$

and

$$\hat{\mathbf{p}}_E^G = -\, \hat{\mathbf{t}}^G - \gamma_\Theta \,\mathrm{grad}\,\Theta - \gamma_V^L \mathbf{v}_{LS} - \gamma_V^G \mathbf{v}_{GS} \ , \tag{7.12}$$

with

$$\hat{\mathbf{t}}^S = \rho^S \frac{\partial \psi^S}{\partial \rho^L} \,\mathrm{grad}\, \rho^L, \quad \hat{\mathbf{t}}^G = \rho^G \frac{\partial \psi^G}{\partial \rho^L} \,\mathrm{grad}\, \rho^L \ . \tag{7.13}$$

In (7.11) and (7.12) the response parameters β_Θ, γ_Θ, β_V^L, β_V^G, and γ_V^G, γ_V^L, which have to reflect test observations, were introduced.

The new constitutive equations for the capillary forces $\hat{\mathbf{t}}^S$ and $\hat{\mathbf{t}}^G$ (see de Boer and Didwania, 2000, 2002, 2004) contain the free Helmholtz energy functions of the solid and gas phases, which depend on the liquid density, as well as the gradient of the liquid density. They reflect the adhesion forces between the solid or gas phases and the liquid constituent in small films between the constituents.

7.3 Uplift

The hydrostatic uplift is a weight reducing force in the direction opposite to the weight which a body immerged in fluid, undergoes, if the fluid is subjected to gravity. The weight reducing force is caused by the pressure of the surrounding medium which increases with the depth of the fluid. The uplift problem in a fluid medium is an old one. It is first investigations go back to Archimedes who developed a principle which carries his name. This principle states that a body in a fluid suffers a lost in weight, which is equal to the amount of fluid displaced by the body.

In the porous media theory it was Fillunger (1913) who pioneered the theoretical treatment of uplift. Although the structure of his uplift formula was correct the result was wrong. This formula was corrected by von Terzaghi and Rendulic (1934). However both authors did not go as far back as Archimedes. Rather they discovered their formulas with the aid of obscure theoretical and experimental verifications.

Today the uplift theory is well based on the fundamentals of the Theory of Porous Media. However, investigations were restricted to a binary model. In recent times a more sophisticated model, namely a ternary model with incompressible and compressible constituents is being used for the theoretical foundations of the uplift problem.

Based on the above-mentioned new findings in the constitutive theory of unsaturated and saturated porous media, the phenomenon uplift will be clarified for a ternary model.

In order to be able to compare the new results with classical solutions we will sum up Cauchy's first equations of motion (7.1). Choosing the notation

$$\mathbf{T} = \mathbf{T}^S + \mathbf{T}^L + \mathbf{T}^G \tag{7.14}$$

and introducing the external acceleration of gravity

$$\mathbf{b}^\alpha = \mathbf{g} \, , \tag{7.15}$$

as well as considering (7.4), we arrive at:

$$\begin{aligned} \operatorname{div} \mathbf{T} + (\rho^S + \rho^L + \rho^G)\mathbf{g} &= \rho^S \mathbf{a}_S + \rho^L \mathbf{a}_L + \rho^G \mathbf{a}_G \\ &+ \hat{\rho}^S \mathbf{v}_S + \hat{\rho}^L \mathbf{v}_L + \hat{\rho}^G \mathbf{v}_G \, . \end{aligned} \tag{7.16}$$

Using (7.3), (7.6), (7.8), (7.9), and (7.14), we obtain from (7.16) that

$$\begin{aligned} &\operatorname{div} \mathbf{T}^S_E - \operatorname{grad} p + \operatorname{grad} p^G_E + (\rho^S + \rho^L + \rho^G)\mathbf{g} \\ &= \rho^S \mathbf{a}_S + \rho^L \mathbf{a}_L + \rho^G \mathbf{a}_G + \hat{\rho}^S \mathbf{v}_S + \hat{\rho}^L \mathbf{v}_L + \hat{\rho}^G \mathbf{v}_G \, . \end{aligned} \tag{7.17}$$

The partial densities ρ^α can be replaced by the real densities $\rho^{\alpha R}$ according to (7.2). In addition, the real densities $\rho^{\alpha R}$ are related to the specific weight $\gamma^{\alpha R}$:

$$\gamma^{\alpha R} = \rho^{\alpha R} \, |\mathbf{g}| \; . \tag{7.18}$$

Using (7.18) and considering the saturation condition (7.3) as well as (7.2), the equation of motion for the mixture body (7.17) can be rearranged:

$$\begin{aligned} \operatorname{div}\mathbf{T}_E^S & - \operatorname{grad}p + \operatorname{grad}p_E^G + \gamma^{LR}\frac{\mathbf{g}}{\mid\mathbf{g}\mid} + n^S(\gamma^{SR} - \gamma^{LR})\frac{\mathbf{g}}{\mid\mathbf{g}\mid} + \\ & + n^G(\gamma^{GR} - \gamma^{LR})\frac{\mathbf{g}}{\mid\mathbf{g}\mid} \\ & = \frac{1}{\mid\mathbf{g}\mid}\gamma^{LR}\mathbf{a}_L + \frac{1}{\mid\mathbf{g}\mid}n^S(\gamma^{SR}\mathbf{a}_S - \gamma^{LR}\mathbf{a}_L) + \\ & + \frac{1}{\mid\mathbf{g}\mid}n^G(\gamma^{GR}\mathbf{a}_G - \gamma^{LR}\mathbf{a}_L) + \hat{\rho}^S\mathbf{v}_S + \hat{\rho}^L\mathbf{v}_L + \hat{\rho}^G\mathbf{v}_G \ . \end{aligned} \tag{7.19}$$

We assume that the gravity field $\mathbf{g}$ can be derived from a potential U:

$$\mathbf{g} = - \operatorname{grad}U \ . \tag{7.20}$$

It is advisable to introduce the pressure head h:

$$h = \frac{p}{\gamma^{LR}} + \frac{U}{\mid\mathbf{g}\mid} \ . \tag{7.21}$$

It should be mentioned that (7.21) is valid in the flow zone with the pressure p. However, Eq. (7.21) is also valid in the capillary zone with a hydrostatic suction if p is replaced by $-\ p^t$ (suction).

From (7.19), considering (7.20) and (7.21), we obtain

$$\begin{aligned} \operatorname{div}\mathbf{T}_E^S & - & \gamma^{LR}\operatorname{grad}h + \operatorname{grad}p_E^G + n^S(\gamma^{SR} - \gamma^{LR})\frac{\mathbf{g}}{\mid\mathbf{g}\mid} + \\ & + & n^G(\gamma^{GR} - \gamma^{LR})\frac{\mathbf{g}}{\mid\mathbf{g}\mid} \\ & = & \rho^{LR}\mathbf{a}_L + n^S(\rho^{SR}\mathbf{a}_S - \rho^{LR}\mathbf{a}_L) + \\ & + & n^G(\rho^{GR}\mathbf{a}_G - \rho^{LR}\mathbf{a}_L) + \hat{\rho}^S\mathbf{v}_S + \hat{\rho}^L\mathbf{v}_L + \hat{\rho}^G\mathbf{v}_G \ . \end{aligned} \tag{7.22}$$

In deriving (7.22) we have assumed that γ^{LR} is spatially constant.

It is recognized that the left-hand side of (7.22) contains two terms describing the specific weights of the solid and gas phases reduced by uplift. Following this, the *uplift forces* $\mathbf{k}_{US}$ and $\mathbf{k}_{UG}$ in the solid and gas media are given by

$$\mathbf{k}_{US} = -\ n^S\gamma^{LR}\frac{\mathbf{g}}{\mid\mathbf{g}\mid} \ , \quad \mathbf{k}_{UG} = -\ n^G\gamma^{LR}\frac{\mathbf{g}}{\mid\mathbf{g}\mid} \ . \tag{7.23}$$

Neglecting the gas phase, Eq.$(7.23)_1$ can be reformulated considering the saturation condition (7.3):

$$\mathbf{k}_{US} = (n^L - 1)\gamma^{LR}\frac{\mathbf{g}}{|\,\mathbf{g}\,|} \,. \tag{7.24}$$

If one assumes $\operatorname{grad} h = \mathbf{0}$, that is, the simplest case in the pure static state, one can conclude from (7.21), (7.20), and (7.24) that

$$\mathbf{k}_{US} = (n^L - 1)\operatorname{grad} p \,, \tag{7.25}$$

$$\operatorname{grad} p = \gamma^{LR}\frac{\mathbf{g}}{|\,\mathbf{g}\,|} = \text{const.} \tag{7.26}$$

It is recognized from (7.22) and (7.25) that the unreduced uplift force is effective according to the principle of Archimedes.

7.4 Friction

The first investigations concerning the friction phenomenon, which occurs during the flow of liquid through a saturated porous solid, was carried out by Fillunger (1914). He stated a formula for the friction force which referred to the volume-element and described the friction force as being dependent on the poreliquid pressure.

Similar formulas were derived by Hoffman (1929) as well as by von Terzaghi and Rendulic (1934).

The *friction* in the flow zone is determined in Eq. (7.22), whereby we neglected the influence of the gas phase:

$$\mathbf{k}_L = -\,\gamma^{LR}\operatorname{grad} h \,, \tag{7.27}$$

or, respectively, by using (7.21) and (7.20),

$$\mathbf{k}_L = -\,\operatorname{grad} p + \gamma^{LR}\frac{\mathbf{g}}{|\,\mathbf{g}\,|} \,. \tag{7.28}$$

The relations (7.27) and (7.28) are however, only valid if the temperature gradient and the inertia effects, see Eq. (7.29), are neglected.

It should be mentioned that the uplift force is always effective. The friction force only occurs if $\operatorname{grad} h \neq \mathbf{0}$, i.e. in the case of flow processes in the flow zone or in the case of vaporization processes at the downstream face of the suction zone.

In order to also include the dynamic behavior of the liquid phase we have to consider the equation of motion of the liquid phase. From (7.1) in combination with the constitutive equations (7.8), $(7.10)_1$, and (7.11) we obtain:

$$\begin{aligned} &-\,\operatorname{grad} p + \gamma^{LR}\frac{\mathbf{g}}{|\,\mathbf{g}\,|} - \frac{1}{n^L}\beta_\Theta \operatorname{grad}\Theta - \\ &-\frac{1}{n^L}\beta_v^L \mathbf{v}_{LS} - \frac{1}{n^L}\beta_v^G \mathbf{v}_{GS} = \rho^{LR}\mathbf{a}_L + \frac{1}{n^L}\hat{\rho}^L \mathbf{v}_L \,. \end{aligned} \tag{7.29}$$

From (7.29), it is recognized for the lingering flow of the liquid (neglecting the inertia effects and the mass supply) that in the flow zone the friction force

$$\mathbf{k}_L = \frac{1}{n^L}\beta_\Theta \operatorname{grad}\Theta + \frac{1}{n^L}\beta_v^L \mathbf{v}_{LS} + \frac{1}{n^L}\beta_v^G \mathbf{v}_{GS} \qquad (7.30)$$

can be expressed by

$$\mathbf{k}_L = - \operatorname{grad} p + \gamma^{LR}\frac{\mathbf{g}}{\mid \mathbf{g} \mid} \,. \qquad (7.31)$$

In this relation the influence of capillarity in small pores has also been neglected. In reality, due to the inclination of the capillary force at the walls of the solid skeleton there is a force perpendicular to the wall, which can cause a considerable friction force according to Coulomb. As of now this effect has not been investigated.

7.5 Capillarity

In physics capillarity is understood as the behavior of moving liquids in narrow tubes and pores owing to external surface stresses. These surface stresses are as a result of the so-called van der Waal's forces or other surface forces acting amongst the molecules. Molecules of the same kind produce inner forces of the same size with the result that the van der Waal's forces in the inner region of the liquid are in equilibrium. These inner forces are responsible for the cohesion of the phase and are denoted with cohesion forces. On the limit surface to another phases the equilibrium is disordered and the phenomenon of surface stresses arises (adhesion). In general, the amount of adhesion- and cohesion forces are different.

Capillarity is a well-known phenomenon in physics and engineering. The term denotes the transport of liquids against the force of gravity in narrow tubes, cracks and pores caused by the intermolecular forces of cohesion and adhesion of the constituents involved. If the forces of adhesion between the liquid and the tube wall are greater than the forces of cohesion between the molecules of the liquid, then the liquid will rise. According to an encyclopedia (Zedler, 1733), in the early stages of the development of mechanics, the capillarity phenomenon was discovered in either France or in Italy. However, the capillary phenomenon was already described by Leonardo da Vinci around 1500 (see Lücke, 1940), but was not physically founded owing to the lack of physical principles. In the early stage of mechanics in the eighteenth century, it was already recognized that in the gravity field, the capillary rise of water in a narrow tube was inversely proportional to the diameter of the tube. Moreover, the cause of capillary motion, namely adhesion, had been known as early as 1733. It was also found that the capillary effect did not occur for non-moistening liquids like mercury. Although all these phenomena and their explanations were known, it was the sharp-witted Laplace (1806) who first calculated the capillary pressure on the microscopic level. A hypothesis was introduced that a jump condition at a surface exists which separates homogenous fluids of different densities. In 1805, Young derived an expression for the static contact angle (Young, 1805). The dynamic aspects of capillarity were first analyzed by West (1912) and later by Washburn (1921) as well as Kozeny (1927).

Further approaches follow, in general, the fundamental work of Kozeny without substantial new improvements (see, e.g., Cammerer, 1963).

In this section we will deal exclusively with the rise of liquids in porous bodies due to the capillarity phenomenon. This occurs in many different branches of engineering and biology ranging from soil mechanics and building physics to biomechanics.

The transient and stationary rise of liquids from a reservoir in porous solids with small pores is complex and has not been described by any satisfactory phenomenological mathematical model until now. One has to pay attention to the fact that capillarity is an effect on the microscopic level. In order to develop a phenomenological mathematical model one has to bring this effect to the macroscopic level. This is relatively simple in the static case, neglecting the inertia effects and the velocities in the determination of the capillarity rise, as has been shown by de Boer and Ehlers (1990 a). For transient problems, however, a mechanical description of the motion is complicated and it seems that this task has not yet been satisfactorily resolved. In the present work we will use a recently developed continuum mechanical description of the capillarity problem (de Boer and Didwania, 2000, 2002, 2004) and show that the classical description is contained in the new approach.

a) Basic Relations

Recalling the basic feature of the capillary problem, namely, the occurence of the rise of water in a narrow tube due to an interaction force between the solid phase of the tube and the water. Such an interaction force must also exist in a saturated porous solid with small interstitial pores. However, the Laplace-Young model is not appropriate for the description of the capillary phenomenon in porous solids, even if the porous solid is replaced by a model consisting of a bundle of narrow tubes. In order to develop a consistent mathematical model, the ensured basic relations of the macroscopic porous media theory should be applied. The only interaction forces between the three constituents φ^α are the interaction forces $\hat{\mathbf{p}}^S$, $\hat{\mathbf{p}}^L$ and $\hat{\mathbf{p}}^G$, which should describe the cause of the capillary rise of water in porous solids. In an extended thermodynamic investigation (see Chapter 6), constitutive equations for the stresses of the three phases, for the heat flux vector and for the interaction forces in particular (see also de Boer and Didwania, 2000, 2002, 2004) have been developed. Their constitutive equations for the interaction forces $\hat{\mathbf{p}}^L$ and $\hat{\mathbf{p}}^G$ read as:

$$\begin{aligned}\hat{\mathbf{p}}^L &= p \text{ grad } n^L - \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} \text{grad } \rho^L - \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \text{grad}\, \rho^L - S_L \mathbf{v}_L \;, \\ \hat{\mathbf{p}}^G &= p\, \text{grad } n^G - \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \text{grad}\, \rho^L - S_G \mathbf{v}_G \;,\end{aligned} \tag{7.32}$$

where $\hat{\psi}^S$ and $\hat{\psi}^G$ denote the free Helmholtz energy function of the solid and gas phases. For convenience we have replaced β_v^L and β_v^G with S_L and S_G In (7.32) it is assumed that the velocity of the rigid porous solid is equal to zero, the temperature is constant, and the mutual influence of both constituents concerning the frictional forces is neglected.

The new constitutive relations represent a remarkable improvement of already existing constitutive equations in the porous media theory. The first terms on the right-hand side of (7.32) are due to the saturation condition (7.3), which is well-known in literature, and the last terms are referred to Darcy's law. The other terms, connected with the density gradient of the liquid, show the interaction between the solid and liquid phases as well as that between the gas and liquid phases. These terms consider the variation of the density in space and are therefore suitable for describing the capillary phenomenon which is caused in narrow tubes by adhesion, which is described macroscopically by a jump in the densities of the constituents involved.

The transient capillary motion of a liquid in a liquid- and gas-filled rigid porous solid is described locally by the equation of motion (7.1):

$$\text{div}\, \mathbf{T}^L + \rho^L \mathbf{g} + \hat{\mathbf{p}}^L = \rho^L \mathbf{a}_L + \hat{\rho}^L \mathbf{v}_L \;. \tag{7.33}$$

With the constitutive equations $(7.8)_1$ and $(7.11)_1$ in consideration of (7.13), Eq. (7.33) can be reformulated:

$$\begin{aligned}&- \text{ grad}(n^L p) + \rho^L \mathbf{g} + p \text{ grad } n^L - \\ &- (\rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} + \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L}) \,\text{grad } \rho^L - S_L \mathbf{v}_L \;= \rho^L \mathbf{a}_L + \hat{\rho}^L \mathbf{v}_L\end{aligned} \tag{7.34}$$

or

$$\begin{aligned}&- \text{ grad } p + \rho^{LR} \mathbf{g} \;-\; \frac{1}{n^L} (\hat{t}^{SL} + \hat{t}^{LG}) \,\text{grad } \rho^L - \frac{1}{n^L} \; S_L \mathbf{v}_L \\ &= \rho^{LR} \mathbf{a}_L + \frac{1}{n^L} \hat{\rho}^L \mathbf{v}_L \;,\end{aligned} \tag{7.35}$$

where the influence of the temperature and the friction between the gas phase and the porous solid has been neglected.

Moreover, the abbreviations

$$\hat{t}^{SL} = \rho^S \frac{\partial \hat{\psi}^S}{\partial \rho^L} \ , \quad \hat{t}^{LG} = \rho^G \frac{\partial \hat{\psi}^G}{\partial \rho^L} \ , \tag{7.36}$$

see (7.13), have been introduced. The scalar quantities $\hat{t}^{SL}$ and $\hat{t}^{LG}$ can be interpreted as hydrostatic stress states between the solid and liquid phases as well as between the gas and liquid phases. As to how the free Helmholtz energy functions ψ^S and ψ^G depend on the density of the liquid, this point will be clarified by experiments and numerical investigations. In order to make the equation of motion (7.35) more transparent, we choose a rectangular coordinate system x_1, x_2, and x_3 as well as the basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$. Applying differential calculus to the equation of motion (7.35) the following vector equation is obtained:

$$\begin{aligned} - p_{,i}\, \mathbf{e}_i - \rho^{LR} g \mathbf{e}_3 \quad &- \quad \frac{1}{n^L}(\hat{t}^{SL} + \hat{t}^{LG})\rho^L_{,i}\, \mathbf{e}_i - \frac{1}{n^L} S_L (x_i)'_L\, \mathbf{e}_i \\ &= \quad \rho^{LR}(x_i)''_L\, \mathbf{e}_i + \frac{1}{n^L}\, \hat{\rho}^L (x_i)'_L\, \mathbf{e}_i \ , \end{aligned} \tag{7.37}$$

where $(\ldots)_{,i}$ denotes the partial derivative with respect to x_i ($i = 1, 2$, and 3). The summation rule is valid, i.e. if the index i appears twice, it must be summed up from 1 to 3. It should be mentioned that the capillary flow is effective in all three directions due to the capillary force. For comparison with Kozeny's (1927) results, we will examine the one-dimensional capillary motion in a tube below .

b) One-Dimensional Capillary Motion

The one-dimensional capillary rise is governed in the $\mathbf{e}_3$-direction by

$$\begin{aligned} - p_{,3} \ - \rho^{LR} g \quad &- \quad \frac{1}{n^L}(\hat{t}^{SL} - \hat{t}^{LG})\rho^L_{,3} \ - \frac{1}{n^L} S_L (x_3)'_L \\ &= \quad \rho^{LR}(x_3)''_L \ + \frac{1}{n^L}\, \hat{\rho}^L (x_3)'_L \ . \end{aligned} \tag{7.38}$$

In the third term on the left-hand side of (7.38) use is made of the fact that the capillary forces in the $\mathbf{e}_3$- direction have opposite directions.

In what follows, we replace the material time derivative $(\ldots)'_L$ with $(\dot{\ldots})$ and $(\ldots)_{,3}$ with $\dfrac{\partial(\ldots)}{\partial x_3}$. Then, for the component in the $\mathbf{e}_3$-direction we have:

$$\begin{aligned} - \frac{\partial p}{\partial x_3} - \rho^{LR} g + \frac{1}{n^L}(\hat{t}^{SL} - \hat{t}^{LG}) \frac{\partial \rho^L}{\partial x_3} \quad &- \quad \frac{1}{n^L} S_L \dot{x}_3 \\ &= \ \rho^{LR} \ddot{x}_3 + \frac{1}{n^L} \hat{\rho}^L \dot{x}_3 \ . \end{aligned} \tag{7.39}$$

Moreover, we exclude mutual mass supplies $\hat{\rho}^\beta$. Thus the equation of motion reduces to

$$\begin{aligned} & -\frac{\partial p}{\partial x_3} - \rho^{LR} g - \frac{1}{n^L}(\hat{t}^{SL} - \hat{t}^{LG})\frac{\partial \rho^L}{\partial x_3} - \frac{1}{n^L} S_L \dot{x}_3 \\ & \qquad = \rho^{LR} \ddot{x}_3 \, . \end{aligned} \tag{7.40}$$

We integrate the equation of motion (7.40) over a cross-section A of the tube from x_3 equal to zero through the temporary capillary rise s:

$$\begin{aligned} & -\int\limits_{(A)}\int\limits_0^s \frac{\partial p}{\partial x_3} dx_3\, dA - \int\limits_{(A)}\int\limits_0^s \rho^{LR} g dx_3\, dA - \\ & -\int\limits_{(A)}\int\limits_0^s \frac{1}{n^L}(\hat{t}^{SL} - \hat{t}^{LG})\frac{\partial \rho^L}{\partial x_3} dx_3\, dA - \int\limits_{(A)}\int\limits_0^s \frac{1}{n^L} S_L \dot{x}_3 dx_3\, dA \\ & = \int\limits_{(A)}\int\limits_0^s \rho^{LR} \ddot{x}_3 dx_3 dA + \int\limits_{(A)}\int\limits_0^s \rho^{LR} \dot{x}_3 \dot{x}_3\, dA \, . \end{aligned} \tag{7.41}$$

The second term on the right-hand side of (7.41) results from a mass supply of a reservoir.

The integration leads to

$$\begin{aligned} & [p(s) - p(0)]A - \rho^{LR} g\; s\; A - \int\limits_{(A)}\int\limits_0^s (\hat{t}^{SL} - \hat{t}^{LG})\frac{\partial \rho^L}{\partial x_3} dx_3\, dA - \\ & -(\int\limits_0^s S_L \dot{x}_3 dx_3)A = \rho^{LR}(\int\limits_0^s \ddot{x}_3 dx_3)A + \rho^{LR}(\int\limits_0^s \dot{x}_3 \dot{x}_3)A \, , \end{aligned} \tag{7.42}$$

where we have assumed that all quantities are constant over the cross-section and over x_3, with the exception of $\hat{t}^{SL}$ and $\hat{t}^{LG}$ in the third term on the left-hand side of (7.42), and where we have considered that n^L is equal to unity. In the next step we elaborate on the third term of the left-hand side of the equation of motion (7.42):

$$I = \int\limits_{(A)}\int\limits_0^s (\hat{t}^{SL} - \hat{t}^{LG})\frac{\partial \rho^L}{\partial x_3} dx_3\, dA \, . \tag{7.43}$$

In the classical capillarity theory it is assumed that a small film which separates homogeneous fluids of different densities, namely liquid and gas, exists at the surface of the liquid in a narrow tube. This means that there is a jump in the

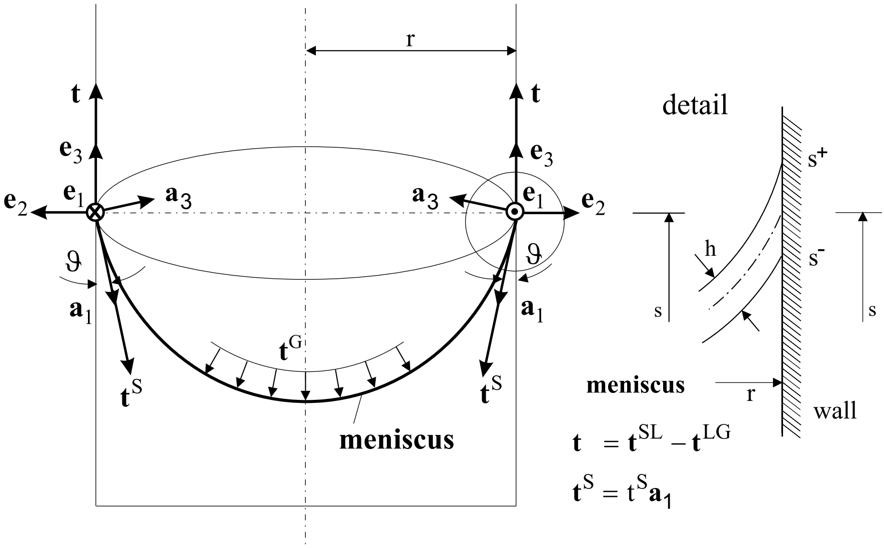

Fig. 7.5.1: Force analysis at the meniscus in a narrow tube.

density distribution (see Fig. 7.5.1). In this case the gradient of the liquid density ρ^L equals unity and the integral exists only in the small film between s^+ and s^- (see Fig. 7.5.1).

With $\operatorname{grad} \rho^L$ equal to plus and minus unity, instead of (7.43), for the inner integral we obtain

$$I_1 = -\int\limits_{s^-}^{s^+} (\hat{t}^{SL} - \hat{t}^{LG}) dx_3 \; . \tag{7.44}$$

In the $\mathbf{e}_3$-direction at the wall the membrane forces are assumed to be constant. This assumption is in accordance with the corresponding assumption in the shell theory for membrane forces. It follows that

$$\begin{aligned} I_1 &= -\{[\hat{t}^{SL}(r) - \hat{t}^{LG}(r)] x_3\}_{s^-}^{s^+} \; , \\ I_1 &= -[\hat{t}^{SL}(r) - \hat{t}^{LG}(r)](s^+ - s^-) \; . \end{aligned} \tag{7.45}$$

We call the components of the external forces in the $\mathbf{e}_3$-direction

$$\hat{t}^{SL}(r)(s^+ - s^-) = t^{SL} \tag{7.46}$$

and

$$\hat{t}^{LG}(r)(s^+ - s^-) = t^{LG} \; . \tag{7.47}$$

In general, the forces $\mathbf{t}^{SL} = t^{SL}\mathbf{e}_3$ and $\mathbf{t}^{LG} = t^{LG}\mathbf{e}_3$ are not measurable. However, the norms of the forces can be replaced equivalently by (see Fig. 7.5.1):

$$t^{SL} - t^{LG} = t^{S}\cos\vartheta \; . \tag{7.48}$$

It should be mentioned that t^{SL}, t^{LG}, and t^{S} are the norms of the forces per unit length in $\mathbf{e}_3$-and $\mathbf{a}_1$-directions, which act only at the boundary of the meniscus, and not over the whole cross-section. In order to determine the total external force at the boundary of the meniscus we have to replace $\int\limits_{(A)} dA$ with $2\pi r$ in (7.43), where r is the radius of the tube. Thus, we can express the integral (7.43) by

$$I = (t^{SL} - t^{LG})2\pi r \tag{7.49}$$

or, considering (7.48), by

$$I = t^{S}\cos\vartheta 2\pi r \; . \tag{7.50}$$

With $A = \pi r^2$ and (7.50), Eqs. (7.44) and (7.42) can be reformulated:

$$\begin{aligned} -[p(s) - p(0)]\pi r^2 - \rho^{LR} g s \pi r^2 + t^{S}\cos\vartheta 2\pi r - \\ - S_L \dot{s} s \pi r^2 = \rho^{LR}\ddot{s} s \pi r^2 + \rho^{LR}\dot{s}\dot{s}\pi r^2 \; , \end{aligned} \tag{7.51}$$

or

$$\begin{aligned} -[p(s) - p(0)] - \rho^{LR} g s \; + \; \frac{2t^{S}}{r}\cos\vartheta - S_L \dot{s} s \\ = \; \rho^{LR}\ddot{s} s + \rho^{LR}\dot{s}\dot{s} \; . \end{aligned} \tag{7.52}$$

After some rearrangement we arrive at

$$\frac{d^2 s^2}{dt^2} + m\frac{ds^2}{dt} + 2gs + n = 0 \tag{7.53}$$

with

$$m = \frac{S_L}{\rho^{LR}} \; , \; n = \frac{2}{\rho^{LR}}[p(s) - p(0)] - \frac{4t^{S}}{r\rho^{LR}}\cos\vartheta \; . \tag{7.54}$$

Eq. (7.53) corresponds to Kozeny's (1927) differential equation, with the exception that $p(s)$ and $p(0)$ are missing in Kozeny's approach. However, if $p(s)$ and $p(0)$ are equal to the atmospheric pressure, the difference between $p(s)$ and $p(0)$ vanishes.

c) Two-Dimensional Capillary Motion (an Example)

The usefulness of the capillarity theory developed in the preceding sections will be shown for a special example, namely the two-dimensional capillary water flow in a brick, see Fig. 7.5.2. It is assumed that the brick is partly saturated and is standing in a water reservoir. The left side of the brick is impermeable to the liquid and gas phases, whereas the right side consists of open pores. This example has been experimentally investigated by Künzel (1994), see Fig. 7.5.2, where the volume fraction of the liquid was depicted after 72 days of capillary rise.

The same example, with the same data has been simulated via the finite element method. For the film between the solid and the water phase it is assumed that the free Helmholtz energy function depends linearly on the density ρ^L and the quantity $\hat{t}^{SL}$ is a material dependent constant containing the surface tension of the water and a characteristic diameter (for more information, see Ricken, 2002). Starting from the governing equations in Section 7.2, in particular (7.35), where the temperature gradient and the friction forces have been neglected, one derives results which fit the test observation quite well (see Fig. 7.5.2).

Beginning with von Terzaghi (1925 a, b) and his colleagues in Vienna, up to this date the subject of capillarity in porous bodies continues to be an area of active research. While there has been many empirical approaches, there has been only a limited number of continuum mechanical treatments. The present

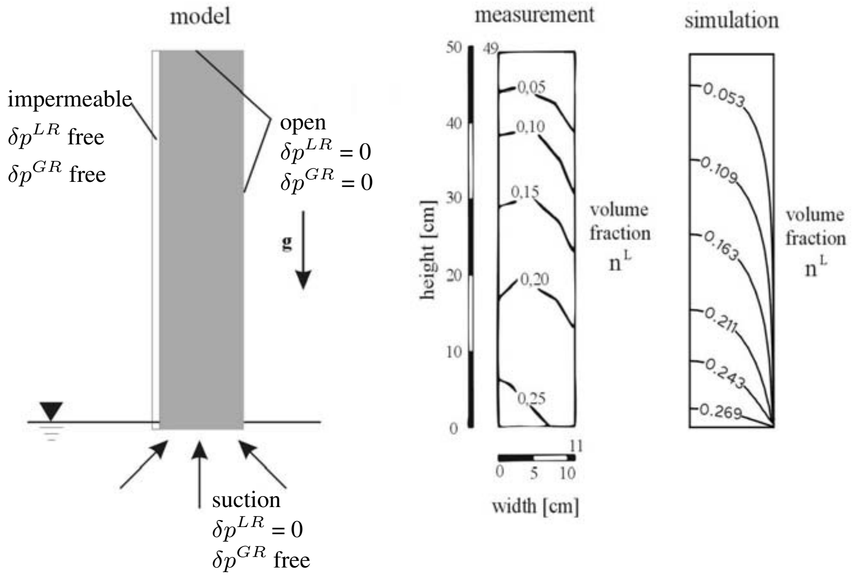

Fig. 7.5.2: Brick model and measurement of the volume fraction after 72 days, by Künzel (1994)

authors' recent treatment of this subject in the framework of a homogenized continuum (porous media theory) has led to very reasonable field equations on proper constitutive equations in particular, for interaction volume forces which cause the capillary flow, which could be demonstrated in examples with one- and two-dimensional motions.

The next step in the investigation of the capillarity problem in porous media will be the development of explicit constitutive equations for interaction forces and their numerical transformation for calculating initial and boundary value problems.

7.6 Effective Stresses

In a saturated porous media, the state of stress in a material point of the partial solid consists of two parts, the weighted liquid pressure and the so-called "extra stress" for which a constitutive equation must be formulated as a function of the respective state of the solid deformation. Thus, only the extra stress, which is well-known in soil mechanics under the notion "effective stress", can describe the strength of the partial solid. In the theory of porous media, the separation of the total stress into effective stress and weighted liquid pressure is founded on the basic assumption of saturation. The separation of stress due to such a constraint as the saturation condition is comparable with the incompressibility condition, a known and theoretical founded effect in classical continuum mechanics for one-component continua (see Truesdell and Noll, 1965). In the theory of mixtures and in the theory of porous media, the saturation condition was considered relatively late. The effect of the effective stresses in binary porous media models could only be clarified in recent times and founded (see de Boer and Ehlers, 1988, 1990 b), although this effect has been a subject of research since the early beginnings of this century in connection with the uplift problem in concrete or masonry dams, the consolidation problem of clay and the shearing resistance of saturated soils. It seems that important contributions which clarify the influence of the liquid pressure are completely ignored in literature and that the theoretical foundations are not well understood.

The concept of *effective stress* for a binary model has been extensively discussed by de Boer (2000 a). Investigations for a ternary model have also been performed by de Boer (1997 a). These investigations were, however, based on simplified constitutive equations. Therefore, the concept of effective stress will be discussed again.

The total stress $\mathbf{T}$ of the mixture body is given by:

$$\mathbf{T} \;=\; \mathbf{T}^S + \mathbf{T}^L + \mathbf{T}^G \,. \tag{7.55}$$

Considering (7.6), (7.8), and (7.3) as well as de Boer (2000 a) and Chapter 8, Eq. (7.55) simplifies to

$$\mathbf{T} = -\,p\mathbf{I} + n^S B^S p\mathbf{I} + \mathbf{T}_E^S + \mathbf{T}_E^G \,. \tag{7.56}$$

where

$$B^S = \frac{K^{SN}}{K^{SR} + K^{SN}} \tag{7.57}$$

with K^{SN} and K^{SR} as the compression moduli of the skeleton and the real material. Besides, Biot (see Biot and Wills, 1957) derived for the compressible solid skeleton in a binary model a similar expression for the effective stresses

$$\mathbf{T}_E^S = \mathbf{T}^S + p(1 - \frac{K^{SN}}{K^{SR}})\mathbf{I} \tag{7.58}$$

For unsaturated porous media (unsaturated means that the pores are not completely filled with liquid) with incompressible solid and liquid phases. Bishop (1960) suggested the following variation of von Terzaghi's expression for the effective stress of the solid phase (see also Skempton, 1962):

$$\mathbf{T}_E^S = [p^{GR} - \chi\,(p^{GR} - p)]\mathbf{I} + \mathbf{T} \,, \tag{7.59}$$

where $\mathbf{T}_E^S$ and $\mathbf{T}$ refer to effective and total stress, respectively, and χ is a parameter related to the degree of saturation; it is equal to unity for fully-saturated porous media.

In order to be able to compare the results of our investigations stated above, with those of Bishop and Biot, we rearrange (7.56) and replace $\mathbf{T}_E^G$ through $(7.8)_2$ in consideration of (7.9) with

$$\mathbf{T}_E^G \quad = \quad n^G(p - p^{GR})\mathbf{I} \,. \tag{7.60}$$

Introducing (7.60) into (7.56) we obtain

$$\mathbf{T}_E^S = [p^{GR} - (n^S + n^L)(p^{GR} - p)]\mathbf{I} - n^S B^S p\mathbf{I} + \mathbf{T} \,, \tag{7.61}$$

where we have used the saturation condition (7.3). It can be stated that (7.61) corresponds to Bishop's formula, if we neglect the compressibility of the real solid material.

If we considered only a binary model with compressible solid and incompressible liquid phases, we arrived at

$$\mathbf{T}_E^S = p(1 - n^S B^S)\mathbf{I} + \mathbf{T} \,, \tag{7.62}$$

a formula, which was already approximately proposed by Suklje(1969).

The sum $\chi = n^S + n^L$, see (7.59) and (7.61) is related to the degree of saturation. If the pores are fully saturated with liquid, the volume fraction n^G

is equal to zero and, according to (7.3), the sum $n^S + n^L$ equals unity. In all other cases the sum $n^S + n^L$ is less than unity.

For fully-saturated porous solids we obtain the classical result

$$\mathbf{T}_E^S = p\mathbf{I} + \mathbf{T} \, , \tag{7.63}$$

founded by Fillunger and von Terzaghi (see de Boer, 2000 a).

7.7 Phase Transitions

A further important characteristic phenomenon in TPM is phase transition in a liquid- and gas-filled porous solid. Well-known examples for this phenomenon are drying as well as freezing and thawing processes. In these processes liquid is transformed to gas as well as liquid to ice and ice to liquid. These effects are partly well documented and the development of consistent theories for these transitions have a long tradition as we will see in the next section.

However, there are other transition effects which have been treated in recent times. These concern mainly growth effects in biomechanics such as growth of tumors and growth owing to different loadings. Moreover, in waste management the problem arises that waste organic parts are contained in the solid skeleton, which biologically degenerate in time and space. These problems are currently under intensive study in Essen. However, there is no experience on the promising results yet. Thus, we will not go deeper into these problems but we would like to concentrate on the theoretical foundation of phase transitions and two applications: drying and freezing processes.

a) Theoretical Foundation

The starting point of our investigations is the inequality (6.34) for the ternary model consisting of a porous solid, an incompressible liquid and a compressible gas, from which we derive the inequality, see de Boer (2000 a):

$$-\hat{\rho}^L(\mu^L - \mu^S) - \hat{\rho}^G(\mu^G - \mu^S) \geq 0 \, , \tag{7.64}$$

where

$$\mu^\alpha = \psi^\alpha - \frac{p^\alpha}{\rho^\alpha} - \frac{1}{2}\mathbf{v}_\alpha \cdot \mathbf{v}_\alpha \tag{7.65}$$

are the chemical potentials.

The inequality (7.64) restricts the constitutive equations for the mass supplies $\hat{\rho}^L$ and $\hat{\rho}^G$. We can always satisfy inequality (7.64) if we choose the constitutive equation

$$\hat{\rho}^L = -G(\mu^L - \mu^S), \quad \hat{\rho}^G = -Q(\mu^G - \mu^S) \, , \tag{7.66}$$

where the material dependent constants G and Q are assumed to be positive. One can recognize from (7.66) that mass exchange processes can only occur if the chemical potentials are different.

In the case of drying processes in a rigid porous solid the constitutive equations (7.66) can be simplified. With the assumption that there is no mass exchange with the solid phase, the supply term $\hat{\rho}^S$ is equal to zero. It follows with Eq. (4.8) that

$$\hat{\rho}^G = -\hat{\rho}^L \tag{7.67}$$

and from (7.64) the inequality

$$-\hat{\rho}^L(\mu^L - \mu^G) \geq 0 \, . \tag{7.68}$$

This inequality can be satisfied if we choose

$$\hat{\rho}^L = -V(\mu^L - \mu^G) \, , \tag{7.69}$$

where V also a material dependent constant.

For freezing and thawing processes we introduce a ternary model consisting of a porous body (S) with poreice (I) and an incompressible liquid (L), where the ice phase moves with the same velocity as the porous body. With the assumption that mass exchange occurs only between the ice and liquid phases, from (7.64) follows

$$-\hat{\rho}^L(\mu^L - \mu^I) \geq 0 \tag{7.70}$$

from which we derive with the preceding arguments

$$-\hat{\rho}^I = \hat{\rho}^L = -U(\mu^L - \mu^I) \tag{7.71}$$

with U as a material dependent constant.

We will elaborate on the constitutive relations (7.69) and (7.71) in the next sections.

b) Drying Processes

With the term drying the extraction of a liquid from a porous solid or from granular materials is described. Thereby a distinction is made between the mechanical drying processes, where the motion of liquid is caused by forces, and the thermic drying processes, where the draining is produced by heat. We have to differentiate between three phenomena concerning the thermic drying processes: first, the transport mechanism of heat into the porous medium, second, the phase transition of the liquid into steam, and third, the transport of the steam. The phase transition, exclusively caused by heat transfer, is denoted with evaporation and can occur in two different forms: At relatively small temperatures the transport of steam is performed via the surface of the liquid phase as diffusion into the gas phase. During the drying process the surfaces of the liquid moves into the inner part of the porous body. This process can be observed in capillary-porous solids if no reservoir is available. Obviously, no phase transition (liquid into gas) exists in the inner part of the porous body (Kneule, 1975).

At higher temperatures at the limit to the boiling point in particular, one can observe a completely different drying behavior in the porous medium. Phase transition occurs on the surface of the porous medium as well as in its inner part. The part of gas produced by phase transition in the form of bubbles, is statistically distributed over the liquid phase. The gas goes up in the form of bubbles contrary to the gravity force caused by uplift forces, and leaves the control volume in this way.

Instead of a description of the first phenomenon (evaporation on the surface of the liquid phase), compare, e.g. Kowalski (2000), in this section we focus exclusively on the second problem of drying, described above, namely on the phenomenon of phase transitions of the liquid phase into the gas phase inside the whole control volume at temperatures in the region of the boiling point. This kind of drying process can be clearly described on the basis of a ternary model consisting of a capillary-porous solid, an incompressible liquid, and a compressible gas. In a capillary-porous medium the drying and mass exchange processes, connected with drying, are influenced by heat propagation and the capillarity rise. Heat propagation will be produced in the inner part of the body by attaching an external heat source at the surface of the porous solid. The exchange of mass between liquid and gas phases occurs at a certain temperature and hydrostatic pressure. Steam leaves the control volume, whereby the motion of the liquid is influenced by capillary forces.

The goal of this section is the development of differential equations, which describe phase transitions. For this purpose we will consider a ternary model consisting of a rigid capillary-porous solid, an incompressible liquid, and a compressible gas.

Mass exchange processes between liquid and gas phases can only occur, if their chemical potentials have reached a determined value. Thereby the phenomenon of condensation (steam condenses at a determined temperature limit leading to a growth in mass in the liquid phase) or of evaporation (liquid vaporizes at a fixed temperature limit, which leads to an exchange in mass in the gas phase).

According to (7.69) and (7.67) the constitutive equation for the change of mass of the gas phase $\hat{\rho}^G$ is given by

$$\hat{\rho}^G = V(\mu^L - \mu^G) \, , \tag{7.72}$$

where V is a material dependent constant and μ^L and μ^G the chemical potentials defined as, see (7.65):

$$\mu^L = \psi^L - \frac{p^L}{\rho^L} - \frac{1}{2}\mathbf{v}_L \cdot \mathbf{v}_L, \quad \mu^G = \psi^G - \frac{p^G}{\rho^G} - \frac{1}{2}\mathbf{v}_G \cdot \mathbf{v}_G \, . \tag{7.73}$$

In the following paragraphs we investigate the constitutive equation (7.72) in connection with (7.73), where we neglect the terms with the velocities. At first we simplify the second term in (7.73)$_1$

$$\frac{p^L}{\rho^L} = \frac{n^L p}{n^L \rho^{LR}} = \frac{p}{\rho^{LR}} \, , \tag{7.74}$$

where we have considered Eq. (7.2). In a similar way we can rearrange the second term in (7.73)$_2$.

$$\frac{p^G}{\rho^G} = \frac{n^G p^{GR}}{n^G \rho^{GR}} = \frac{p^{GR}}{\rho^{GR}} \, . \tag{7.75}$$

In consideration of the constitutive law for ideal gases we have:

$$p^{GR} = \bar{R}\rho^{GR}\Theta, \quad \bar{R} = \frac{R}{M_r}, \quad M_r = \frac{\mu}{\mu_0} \, . \tag{7.76}$$

R is the ideal gas constant, M_r the relative molecular mass, μ the absolute mass and μ_0 the reference mass (see Müller, 1994). The above equation (7.75) can be reformulated and we obtain

$$\frac{p^G}{\rho^G} = \bar{R}\Theta \, . \tag{7.77}$$

From thermodynamic investigations (see de Boer and Didwania, 2004) it follows that

$$\psi^L = \hat{\psi}^L(\Theta), \quad \psi^G = \hat{\psi}^G(\rho^{GR}, n^G, \Theta) \, . \tag{7.78}$$

For the free Helmholtz energy function we choose the ansatz

$$\hat{\psi}^L(\Theta) = c_\nu^L \Theta \ln\Big(\frac{\Theta}{\Theta_0}\Big) \, , \tag{7.79}$$

where c_ν^L is the specific heat of the liquid and Θ_0 the reference temperature.

For the free Helmholtz energy function ψ^G of the gas phase considering $(7.78)_2$, $(7.9)_2$, and (7.76) we introduce

$$\hat{\psi}^G = \bar{R}\Theta \ln\Big(\frac{\rho^{GR}}{\rho_0^{GR}}\Big) + C(\Theta, n^G) \; , \tag{7.80}$$

where $C(\Theta, n^G)$ is an integration function and ρ_0^{GR} the density of the gas phase in the reference placement. The ansatz (7.80) can be reformulated in consideration of (7.76) to

$$\hat{\psi}^G = \bar{R}\Theta \ln\Big(\frac{\Theta_0 p^{GR}}{\Theta p_0^{GR}}\Big) + C(\Theta, n^G) \; , \tag{7.81}$$

with the initial temperature Θ_0 and the initial pressure p_0^{GR} of the gas phase in reference placement. With (7.73) and (7.81) as well as (7.77) we get the chemical potential of the gas phase in the form

$$\mu^G = \bar{R}\Theta \ln\Big(\frac{\Theta_0 p^{GR}}{\Theta p_0^{GR}}\Big) + C(\Theta, n^G) + \bar{R}\Theta \; , \tag{7.82}$$

which corresponds to the results of de Groot and Mazur (1969). In (7.82) we have neglected the term with the velocities. Finally, proceeding from (7.72) in connection with the equation (7.73) to (7.76) and (7.82) we obtain for the mass supply $\hat{\rho}^G$

$$\hat{\rho}^G = V[-\bar{R}\Theta \ln\Big(\frac{\Theta_0 p^{GR}}{\Theta p_0^{GR}}\Big) - C(\Theta, n^G) - \frac{p}{\rho^{LR}} - \bar{R}\Theta + f(\Theta)] \; , \tag{7.83}$$

where

$$f(\Theta) = \hat{\psi}^L(\Theta) \; . \tag{7.84}$$

In consideration of (7.84) Eq. (7.83) reads as:

$$\hat{\rho}^G = V[-\bar{R}\Theta \ln\Big(\frac{\Theta_0 p^{GR}}{\Theta p_0^{GR}}\Big) - C(\Theta, n^G) - \frac{p}{\rho^{LR}} + \hat{\psi}^L(\Theta) + \bar{R}\Theta] \; . \tag{7.85}$$

For the function $C(n^G)$ as a first approach we choose

$$C(n^G) = a \ln\Big(\frac{n^G}{n_{0G}^G}\Big) \tag{7.86}$$

with the letter a being a material dependent quantity and n_{0G}^G the volume fraction in the reference placement. Eq. (7.85) can be transformed with (7.79) and (7.86) into

$$\begin{aligned}\hat{\rho}^G = V[-\bar{R}\Theta\ln\Big(\frac{\Theta_0 p^{GR}}{\Theta p_0^{GR}}\Big) - a\ln\Big(\frac{n^G}{n_{0G}^G}\Big) - \frac{p}{\rho^{LR}} + \\ +c_\nu^L\Theta\ln\Big(\frac{\Theta}{\Theta_0}\Big) + \bar{R}\Theta]\ .\end{aligned} \tag{7.87}$$

With (7.87) initial and boundary value problems can be solved within the framework of FEM-formulation. The mass transfer to the gas phase ends when the liquid is completely evaporated. At this point $\hat{\rho}^G$ is equal to zero.

The numerical evaluation of the concept for the simulation of the thermal induced phase transition with temperatures at the boiling point is currently under study at the Institute of Mechanics in Essen.

The removal of liquid from the porous body causes shrinking and is the main reason for the stress induced by drying (see Kowalski, 2000). One can say that the saturated porous solid is a swelling, however, a stress free continuum. The interactions amongst the particles of granular materials are small in a complete saturated state. Their values increase, however, in the course of the drying process because the cohesion amongst the solid particles become larger. The moist porous solid contract during drying and the inhomogeneous shrinking causes stress in the body. This critical state occurs, when the boundary layer drys very fast and the inner part remains in a moist state. In such a case large tension stresses arise in the boundary layer and the surface wharpens and cracks. For the assessment of the danger of cracks a failure condition is necessary. This can be derived from a condition developed by de Boer (1988) and extended by de Boer and co-workers in several papers (see de Boer, 2000b). This condition contains the yield conditions proposed by von Mises (1913) and Green (1972) as special cases. It can also be applied to purely hydrostatic stress states appearing partly during drying processes in porous bodies. So far, the first investigations have been very promising.

c) Freezing Processes

A ternary model consisting of a porous solid, a further solid (ice) and a poreliquid will be treated. All constituents are assumed to be materially incompressible and the ice-phase φ^I possesses the same motion function as the porous solid phase.

Mass exchange occurs only between the constituent liquid and ice, i.e.

$$\begin{aligned}\hat{\rho}^L = -\hat{\rho}^I = -U(\mu^L - \mu^I)\ ,\\ \hat{\rho}^S = 0\ ,\end{aligned} \tag{7.88}$$

where U is a material dependent number, which depends on temperature.

Numerical treatment has shown that the influence of velocities and also of the poreliquid pressure (poresize: 1mm to 100 nm) in chemical potentials phase transition is negligible small. Therefore, for the mass supply of the liquid one can choose the following simplified ansatz:

$$\hat{\rho}^L = -\hat{\rho}^I = -U(\psi^L - \psi^I) \, . \tag{7.89}$$

With the assumption that the free Helmholtz energy functions concerning the phase transition depends only on temperature, Eq. (7.89) can be written as (see de Boer *et al.*, 2003)

$$\hat{\rho}^L = -\hat{\rho}^I = -U(c^L - c^I)\Big(\Theta \ln \frac{\Theta}{\Theta_0} - \Theta + \Theta_0\Big) + C \, . \tag{7.90}$$

with c^L and c^I as the heat storage capacities.
The constant C is founded in the fact that the process of ice formation occurs spontaneously. The cause for this spontaneous process lies in the change of entropy in the surroundings of the liquid. It is known that a transition from a disordered state (liquid) into a state of higher order (ice) can never occur spontaneously, see Atkins (1993). The solidification of liquid is an exothermal process, whereby heat propagates from the system into the surroundings and, thus, increases the disorder. When the disorder in the surroundings is larger than a decrease in the disorder of a system, the total disorder increases and the process is spontaneous. This spontaneity is described by the constant C in this work, which contains a material dependent value by the occurrence of the phase transition, otherwise becomes zero. When ice production takes place the energy of the liquid is reduced abruptly. In order to get an idea of the course of mass supply of the liquid constituent, we will focus on the factor U and on the mass supply $\hat{\rho}^L$.

It is easily understandable that the production of ice or a phase transition occurs only when the zero-grade-Celsius-limit is reached, i. e. the factor U may not be zero at a temperature of 273.15 K. We choose the following ansatz for U:

$$U = \frac{1}{\mathrm{g}}(273.2 - \Theta)^{\mathrm{d}} \, . \tag{7.91}$$

At temperatures above the freezing point no phase transition can be observed between the constituents water and ice; the factor U becomes zero in this region. In (7.91) g describes a material dependent constant, which governs the amount of mass supply and the exponent d describes the course of the function. Finally, with (7.91) the Eq. (7.90) delivers the growth of mass in dependence of the temperature. In order to be able to describe the exact extension of the ice constituent, a constitutive equation for volume strain is necessary. We choose

$$V(\Theta) = V_0[1 + 3\alpha_0^I(\Theta - \Theta_0)] \tag{7.92}$$

and for α_0^I:

$$\alpha_0^I(\Theta) = -7 \cdot 10^{-3} - 3 \cdot 10^{-7}(\Theta - 291.15) \quad \left[\frac{1}{K}\right]. \tag{7.93}$$

For further information see de Boer *et al.* (2003).

For the solution of initial and boundary value problems the above derived constitutive equations in connection with special constitutive equations for special models and the material-independent balance equations are required. Analytical solutions are seldom, therefore, in most cases, numerical solutions are preferred with the aid of the finite element method. The necessary equations are derived with the standard Galerkin-method. In this way the course of ice-zones is calculated for different times compared with experimental data in de Boer *et al.* (2003), see Figures 7.7.1 through 7.7.4.

The numerical results, represented in Figures 7.7.2 through 7.7.4 show that thermodynamical processes in porous media can be well simulated with the introduced assumptions concerning the phase transition. With the computation of the ice formation between two freezing pipes it could be shown that concrete initial and boundary value problems can be described and solved with the proposed model. This opens further perspectives for the computation, e.g., of front propagations, if different soil layers exist.

Due to new findings in the constitutive theory of saturated and unsaturated capillary-porous and non-capillary-porous solids, namely the identification of the capillary volume forces, the treatment of the compressibility of the constituents as well as the development of ternary models, the fundamental effects in porous media, namely uplift, friction, capillarity, effective stress, and phase transitions have been revisited.

Some classical result of the uplift and friction problems could be confirmed, others have to be extended such as the effective stress concept, and finally the results of the investigations of the capillary phenomenon and the phase transitions have led to new findings. It seems that the fundamental effects in porous media are completely cleared up now.

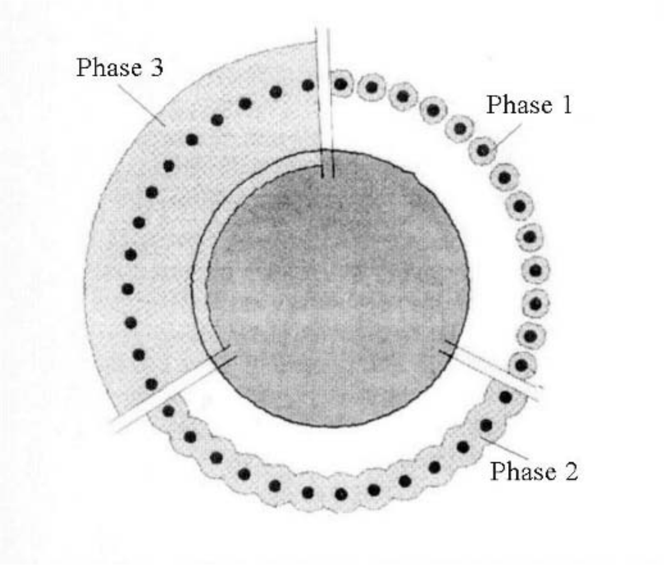

Fig. 7.7.1: Course of the frost zones at different times (experimental datas) (phase 1: 0-20 h, phase 2: 20-40 h, phase 3 $>$ 40 h).

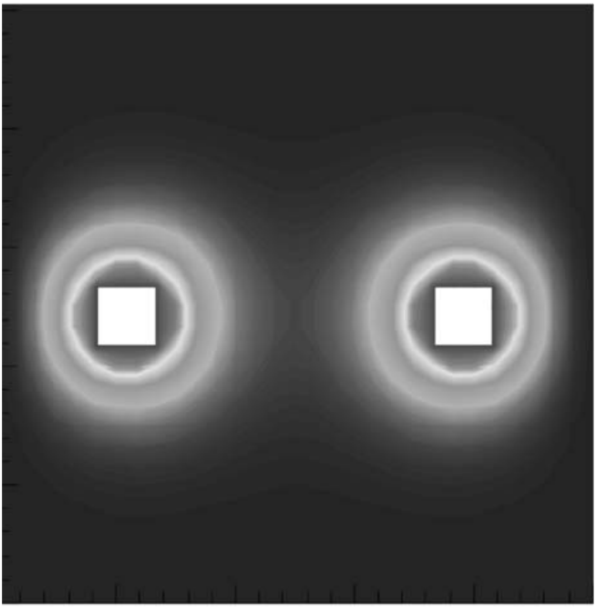

Fig. 7.7.2: Course of the frost zones after ten hours (numerical simulation).

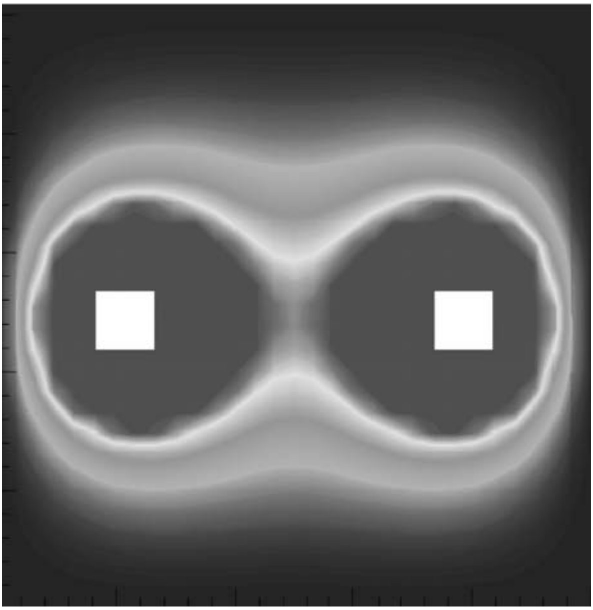

Fig. 7.7.3: Course of the frost zones after twenty hours (numerical simulation).

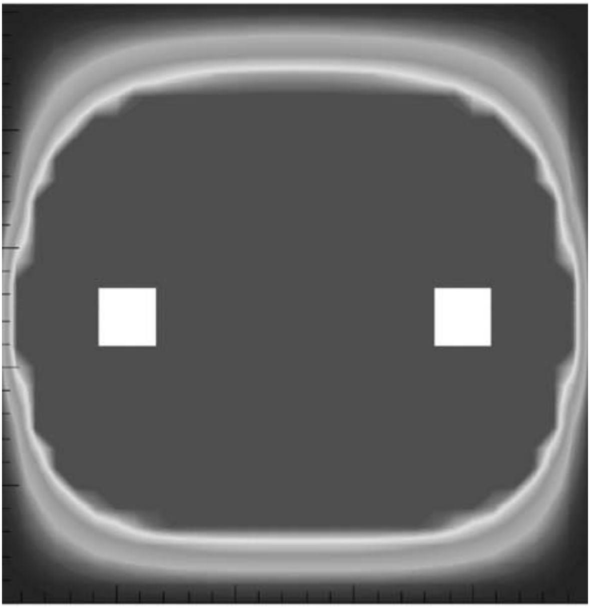

Fig. 7.7.4: Course of the frost zones after 2.5 days (numerical simulation).

Chapter 8

POROELASTICITY

8.1 Introduction

In general, poroelasticity is understood as the pendant to linear elasticity of a one-component elastic solid. That means that poroelasticty is devoted exclusively to saturated and/or unsaturated elastic porous solids within the framework of the geometrically and physically linear theory with all its consequences. Poroelasticity was obviously generated by Biot in several articles (see Biot and Willis, 1957). He combined stress strain relations of the classical linear elasticity theory for the empty porous solid with some constitutive equations of the contents in the pores and interaction effects. However, Biot's poroelasticity is an ad-hoc theory, which is not mainly based on profound mechanical axioms and principles. Clear definitions and terms are missing in certain parts. In addition, terms are omitted in the balance equations, and unnecessary approximations are used which greatly restrict the theory. Moreover, some derivations are obscure. Therefore we proceed from the ensured fundamental equations of the theory of porous media (TPM), derived in the preceding sections.

As already stated above poroelasticity is based on the geometrically linear theory. This means that in the strain tensor only linear terms of the displacement gradients are contained and in the balance equations the dependence of all introduced terms on the actual placement can be approximately replaced with the reference placement. Thus, Cauchy's stress tensor, e.g., is approximately equal to the first and second (symmetric) Piola-Kirchhoff stress tensors. Bearing these approximations in mind we disclaim extra notations of the mechanical terms and retain the notations used in the proceedings sections, although all definitions in poroelasticity depend on the reference placement. This statement is also valid for the mathematical operations such as gradient, divergence, curl, and Laplace operation.

We will develop the fundamental equations for a ternary model consisting of a capillary compressible elastic porous solid as well as compressible and incompressible gas and liquid phases.

However, for the investigations of basic solutions and dynamic problems, we restrict our investigations to incompressible elastic porous solids because the new constitutive equations for compressible constituents and capillary porous solids have been used only for special problems and the corresponding fundamental solutions for the field equations have not been developed yet.

Therefore, only this model will be addressed with some investigations of basic solutions and dynamic problems (wave propagations), which had been analytically solved at the University of Essen since the 1990s.

The main result of the constitutive theory are the following constitutive relations for the stress of the compressible elastic solid phase $\mathbf{T}^S$ (partial stress) and p^{SR} (real hydrostatic stress of solid material), the liquid and gas phases and the interaction forces $\hat{\mathbf{p}}^S$, $\hat{\mathbf{p}}^L$, $\hat{\mathbf{p}}^G$, for further information see de Boer (2000 a), de Boer and Didwania (2000, 2002, 2004):

$$\mathbf{T}^S \;=\; -n^S p\,\mathbf{I} + \mathbf{T}^S_E\,, \tag{8.1}$$

where p is the liquid or gas pressure, and the effective stresses $\mathbf{T}^S_E$ are restricted by the constitutive equation

$$\mathbf{T}^S_E \;=\; -\rho^S\,\frac{\partial\psi^S}{\partial\mathbf{E}_S}\,, \tag{8.2}$$

where ψ^S is the free Helmholtz energy function. The additive decomposion of the total stress $\mathbf{T}^S$ in a part of the liquid or gas pressure p and the effective stress, which depends on the motion of the partial solid, is a consequence of the saturation constraint (see de Boer, 2000 a).

With an appropriate ansatz for the free Helmholtz energy function and the additive decomposition of the strain tensor $\mathbf{E}_S$ in a deviatoric part $\mathbf{E}^D_S$ and a volumetric part $(\mathbf{E}_S\cdot\mathbf{I})\mathbf{I}$, we obtain

$$\mathbf{T}^S_E \;=\; 2\mu^S\mathbf{E}^D_S + K^S(\mathbf{E}_S\cdot\mathbf{I})\mathbf{I}\,, \tag{8.3}$$

where the partial compression modulus K^S is connected with the Lamé constants μ^S, λ^S through

$$K^S \;=\; \frac{2}{3}\mu^S + \lambda^S\,. \tag{8.4}$$

It is desirable to express the partial material constant K^S through quantities of the real solid material and of the empty porous solid. On the one hand these material parameters can in many cases be determined experimentally in a simple way, on the other hand this procedure points the immediate consideration of the incompressibility of the real material. This is however not possible for the Lamé constant μ^S because this would require the consideration of the shearing of the real material. Because there is no balance principle or constitutive relation for the real shearing a split of $\mathbf{E}_S^D$ is not meaningful.

However, the situation concerning the compression modulus is completely different because there exists a constitutive equation for the volumetric strain of the real solid material. This important feature of the constitutive theory will be revealed in the following paragraphs.

The constitutive equation (8.3) is a result of the exploitation of the entropy inequality if the free Helmholtz energy function is assumed to be dependent on the strain tensor $\mathbf{E}_S$. The volumetric strain $e_S = \mathbf{E}_S \cdot \mathbf{I}$ can be additively decomposed according to (3.25) into e_{SR} and e_{SN} within the frame of the geometrically linear theory. These volumetric strains, namely of the real material and of the change of the pores in size, are completely independent and it is advisable to require that the free Helmholtz energy function depends on e_{SR} and e_{SN} instead of e_S, because it can be expected that one gets more information in the new constitutive equations gained from the evaluation of the entropy inequality. From the entropy inequality we obtain with the new parameters e_{SR} and e_{SN} the following two relations for the hydrostatic stress and strain states (see de Boer, 2000 a):

$$p^S \;=\; \frac{1}{3}(\mathbf{T}^S \cdot \mathbf{I}) \;=\; -n^S p + n^S p_E^{SN} \;, \tag{8.5}$$

where p_E^{SN} depends on e_{SN} and

$$p^S \;=\; n^S p_E^{SR} \;, \tag{8.6}$$

where p_E^{SR} is dependent on e_{SR}.

Within the geometrical linear theory the following constitutive equations are proposed

$$p_E^{SN} \;=\; K^{SN} e_{SN} \tag{8.7}$$

and

$$p_E^{SR} \;=\; K^{SR} e_{SR} \;. \tag{8.8}$$

The compression moduli K^{SN} and K^{SR}, representing the compression modulus of the empty porous solid and the compression modulus of the real solid material which can be experimentally determined easily (see Biot and Willis 1957). With (8.7) and (8.8) the constitutive equations (8.5) and (8.6) can be rewritten:

$$p^S \;=\; -n^S p + n^S K^{SN} e_{SN} \;, \tag{8.9}$$

$$p^S \;=\; n^S K^{SR} e_{SR} \;. \tag{8.10}$$

If the liquid and gas phases are compressible, then the hydrostatic stress states of the real materials can be expressed by constitutive equations (see de Boer, 2000 a).

In order to bring the constitutive equations (8.9) and (8.10) into the form (8.3), where the stress and the strain tensors are partial quantities, some manipulations must be performed.

At first the real volumetric strain e_{SR} will be replaced with e_S and e_{SN} owing to the additive decomposition of e_S.

$$e_{SR} \;=\; e_S - e_{SN} \;. \tag{8.11}$$

Then, from (8.9) and (8.10) combined with (8.11) we have

$$n^S K^{SR} e_S - n^S K^{SR} e_{SN} \;=\; -n^S p + n^S K^{SN} e_{SN} \tag{8.12}$$

or

$$e_{SN} \;=\; \frac{1}{K^{SR} + K^{SN}}\,(p + K^{SR} e_S)\;. \tag{8.13}$$

With (8.13) we can determine the partial hydrostatic pressure p^S (8.9)

$$p^S \;=\; -n^S p \left(1 - \frac{K^{SN}}{K^{SR} + K^{SN}}\right) + n^S\,\frac{K^{SN} K^{SR}}{K^{SR} + K^{SN}}\,e_S \tag{8.14}$$

or

$$p^S \;=\; -n^S\,\frac{1}{1 + \dfrac{K^{SN}}{K^{SR}}}\,p \;+\; n^S\,\frac{K^{SN}}{1 + \dfrac{K^{SN}}{K^{SR}}}\,e_S\;. \tag{8.15}$$

In the special case, if the solid material is incompressible – that means $K^{SR} \to \infty$ – the relations (8.14) or (8.15) turn to

$$p^S \;=\; -n^S p \;+\; n^S K^{SN} e_S\;. \tag{8.16}$$

Finally, we obtain from (8.1), (8.3), (8.5) and (8.14) with

$$\mathbf{T}^S \;=\; \mathbf{T}^{SD} \;+\; \frac{1}{3}(\mathbf{T}^S \cdot \mathbf{I}) \;=\; \mathbf{T}^{SD} \;+ p^S \mathbf{I}\,, \tag{8.17}$$

$$\begin{aligned}\mathbf{T}^S \;=\; & -n^S p \left(1 - \frac{K^{SN}}{K^{SR} + K^{SN}}\right)\mathbf{I} \;+\; 2\mu^S \mathbf{E}_S^D \;+ \\ & + n^S\,\frac{K^{SN} K^{SR}}{K^{SR} + K^{SN}}\,e_S\;.\end{aligned} \tag{8.18}$$

or

$$\mathbf{T}^S \;=\; -n^S p\,(1 - B^S)\mathbf{I} \;+\; 2\mu^S \mathbf{E}_S^D \;+\; \hat{K}^S e_S\,, \tag{8.19}$$

where we have introduced the abbreviations

$$B^S = \frac{K^{SN}}{K^{SR} + K^{SN}} , \tag{8.20}$$

$$\hat{K}^S = n^S K^{SR} B^S . \tag{8.21}$$

For further considerations, namely for the development of the fundamental field equations in elastodynamics a slight reformulation of (8.19) is necessary in order to be able to compare the constitutive equation for the elastic behavior of the solid skeleton and the field equations with those of the classical linear elasticity theory. For this reason we replace $\mathbf{E}_S^D$ with $\mathbf{E}_S$ in (8.19). This procedure leads to

$$\mathbf{T}^S = -n^S p\,(1 - B^S)\mathbf{I} + 2\mu^S \mathbf{E}_S + \hat{\lambda}^S e_S \mathbf{I} , \tag{8.22}$$

with

$$\hat{\lambda}^S = \hat{K}^S - \frac{2}{3}\mu^S . \tag{8.23}$$

It should be mentioned that the quantity B^S (8.20) in the first term on the right-hand of (8.19) and (8.22) is similar to the Biot factor (see Biot and Willes, 1957) with the difference being that the denominator in the Biot factor contains only the compression modulus of the real material K^{SR} whereas in (8.20) the denominator consist of the sum of the compression modulus of the real materials K^{SR} and the compression modulus of the solid skeleton K^{SN}. Only in the case where K^{SN} is very small in comparison with K^{SR} are both results approximately equal.

Moreover, for the inviscid liquid and gas phases thermodynamic investigations reveal the following constitutive equations, if the liquid phase is considered as the carrier of the gas bubbles:

$$\mathbf{T}^L = -n^L p\,\mathbf{I} \tag{8.24}$$

and

$$\mathbf{T}^G = -n^G p\,\mathbf{I} + \mathbf{T}_E^G , \tag{8.25}$$

where

$$\mathbf{T}_E^G = p_E^G\,\mathbf{I} \tag{8.26}$$

with

$$\mathbf{p}_E^G = \rho^G \frac{\partial \psi^G}{\partial n^G} \tag{8.27}$$

is the effective stress for the gas phase.

With appropriate constitutive ansätze for (8.27), p_E^G can be expressed by the displacement vector $\mathbf{u}_G$. As of now, there is little experience with these constitutive equations. Thus, we maintain p_E^G in the fundamental equations.

In order to be able to formulate the fundamental field equations in poroelasticity in the next section the constitutive equations for the interaction force $\hat{\mathbf{p}}^S$ must also be addressed. This has been done by de Boer and Didwania (2000, 2002, 2004) for a capillary-porous solid field with liquid and gas

$$\begin{aligned} \hat{\mathbf{p}}^S \quad = \quad & p \text{ grad } n^S - \hat{t}^{SL} \text{grad } \rho^L + \\ & + S_L(\mathbf{v}_L - \mathbf{v}_S) - S_G(\mathbf{v}_G - \mathbf{v}_S) \;, \end{aligned} \tag{8.28}$$

$$\begin{aligned} \hat{\mathbf{p}}^L \; = \; & p \text{ grad } \; n^L + (\hat{t}^{SL} + \hat{t}^{LG}) \text{grad } \rho^L - \\ & - S_L(\mathbf{v}_L - \mathbf{v}_S) \;, \end{aligned} \tag{8.29}$$

$$\begin{aligned} \hat{\mathbf{p}}^G \; = \; & p \text{ grad } \; n^G - \hat{t}^{LG} \text{grad } \rho^L + \\ & + S_G(\mathbf{v}_G - \mathbf{v}_S) \;. \end{aligned} \tag{8.30}$$

In (8.28) to (8.30) the second term on the right-hand side are the capillary volume forces depending on the gradient of the liquid density ρ^L and the interaction effects between solid and liquid as well as between liquid and gas manifested by $\hat{t}^{SL}$ and $\hat{t}^{GL}$. The last term on the right-hand side describes the friction volume force between liquid and solid as well as between gas and solid, where S_L, S_G are scalar response parameters. The corresponding friction forces between liquid and gas can be neglected as a result of their small effects.

8.2 The Fundamental Field Equations of Poroelasticity

In this section we summarize briefly the fundamental field equations of poroelasticity, whereby we do not differentiate in the mathematical operations between the actual and reference placements, because within the framework of the geomerically linear theory both placements are approximately equal (see, e.g., de Boer, 1982). These field equations describing the motion of a saturated elastic porous body consist of the

strain-displacement relation according to $(3.52)_1$

$$\begin{aligned} \mathbf{E}_S \; &= \; \frac{1}{2}\,(\nabla \mathbf{u}_S \; + \; \nabla^T \mathbf{u}_S), \\ e_S \; &= \; \mathbf{E}_S \cdot \mathbf{I} \; = \; \operatorname{div} \mathbf{u}_S \;, \end{aligned} \tag{8.31}$$

were ∇ stands for grad,

the density-displacement relations

$$\text{grad } \rho^L \; = \; -\rho_{0L}^L (\nabla\nabla \mathbf{u}_L)^{\overset{13}{T}} \, \mathbf{I} \;, \tag{8.32}$$

which follows from (3.16), (4.9) and (8.31),

the equation of motion for the partial phases (4.15)

$$\operatorname{div}\mathbf{T}^\alpha + \rho^\alpha\mathbf{b} + \hat{\mathbf{p}}^\alpha = \rho^\alpha(\mathbf{u}_\alpha)''_\alpha \,, \tag{8.33}$$

thereby we have assumed a common external acceleration $\mathbf{b}$, and have expressed acceleration $\mathbf{a}_\alpha$ by the material time derivative of the displacement vectors $\mathbf{u}_\alpha$, see de Boer (2000 a),

the constitutive relations (8.22) *and* (8.24) *to* (8.30)

$$\mathbf{T}^S \; = \; -n^S p\,(1 - B^S)\mathbf{I} \; + \; 2\mu^S\mathbf{E}_S \; + \; \hat{\lambda}^S e_S\mathbf{I} \,, \tag{8.34}$$

$$\mathbf{T}^L \; = \; -n^L p \,, \tag{8.35}$$

$$\mathbf{T}^G \; = \; -n^G p\,\mathbf{I} \; + \; \mathbf{p}_E^G\,\mathbf{I} \,, \tag{8.36}$$

$$\begin{aligned}\hat{\mathbf{p}}^S \; = \; p \text{ grad } \; n^S - \hat{t}^{SL}\,\text{grad } \rho^L \; & - S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] \\ & + S_G[(\mathbf{u}_G)'_G - (\mathbf{u}_S)'_S] \,,\end{aligned} \tag{8.37}$$

$$\begin{aligned}\hat{\mathbf{p}}^L \; = \; p \text{ grad } \; n^L \; & + (\hat{t}^{SL} + \hat{t}^{LG})\,\text{grad } \rho^L \; + \\ & + S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] \,,\end{aligned} \tag{8.38}$$

$$\begin{aligned}\hat{\mathbf{p}}^G \; = \; p \text{ grad } \; n^G \; & - \; \hat{t}^{LG}\,\text{grad } \rho^L \; - \\ & - \; S_G[(\mathbf{u}_G)'_G - (\mathbf{u}_S)'_S] \,,\end{aligned} \tag{8.39}$$

whereby the different velocities and accelerations have been replaced with the time derivatives of the displacement vectors $\mathbf{u}_\alpha$.

We assume that the porous solid is homogenous and isotropic; therefore the material response parameters are not dependent on the position $\mathbf{x}$. With the calculation rules

$$\operatorname{div}(\mu^S\nabla\mathbf{u}_S) \; = \; \mu^S\triangle\mathbf{u}_S \,, \tag{8.40}$$

$$\operatorname{div}(\mu^S\nabla^T\mathbf{u}_S) \; = \; \mu^S\nabla\operatorname{div}\mathbf{u}_S \,, \tag{8.41}$$

$$\operatorname{div}\,[\hat{\lambda}^S\nabla(\operatorname{div}\mathbf{u}_S)\,\mathbf{I}] \; = \; \hat{\lambda}^S\triangle\operatorname{div}\mathbf{u}_S \,, \tag{8.42}$$

$$\mathrm{div}\,[n^S p(1-B^S)\mathbf{I}] \;=\; n^S(1-B^S)\nabla p\,, \tag{8.43}$$

wherein $\triangle(...)$ denotes the Laplace and ∇ the gradient operator, the fundamental field equations take the following forms inserting the constitutiv equations (8.34) through (8.39) into the equations of motion (8.33):
for the porous solid

$$\begin{aligned} &(\hat{\lambda}^S \;+\; \mu^S)\,\mathrm{grad\,div}\,\mathbf{u}_S \;+\; \mu^S\triangle\,\mathbf{u}_S \;-\; n^S(1-B^S)\nabla p \,+ \\ &+\; B^S p\nabla n^S \;-\; \hat{t}^{SL}\rho^L_{0L}(\nabla\nabla\mathbf{u}_L)^{\overset{13}{T}}\,\mathbf{I} \;-\; S_L[(\mathbf{u}_S)'_S - (\mathbf{u}_L)'_L] \;- \\ &-\; S_G[(\mathbf{u}_S)'_S - (\mathbf{u}_G)'_G] \;+\; \rho^S\mathbf{b} \;=\; \rho^S(\mathbf{u}_S)''_S\,, \end{aligned} \tag{8.44}$$

for the liquid

$$\begin{aligned} &-\; n^L\nabla p \;+\; (\hat{t}^{SL}+\hat{t}^{LG})\rho^L_{0L}(\nabla\nabla\mathbf{u}_L)^{\overset{13}{T}}\mathbf{I} \;- \\ &-\; S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] \;+\rho^L\mathbf{b} \;=\; \rho^L(\mathbf{u}_L)''_L\,, \end{aligned} \tag{8.45}$$

for the gas

$$\begin{aligned} &-\; n^G\nabla p \;+\; \nabla p^G_E \;-\; \hat{t}^{LG}\rho^L_{0L}(\nabla\nabla\mathbf{u}_L)^{\overset{13}{T}}\mathbf{I} \;- \\ &-\; S_G[(\mathbf{u}_G)'_G - (\mathbf{u}_S)'_S] \;+\rho^G\mathbf{b} \;=\; \rho^G(\mathbf{u}_G)''_G\,. \end{aligned} \tag{8.46}$$

In addition, the saturation condition (2.2) has the differential form

$$\begin{aligned} &(n^S)'_S \;+\; (n^L)'_L \;+\; (n^G)'_G \;-\; \nabla n^L\cdot[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] \;- \\ &-\; \nabla n^G\cdot[(\mathbf{u}_G)'_G - (\mathbf{u}_S)'_S] \;=\; 0\,. \end{aligned} \tag{8.47}$$

The set of fundamental equations (8.44) to (8.47) constitute the governing equations of the boundary and initial value problems for liquid and gas filled linear elastic porous solids if appropriate boundary conditions and initial conditions are presented. The system of the fundamental equations represents a counterpart to the linear elasticity theory of one-component elastic bodies. The equations serve particularly as the basic equations for numerical investigations. From the above set of equations the weak formulations of dynamics within the framework of the finite element method can easily be derived and can be introduced into existing computer programs. Eqs. (8.44),(8.45), (8.46), and (8.47) turn over to the corresponding fundamental equations, of a binary incompressible elastic, non-capillary porous medium, already widely used (see, e.g, Liu and de Boer, 1999)

$$\begin{aligned}&(\lambda^S + \mu^S)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}_S + \mu^S \triangle \mathbf{u}_S - n^S \nabla p - \\ &- S_L[(\mathbf{u}_S)'_S - (\mathbf{u}_L)'_L] + \rho^S \mathbf{b} = \rho^S (\mathbf{u}_S)''_S \,,\end{aligned} \tag{8.48}$$

$$-n^L \nabla p - S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] + \rho^L \mathbf{b} = \rho^L (\mathbf{u}_L)''_L \,, \tag{8.49}$$

$$\mathrm{div}\left(n^S \frac{\partial \mathbf{u}_S}{\partial t} + n^L \frac{\partial \mathbf{u}_L}{\partial t} \right) = 0 \,, \tag{8.50}$$

where λ^S and μ^S are the Lamé constants setting $K^{SR} \rightarrow \infty$ in (8.20) through (8.23) and where the convective terms in (8.50) have been neglected.

In the next section the main equations of the binary incompressible elastic model are summarized and developed.

8.3 Main Equations for an Incompressible Binary Model

We will investigate a saturated non-capillary incompressible elastic porous solid filled with an incompressible liquid. First we will summarize all basic equations needed for the present investigations, excluding any mass exchange and considering only isothermal deformations.

1. Concept of volume fraction

$$n^S + n^L = 1. \tag{8.51}$$

2. Balance of mass

$$(\rho^\alpha)'_\alpha + \rho^\alpha \mathrm{div}\ \mathbf{x}'_\alpha = 0. \tag{8.52}$$

3. Balance of momentum

$$\mathrm{div}\ \mathbf{T}^\alpha + \rho^\alpha(\mathbf{b}^\alpha - \mathbf{x}''_\alpha) + \hat{\mathbf{p}}^\alpha = 0 \,. \tag{8.53}$$

with

$$\hat{\mathbf{p}}^S + \hat{\mathbf{p}}^L = 0 \,. \tag{8.54}$$

4. Balance of moment of momentum

$$\mathbf{T}^\alpha = (\mathbf{T}^\alpha)^T \,. \tag{8.55}$$

With the incompressibility condition

$$\rho^{\alpha R} = \text{const.} \tag{8.56}$$

the combination of (2.4), (8.52) and (8.56) yields

$$(n^\alpha)'_\alpha + n^\alpha \text{div}\, \mathbf{x}'_\alpha = 0 \;, \tag{8.57}$$

i.e., the equations of balance of mass reduced to equations of balance of volume fractions.

Then, by use of (8.51) and (3.9) the relation

$$(n^L)'_S = (n^L)'_L - \text{grad}\, n^L \cdot (\mathbf{x}'_L - \mathbf{x}'_S) \tag{8.58}$$

holds. Alternatively, one obtains in consideration of (8.57)

$$\text{div}(n^S \mathbf{x}'_S + n^F \mathbf{x}'_F) = 0 \;. \tag{8.59}$$

In the case of a common external acceleration $\mathbf{b} = \mathbf{b}^S = \mathbf{b}^L$ the balance equations of momentum (8.53) can be rewritten in the forms

$$\text{div}\, \mathbf{T}^S + \rho^S(\mathbf{b} - \mathbf{x}''_S) - \hat{\mathbf{p}}^L = 0 \;, \tag{8.60}$$

$$\text{div}\, \mathbf{T}^L + \rho^L(\mathbf{b} - \mathbf{x}''_L) + \hat{\mathbf{p}}^L = 0 \;, \tag{8.61}$$

where, Eqs. (8.54) has been used in addition.

As a consequence of the incompressibility constraint (8.56), the stress tensor and the interaction force are additively decomposed into two terms

$$\mathbf{T}^S = -\, n^S p\, \mathbf{I} + \mathbf{T}^S_E \;, \tag{8.62}$$

$$\mathbf{T}^L = -\, n^L p\, \mathbf{I} + \mathbf{T}^L_E \;, \tag{8.63}$$

$$\hat{\mathbf{p}}^L = p\ \text{grad}\ n^L + \hat{\mathbf{p}}^L_E \;, \tag{8.64}$$

where p denotes the poreliquid pressure of the incompressible poreliquid. In (8.62) to (8.64) the index $(...)_E$ expresses the effective quantities for which constitutive equations must be formulated. The insertion of Eqs. (8.62) to (8.64) into (8.60) and (8.61) leads to

$$\text{div}\, \mathbf{T}^S_E - n^S\ \text{grad}\ p + \rho^S(\mathbf{b} - \mathbf{x}''_S) - \hat{\mathbf{p}}^L_E = \mathbf{0} \;, \tag{8.65}$$

$$\text{div}\, \mathbf{T}^L_E - n^L\ \text{grad}\ p + \rho^L(\mathbf{b} - \mathbf{x}''_L) + \hat{\mathbf{p}}^L_E = \mathbf{0} \;. \tag{8.66}$$

For further considerations, it is convenient to replace the velocity $\mathbf{x}'_\alpha$ and acceleration $\mathbf{x}''_\alpha$ with the corresponding derivatives of the displacement vector

$$\mathbf{x}'_\alpha = (\mathbf{u}_\alpha)'_\alpha \,, \quad \mathbf{x}''_\alpha = (\mathbf{u}_\alpha)''_\alpha \,. \tag{8.67}$$

The following investigations are restricted to an isotropic, linear elastic porous solid filled with an incompressible liquid with negligible viscous properties. The constitutive equations for the effective stresses and the effective interaction force are:

$$\mathbf{T}^S_E = 2\mu^S \mathbf{E}_S + \lambda^S (\mathbf{E}_S \cdot \mathbf{I})\mathbf{I} \,, \tag{8.68}$$

$$\mathbf{T}^L_E \simeq \mathbf{O} \,, \tag{8.69}$$

$$\hat{\mathbf{p}}^L_E = -S_L(\mathbf{u}'_L - \mathbf{u}'_S) \,, \tag{8.70}$$

where μ^S and λ^S are the macroscopic Lamé constants of the porous solid; S_L describes the coupled interaction between the solid and fluid phases and is (Heinrich, 1938):

$$S_L = \frac{(n^L)^2 \gamma^{LR}}{k^L} \,. \tag{8.71}$$

The quantity k^L is the Darcy permeability coefficient of the porous medium and γ^{LR} the real density of the liquid.

The volume fraction n^S can be expressed by the Lagrangian strain tensor $\mathbf{E}_S$ by the integration of (8.57):

$$n^S = n^S_{0S} (\det \mathbf{F}_S)^{-1} = n^S_{0S} (1 + \mathbf{E}_S \cdot \mathbf{I})^{-1} \tag{8.72}$$

in the scope of infinitesimal deformations, all terms of higher order are neglected. Moreover, since $\mathbf{E}_S \cdot \mathbf{I} \ll 1$, n^S may be approximated as n^S_{0S} which is the solid volume fraction in the initial placement.

Now, inserting Eqs. (8.67) to (8.70) into (8.65) and (8.66) in consideration of (8.31) we may write the field of main equations as follows:

$$\begin{gathered}
(\mu^S + \lambda^S)\,\mathrm{grad}\ \mathrm{div}\,\mathbf{u}_S + \mu^S\,\mathrm{div}\,\mathrm{grad}\ \mathbf{u}_S - n^S\,\mathrm{grad}\ p\ + \\
+\rho^S[\mathbf{b} - (\mathbf{u}_S)''_S] + S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0} \,, \\
-n^L\,\mathrm{grad}\ p + \rho^L[\mathbf{b} - (\mathbf{u}_L)''_L] - S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0} \,, \\
\mathrm{div}[n^S(\mathbf{u}_S)'_S - n^L(\mathbf{u}_S)'_S] = 0 \,.
\end{gathered} \tag{8.73}$$

Within the framework of the infinitesimal theory, the superposition principle holds, i.e., loading through body forces and external forces can be treated separately. Thus, it may sometimes be convenient to write the main equations (8.73) without the body forces:

$$
\begin{aligned}
&(\mu^S + \lambda^S)\,\mathrm{grad}\ \mathrm{div}\,\mathbf{u}_S + \mu^S\,\mathrm{div}\,\mathrm{grad}\ \mathbf{u}_S - n^S\,\mathrm{grad}\ p - \\
&\quad -\rho^S(\mathbf{u}_S)''_S + S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0}\ , \\
&-n^L\,\mathrm{grad}\ p - \rho^L(\mathbf{u}_L)''_L - S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0}\ , \\
&\mathrm{div}[n^S(\mathbf{u}_S)'_S - n^L(\mathbf{u}_S)'_S] = 0\ .
\end{aligned}
\tag{8.74}
$$

The set of equations (8.73), or without body forces, the versions (8.74) are the fundamental statement of boundary and initial value problems for saturated elastic porous media if appropriate boundary and initial conditions are given.

8.4 Basic Solutions for an Incompressible Binary Model

For investigating boundary value problems of the theory of elasticity and thermoelasticity with the boundary integral equation method (potential method) it is necessary to construct fundamental solutions of systems of partial differential equations and establish their basic properties.

Several methods to construct fundamental solutions of systems of differential equations of the classical theory of elasticity and thermoelasticity are known (see, Kupradze *at al.*, 1979 and Nowacki, 1962). Useful information on fundamental solutions of differential equations is contained in the book by Hörmander (1983).

The fundamental solutions in the linear theory of porous-elasticity (Biot theory) are established by Cleary (1977), Cheng and Liggett (1984) (see, also Rudnicki, 1981). The fundamental solution in the linear theory of elastic materials with voids is constructed by Pompei and Scalia (1994). The fundamental solutions of equations of the linear theory of binary mixtures for elastic and thermoelastic solids are constructed by Svanadze (1988, 1990, 1996).

The following investigations are based on two papers by de Boer and Svanadze (2004) and Svanadze and de Boer (2005).

a) Fundamental Solution of the System of Equations of Steady Oscillations in the Theory of Fluidsaturated Porous Media

In this section the linear theory of the incompressible liquidsaturated porous medium is considered, and a brief introduction to the incompressible porous media model by Bowen (1980) and de Boer (1992) is presented. Then, the fundamental solution of equations of the steady oscillations of the theory of incompressible fluidsaturated poroelastic solids is constructed by means of elementary functions, and basic properties are established.

Field equations

Within the framework of the general porous media theory, see Bowen (1980) and de Boer (1992), the liquidsaturated porous medium is viewed as a mixture of superimposed, but immiscible, incompressible constituents with particles of solid and liquid phases.

In the following the porous medium is restricted to the range of small deformations with all consequences concerning the stress tensors and the equation of motion and the investigations are limited to an isotropic, linear-elastic porous solid filled with an ideal liquid; in other words, the investigations on the porous solid are performed within the framework of the geometrically and physically linear theory.

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the points of the Euclidean three-dimensional space E^3, $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $\mathbf{D}_{\mathbf{x}} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, and let t denote the time variable.

The balance equations of mass and momentum of the theory of incompressible fluidsaturated poroelastic solids can be written as (de Boer, 1992, 2000 a, de Boer and Liu, 1994, 1995)

$$
\begin{gathered}
\mu^S \triangle \mathbf{u}_S + (\lambda^S + \mu^S)\,\mathrm{grad\,div}\;\mathbf{u}_S - n^S\,\mathrm{grad}\,p + \\
+\rho^S[\mathbf{b}_S - (\mathbf{u}_S)''_S] + S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0}\ , \\
-n^L\,\mathrm{grad}\,p + \rho^L[\mathbf{b}_L - (\mathbf{u}_L)''_L] - S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0}\ , \\
\mathrm{div}[n^S(\mathbf{u}^S)'_S + n^L(\mathbf{u}_L)'_L] = 0\ .
\end{gathered}
\tag{8.75}
$$

If the external accelerations $\mathbf{b}_S$ and $\mathbf{b}_L$ are assumed to be absent and the convective terms in the material time derivatives are neglected as well as the displacements $\mathbf{u}_S$, $\mathbf{u}_L$ and the pressure p are postulated to have a harmonic time variation, i.e.

$$
\begin{aligned}
\mathbf{u}_S(\mathbf{x}, t) &= \mathbf{u}(\mathbf{x})e^{-i\omega t}\ , \\
\mathbf{u}_L(\mathbf{x}, t) &= \mathbf{w}(\mathbf{x})e^{-i\omega t}\ , \\
p(\mathbf{x}, t) &= \tilde{p}(\mathbf{x})e^{-i\omega t}\ ,
\end{aligned}
\tag{8.76}
$$

then from the system of equations of motion (8.75) we obtain the following system of equations of steady oscillations

$$\begin{gathered}\mu^S \,\Delta\mathbf{u} + (\lambda^S + \mu^S)\ \operatorname{grad}\operatorname{div}\ \mathbf{u} + \omega^2\rho^S\ \mathbf{u} + \\ + i\,\omega S_L(\mathbf{u}-\mathbf{w}) - n^S\ \operatorname{grad}\tilde{p} = \mathbf{0}\ , \\ \omega^2\rho^L\ \mathbf{w} - i\,\omega S_L(\mathbf{u}-\mathbf{w}) - n^L\ \operatorname{grad}\tilde{p} = \mathbf{0}\ , \\ n^S\ \operatorname{div}\ \mathbf{u} + n^L\ \operatorname{div}\ \mathbf{w} = 0\ ,\end{gathered} \tag{8.77}$$

where ω is the oscillation frequency $(\omega > 0)$ and where in (8.77)$_3$ the rates of volume fractions have been neglected.

We introduce the matrix differential operator

$$\mathbf{A}(\mathbf{D_x}) = \|A_{lj}(\mathbf{D_x})\|_{7\times7}\ , \tag{8.78}$$

where

$$\begin{gathered}A_{lk}(\mathbf{D_x}) = (\mu^S\Delta + \beta_S)\delta_{lk} + (\lambda^S + \mu^S)\frac{\partial^2}{\partial x_l \partial x_k}\ , \\ A_{l,k+3}(\mathbf{D_x}) = A_{l+3,k}(\mathbf{D_x}) = -\beta_3\delta_{lk}\ , \\ A_{l+3,k+3}(\mathbf{D_x}) = \beta_L\delta_{lk}, \\ A_{7l}(\mathbf{D_x}) = -A_{l7}(\mathbf{D_x}) = n^S\frac{\partial}{\partial x_l}\ , \\ A_{7,l+3}(\mathbf{D_x}) = -A_{l+3,7}(\mathbf{D_x}) = n^L\frac{\partial}{\partial x_l}\ , \\ A_{77}(\mathbf{D_x}) = 0, \qquad l,k = 1,2,3,\end{gathered} \tag{8.79}$$

and δ_{lp} is the Kronecker delta, $\beta_j = \omega^2\rho^j + \beta_3\ (j = S, L),\ \beta_3 = i\omega S_L$.

The system (8.77) can be written as

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = 0\ , \tag{8.80}$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \tilde{p})$.

Definition. The fundamental matrix of operator $\mathbf{A}(\mathbf{D_x})$ (the fundamental solution of system (8.77)) is matrix $\mathbf{\Gamma}(\mathbf{x}) = \|\Gamma_{lj}(\mathbf{x})\|_{7\times7}$ satisfying the condition (Hörmander, 1983)

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \qquad \mathbf{x} \in E^3\ , \tag{8.81}$$

where $\delta(\mathbf{x})$ is the Dirac delta and $\mathbf{J} = \|\delta_{lj}\|_{7\times7}$ stands for unit matrix.

In this article the matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed in terms of elementary functions, and some basic properties are established.

Fundamental solution of the system of equations of steady oscillations

We introduce the notations

$$\begin{aligned}
&a = \lambda^S + 2\mu^S, \qquad \beta_0 = \beta_S\beta_L - \beta_3^2,\\
&d_1 = \frac{1}{a(n^S)^2}, \qquad d_2 = \frac{1}{\mu^S\beta_L}\ ,\\
&m_1 = n^S\beta_L + n^L\beta_3, \qquad m_2 = n^S\beta_3 + n^L\beta_S,\\
&k_1^2 = d_1[\beta_L\,(n^S)^2 + 2\beta_3\,n^Sn^L + \beta_S\,(n^L)^2]\ , \qquad k_2^2 = d_2\,\beta_0\ ,\\
&d = a\,m_1n^L = an^L(n^S\omega^2\rho^L - i\,\omega S_L) \neq 0\ .
\end{aligned} \tag{8.82}$$

Then we have

$$k_1^2 = (m_1n^S + m_2n^L)d_1 = \frac{m_2 + d_1n^S}{an^L}\,d\ . \tag{8.83}$$

We consider the system of equations

$$\begin{aligned}
\mu^S\Delta\mathbf{u} + (\lambda^S + \mu^S)\ \operatorname{grad}\operatorname{div}\ \mathbf{u} + \beta_S\mathbf{u} - \beta_3\mathbf{w} + n^S\ \operatorname{grad}\tilde{p} &= \mathbf{F}'\ ,\\
-\beta_3\mathbf{u} + \beta_L\mathbf{w} + n^L\ \operatorname{grad}\tilde{p} &= \mathbf{F}''\ ,\\
-n^S\ \operatorname{div}\ \mathbf{u} - n^L\ \operatorname{div}\ \mathbf{w} &= f\ ,
\end{aligned} \tag{8.84}$$

where $\mathbf{F}'$ and $\mathbf{F}''$ are three-component vector functions of E^3 and f is a scalar function in E^3.

As one can easily verify, the system (8.84) may be written in the form

$$\mathbf{A}^\top(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \tilde{\mathbf{F}}(\mathbf{x})\ , \tag{8.85}$$

where $\mathbf{A}^\top$ is transposed from the matrix $\mathbf{A}$, $\tilde{\mathbf{F}} = (\mathbf{F}', \mathbf{F}'', f)$ and $\mathbf{x} \in E^3$.

Applying the operator div to Eqs. $(8.84)_1$, $(8.84)_2$ from Eq. (8.84) we have

$$\begin{aligned}
(a\,\Delta + \beta_S)\ \operatorname{div}\ \mathbf{u} - \beta_3\ \operatorname{div}\ \mathbf{w} + n^S\,\Delta\,\tilde{p} &= \operatorname{div}\ \mathbf{F}'\ ,\\
-\beta_3\ \operatorname{div}\ \mathbf{u} + \beta_L\ \operatorname{div}\ \mathbf{w} + n^L\,\Delta\,\tilde{p} &= \operatorname{div}\ \mathbf{F}''\ ,\\
-n^S\ \operatorname{div}\ \mathbf{u} - n^L\ \operatorname{div}\ \mathbf{w} &= f\ .
\end{aligned} \tag{8.86}$$

By virtue of the identities (8.82) from (8.86) we obtain

$$\begin{aligned}
(\Delta + k_1^2)\ \operatorname{div}\ \mathbf{u} &= \varphi_1\ ,\\
(\Delta + k_1^2)\ \operatorname{div}\ \mathbf{w} &= \varphi_2\ ,\\
\Delta\,(\Delta + k_1^2)\,p &= \varphi_3\ ,
\end{aligned} \tag{8.87}$$

where

$$\begin{aligned}\varphi_1 &= d_1[(n^L)^2\ \operatorname{div}\ \mathbf{F}' - n^S n^L\ \operatorname{div}\mathbf{F}'' - m_1 f]\ ,\\ \varphi_2 &= d_1[-n^S n^L\ \operatorname{div}\ \mathbf{F}' + (n^S)^2\ \operatorname{div}\mathbf{F}'' - (an^L\Delta + m_2)f]\ ,\\ \varphi_3 &= d_1\{m_1\ \operatorname{div}\ \mathbf{F}' + (an^L\Delta + m_2)\ \operatorname{div}\mathbf{F}'' +\\ &+[(a\Delta + \beta_S)\beta_L - \beta_3^2]f\}\ .\end{aligned} \tag{8.88}$$

Applying the operator $\Delta(\Delta + k_1^2)$ to Eqs. $(8.84)_1, (8.84)_2$ and taking into account Eq. (8.87), we obtain

$$\begin{aligned}\Delta(\Delta + k_1^2)[(\mu^S\Delta + \beta_S)\ \mathbf{u} - \beta_3\ \mathbf{w}] &= \mathbf{f}_1\ ,\\ \Delta(\Delta + k_1^2)(-\beta_3\ \mathbf{u} + \beta_L\ \mathbf{w}) &= \mathbf{f}_2\ ,\end{aligned} \tag{8.89}$$

where

$$\begin{aligned}\mathbf{f}_1 &= \Delta(\Delta + k_1^2)\mathbf{F}' - (\lambda^S + \mu^S)\ \Delta\ \operatorname{grad}\ \ \varphi_1 - n^S\ \operatorname{grad}\ \ \varphi_3\ ,\\ \mathbf{f}_2 &= \Delta(\Delta + k_1^2)\mathbf{F}'' - n^L\ \operatorname{grad}\ \ \varphi_3\ .\end{aligned} \tag{8.90}$$

From system (8.89) we have

$$\Delta(\Delta + k_1^2)(\Delta + k_2^2)\ \mathbf{u} = \mathbf{\Phi}_1\ , \tag{8.91}$$

$$\Delta(\Delta + k_1^2)(\Delta + k_2^2)\ \mathbf{w} = \mathbf{\Phi}_2\ , \tag{8.92}$$

where

$$\begin{aligned}\mathbf{\Phi}_1 &= d_2(\beta_L\ \mathbf{f}_1 + \beta_3\ \mathbf{f}_2)\ ,\\ \mathbf{\Phi}_2 &= d_2[\beta_3\ \mathbf{f}_1 + (\mu^S\Delta + \beta_S)\ \mathbf{f}_2]\ .\end{aligned} \tag{8.93}$$

On the basis of Eqs. $(8.87)_3$ and (8.91), and (8.92) we get

$$\mathbf{\Lambda}^1(\Delta)\mathbf{U}(\mathbf{x}) = \tilde{\mathbf{\Phi}}(\mathbf{x}), \tag{8.94}$$

where $\tilde{\mathbf{\Phi}} = (\mathbf{\Phi}_1, \mathbf{\Phi}_2, \varphi_3)$ and

$$\begin{aligned}\mathbf{\Lambda}(\Delta) &= \|\Lambda_{lj}(\Delta)\|_{7\times7}\ ,\\ \Lambda_{qq}(\Delta) &= \Delta(\Delta + k_1^2)(\Delta + k_2^2)\ ,\\ \Lambda_{77}(\Delta) &= \Delta(\Delta + k_1^2)\ ,\\ \Lambda_{lj}(\Delta) &= 0, \qquad q = 1, 2, \cdots 6\ ,\\ l, j &= 1, 2, \cdots, 7, \qquad l \neq j\ .\end{aligned} \tag{8.95}$$

---

[1] This Λ has nothing to do with the same notation in (6.115). A confusion, however, seems to be improbable.

In what follows we use the notations

$$\mathbf{L} = \|L_{lj}\|_{7\times 7} = \begin{Vmatrix} \mathbf{L}^{(1)} & \mathbf{L}^{(2)} & \mathbf{L}^{(5)} \\ \mathbf{L}^{(3)} & \mathbf{L}^{(4)} & \mathbf{L}^{(6)} \\ \mathbf{L}^{(7)} & \mathbf{L}^{(8)} & L^{(9)} \end{Vmatrix}_{7\times 7},$$

$$\mathbf{L}^{(m)} = \|L_{lj}^{(m)}\|_{3\times 3}, \qquad \mathbf{L}^{(k)} = \|L_{lj}^{(k)}\|_{3\times 1}, \tag{8.96}$$

$$\mathbf{L}^{(q)} = \|L_{lj}^{(q)}\|_{1\times 3}, \qquad L^{(9)} = L_{77},$$

$$\mathbf{L}^{(1)}(\mathbf{D_x}) = d_1(n^L)^2(\Delta + k_2^2)\ \operatorname{grad}\operatorname{div} - d_2\beta_L(\Delta + k_1^2)\ \operatorname{curl}\operatorname{curl}\,,$$

$$\mathbf{L}^{(2)}(\mathbf{D_x}) = \mathbf{L}^{(3)}(\mathbf{D_x}) = -d_1 n^S n^L(\Delta + k_2^2)\ \operatorname{grad}\operatorname{div}\ -$$

$$-d_2\beta_3(\Delta + k_1^2)\ \operatorname{curl}\operatorname{curl}\,,$$

$$\mathbf{L}^{(4)}(\mathbf{D_x}) = d_1(n^S)^2(\Delta + k_2^2)\ \operatorname{grad}\operatorname{div}\ -$$

$$-d_2(\mu^S\Delta + \beta_S)(\Delta + k_1^2)\ \operatorname{curl}\operatorname{curl}\,,$$

$$L_{l1}^{(5)}(\mathbf{D_x}) = d_1 m_1 \frac{\partial}{\partial x_l}\,,$$

$$L_{l1}^{(6)}(\mathbf{D_x}) = d_1(a n^L\Delta + m_2)\frac{\partial}{\partial x_l}\,,$$

$$L_{1l}^{(7)}(\mathbf{D_x}) = -d_1 m_1(\Delta + k_2^2)\frac{\partial}{\partial x_l}\,,$$

$$L_{1l}^{(8)}(\mathbf{D_x}) = -d_1(a n^L\Delta + m_2)(\Delta + k_2^2)\frac{\partial}{\partial x_l}\,,$$

$$L^{(9)}(\mathbf{D_x}) = d_1[(a\Delta + \beta_S)\beta_L - \beta_3^2]\,,$$

$$l = 1,2,3, \qquad m = 1,2,3,4, \qquad k = 5,6, \qquad q = 7,8\,.$$

In view of Eqs. $(8.88)_{1,2}$, (8.90) and (8.96) from Eqs. (8.93), $(8.88)_3$ we have

$$\begin{aligned}
\mathbf{\Phi}_1 &= \mathbf{L}^{(1)}(\mathbf{D_x})\mathbf{F}' + \mathbf{L}^{(3)}(\mathbf{D_x})\mathbf{F}'' + \mathbf{L}^{(7)}(\mathbf{D_x})f\,,\\
\mathbf{\Phi}_2 &= \mathbf{L}^{(2)}(\mathbf{D_x})\mathbf{F}' + \mathbf{L}^{(4)}(\mathbf{D_x})\mathbf{F}'' + \mathbf{L}^{(8)}(\mathbf{D_x})f\,,\\
\varphi_3 &= \mathbf{L}^{(5)}(\mathbf{D_x})\mathbf{F}' + \mathbf{L}^{(6)}(\mathbf{D_x})\mathbf{F}'' + L^{(9)}(\mathbf{D_x})f\,.
\end{aligned} \tag{8.97}$$

Thus, from Eq. (8.97) we have

$$\tilde{\mathbf{\Phi}}(\mathbf{x}) = \mathbf{L}^\top(\mathbf{D_x})\tilde{\mathbf{F}}(\mathbf{x})\,. \tag{8.98}$$

By virtue of Eqs. (8.85), (8.97) from Eq. (8.94) it follows that $\mathbf{\Lambda U} = \mathbf{L}^\top \mathbf{A}^\top \mathbf{U}$. It is obvious that $\mathbf{L}^\top \mathbf{A}^\top = \mathbf{\Lambda}$ and hence

$$\mathbf{A}(\mathbf{D_x})\mathbf{L}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta) \ . \tag{8.99}$$

We assume that $k_1 k_2 (k_1^2 - k_2^2) \neq 0$.
Let

$$\begin{gathered}
\boldsymbol{\Upsilon}(\mathbf{x}) = \|\Upsilon_{mj}(\mathbf{x})\|_{7\times 7}, \qquad \Upsilon_{ll}(\mathbf{x}) = \sum_{q=1}^{3} \eta_{1q}\gamma_q(\mathbf{x}) \ , \\
\Upsilon_{77}(\mathbf{x}) = \eta_{21}\gamma_1(\mathbf{x}) + \eta_{23}\gamma_3(\mathbf{x}), \qquad \Upsilon_{mj}(\mathbf{x}) = 0 \ , \\
l = 1, 2, \ldots 6, \qquad m, j = 1, 2, \cdots, 7, \qquad m \neq j \ ,
\end{gathered} \tag{8.100}$$

where

$$\begin{gathered}
\gamma_l(\mathbf{x}) = -\frac{e^{ik_l|\mathbf{x}|}}{4\pi|\mathbf{x}|}, \qquad \gamma_3(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|} \ , \\
\eta_{11} = \frac{1}{k_1^2(k_1^2 - k_2^2)}, \qquad \eta_{12} = \frac{1}{k_2^2(k_2^2 - k_1^2)}, \qquad \eta_{13} = \frac{1}{k_1^2 k_2^2} \ , \\
\eta_{21} = -\eta_{23} = -\frac{1}{k_1^2}, \qquad l = 1, 2 \ .
\end{gathered} \tag{8.101}$$

Obviously (see, Kupradze *et al.*, 1979)

$$(\Delta + k_l^2)\gamma_l(\mathbf{x}) = \delta(\mathbf{x}), \qquad \Delta\gamma_3(\mathbf{x}) = \delta(\mathbf{x}), \qquad l = 1, 2 \ . \tag{8.102}$$

Lemma 1. The matrix $\boldsymbol{\Upsilon}$ is the fundamental matrix of operator $\mathbf{\Lambda}(\Delta)$, that is,

$$\Lambda(\triangle)\boldsymbol{\Upsilon}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J} \ . \tag{8.103}$$

Proof. It is sufficient to show

$$\begin{gathered}
\Delta(\Delta + k_1^2)(\Delta + k_2^2)\Upsilon_{11}(\mathbf{x}) = \delta(\mathbf{x}) \ , \\
\Delta(\Delta + k_1^2)\Upsilon_{77}(\mathbf{x}) = \delta(\mathbf{x}) \ .
\end{gathered} \tag{8.104}$$

Taking into account the equalities

$$\eta_{11} + \eta_{12} + \eta_{13} = 0 \ , \qquad \eta_{11}k_1^2 + \eta_{12}k_2^2 = 0 \ , \tag{8.105}$$

we have

$$\begin{aligned}
&\Delta(\Delta+k_1^2)(\Delta+k_2^2)\Upsilon_{11}\\
=&\ (\Delta+k_1^2)(\Delta+k_2^2)[\eta_{11}(\delta-k_1^2\gamma_1)+\eta_{12}(\delta-k_2^2\gamma_2)+\eta_{13}\delta]\\
&=-\ (\Delta+k_1^2)(\Delta+k_2^2)[\eta_{11}k_1^2\gamma_1+\eta_{12}k_2^2\gamma_2]\\
&=-\ (\Delta+k_1^2)\{\eta_{11}k_1^2[\delta+(k_2^2-k_1^2)]\gamma_1+\eta_{12}k_2^2\delta\}\\
&=-\ \eta_{11}k_1^2(k_2^2-k_1^2)(\Delta+k_1^2)\gamma_1=\delta\ .
\end{aligned} \tag{8.106}$$

Eq. $(8.104)_2$ is proved in a quiet similar manner.

We introduce the matrix

$$\mathbf{\Gamma}(\mathbf{x})=\mathbf{L}(\mathbf{D}_\mathbf{x})\mathbf{\Upsilon}(\mathbf{x})\ . \tag{8.107}$$

Using the identities (8.107), (8.99), (8.103) we get

$$\mathbf{A}(\mathbf{D}_\mathbf{x})\mathbf{\Gamma}(\mathbf{x})=\mathbf{A}(\mathbf{D}_\mathbf{x})\mathbf{L}(\mathbf{D}_\mathbf{x})\mathbf{\Upsilon}(\mathbf{x})=\mathbf{\Lambda}(\Delta)\mathbf{\Upsilon}(\mathbf{x})=\delta(\mathbf{x})\mathbf{J}\ . \tag{8.108}$$

Hence $\mathbf{\Gamma}(\mathbf{x})$ is a solution to Eq. (8.81).

We have thereby proven the following theorem:

Theorem 1. The matrix $\mathbf{\Gamma}(\mathbf{x})$ defined by Eq. (8.107) is the fundamental solution of system (8.77).

Theorem 1 leads to the following results:

Corollary 1. Each column of the matrix $\mathbf{\Gamma}(\mathbf{x})$ is solution to the system (2) at every point $\mathbf{x}\in E^3$ except the origin.

Corollary 2. If $\mathbf{x}\in E^3\backslash\{0\}$, then each element $\Gamma_{kl}(\mathbf{x})$ of the matrix $\mathbf{\Gamma}(\mathbf{x})$ has the form

$$\begin{gathered}
\Gamma_{mj}(\mathbf{x})=\sum_{q=1}^{3}\Gamma_{mjq}(\mathbf{x})\ ,\\
\Gamma_{m7}(\mathbf{x})=\Gamma_{m71}(\mathbf{x})+\Gamma_{m73}(\mathbf{x})\ ,
\end{gathered} \tag{8.109}$$

where

$$\begin{gathered}
(\Delta+k_l^2)\,\Gamma_{mkl}(\mathbf{x})=0,\qquad \Delta\,\Gamma_{mk3}(\mathbf{x})=0\ ,\\
m,k=1,2,\cdots,7,\qquad j=1,2,\cdots,6,\qquad l=1,2\ .
\end{gathered} \tag{8.110}$$

The fundamental solution $\mathbf{\Gamma}(\mathbf{x})$ of the system (8.77) makes it possible to investigate three-dimensional boundary value problems of the theory of fluid-saturated porous media with the boundary integral method (potential method). The main results obtained in the classical theory of elasticity and thermoelasticity with the potential method are given in the book of Kupradze *et al.* (1979).

The method, presented above for constructing the fundamental solution of the system (8.77) is more simple, than the methods used in the classical theory of elasticity and thermoelasticity for isotropic elastic materials (see, Kupradze *et al.*, 1979 and Nowacki, 1962).

From the method, developed in this Section, it is possible to construct the fundamental solutions of the systems of equations in the linear theories of isotropic elastic materials with microstructure.

b) On the Representations of Solutions in the Theory of Fluidsaturated Porous Media

Contemporary treatment of the various boundary value problems of the elasticity theory usually begins with the representation of a general solution of field equations in terms of elementary (harmonic, biharmonic, metaharmonic and etc.) functions. In the classical theory of elasticity the Boussinesq-Somigliana-Galerkin, Boussinesq-Papkovitch-Neuber (BPN), Green-Lamé (GL) and Cauchy-Kovalevski-Somigliana (CKS) solutions are well-known (see Gurtin, 1972; Nowacki, 1962, 1975; Kupradze *et al.*, 1979). An excellent review of the history of these solutions is given in Gurtin (1972).

The Galerkin type solution (see Galerkin, 1930) of equations of classical elastokinetics was obtained by Iacovache (1949). The classical theory of thermodynamics and the theory of micropolar elasticity representations of solutions are presented in Nowacki (1964, 1969) and Sandru (1966). The representation theorem of Galerkin type in the theory of thermoelastic materials with voids was proven by Ciarlleta (1991). In the case of the isothermal theory of materials with voids, the BPN, GL and CKS type solutions were obtained by Chandrasekharaiah (1987, 1989).

The representations of Galerkin type solutions in the theory of micropolar thermoelasticity without energy dissipation and in the dynamical theory of binary mixture consisting of a gas and an elastic solid were established by Ciarlleta (1999, 1995). The theory of binary mixtures of elastic solids, the representations of general solutions, are presented by Basheleishvili (1999) and Svanadze (1993).

In the next section the representation of the Galerkin type solution of equations of motion is obtained. Then, the representation theorem of the Galerkin type of system of the inhomogeneous equations of the steady oscillations is proved. Finally, the general solution of system of the homogeneous equations of steady oscillations in terms of a harmonic function and four metaharmonic functions is established.

Field equations

The system of equations of motion in the linear theory of incompressible fluidsaturated poroelastic solids can be written as, see Eq. (8.75)

$$\begin{gathered}
\mu^S \Delta \mathbf{u}_S + (\lambda^S + \mu^S)\ \operatorname{grad}\operatorname{div}\mathbf{u}_S + S_L(\dot{\mathbf{u}}_L - \dot{\mathbf{u}}_S) - \\
- n^S\ \operatorname{grad} p = \rho^S\ (\ddot{\mathbf{u}} - \mathbf{b}_S)\ , \\
- S_L(\dot{\mathbf{u}}^L - \dot{\mathbf{u}}^S) - n^L\ \operatorname{grad} p = \rho^L\ (\ddot{\mathbf{u}} - \mathbf{b}_L)\ , \\
n^S\ \operatorname{div}\ \dot{\mathbf{u}}_S + n^L\ \operatorname{div}\ \dot{\mathbf{u}}_L = 0\ ,
\end{gathered} \tag{8.111}$$

where $\mathbf{u}_S(\mathbf{x},t),\ \mathbf{u}_L(\mathbf{x},t), \mathbf{b}_S(\mathbf{x},t)$ and $\mathbf{b}_L(\mathbf{x},t)$ are three-component vector functions, $p(\mathbf{x},t)$ is interpreted as the true poreliquid pressure of the incompressible liquid, $(\mathbf{x},t) \in \Omega \times I,\ I = (0,+\infty);\ n^j > 0,\ \rho^j > 0\,(j = S, L);$ a superposed dot denotes differentiation with respect to $t:\ \dot{\mathbf{u}} = \dfrac{\partial\,\mathbf{u}}{\partial\,t},\ \ddot{\mathbf{u}} = \dfrac{\partial^{\,2}\mathbf{u}}{\partial\,t^{\,2}},$ where we have neglected the convective terms and the rates of the volume fractions. We consider a body which occupies the region Ω of E^3.

If we assume that

$$\begin{gathered}
\mathbf{u}_S(\mathbf{x},t) = Re\ [\mathbf{u}(\mathbf{x})\, e^{-i\,\omega\,t}]\ , \\
\mathbf{u}_L(\mathbf{x},t) = Re\ [\mathbf{w}(\mathbf{x})\, e^{-i\,\omega\,t}]\ , \\
p(\mathbf{x},t) = Re\ [\,\tilde{p}(\mathbf{x})\, e^{-i\,\omega\,t}]\ , \\
\mathbf{b}_j(\mathbf{x},t) = Re\ [\,\tilde{\mathbf{b}}_j(\mathbf{x})\, e^{-i\,\omega\,t}]\ , \\
j = S, L,
\end{gathered} \tag{8.112}$$

then from (8.111) the system of equations of the steady oscillations of incompressible fluidsaturated poroelastic solids follows as

$$\begin{gathered}
\mu^S \Delta \mathbf{u} + (\lambda^S + \mu^S)\ \operatorname{grad}\operatorname{div}\ \mathbf{u} + \beta_S\ \mathbf{u} - \\
- \beta_3\ \mathbf{w} - n^S\ \operatorname{grad}\tilde{p} = -\rho^S \tilde{\mathbf{b}}\ , \\
- \beta_3\ \mathbf{u} + \beta_L \mathbf{w} - n^L\ \operatorname{grad}\tilde{p} = -\rho_L \tilde{\mathbf{b}}_L\ , \\
n^S\ \operatorname{div}\mathbf{u} + n^L\ \operatorname{div}\ \mathbf{w} = 0\ ,
\end{gathered} \tag{8.113}$$

where $\beta_j = \omega^2 \rho^j + \beta_3\ (j = S, L),\ \beta_3 = i\omega\, S_L,\ i = \sqrt{-1},$ and ω is the oscillation frequency $(\omega > 0)$.

If the external body forces $\tilde{\mathbf{b}}_S$ and $\tilde{\mathbf{b}}_L$ are assumed to be absent, the system (8.113) is

$$\begin{gathered}
\mu^S \Delta \mathbf{u} + (\lambda^S + \mu^S)\ \operatorname{grad}\ \operatorname{div}\ \mathbf{u} + \beta_S\ \mathbf{u} - \beta_3\ \mathbf{w} - n^S\ \operatorname{grad}\ \ \tilde{p} = \mathbf{0}\ , \\
- \beta_3\ \mathbf{u} + \beta_L \mathbf{w} - n^L\ \operatorname{grad}\tilde{p} = \mathbf{0}\ , \\
n^S\ \operatorname{div}\ \mathbf{u} + n^L\ \operatorname{div}\ \mathbf{w} = 0\ .
\end{gathered} \tag{8.114}$$

Throughout this section, it is assumed that all functions are continuous and differentiable up to the required order of $\Omega \times I$ or Ω.

In this section the Galerkin type solutions of systems (8.111) and (8.113) are presented. The general solution of system (8.114) in terms of a harmonic function and four metaharmonic functions are established.

Galerkin type solution of equations of motion

We introduce the matrix differential operator

$$\begin{aligned}
&\mathbf{B}(\mathbf{D}_{\mathbf{x}},T) = \|B_{lj}(\mathbf{D}_{\mathbf{x}},T)\|_{7\times 7},\\
&B_{lk}(\mathbf{D}_{\mathbf{x}},T) = (\mu^S\Delta - S_L T - \rho^S T^2)\delta_{lk} + (\lambda^S+\mu^S)\frac{\partial^2}{\partial x_l \partial x_k},\\
&B_{l,k+3}(\mathbf{D}_{\mathbf{x}},T) = B_{l+3,k}(\mathbf{D}_{\mathbf{x}},T) = S_L T\delta_{lk},\\
&B_{l+3,k+3}(\mathbf{D}_{\mathbf{x}},T) = -(S_L + \rho^L T)T\delta_{lk},\\
&B_{l7}(\mathbf{D}_{\mathbf{x}},T) = -n^S\frac{\partial}{\partial x_l}, \qquad B_{l+3,7}(\mathbf{D}_{\mathbf{x}},T) = -n^L\frac{\partial}{\partial x_l},\\
&B_{7l}(\mathbf{D}_{\mathbf{x}},T) = n^S T\frac{\partial}{\partial x_l}, \qquad B_{7,l+3}(\mathbf{D}_{\mathbf{x}},T) = n^L T\frac{\partial}{\partial x_l},\\
&B_{77}(\mathbf{D}_{\mathbf{x}},T) = 0, \qquad l,k = 1,2,3,
\end{aligned} \tag{8.115}$$

where, $T = \dfrac{\partial}{\partial t}$, $T^2 = \dfrac{\partial^2}{\partial t^2}$.

The system (8.111) can be written as

$$\mathbf{B}(\mathbf{D}_{\mathbf{x}},T)\mathbf{V}(\mathbf{x},t) = \mathbf{F}(\mathbf{x},t), \tag{8.116}$$

where $\mathbf{V} = (\mathbf{u},\mathbf{w},p)$, $\mathbf{F} = (\mathbf{F}_1,\mathbf{F}_2,0)$, $\mathbf{F}_j = -\rho_j \mathbf{b}_j$ $(j = S,L)$, $(\mathbf{x},\, t) \in \Omega \times I$.

We consider the system of equations

$$\begin{gathered}
[\mu^S\Delta + (\lambda^S+\mu^S)\ \operatorname{grad}\operatorname{div} - (S_L + \rho^S T)\,T\,]\mathbf{u}_S\ + \\
+ S_L T\mathbf{u}_L + n^S T\operatorname{grad} p = \mathbf{\Phi}',\\
S_L T\mathbf{u}_S - (S_L + \rho^L T)\,T\,\mathbf{u}_L + n^L T\ \operatorname{grad} p = \mathbf{\Phi}'',\\
-n^S\ \operatorname{div}\mathbf{u}_S - n^L\ \operatorname{div}\mathbf{u}_L = \varphi,
\end{gathered} \tag{8.117}$$

where $\mathbf{\Phi}'$ and $\mathbf{\Phi}''$ are three-component vector functions of $\Omega \times I$ and φ is a scalar function of $\Omega \times I$.

As one can easily verify, the system (8.117) may be written in the form

$$\mathbf{B}^\top(\mathbf{D_x}, T)\mathbf{V}(\mathbf{x}, t) = \mathbf{\Phi}(\mathbf{x}, t), \tag{8.118}$$

where $\mathbf{B}^\top$ is the transpose of matrix $\mathbf{B}$, and $\mathbf{\Phi} = (\mathbf{\Phi}', \mathbf{\Phi}'', \varphi)$.

Applying the operator div to Eqs. $(8.117)_{1,2}$ we have

$$\begin{aligned} &C_1 \ \operatorname{div} \mathbf{u}_S + S_L T \ \operatorname{div} \mathbf{u}_L + n^S T \,\Delta\, p = \operatorname{div} \mathbf{\Phi}', \\ &S_L T \ \operatorname{div} \mathbf{u}_S + C_2 \ \operatorname{div} \mathbf{u}_L + n^L T \,\Delta\, p = \operatorname{div} \mathbf{\Phi}''. \end{aligned} \tag{8.119}$$

where

$$C_1 = a\Delta - (S_L + \rho^S T)T, \qquad C_2 = -(S_L + \rho^L T)T. \tag{8.120}$$

From $(8.117)_3$ we get

$$\operatorname{div}\ \mathbf{u}_L = m \ \operatorname{div}\ \mathbf{u}_S - \frac{1}{n^L}\, \varphi, \tag{8.121}$$

where $m = -\dfrac{n^S}{n^L}$.

By virtue of Eq. (8.121) from (8.119) follows that

$$\begin{aligned} &(C_1 + m S_L T) \ \operatorname{div} \mathbf{u}_S + n^S T \,\Delta\, p = \psi_1, \\ &(m C_2 + S_L T) \ \operatorname{div} \mathbf{u}_S + n^L T \,\Delta\, p = \psi_2, \end{aligned} \tag{8.122}$$

where

$$\psi_1 = \ \operatorname{div}\ \mathbf{\Phi}' + \frac{1}{n^L} S_L T \varphi, \qquad \psi_2 = \ \operatorname{div}\ \mathbf{\Phi}'' + \frac{1}{n^L} C_2 \varphi. \tag{8.123}$$

From system (8.122) and Eq. (8.121) we have

$$\Lambda_1 \ \operatorname{div}\ \mathbf{u}_S = \varphi_1, \qquad \Lambda_1 \ \operatorname{div}\ \mathbf{u}_L = \varphi_2, \qquad \Delta\,\Lambda_1\, T\, p = \varphi_3, \tag{8.124}$$

where

$$\begin{aligned} &\varphi_1 = n^L \psi_1 - n^S \psi_2, \\ &\varphi_2 = m(n^L \psi_1 - n^S \psi_2) - \frac{1}{n^L} \Lambda_1 \varphi, \\ &\varphi_3 = -(m C_2 + S_L T)\psi_1 + (C_1 + S_L m T)\psi_2, \end{aligned} \tag{8.125}$$

and

$$\Lambda_1 = n^L C_1 - m n^S C_2 - 2 n^S S_L T. \tag{8.126}$$

Applying the operator $\Delta\Lambda_1$ to Eqs. $(8.117)_{1,2}$ and taking into account Eq. (8.121), we obtain

$$\Delta\Lambda_1(B_1\, \mathbf{u}_S + S_L T\ \mathbf{u}_L) = \ \mathbf{f}_1, \qquad \Delta\Lambda_1(S_L T \mathbf{u}_S - B_2 T\ \mathbf{u}_L) = \ \mathbf{f}_2, \tag{8.127}$$

where

$$B_1 = \mu^S \Delta - (S_L + \rho^S T)T, \qquad B_2 = S_L + \rho^L T, \tag{8.128}$$

and

$$\begin{aligned} \mathbf{f}_1 &= \Delta \Lambda_1 \mathbf{\Phi}' - (\lambda^S + \mu^S)\, \Delta \ \operatorname{grad} \varphi_1 - n^S \ \operatorname{grad} \varphi_3, \\ \mathbf{f}_2 &= \Delta \Lambda_1 \mathbf{\Phi}'' - n^L \ \operatorname{grad} \varphi_3. \end{aligned} \tag{8.129}$$

From system (8.127) we have

$$\Delta \Lambda_1 \Lambda_2 \, \mathbf{u}_S = \mathbf{\Phi}_1, \qquad \Delta \Lambda_1 \Lambda_2 T \, \mathbf{u}_L = \mathbf{\Phi}_2, \tag{8.130}$$

where

$$\Lambda_2 = B_1 B_2 + S_L^2 T = \mu^S (S_L + \rho^L T)\Delta - [(\rho^S + \rho^L) S_L + \rho^S \rho^L T] T^2, \tag{8.131}$$

and

$$\mathbf{\Phi}_1 = B_2 \mathbf{f}_1 + S_L \mathbf{f}_2, \qquad \mathbf{\Phi}_2 = S_L T \mathbf{f}_1 + B_1 \mathbf{f}_2. \tag{8.132}$$

On the basis of Eqs. $(8.124)_3$ and (8.130) we get

$$\tilde{\mathbf{\Lambda}}(\Delta, T) \mathbf{V}(\mathbf{x}, t) = \tilde{\mathbf{\Phi}}(\mathbf{x}, t), \tag{8.133}$$

where $\tilde{\mathbf{\Phi}} = (\mathbf{\Phi}_1, \mathbf{\Phi}_2, \varphi_3)$ and

$$\begin{gathered} \tilde{\mathbf{\Lambda}}(\Delta, T) = \|\tilde{\Lambda}_{lj}(\Delta, T)\|_{7\times 7}, \qquad \tilde{\Lambda}_{kk} = \Delta \Lambda_1 \Lambda_2, \\ \tilde{\Lambda}_{k+3,k+3} = \Delta \Lambda_1 \Lambda_2 T, \qquad \tilde{\Lambda}_{77} = \Delta \Lambda_1 T, \qquad \Lambda_{lj} = 0, \\ k = 1,2,3, \qquad l, j = 1, 2, \cdots, 7, \qquad l \neq j. \end{gathered} \tag{8.134}$$

In the following we use the notations

$$\begin{aligned} &\mathbf{M} = \|M_{lj}\|_{7\times 7} = \left\| \begin{matrix} \mathbf{M}^{(1)} & \mathbf{M}^{(2)} & \mathbf{M}^{(5)} \\ \mathbf{M}^{(3)} & \mathbf{M}^{(4)} & \mathbf{M}^{(6)} \\ \mathbf{M}^{(7)} & \mathbf{M}^{(8)} & M^{(9)} \end{matrix} \right\|_{7\times 7}, \\ &\mathbf{M}^{(m)} = \|M_{lj}^{(m)}\|_{3\times 3}, \quad \mathbf{M}^{(k)} = \|M_{lj}^{(k)}\|_{3\times 1}\ , \\ &\quad \mathbf{M}^{(q)} = \|M_{lj}^{(q)}\|_{1\times 3}, \quad M^{(9)} = M_{77}\ , \\ &\mathbf{M}^{(1)}(\mathbf{D_x}, T) = n^L \Lambda_2 \ \operatorname{grad}\operatorname{div} - B_2 \Lambda_1 \ \operatorname{curl}\operatorname{curl}\ , \\ &\mathbf{M}^{(2)}(\mathbf{D_x}, T) = -n^S \Lambda_2 T \ \operatorname{grad}\operatorname{div} - S_L T \Lambda_1 \ \operatorname{curl}\operatorname{curl}\ , \\ &\mathbf{M}^{(3)}(\mathbf{D_x}, T) = -n^S \Lambda_2 \ \operatorname{grad}\operatorname{div} - S_L \Lambda_1 \ \operatorname{curl}\operatorname{curl}\ , \end{aligned} \tag{8.135}$$

$$
\begin{aligned}
&\mathbf{M}^{(4)}(\mathbf{D_x},T) = -mn^S\Lambda_2 T \ \operatorname{grad}\operatorname{div} + B_1\Lambda_1 \operatorname{curl}\operatorname{curl}\,,\\
&M_{l1}^{(5)}(\mathbf{D_x},T) = -(mC_2+S_LT)\frac{\partial}{\partial x_l}\,,\\
&M_{l1}^{(6)}(\mathbf{D_x},T) = (C_1+mS_LT)\frac{\partial}{\partial x_l}\,,\\
&M_{1l}^{(7)}(\mathbf{D_x},T) = (mC_2+S_LT)\Lambda_2\frac{\partial}{\partial x_l}\,,\\
&M_{1l}^{(8)}(\mathbf{D_x},T) = -(C_1+mS_LT)T\Lambda_2\frac{\partial}{\partial x_l}\,,\\
&M^{(9)}(\mathbf{D_x},T) = \frac{1}{n^L}(C_1C_2 - S_L^2T^2)\,,\\
&l=1,2,3,\quad m=1,2,3,4,\quad k=5,6,\quad q=7,8\,.
\end{aligned}
$$

In view of Eqs. (8.123), $(8.125)_{1,2}$ and (8.129) from Eq. (8.132) and $(8.125)_3$ we have

$$
\begin{aligned}
\mathbf{\Phi}_1 &= \mathbf{M}^{(1)}\mathbf{\Phi}' + \mathbf{M}^{(3)}\mathbf{\Phi}'' + \mathbf{M}^{(7)}\varphi\,,\\
\mathbf{\Phi}_2 &= \mathbf{M}^{(2)}\mathbf{\Phi}' + \mathbf{M}^{(4)}\mathbf{\Phi}'' + \mathbf{M}^{(8)}\varphi\,,\\
\varphi_3 &= \mathbf{M}^{(5)}\mathbf{\Phi}' + \mathbf{M}^{(6)}\mathbf{\Phi}'' + M^{(9)}\varphi\,.
\end{aligned}
\tag{8.136}
$$

Thus, from Eq. (8.136) we obtain

$$
\tilde{\mathbf{\Phi}}(\mathbf{x},t) = \mathbf{M}^\top(\mathbf{D_x},T)\,\mathbf{\Phi}(\mathbf{x},t)\,. \tag{8.137}
$$

By virtue of Eqs. (8.121), (8.125) from Eq. (8.133) it follows that

$$
\tilde{\mathbf{\Lambda}}\mathbf{V} = \mathbf{M}^\top\mathbf{\Phi} = \mathbf{M}^\top\mathbf{B}^\top\mathbf{V}\,. \tag{8.138}
$$

It is obvious that $\mathbf{M}^\top\mathbf{A}^\top = \tilde{\mathbf{\Lambda}}$ and hence

$$
\mathbf{B}(\mathbf{D_x},T)\mathbf{M}(\mathbf{D_x},T) = \tilde{\mathbf{\Lambda}}(\Delta,T)\,. \tag{8.139}
$$

We have thereby proven the following Lemma:

Lemma 2. The matrix differential operators $\mathbf{B}$, $\mathbf{M}$ and $\tilde{\mathbf{\Lambda}}$ satisfy Eq. (8.139), where $\mathbf{B}$, $\mathbf{M}$ and $\tilde{\mathbf{\Lambda}}$ are defined by Eqs. (8.115), (8.135) and (8.134), respectively.

Let $G_j'(\mathbf{x},t)$, $G_j''(\mathbf{x},t)(j=1,2,3)$ and $g(\mathbf{x},t)$ be functions on $\Omega\times I$, $\mathbf{G}' = (G_1',G_2',G_3')$, $\mathbf{G}'' = (G_1'',G_2'',G_3'')$, $\mathbf{G} = (\mathbf{G}',\,\mathbf{G}'',\,g)$.

The next theorem provides a Galerkin type solution to system (8.111).

Theorem 2.
Let

$$\begin{aligned} \mathbf{u}_S &= \mathbf{M}^{(1)}\,\mathbf{G}' + \mathbf{M}^{(2)}\,\mathbf{G}'' + \mathbf{M}^{(5)}\,g\ , \\ \mathbf{u}_L &= \mathbf{M}^{(3)}\,\mathbf{G}' + \mathbf{M}^{(4)}\,\mathbf{G}'' + \mathbf{M}^{(6)}\,g\ , \\ p &= \mathbf{M}^{(7)}\,\mathbf{G}' + \mathbf{M}^{(8)}\,\mathbf{G}'' + \mathbf{M}^{(9)}\,g\ , \end{aligned} \tag{8.140}$$

where the fields G_j' of class C^7, G_j'' of class C^8, and g of class C^5 satisfy

$$\begin{aligned} &\Delta\,\Lambda_1\,\Lambda_2\,\mathbf{G}' = -\rho^S\,\mathbf{b}_S\ , \\ &\Delta\,\Lambda_1\,\Lambda_2\,T\,\mathbf{G}'' = -\rho^L\,\mathbf{b}_L\ , \\ &\Delta\,\Lambda_1\,T\,g = 0, \qquad\qquad \text{on}\ \ \Omega\times I\ , \end{aligned} \tag{8.141}$$

then $(\mathbf{u}_S,\ \mathbf{u}_L,\ p)$ is the solution of system (8.111).

Proof. From (8.140) we have

$$\mathbf{V} = \mathbf{M}\,\mathbf{G}\ . \tag{8.142}$$

On the other hand, from (8.141) we get

$$\tilde{\mathbf{\Lambda}}\,G\ =\ F\ . \tag{8.143}$$

By virtue of Eqs. (8.139), (8.142), (8.143) we obtain

$$\mathbf{B}\,\mathbf{V}\ =\ \mathbf{B}\,\mathbf{M}\,\mathbf{G}\ =\ \tilde{\mathbf{\Lambda}}\,\mathbf{G}\ =\ \mathbf{F}\ . \tag{8.144}$$

Remark 1.
If $\mathbf{u}_S$, $\mathbf{u}_L$, and p are given by Eq. (8.140), then the vector $(\mathbf{u}_S,\ \mathbf{u}_L,\ p)$ is a Galerkin type solution of the system of equations of motion in the linear theory of the incompressible fluidsaturated poroelastic solid. In view of Theorem 2 the system of coupled equations (8.111) in the system of uncoupled equations (8.141) is deduced.

Galerkin type solution of equations of steady oscillations

Let Q_j', $Q_j''(j=1,2,3)$ and q be functions on Ω , $\mathbf{Q}' = (Q_1', Q_2', Q_3')$, $\mathbf{Q}'' = (Q_1'', Q_2'', Q_3'')$, $\mathbf{Q} = (\mathbf{Q}',\ \mathbf{Q}'',\ q)$.

The next theorem provides a Galerkin type solution to system (8.113).

Theorem 2 leads to the following.

Theorem 3.
Let

$$
\begin{aligned}
\mathbf{u} &= \mathbf{L}^{(1)}\,\mathbf{Q}' + \mathbf{L}^{(2)}\,\mathbf{Q}'' + \mathbf{L}^{(5)}\,q\ ,\\
\mathbf{w} &= \mathbf{L}^{(3)}\,\mathbf{Q}' + \mathbf{L}^{(4)}\,\mathbf{Q}'' + \mathbf{L}^{(6)}\,q\ ,\\
\tilde{p} &= \mathbf{L}^{(7)}\,\mathbf{Q}' + \mathbf{L}^{(8)}\,\mathbf{Q}'' + M^{(9)}\,q\ ,
\end{aligned}
\tag{8.145}
$$

where $\mathbf{L}^{(m)}(\mathbf{D}x)(m = 1, 2, ..., 9)$ is definied by (8.96); the fields Q'_j and Q''_j of class C^6, and q of class C^4 on Ω satisfy

$$
\begin{aligned}
&\Delta\,(\Delta + k_1^2)\,(\Delta + k_2^2)\,\mathbf{Q}' = -\rho^S\,\tilde{\mathbf{b}}_S\ ,\\
&\Delta\,(\Delta + k_1^2)\,(\Delta + k_2^2)\,\mathbf{Q}'' = -\rho^L\,\tilde{\mathbf{b}}_L\ ,\\
&\Delta\,(\Delta + k_1^2)\,q = 0, \qquad\qquad \text{on}\ \ \Omega\ ,
\end{aligned}
\tag{8.146}
$$

then $(\mathbf{u},\ \mathbf{w}, \tilde{p})$ is the solution of system (8.113).

Remark 2. If $\mathbf{u}$, $\mathbf{w}$ and $\tilde{p}$ are given by Eq. (8.145), then the vector $(\mathbf{u},\ \mathbf{w},\ \tilde{p})$ is a Galerkin type solution of the system of equations of the steady oscillations in the linear theory of an incompressible fluidsaturated poroelastic solid. In view of Theorem 3 the system of coupled equations (8.113) on the system of uncoupled equations (8.141) is deduced.

General solution of equations of steady oscillations

In the sequel we will use the following lemmas.
Lemma 3.
If

$$
\begin{aligned}
&\eta_1 = -\frac{1}{\beta_0}a\beta_L, \qquad \eta_2 = -\frac{1}{\beta_0}a\beta_3\ ,\\
&d = a\,m_1 n^L = n^S n^L\,\omega^2 a\,\rho^L - i\,\omega S_L a\,n^L\ ,
\end{aligned}
\tag{8.147}
$$

then

$$
\begin{aligned}
&m_2 + n^S d_1 d = a n^L k_1^2, \qquad m_1 - n^L d_1 d = 0\ ,\\
&\eta_1 n^L k_1^2 + \frac{1}{\beta_0} m_1 d_1 d = -n^L, \qquad \eta_2 n^L k_1^2 + \frac{1}{\beta_0} m_2 d_1 d = n^S\ .
\end{aligned}
\tag{8.148}
$$

Proof. By virtue of Eqs. $(8.82)_2$, $(8.82)_7$, and $(8.147)_3$ we have

$$
\begin{aligned}
m_2 + n^S d_1 d \ &=\ d_1[m_2 a(n^L)^2 + n^S(a m_1 n^L)]\\
&=\ d_1 a n^L(m_1 n^S + m_2 n^L) = a n^L k_1^2\ ,
\end{aligned}
\tag{8.149}
$$

and

$$m_1 - n^L d_1 d \;=\; d_1[m_1 a (n^L)^2 - n^L a m_1\, n^L] = 0\ . \tag{8.150}$$

In view of Eqs. $(8.82)_5, (8.147)_1$ and $(8.148)_{1,2}$ we get

$$\begin{aligned} \eta_1 n^L k_1^2 + \frac{1}{\beta_0} m_1 d_1 d &= \frac{1}{\beta_0}[m_1 d_1 d - a n^L \beta_L k_1^2] \\ &= \frac{1}{\beta_0}[(n^S \beta_L + n^L \beta_3) d_1 d - a n^L \beta_L k_1^2] \\ = \frac{1}{\beta_0}[\beta_L (n^S d_1 d - a n^L \beta_L) k_1^2)] &= \frac{1}{\beta_0}(m_1 \beta_3 - m_2 \beta_L)\ . \end{aligned} \tag{8.151}$$

Taking into account the equality

$$m_1 \beta_3 - m_2 \beta_L = -n^L \beta_0\ , \tag{8.152}$$

from (8.151) we obtain Eq. $(8.148)_3$.

Equation $(8.148)_4$ is proven in quite a similar manner.

Lemma 4.

If $(\mathbf{u},\ \mathbf{w}\,\tilde{p})$ is a solution of system (8.114), then

$$(\Delta + k_1^2)\ \mathrm{div}\,\mathbf{u} = 0, \qquad (\Delta + k_2^2)\,\mathrm{curl}\ \mathbf{u} = \mathbf{0}\ , \tag{8.153}$$

and

$$(\Delta + k_1^2)\,\Delta \tilde{p} = 0\ , \tag{8.154}$$

Proof. Applying the operator div to Eqs. $(8.114)_{1,2}$ we have

$$\begin{aligned} (a\,\Delta + \beta_S)\,\mathrm{div}\,\mathbf{u} - \beta_3\,\mathrm{div}\,\mathbf{w} - n^S\ \Delta\,\tilde{p} = 0\ , \\ -\beta_3\,\mathrm{div}\,\mathbf{u} + \beta_L\,\mathrm{div}\,\mathbf{w} - n^L\ \Delta\,\tilde{p} = 0\ . \end{aligned} \tag{8.155}$$

By virtue of Eq. $(8.114)_3$ from (8.155) it follows that

$$\begin{aligned} (a n^L\ \Delta + m_2)\,\mathrm{div}\,\mathbf{u} - n^S n^L\ \Delta\,\tilde{p} = 0\ , \\ m_1\,\mathrm{div}\,\mathbf{u} + (n^L)^2\ \Delta\,\tilde{p} = 0\ . \end{aligned} \tag{8.156}$$

From system (8.156) we get

$$[a (n^L)^2\ \Delta + (m_1 n^S + m_2 n^L)]\,\mathrm{div}\ \mathbf{u} = 0\ , \tag{8.157}$$

and

$$[a(n^L)^2\Delta + (m_1 n^S + m_2 n^L)]\Delta\,\tilde{p} = 0\ . \tag{8.158}$$

Obviously, from (8.157) and (8.158) we have Eq. $(8.153)_1$ and (8.154), respectively.

Now applying the operator curl to Eqs. $(8.114)_{1,2}$ we obtain

$$\begin{gathered}(\mu^S\Delta + \beta_S)\ \mathrm{curl}\ \mathbf{u} - \beta_3\ \mathrm{curl}\ \mathbf{w} = \mathbf{0}\ ,\\ -\beta_3\ \mathrm{curl}\ \mathbf{u} + \beta_L\ \mathrm{curl}\ \mathbf{w} = \mathbf{0}\ .\end{gathered} \tag{8.159}$$

The system (8.159) implies that

$$(\mu^S\beta_L\,\Delta + \beta_0)\ \mathrm{curl}\,\mathbf{u} = \mathbf{0}\ . \tag{8.160}$$

Hence, from (8.160) we have Eq. $(8.153)_2$.

Theorem 4.

If $(\mathbf{u},\ \mathbf{w}, \tilde{p})$ is a solution of system (8.114), then

$$\begin{aligned}\mathbf{u} &= \mathrm{grad}\,(m_1\,\varphi - n^L\,\psi) + \beta_L\boldsymbol{\Psi}\ ,\\ \mathbf{w} &= \mathrm{grad}\,(m_2\,\varphi + n^S\,\psi) + \beta_3\boldsymbol{\Psi}\ ,\\ \tilde{p} &= \beta_0\,\varphi + d_1 d\,\psi, \qquad \text{on } \Omega\ ,\end{aligned} \tag{8.161}$$

where φ, ψ and $\boldsymbol{\Psi}$ satisfy following equations

$$\begin{gathered}\Delta\,\varphi(\mathbf{x}) = 0, \qquad (\Delta + k_1^2)\,\psi(\mathbf{x}) = 0\ ,\\ (\Delta + k_2^2)\,\boldsymbol{\Psi}(\mathbf{x}) = \mathbf{0}, \qquad \mathrm{div}\,\boldsymbol{\Psi}(\mathbf{x}) = 0, \qquad \mathbf{x}\in\Omega\ .\end{gathered} \tag{8.162}$$

Proof. Let $(\mathbf{u},\ \mathbf{w},\ \tilde{p})$ be a solution of system (8.114). From Eq. $(8.114)_3$ we get

$$\mathrm{div}\,\mathbf{w} = m\,\mathrm{div}\,\mathbf{u}\ . \tag{8.163}$$

Taking into account Eq. (8.163) and

$$\Delta\,\mathbf{u} = \mathrm{grad}\,\mathrm{div}\ \mathbf{u} - \mathrm{curl}\,\mathrm{curl}\ \mathbf{u}\ , \tag{8.164}$$

from $(8.114)_{1,2}$ we have

$$\begin{gathered}\beta_S\mathbf{u} - \beta_3\mathbf{w} = \mathrm{grad}\ (-a\,\mathrm{div}\,\mathbf{u} + n^S\,\tilde{p}) + \mu^S\ \mathrm{curl}\ \mathrm{curl}\ \mathbf{u}\ ,\\ -\beta_3\mathbf{u} + \beta_L\mathbf{w} = \mathrm{grad}\,n^L\,\tilde{p}\ .\end{gathered} \tag{8.165}$$

Obviously, Eq. (8.165) implies

$$\begin{aligned}\mathbf{u} &= \operatorname{grad}(\eta_1 \operatorname{div}\mathbf{u} + \frac{m_1}{\beta_0}\tilde{p}) + \frac{\mu^S \beta_L}{\beta_0} \operatorname{curl}\operatorname{curl}\mathbf{u}\,,\\ \mathbf{w} &= \operatorname{grad}(\eta_2 \operatorname{div}\mathbf{u} + \frac{m_2}{\beta_0}\tilde{p}) + \frac{\mu^S \beta_3}{\beta_0} \operatorname{curl}\operatorname{curl}\mathbf{u}\,.\end{aligned} \tag{8.166}$$

We introduce the notations

$$\begin{gathered}\psi(\mathbf{x}) = \frac{1}{n^L k_1^2} \operatorname{div}\mathbf{u}(\mathbf{x}), \qquad \mathbf{\Psi}(\mathbf{x}) = \frac{\mu^S}{\beta_0} \operatorname{curl}\operatorname{curl}\mathbf{u}(\mathbf{x})\,,\\ \mathbf{\Psi} = (\Psi_1,\ \Psi_2,\ \Psi_3), \qquad \mathbf{x} \in \Omega\,.\end{gathered} \tag{8.167}$$

On the basis of Eqs. $(8.153)_{1,2}$ and

$$\operatorname{div}\ \operatorname{curl}\mathbf{u}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega \tag{8.168}$$

we obtain Eqs. $(8.162)_2 - (8.162)_4$.

In view of $(8.162)_2$ and $(8.167)_1$ from $(8.156)_1$ we get

$$\Delta\tilde{p} = \frac{1}{n^S n^L}(a n^L \Delta + m_2) \operatorname{div}\mathbf{u} = \frac{k_1^2}{n^S}(m_2 - a n^L k_1^2)\,\psi\,. \tag{8.169}$$

By virtue of $(8.148)_1$ from (8.169) it follows that

$$\Delta\tilde{p}(\mathbf{x}) = -k_1^2 d_1 d\,\psi(\mathbf{x}), \qquad \mathbf{x} \in \Omega\,. \tag{8.170}$$

Hence, $\tilde{p}$ satisfies Eq. $(8.161)_3$, where φ is a solution of $(8.162)_1$ in Ω.

Using Eqs. $(8.161)_3$, $(8.167)_{1,2}$ from (8.166) we have

$$\begin{aligned}\mathbf{u} &= \operatorname{grad}[m_1\,\varphi + (\eta_1 n^L k_1^2 + \frac{1}{\beta_0} m_1 d_1 d)\,\psi] + \beta_L\,\mathbf{\Psi}\,,\\ \mathbf{w} &= \operatorname{grad}[m_2\,\varphi + (\eta_2 n^L k_1^2 + \frac{1}{\beta_0} m_2 d_1 d)\,\psi] + \beta_3\,\mathbf{\Psi}\,.\end{aligned} \tag{8.171}$$

In view of the identities $(8.148)_{3,4}$ from (8.171) we obtain Eqs. $(8.161)_{1,2}$.

Theorem 5. If $\mathbf{u}$, $\mathbf{w}$, and $\tilde{p}$ given by Eq. (8.161), where φ, ψ and $\mathbf{\Psi}$ satisfies Eq. (8.162), then $(\mathbf{u},\ \mathbf{w},\ \tilde{p})$ is the solution of system (8.114) on Ω.

Proof. From (8.161) we get

$$\Delta\mathbf{u} = n^L k_1^2 \operatorname{grad}\psi - \beta_L k_2^2\,\mathbf{\Psi}, \quad \operatorname{grad}\ \operatorname{div}\mathbf{u} = n^L k_1^2 \operatorname{grad}\psi\,. \tag{8.172}$$

We replace $\mathbf{u}$, $\mathbf{w}$, and $\tilde{p}$ given by Eqs. $(8.161)_{1,2,3}$ on the left-hand side of Eq. $(8.114)_1$. On the basis of relations (8.82), $(8.148)_1$, (8.162), (8.172) and $\beta_S m_1 - \beta_3 m_2 = n^S \beta_0$ we obtain

$$\begin{gathered}\mu^S\Delta\mathbf{u}+(\lambda^S+\mu^S)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+\beta_S\,\mathbf{u}-\beta_3\,\mathbf{w}-n^S\,\mathrm{grad}\;p\\ =(m_1\beta_S-m_2\beta_3-n^S\beta_0)\,\mathrm{grad}\;\varphi+\\ +(an^Lk_1^2-n^Sd_1d-m_2)\;\mathrm{grad}\,\psi+(\beta_0-\mu^S\beta_Lk_2^2)\,\mathbf{\Psi}=\mathbf{0}\;.\end{gathered}\tag{8.173}$$

Similarly, in view of Eqs. $(8.148)_2$, (8.152), (8.162)
and of

$$\mathrm{grad}\,\mathrm{div}\;\mathbf{w}=-n^Sk_1^2\;\mathrm{grad}\;\psi\;,\tag{8.174}$$

we obtain

$$\begin{gathered}-\beta_3\,\mathbf{u}+\beta_L\mathbf{w}-n^L\,\mathrm{grad}\,\tilde{p}\\ =(m_2\beta_L-m_1\beta_3-n^L\beta_0)\;\mathrm{grad}\,\varphi+(m_1-n^Ld_1d)\;\mathrm{grad}\,\psi=\mathbf{0}\end{gathered}\tag{8.175}$$

Obviously, from (8.161) we get

$$\mathrm{div}\,\mathbf{u}=n^Lk_1^2\,\psi,\qquad \mathrm{div}\,\mathbf{w}=-n^Sk_1^2\,\psi\;.\tag{8.176}$$

Hence, we have Eq. $(8.114)_3$. The proof is complete.

Thus, the general solution of system of homogeneous equations (8.114) in terms of one harmonic function φ and four metaharmonic functions ψ, Ψ_1, Ψ_2, Ψ_3 is obtained. In view of Theorem 4 the system of coupled equations (8.114) on the system of uncoupled equations (8.162) is deduced.

Let

$$\begin{gathered}\Omega_1=\{\mathbf{x}:|\mathbf{x}|<r\},\;\;\Omega_2=\{\mathbf{x}:|\mathbf{x}|>r\},\;\;\Omega_3=\{\mathbf{x}:r_1<|\mathbf{x}|<r_2\}\;,\\ \text{where}\\ |\mathbf{x}|=(x_1^2+x_2^2+x_3^2)^{1/2},\;r>0,\;r_j>0\,(j=1,2)\;.\end{gathered}\tag{8.177}$$

If $\mathbf{x}\in\Omega_j$ $(j=1,2,3)$, then from Eqs. $(8.162)_{3,4}$, we have (for details see Natroshvili and Svanadze (1981))

$$\mathbf{\Psi}(\mathbf{x})=[\mathbf{x}\times\mathbf{D_x}]\widetilde{\Psi}_1(\mathbf{x})+\mathrm{curl}\;[\mathbf{x}\times\mathbf{D_x}]\widetilde{\Psi}_2(\mathbf{x})\;,\tag{8.178}$$

where $[\mathbf{x}\times\mathbf{D_x}]$ is the vector product of vectors $\mathbf{x}$ and $\mathbf{D_x}$, and

$$(\Delta+k_2^2)\,\tilde{\Psi}_l(\mathbf{x})=0,\qquad l=1,2\;.\tag{8.179}$$

Theorem 4 and Eq. (8.178) leads to the following Theorem.

Theorem 6.

If $(\mathbf{u},\, \mathbf{w}, \tilde{p})$ is a solution of system (8.114) on Ω_j, then

$$\begin{aligned} \mathbf{u}(\mathbf{x}) \ &= \mathrm{grad}\,[m_1\,\varphi(\mathbf{x}) - n^L\,\psi(\mathbf{x})] + \beta_L[\mathbf{x}\times\mathbf{D_x}]\widetilde{\Psi}_1(\mathbf{x}) + \\ &+\beta_L\,\mathrm{curl}\ \ [\mathbf{x}\times\mathbf{D_x}]\widetilde{\Psi}_2(\mathbf{x})\ , \end{aligned} \tag{8.180}$$

$$\begin{aligned} \mathbf{w}(\mathbf{x}) \ &= \mathrm{grad}\ [m_2\,\varphi(\mathbf{x}) + n^S\,\psi(\mathbf{x})] + \beta_3[\mathbf{x}\times\mathbf{D_x}]\widetilde{\Psi}_1(\mathbf{x}) + \\ &+\beta_3\,\mathrm{curl}\ \ [\mathbf{x}\times\mathbf{D_x}]\widetilde{\Psi}_2(\mathbf{x})\ , \end{aligned} \tag{8.181}$$

$$\tilde{p}(\mathbf{x}) \ \ = \beta_0\,\varphi(\mathbf{x}) + d_1 d\,\psi(\mathbf{x})\ , \tag{8.182}$$

where $\varphi,\ \psi,\ \widetilde{\Psi}_l$ satisfy Eqs. $(8.162)_1, (8.162)_2, (8.179)$, respectively, and $\mathbf{x}\in\Omega_j,\ j = 1, 2, 3,\ l = 1, 2.$

Hence, in the region $\Omega_j\ (j = 1, 2, 3)$ the general solution of system (8.114) in terms of one harmonic function φ and three metaharmonic functions $\psi,\ \widetilde{\Psi}_1,\ \widetilde{\Psi}_2$ is established.

As in the classical theory of elasticity, by virtue of Theorems 1 to 6 it is possible to construct the solutions of boundary value problems in the linear theory of incompressible fluidsaturated poroelastic solids.

8.5 Wave Propagation

Due to the complexity of the dynamic behavior of saturated porous solids, exact analytical solutions of initial and boundary value problems are not often found. Fortunately, some problems can be solved exactly. This has been done by de Boer and Liu on the basis of the Theory of Porous Media in several papers which concerns the one-dimensional wave propagation, plane waves propagation as well as growth and decay of acceleration waves, dispersion and attenuation of surface waves, inhomogeneous plane waves, mechanical energy flux, and energy dissipation and, finally, propagation and evolution of wave fronts.

With these contributions excellent tools have been provided for comparing numerical results for complicated practical problems gained by the finite or boundary element methods. This has already been done by several scientists, especially when comparing their results with the exact solution of the one-dimensional wave propagation, Breuer, 1997 a, b, c.

The one-dimensional wave propagation with its exact solution is well documented in the book by de Boer (2000 a) and will not be addressed again. Rather some other contributions to the dynamics of saturated porous solids with exact solutions will be considered in the next sections.

a) Plane Waves in a Semi-Infinite Liquidsaturated Porous Medium

The problem of the reflection and refraction of plane waves on a plane boundary has been studied extensively by several authors (see, e.g., Deresiewicz, 1962 a, b; Hajra and Mukhopadhyay, 1982), who adopted the field formulations given by Biot (1956), which describe the propagation of small amplitude disturbances in an isotropic, elastic, liquid-filled porous medium.

In this investigation, the approach used to describe fluidsaturated porous solids is applied to deal with elastic body waves in a liquidsaturated porous medium and the problem of their reflections on a plane, traction-free boundary, and the forthcoming reflections back into the medium. Also in this section the geometrically and physically linear theory is used for an incompressible binary model. As shown in the preceding section porous media are governed by a set of linear coupled partial differential equations. In this case, it has been revealed that a coupled longitudinal and transverse wave in the binary model consists. Their wave numbers or propagation waves receive clear physical meanings and are comparable with the classical solution of linear-elastically deformed one-component continua. Moreover, it is not necessary to approximate the dispersion relationship for high and low frequencies. Generally, the presence of a plane traction-free boundary results in two types of reflected wave for each incident wave. In addition, poreliquid pressure is also accompanied with the reflection of waves, to which it is worthwhile to pay special attention.

We repeat the fundamental main equation (8.75):

$$\begin{gathered}\mu^S \triangle \mathbf{u}_S + (\lambda^S + \mu^S)\,\mathrm{grad}\ \mathrm{div}\,\mathbf{u}_S - n^S\,\mathrm{grad}\ p + \\ +\rho^S[\mathbf{b}_S - (\mathbf{u}_S)''_S] + S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0}\ ,\end{gathered} \tag{8.183}$$

$$-n^L\,\mathrm{grad}\ p + \rho^L[\mathbf{b}_L - (\mathbf{u}_L)''_L] - S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0}\ , \tag{8.184}$$

$$\mathrm{div}[n^S(\mathbf{u}_S)'_S + n^L(\mathbf{u}_L)'_L] = 0\ . \tag{8.185}$$

We neglect the external accelerations $\mathbf{b}_S$ and $\mathbf{b}_L$; then the equations of motion (8.183) and (8.184) read as

$$\begin{gathered}(\lambda^S + \mu^S)\,\mathrm{grad}\ \mathrm{div}\,\mathbf{u}_S + \mu^S\,\mathrm{div}\,\mathrm{grad}\ \mathbf{u}_S - n^S\,\mathrm{grad}\ p - \\ -\rho^S(\mathbf{u}_S)''_S + S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0}\ ,\end{gathered} \tag{8.186}$$

$$-n^L\,\mathrm{grad}\ p + \rho^L(\mathbf{u}_L)'' - S_L[(\mathbf{u}_L)'_L - (\mathbf{u}_S)'_S] = \mathbf{0}\ . \tag{8.187}$$

Helmholtz's resolutions for each of the two displacement vectors $\mathbf{u}_S$ and $\mathbf{u}_L$ are considered in the forms :

$$\mathbf{u}_S = \text{grad } \Phi + \text{curl } \mathbf{h} \; , \tag{8.188}$$

$$\mathbf{u}_L = \text{grad } \Psi + \text{curl } \mathbf{g} \; , \tag{8.189}$$

where Φ, Ψ, $\mathbf{h}$ and $\mathbf{g}$ are functions of $\mathbf{x}$ and t. Inserting (8.188) and (8.189) into (8.186), (8.187) and (8.185) leads with the Laplace operator

$$\triangle(...) = \text{div}\,\text{grad}\,(...) \tag{8.190}$$

to

$$\begin{aligned} &\text{grad } [(\lambda^S + 2\mu^S)\triangle\Phi - n^S p - \rho^S \frac{\partial^2\Phi}{\partial t^2} + S_L(\frac{\partial\Psi}{\partial t} - \frac{\partial\Phi}{\partial t})] + \\ &+\text{curl } [\mu^S\triangle\mathbf{h} - \rho^S\frac{\partial^2\mathbf{h}}{\partial t^2} + S_L(\frac{\partial\mathbf{g}}{\partial t} - \frac{\partial\mathbf{h}}{\partial t})] = \mathbf{0} \; , \end{aligned} \tag{8.191}$$

$$\begin{aligned} &\text{grad } [-n^L p - \rho^L\frac{\partial^2\Psi}{\partial t^2} - S_L(\frac{\partial\Psi}{\partial t} - \frac{\partial\Phi}{\partial t})] + \\ &+\text{curl } [-\rho^L\frac{\partial^2\mathbf{g}}{\partial t^2} - S_L(\frac{\partial\mathbf{g}}{\partial t} - \frac{\partial\mathbf{h}}{\partial t})] = \mathbf{0} \; , \end{aligned} \tag{8.192}$$

$$\triangle(n^S\frac{\partial\Phi}{\partial t} + n^L\frac{\partial\Psi}{\partial t}) = 0 \; , \tag{8.193}$$

if the convective terms in the material time derivatives, are neglected.

The above mentioned equations (8.191) to (8.193) may be identically satisfied by setting

$$(\lambda^S + 2\mu^S)\triangle\Phi - n^S p = \rho^S\frac{\partial^2\Phi}{\partial t^2} - S_L(\frac{\partial\Psi}{\partial t} - \frac{\partial\Phi}{\partial t}) \; , \tag{8.194}$$

$$-n^L p = \rho^L\frac{\partial^2\Psi}{\partial t^2} + S_L(\frac{\partial\Psi}{\partial t} - \frac{\partial\Phi}{\partial t}) \; , \tag{8.195}$$

$$\triangle(n^S\frac{\partial\Phi}{\partial t} + n^L\frac{\partial\Psi}{\partial t}) = 0 \tag{8.196}$$

and

$$\mu^S\triangle\mathbf{h} = \rho^S\frac{\partial^2\mathbf{h}}{\partial t^2} - S_L(\frac{\partial\mathbf{g}}{\partial t} - \frac{\partial\mathbf{h}}{\partial t}) \; , \tag{8.197}$$

$$\rho^L\frac{\partial^2\mathbf{g}}{\partial t^2} + S_L(\frac{\partial\mathbf{g}}{\partial t} - \frac{\partial\mathbf{h}}{\partial t}) = \mathbf{0} \; . \tag{8.198}$$

By eliminating the unknown poreliquid pressure from (8.194) and (8.195), the following field equation is obtained:

$$(\lambda^S + 2\mu^S)\triangle\Phi = \rho^S \frac{\partial^2\Phi}{\partial t^2} - \frac{n^S}{n^L}\rho^L \frac{\partial^2\Psi}{\partial t^2} - \frac{S_L}{n^L}(\frac{\partial\Psi}{\partial t} - \frac{\partial\Phi}{\partial t}) \ . \tag{8.199}$$

Scalar potentials are postulated to have a harmonic time variation, i.e.

$$\Phi(\mathbf{x}, t) = \bar{\Phi}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}} \ , \tag{8.200}$$

$$\Psi(\mathbf{x}, t) = \bar{\Psi}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}} \ , \tag{8.201}$$

where ω is the frequency. Thus, by inserting (8.200) and (8.201) into (8.199) and (8.194), the following relations are obtained:

$$\begin{aligned}(\lambda^S + 2\mu^S)\triangle\bar{\Phi} = & \ -\rho^S\omega^2\bar{\Phi} + \frac{n^S}{n^L}\rho^L\omega^2\bar{\Psi} + \\ & + \frac{S_L i\omega}{n^L}\bar{\Psi} - \frac{S_L i\omega}{n^L}\bar{\Phi} \ ,\end{aligned} \tag{8.202}$$

$$n^S\triangle\bar{\Phi} + n^L\triangle\bar{\Psi} = 0 \ . \tag{8.203}$$

Also, from (8.202) and (8.203) we have

$$\triangle(\triangle + k_1^2)\bar{\Phi} = 0 \ , \tag{8.204}$$

where k_1 is the wave number (here complex). Thus, the so-called dispersion relationship is

$$k_1^2 = A_1(\omega) + B_1(\omega)i \ , \tag{8.205}$$

where

$$A_1 = \frac{\rho^S(n^L)^2 + \rho^L(n^S)^2}{(n^L)^2\lambda^S + 2(n^L)^2\mu^S} \ \omega^2 \ , \tag{8.206}$$

$$B_1 = \frac{S_L \ \omega}{(n^L)^2\lambda^S + 2(n^L)^2\mu^S} \ . \tag{8.207}$$

One physically reasonable solution of(8.204) is

$$\triangle\bar{\Phi} + k_1^2\bar{\Phi} = 0 \ , \tag{8.208}$$

or

$$(\triangle + k_1^2)\bar{\Phi} = 0 \ . \tag{8.209}$$

Otherwise we can write

$$k_1 = a_1(\omega) + b_1(\omega)i \ , \tag{8.210}$$

where

$$\begin{aligned} a_1^2 &= \frac{\sqrt{[(n^S)^2\rho^L\omega^2 + (n^L)^2\rho^S\omega^2]^2 + (S_L\,\omega)^2}}{2[(n^L)^2\lambda^S + 2(n^L)^2\mu^S]} + \\ &\quad + \frac{(n^S)^2\rho^L\omega^2 + (n^L)^2\rho^S\omega^2}{2[(n^L)^2\lambda^S + 2(n^L)^2\mu^S]} \,. \end{aligned} \tag{8.211}$$

Then, the phase velocity of the longitudinal wave is definied as:

$$\begin{aligned} c_1^2 &= c_1^2(\omega) = \frac{\omega^2}{a_1^2} \\ &= \frac{2[(n^L)^2(\lambda^S + 2\mu^S)]}{\sqrt{[(n^S)^2\rho^L + (n^L)^2\rho^S]^2 + \dfrac{S_L^2}{\omega^2}} + (n^S)^2\rho^L + (n^L)^2\rho^S} \,. \end{aligned} \tag{8.212}$$

For high frequency waves, i.e. $\omega \to \infty$, or in the case of non-dissipation ($S_L = 0$), the velocity c_1^2 is reduced to

$$\bar{c}_1^2 = \frac{(n^L)^2(\lambda^S + 2\mu^S)}{(n^S)^2\rho^L + (n^L)^2\rho^S} \,. \tag{8.213}$$

Note in passing that the above result is the same as that given by de Boer *et al.* (1993).

The attenuation coefficient is defined by

$$\begin{aligned} b_1^2 &= \frac{\sqrt{[(n^S)^2\rho^L\omega^2 + (n^L)^2\rho^S\omega^2]^2 + (S_L\omega)^2}}{2(n^L)^2(\lambda^S + 2\mu^S)} + \\ &\quad + \frac{-\,(n^S)^2\rho^L\omega^2 + (n^L)^2\rho^S\omega^2}{2(n^L)^2(\lambda^S + 2\mu^S)} \,. \end{aligned} \tag{8.214}$$

The scalar field $\bar{\Psi}$ is gained from (8.203) by an integration process

$$\bar{\Psi} = -\frac{n^S}{n^L}\bar{\Phi} \,. \tag{8.215}$$

The vector fields $\mathbf{h}$ (8.197) and $\mathbf{g}$ (8.198) can be treated with arguments similar to those stated above, i.e.

$$\mathbf{h}(\mathbf{x}, t) = \bar{\mathbf{h}}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}} \,, \tag{8.216}$$

$$\mathbf{g}(\mathbf{x}, t) = \bar{\mathbf{g}}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}} \,. \tag{8.217}$$

Then,

$$(\triangle + k_2^2)\bar{\mathbf{h}} = \mathbf{0}\ , \tag{8.218}$$

$$\bar{\mathbf{g}}(\mathbf{x}) = \frac{\rho^L S_L \omega i + S_L^2}{(\rho^L)^2\omega^2 + S_L^2}\bar{\mathbf{h}}(\mathbf{x}) \tag{8.219}$$

are derived. In (8.218) the abbreviation

$$k_2^2 = A_2(\omega) + B_2(\omega)i \tag{8.220}$$

with

$$A_2 = \frac{\rho^S(\rho^L)^2\omega^4 + \rho^S S_L^2\omega^2 + \rho^L S_L^2\omega^2}{\mu^S[(\rho^L)^2\omega^2 + S_L^2]}\ , \tag{8.221}$$

$$B_2 = \frac{(\rho^L)^2 S_L\omega^3}{\mu^S[(\rho^L)^2\omega^2 + S_L^2]} \tag{8.222}$$

has been introduced.

Let k_2 be denoted by

$$k_2 = a_2(\omega) + b_2(\omega)i\ , \tag{8.223}$$

then from (8.223) we find the phase velocity of the transverse waves

$$\begin{aligned} c_2^2 &= \frac{\omega^2}{a_2^2} \\ &= \frac{2\mu^S[(\rho^L)^2\omega^2 + S_L^2]}{\sqrt{\left(\frac{q}{\omega^2}\right)^2 + [(\rho^L)^2 S_L\omega]^2} + \frac{q}{\omega^2}}\ , \end{aligned} \tag{8.224}$$

where

$$q = \rho^S(\rho^L)^2\omega^4 + \rho^S S_L\omega^2 + \rho^L S_L^2\omega^2\ . \tag{8.225}$$

If the frequency tends to infinity or the porous material is assumed to be non-dissipative, the transverse wave velocity (8.224) becomes

$$c_2^2 = \frac{\mu^S}{\rho^S}\ . \tag{8.226}$$

The attenuation coefficient for rotational waves is expressed by

$$b_2^2 = \frac{\sqrt{q^2 + [(\rho^L)^2 S_L\omega^3]^2} - q}{2\mu^S[(\rho^L)^2\omega^2 + S_L^2]}\ . \tag{8.227}$$

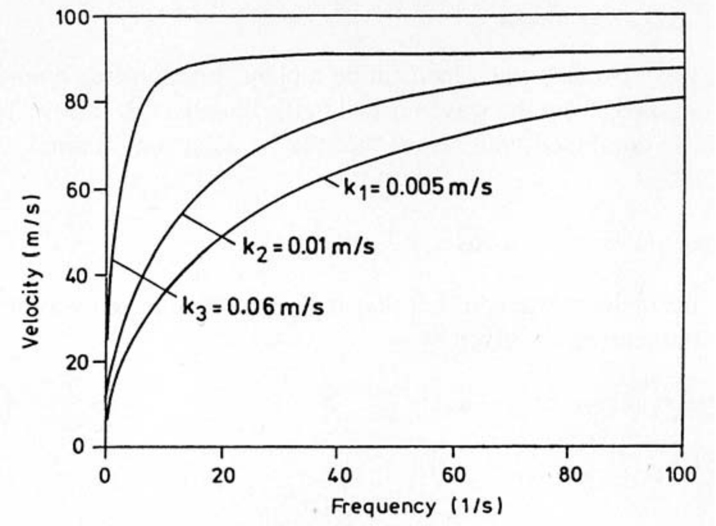

Fig. 8.5.1: Phase velocity of longitudinal waves vs. frequency.

It is indicated in (8.208), (8.215), (8.218) and (8.219) that there is one coupled longitudinal wave and one coupled transverse wave traveling in the solid and liquid of a saturated porous medium. The disturbances occurring in the solid and liquid with respect to longitudinal waves are described by the same motion equation (8.209) with (8.210) and propagate with the same velocity (8.212), but the phases and the amplitudes amongst the disturbances traveling in the solid and liquid are different. Obviously, they represent a coupled longitudinal wave. This is also true for the coupled transverse wave in the solid and liquid.

Finally, the reflection of plane waves is investigated. A half space occupied by a liquidsaturated porous medium and bounded by a horizontal plane is considered. A rectangular Cartesian base system (x, y, z) is introduced and the origin is placed at the boundary, such that $z = 0$ stands for the equation of the boundary with the z-axis being directed into the interior of the medium. The wave front is taken to be parallel to the y-axis and only plane strain waves in the xz-plane are dealt with. It is regarded that the boundary is of adequate permeability. Thus, for the two-dimensional wave propagation, the boundary conditions, corresponding to the fraction-free surface $z = 0$ are such that any force vector $\mathbf{t}$ on the boundary with the surface orthonormal vector $\mathbf{n}$

$$\mathbf{t} = (-p\mathbf{I} + \mathbf{T}_E^S)\mathbf{n} = \mathbf{0} \tag{8.228}$$

and

$$p = 0\ . \tag{8.229}$$

These boundary conditions indicate that

$$(\mathbf{T}_E^S)_{zx} = 0 \ , \quad (\mathbf{T}_E^S)_{zy} = 0, \quad (\mathbf{T}_E^S)_{zz} = 0 \ . \tag{8.230}$$

First, an *incident longitudinal wave* is considered. A plane harmonic wave is taken to be incident on a plane, traction-free boundary with an angle α_1 measured from the wave normal to the boundary. It follows from the solution of (8.208) combined with (8.200) that the incident longitudinal wave reads as

$$\Phi_1 = a_1 \mathrm{exp}\ i[k_1(x\mathrm{sin}\alpha_1 - z\mathrm{cos}\alpha_1) - \omega_1 t] \ . \tag{8.231}$$

Corresponding to this incident wave, it is anticipated that two reflected waves will be generated. Their potentials are given by

$$\begin{aligned} \Phi_1' &= A_1' \mathrm{exp}\ i[k_1'(x\xi_1' + z\eta_1') - \omega_1' t] \ , \\ h_2' &= A_2' \mathrm{exp}\ i[k_2'(x\xi_2' + z\eta_2') - \omega_2' t] \ , \end{aligned} \tag{8.232}$$

where

$$\xi_1'^2 + \eta_1'^2 = 1 \ , \tag{8.233}$$

$$\xi_2'^2 + \eta_2'^2 = 1 \ . \tag{8.234}$$

In (3.232) h_2' is the unique non-zero component of the vector $\mathbf{h}'$.

According to the superposition principle, with the aid of (8.188) and (8.189), the displacement components of the solid skeleton are:

$$(\mathbf{u}_S)_x = \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_1'}{\partial x} - \frac{\partial h_2'}{\partial z} \ , \tag{8.235}$$

$$(\mathbf{u}_S)_y = 0 \ , \tag{8.236}$$

$$(\mathbf{u}_S)_z = \frac{\partial \Phi_1}{\partial z} + \frac{\partial \Phi_1'}{\partial z} + \frac{\partial h_2'}{\partial x} \ . \tag{8.237}$$

The displacement components are inserted into the boundary conditions (8.230) and, in connection with (8.3) and with (8.4) and (8.31), we obtain:

$$k_1^2 \mathrm{sin}2\alpha_1 \Phi_1 - 2k_1'^2 \xi_1' \eta_1' \Phi_1' + k_2'^2(\eta_2'^2 - \xi_2'^2) h_2' \Big|_{z=0} = 0 \ , \tag{8.238}$$

$$\begin{aligned} k_1^2(\lambda^S + 2\mu^S \mathrm{cos}2\alpha_1)\Phi_1 \quad &+ \quad k_1'^2(\lambda^S + 2\mu^S \eta_1'^2)\Phi_1' \ + \\ &+ \quad 2k_2'^2 \mu^S \xi_2' \eta_2' h_2' \Big|_{z=0} = 0 \ . \end{aligned} \tag{8.239}$$

Note that the condition $(8.230)_2$ is identically satisfied with the consideration of a plane strain.

In order to meet the demands (8.238) and (8.239), the adequate condition should be

$$\omega_1 = \omega_1' = \omega_2' = \omega \ , \tag{8.240}$$

$$k_1 \text{sin}\alpha_1 = k_1' \xi_1' \ , \tag{8.241}$$

$$k_1 \text{sin}\alpha_1 = k_2' \xi_2' \ . \tag{8.242}$$

From (8.240) with (8.209), (8.210) and (8.218) we can write

$$k_1' = k_1 \ , \tag{8.243}$$

$$k_2' = k_2 \ . \tag{8.244}$$

Let α_1' denote the reflection angle of reflected longitudinal waves. Thus, due to (8.243) and (8.241) we have

$$\begin{aligned} \xi_1' &= \text{sin}\alpha_1' \\ &= \text{sin}\alpha_1 \ . \end{aligned} \tag{8.245}$$

Moreover, it follows from (8.233) that

$$\begin{aligned} \eta_1' &= \text{cos}\alpha_1' \\ &= \text{cos}\alpha_1 \end{aligned} \tag{8.246}$$

and

$$\alpha_1' = \alpha_1 \tag{8.247}$$

holds. This implies that the angles of incidence and reflection are equal.

From (8.240) and (8.242) with (8.210) and (8.218) we can write

$$\xi_2' = l_2' + l_2'' i \ . \tag{8.248}$$

Thus, it follows from (8.242) that

$$l_2' = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \ \text{sin}\alpha_1 \ , \tag{8.249}$$

$$l_2'' = \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \ \text{sin}\alpha_1 \ . \tag{8.250}$$

The other component of the unit vector is

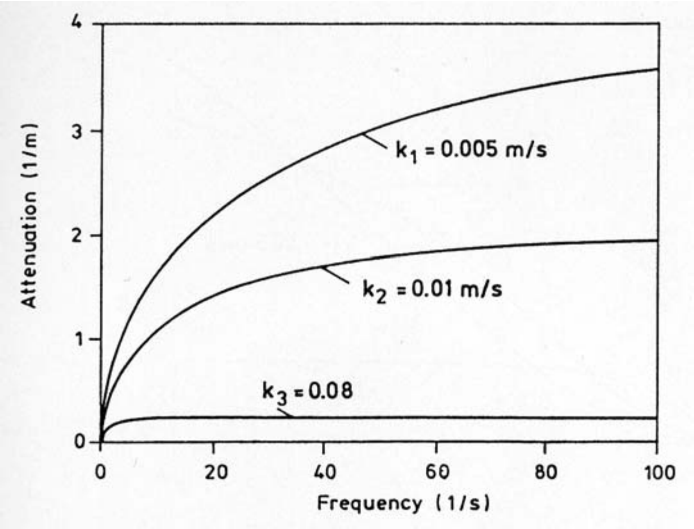

Fig. 8.5.2: Attenuation coefficient of longitudinal waves vs. frequency.

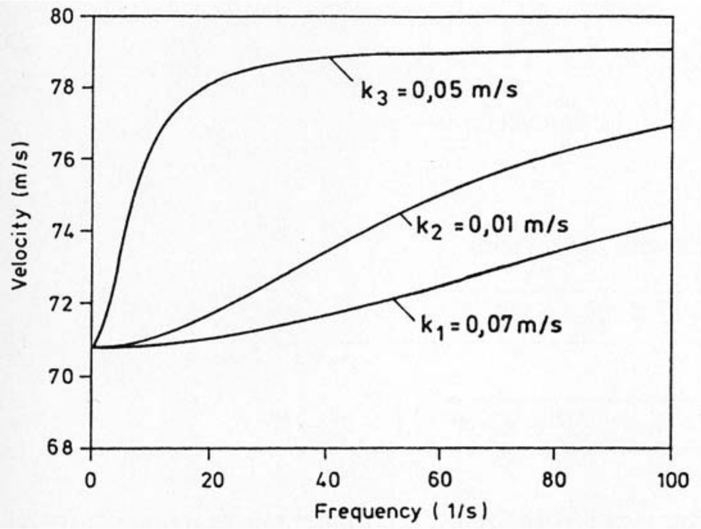

Fig. 8.5.3: Phase velocity of transverse waves vs. frequency.

$$\eta_2' = m_2' + m_2'' i \; , \tag{8.251}$$

which in connection with (8.234) yields

$$m_2'^2 = \frac{\sqrt{(1 - l_2'^2 + l_2''^2)^2 + 4l_2'^2 l_2''^2} + 1 - l_2'^2 + l_2''^2}{2} \,, \tag{8.252}$$

$$m_2''^2 = \frac{\sqrt{(1 - l_2'^2 + l_2''^2)^2 + 4l_2'^2 l_2''^2} - 1 + l_2'^2 - l_2''^2}{2} \,. \tag{8.253}$$

Therefore, from (8.232), (8.252) and considering the above results the explicit expression for reflected waves may be written as follows:

$$\Phi_1' = A_1' \exp i[k_1(x \sin\alpha_1 + z\cos\alpha_1) - \omega t] \,, \tag{8.254}$$

$$\begin{aligned} h_2' \;&=\; A_2' \exp[-(a_2 l_2'' + b_2 l_2')x - (a_2 m_2'' + b_2 m_2')z] \times \\ &\quad \times \exp i[(a_2 l_2' - b_2 l_2'')x + (a_2 m_2' - b_2 m_2'')z - \omega t] \,. \end{aligned} \tag{8.255}$$

The latter reflected waves represent inhomogeneous plane waves, because the planes of constant amplitude

$$(a_2 l_2'' + b_2 l_2')x + (a_2 m_2'' + b_2 m_2')z = \text{const.} \tag{8.256}$$

are not parallel to the planes of constant phase

$$(a_2 l_2' - b_2 l_2'')x + (a_2 m_2' - b_2 m_2'')z - \omega t = \text{const.} \tag{8.257}$$

By virtue of $(3.255)_2$, the special attenuation coefficient for the reflected waves are given by

(a) along the x-axis:

$$a_1 l_2'' + b_2 l_2' \,, \tag{8.258}$$

(b) along the z-axis:

$$a_2 m_2'' + b_2 m_2' \,. \tag{8.259}$$

In addition, the reflection angle for the shear waves are given by

$$\tan\alpha_2' = \frac{a_2 l_2' - b_2 l_2''}{a_2 m_2' - b_2 m_2''} \tag{8.260}$$

and the propagation velocity of the planes of constant phase for the reflected S_r waves can be obtained as follows:

$$c_2' = \frac{\omega \sin\alpha_2'}{a_2 l_2 - b_2 l_2''} \,. \tag{8.261}$$

It follows from (8.238) and (8.239) together with (8.240) to (8.251) that the boundary conditions become

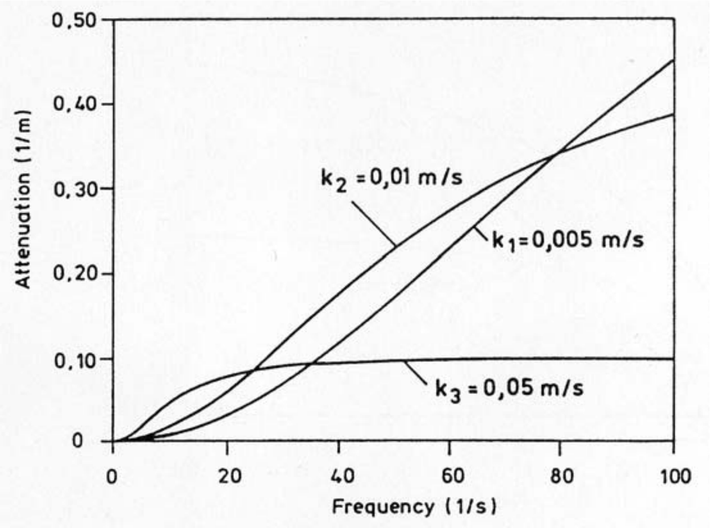

Fig. 8.5.4: Attenuation coefficient of transverse waves vs. frequency.

$$k_1^2 \sin 2\alpha_1 e_1 + k_2^2({\xi'}^2 - \eta_2')e_2 = k_1^2 \sin 2\alpha_1 \ , \tag{8.262}$$

$$\begin{aligned} &k_1^2(\lambda^S + 2\mu^S \cos^2 \alpha_1)e_1 + 2k_2^2 \mu^S \xi_2' \eta_2' e_2 \\ &\qquad = -k_1^2(\lambda^S + 2\mu^S \cos^2 \alpha_1) \ , \end{aligned} \tag{8.263}$$

where the amplitude ratios are defined

$$e_1 = \frac{A_1'}{A_1}, \quad e_2 = \frac{A_2'}{A_1} \ . \tag{8.264}$$

These amplitude ratios are complex and thus described in the form:

$$e_1 = R_1 \exp(i\Theta_1), \quad e_2 = R_2 \exp(i\Theta_2) \ , \tag{8.265}$$

where

$$R_1 = \sqrt{E_1^2 + E_2^2}, \quad \Theta_1 = \arctan \frac{E_2}{E_1} \ , \tag{8.266}$$

$$R_2 = \sqrt{F_1^2 + F_2^2}, \quad \Theta_2 = \arctan \frac{F_2}{F_1} \ . \tag{8.267}$$

In the above formulas, the following expressions are specified:

$$E_1 = \frac{E_{11}E_{13} + E_{12}E_{14}}{E_{13}^2 + E_{14}^2}, \quad E_2 = \frac{E_{12}E_{13} - E_{11}E_{14}}{E_{13}^2 + E_{14}^2} \ , \tag{8.268}$$

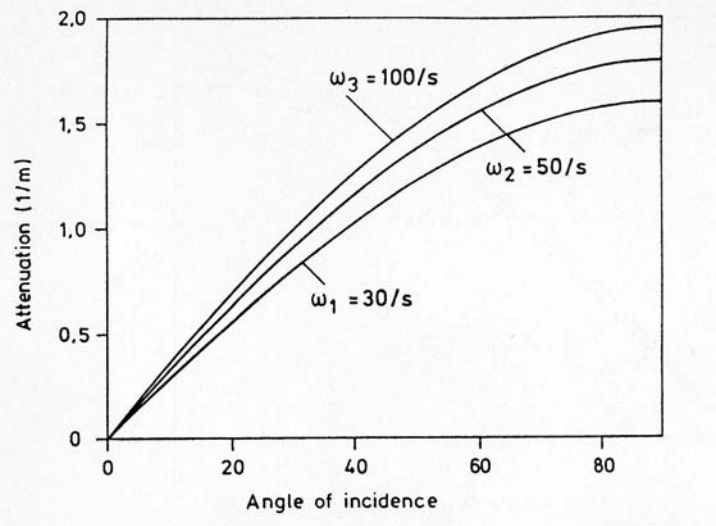

Fig. 8.5.5: Modulus of amplitude ratio of scalar potential for reflected longitudinal waves vs. angle of incidence of longitudinal waves.

$$F_1 = \frac{F_{11}F_{13} + F_{12}F_{14}}{F_{13}^2 + F_{14}^2}, \quad F_2 = \frac{F_{12}F_{13} - F_{11}F_{14}}{F_{13}^2 + F_{14}^2}\ , \tag{8.269}$$

where

$$\begin{aligned}
E_{11} &= 2\mu^S \sin 2\alpha_1 (l_2' m_2' - l_2'' m_2'') + \\
&\quad +(\lambda^S + \mu^S \cos^2\alpha_1)(2l_2'^2 - 2l_2''^2 - 1)\ , \\
E_{12} &= 2\mu^S \sin 2\alpha_1 (l_2' m_2'' - l_2'' m_2') + \\
&\quad +4(\lambda^S + 2\mu^S \cos^2\alpha_1) l_2' l_2''\ , \\
E_{13} &= 2\mu^S \sin 2\alpha_1 (l_2' m_2' - l_2'' m_2'') - \\
&\quad -(\lambda^S + \mu^S \cos^2\alpha_1)(2l_2'^2 - 2l_2''^2 - 1)\ , \\
E_{14} &= 2\mu^S \sin 2\alpha_1 (l_2' m_2'' - l_2'' m_2') - \\
&\quad -4(\lambda^S + 2\mu^S \cos^2\alpha_1) l_2' l_2''\ ,
\end{aligned} \tag{8.270}$$

$$\begin{aligned}
F_{11} &= -2A_1 \sin 2\alpha_1 (\lambda^S + 2\mu^S \cos^2 \alpha_1) \ , \\
F_{12} &= -2B_1 \sin 2\alpha_1 (\lambda^S + 2\mu^S \cos^2 \alpha_1) \ , \\
F_{13} &= A_2[2\mu^S \sin 2\alpha_1 (l_2' m_2' - l_2'' m_2'') - \\
&\quad -(\lambda^S + \mu^S \cos^2 \alpha_1)(2l_2'^2 - 2l_2''^2 - 1) + \\
&\quad +B_2[4(\lambda^S + 2\mu^S \cos^2 \alpha_1) l_2' l_2'' - \\
&\quad -2\mu^S \sin 2\alpha_1 (l_2' m_2'' + l_2'' m_2')] \ , \\
F_{14} &= A_2[2\mu^S \sin 2\alpha_1 (l_2' m_2'' + l_2'' m_2') - \\
&\quad -4(\lambda^S + 2\mu^S \cos^2 \alpha_1) l_2' l_2''] + \\
&\quad +B_2[2\mu^S \sin 2\alpha_1 (l_2' m_2' - l_2'' m_2'') - \\
&\quad -(\lambda^S + 2\mu^S \cos^2 \alpha_1)(2l_2'^2 - 2l_2''^2 - 1)] \ .
\end{aligned} \tag{8.271}$$

In the case of normal incidence of a longitudinal wave, $\alpha_1 = 0$, it is concluded from (8.265) to (8.271) that

$$e_1 = -1, \quad e_2 = 0 \ . \tag{8.272}$$

This result demonstrates that the reflected waves are only incident longitudinal waves without transverse waves.

There is a possibility of complete mode conversion if the ratio e_1 disappears, thus, only the shear waves reflected and the amplitude ratio is

$$e_2 = \frac{k_1^2 \sin 2\alpha_1}{k_2^2(\xi'^2 - \eta_2'^2)} \ . \tag{8.273}$$

The variation of the poreliquid pressure p can be attained from the motion equations (8.186) and (8.187). Note in passing that the variation of p is related only to the scalar potentials. It follows from (8.187), (8.188), and (8.215) that

$$\frac{\partial p}{\partial z} = \frac{n^S \rho^L}{(n^L)^2} \frac{\partial^2 (u_S)_z}{\partial t^2} + \frac{S_L}{(n^L)^2} \frac{\partial (u_S)_z}{\partial t} \ , \tag{8.274}$$

where

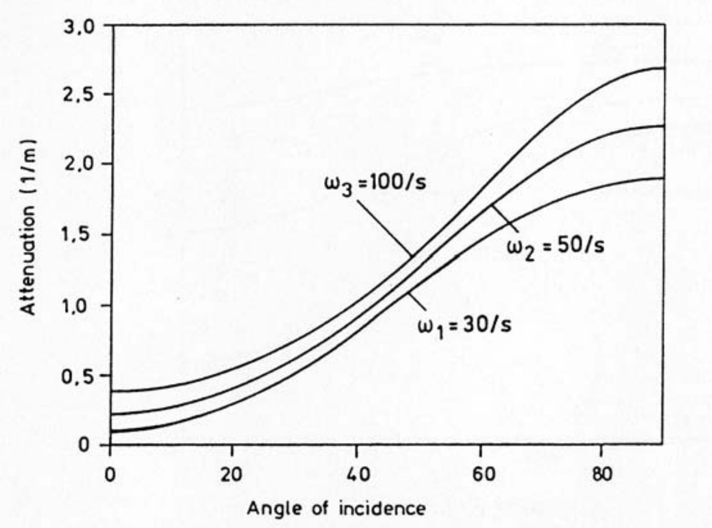

Fig. 8.5.6: Phase angle of amplitude ratio of scalar potential for reflected longitudinal waves vs. angle of incidence of longitudinal waves.

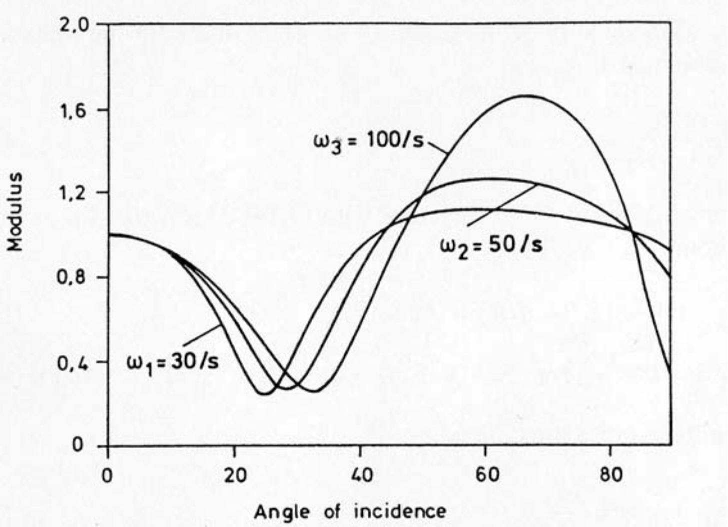

Fig. 8.5.7: Angle of reflection of transverse waves vs. angle of incidence of longitudinal waves.

$$(u_S)_z = \frac{\partial \Phi_1}{\partial z} + \frac{\partial \Phi_1'}{\partial z} \; . \tag{8.275}$$

With the aid of (8.231), (8.264), and the boundary condition (8.230) we have

$$\begin{aligned} p \;=\; & \frac{n^S \rho^2 \omega^2 + S_L i\omega}{(n^L)^2} A_1 [1 + e_1 - \exp(-ik_1 \cos\alpha_1 z) - \\ & -e_1 \exp(ik_1 \cos\alpha_1 z)] \exp[i(k_1 \sin\alpha_1 x - \omega t)] \; . \end{aligned} \tag{8.276}$$

Let *transverse waves* be *incident* on the plane and traction-free boundary at an angle β_2, then the corresponding potential is

$$h_2 = M_2 \exp i[k_2(x \sin\beta_2 - z \cos\beta_2) - \omega_2 t] \; . \tag{8.277}$$

Then, according to the above-mentioned argument, it is anticipated that the longitudinal waves at an angle β_1' and transverse waves at an angle β_2' may reflect and we obtain

$$\omega_2 = \omega_2' = \omega_1' = \omega \; , \tag{8.278}$$

$$\beta_2 = \beta_2' \; , \quad k_2 \mathrm{sin}\beta_2 = k_1' m \; , \tag{8.279}$$

where m is one component of the wave normal along the x-axis for the reflected longitudinal waves. It is evident from (8.278) and (8.279) that each of the reflected waves travels at the same frequency as the incident waves. The transverse waves reflect at the angle of incidence and with the same velocity. Thus, the potential of reflected transverse waves are

$$h_2' = M_2' \exp i[k_2(x \sin\beta_2 + z \cos\beta_2) - \omega t] \; , \tag{8.280}$$

and the reflected longitudinal waves become inhomogeneous planes waves. Its potential writes

$$\begin{aligned} \Phi_1' \;=\; & M_1' \exp[-(a_1 m_2 + b_1 m_1)x - (a_1 n_2 + b_1 n_1)z] \times \\ \times \;\; & \exp i[(a_1 m_1 - b_1 m_2)x - (a_1 n_1 - b_1 n_2)z - \omega t] \; , \end{aligned} \tag{8.281}$$

where the following expressions

$$m = m_1 + m_2 i \; , \tag{8.282}$$

$$m_1 = \frac{a_1 a_2 + b_1 b_2}{a_1^2 + b_1^2} \sin\beta_2 \; , \tag{8.283}$$

$$m_2 = \frac{a_1 b_2 - a_2 b_1}{a_1^2 + b_1^2} \sin\beta_2 \; , \tag{8.284}$$

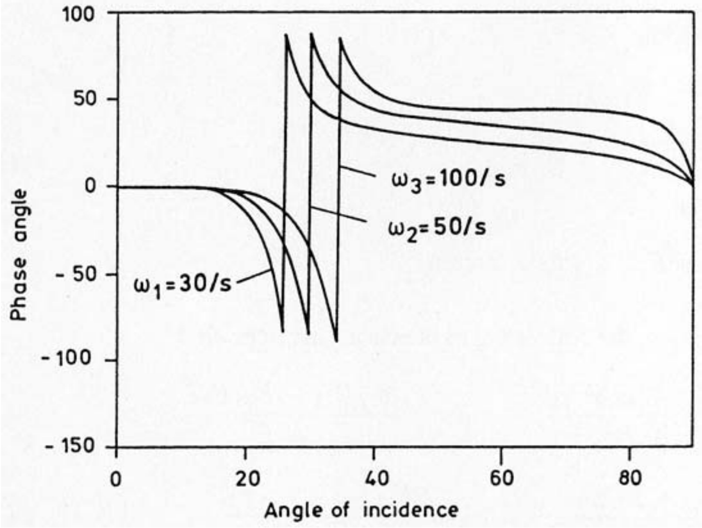

Fig. 8.5.8: Phase velocity of transverse waves vs. angle of incidence of longitudinal waves.

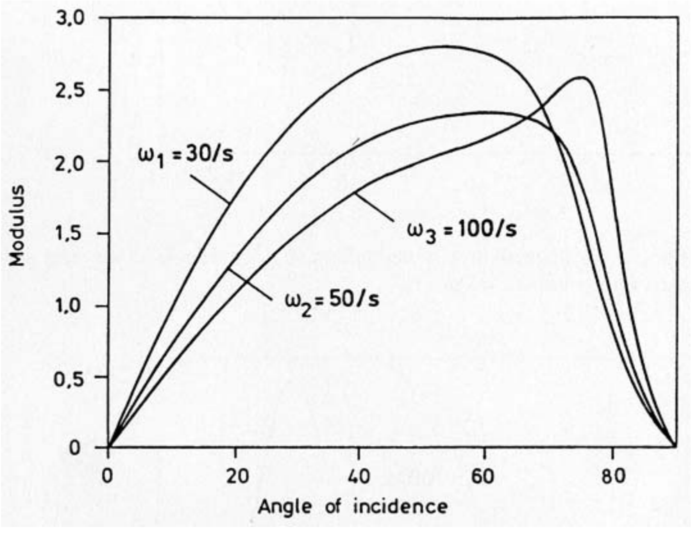

Fig. 8.5.9: Modulus of amplitude ratio of vector potential for reflected transverse waves vs. angle of incidence of longitudinal waves.

$$(n_1)^2 = \frac{\sqrt{(1 - m_1^2 + m_2^2)^2 + 4m_1^2 m_2^2} + 1 - m_1^2 + m_2^2}{2} \,, \qquad (8.285)$$

$$(n_2)^2 = \frac{\sqrt{(1 - m_1^2 + m_2^2)^2 + 4m_1^2 m_2^2} - 1 + m_1^2 + m_2^2}{2}\,, \tag{8.286}$$

are introduced. Similarly, the reflected longitudinal waves are inhomogeneous plane waves. The planes of constant amplitudes are

$$(a_1 m_2 + b_1 m_1)x + (a_1 n_2 + b_1 n_1)z = \text{const.}\,, \tag{8.287}$$

which do not coincide with the planes of constant phases

$$(a_1 m_1 - b_1 m_2)x + (a_1 n_1 - b_1 n_2)z - \omega t = \text{const.} \tag{8.288}$$

The reflected longitudinal waves propagate at the velocity

$$c_1' = \frac{\omega \sin\beta}{a_1 m_1 - b_1 m_2}\,, \tag{8.289}$$

where the reflection angle β_1 is determined by

$$\tan\beta_1 = \frac{a_1 m_1 - b_1 m_2}{a_1 n_1 - b_1 n_2}\,. \tag{8.290}$$

The attenuation coefficient parallel to the boundary is

$$a_1 m_2 + b_1 m_1\,, \tag{8.291}$$

and the attenuation coefficient perpendicular to the boundary is given by

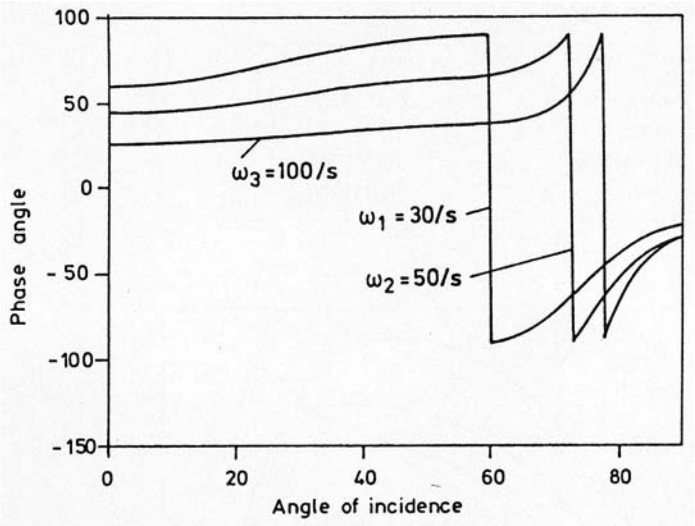

Fig. 8.5.10: Phase angle of amplitude ratio of vector potential for reflected transverse waves vs. angle of incidence of longitudinal waves.

$$a_1 n_2 + b_1 n_1 \ . \tag{8.292}$$

The amplitude ratios of the potentials in (8.277), (8.280) and (8.281) are defined by

$$r_1 = \frac{M_1'}{M_2}, \quad r_2 = \frac{M_2'}{M_2} \ . \tag{8.293}$$

Since the amplitude ratios are complex, we obtain

$$r_1 = Q_1 \exp(i\delta_1), \quad r_2 = Q_2 \exp(i\delta_2) \ , \tag{8.294}$$

where the moduli and phase angles of the amplitude ratios are determined as follows:

$$Q_1 = \sqrt{C_1^2 + C_2^2}, \quad \delta_1 = \arctan \frac{C_2}{C_1} \ , \tag{8.295}$$

$$C_1 = \frac{C_{11}C_{13} + C_{12}C_{14}}{C_{13}^2 + C_{14}^2}, \quad C_2 = \frac{C_{12}C_{13} - C_{11}C_{14}}{C_{13}^2 + C_{14}^2} \ , \tag{8.296}$$

$$Q_2 = \sqrt{D_1^2 + D_2^2}, \quad \delta_2 = \arctan \frac{D_2}{D_1} \ , \tag{8.297}$$

$$D_1 = \frac{D_{11}D_{13} + D_{12}D_{14}}{D_{13}^2 + D_{14}^2}, \quad D_2 = \frac{D_{12}D_{13} - D_{11}D_{14}}{D_{13}^2 + D_{14}^2} \tag{8.298}$$

and

$$C_{11} = A_2 \mu^S \sin 4\beta_2 \ , \tag{8.299}$$

$$C_{12} = B_2 \mu^S \sin 4\beta_2 \ , \tag{8.300}$$

$$\begin{aligned} C_{13} \quad = \quad & 2\mu^S \sin 2\beta_2 [A_1(m_1 n_1 - m_2 n_2) - B_1(m_1 n_2 + m_2 n_1)] + \\ & + A_1[\lambda^S + 2\mu^S(n_1^2 - n_2^2)] \cos 2\beta_2 - 4B_1 \mu^S n_1 n_2 \cos 2\beta_2 \ , \end{aligned} \tag{8.301}$$

$$\begin{aligned} C_{14} \quad = \quad & 2\mu^S [A_1(m_1 n_2 + m_2 n_1) + B_2(m_1 n_1 - m_2 n_2)] + \\ & + 4A_1 \mu^S n_1 n_2 \cos 2\beta_2 + B_1[\lambda^S + 2\mu^S(n_1^2 - n_2^2)] \cos 2\beta_2 \ , \end{aligned} \tag{8.302}$$

$$\begin{aligned} D_{11} \quad = \quad & 2\mu^S (m_1 n_1 - m_2 n_2) \sin 2\beta_2 - \\ & - [\lambda^S + 2\mu^S(n_1^2 - n_2^2)] \cos 2\beta_2 \ , \end{aligned} \tag{8.303}$$

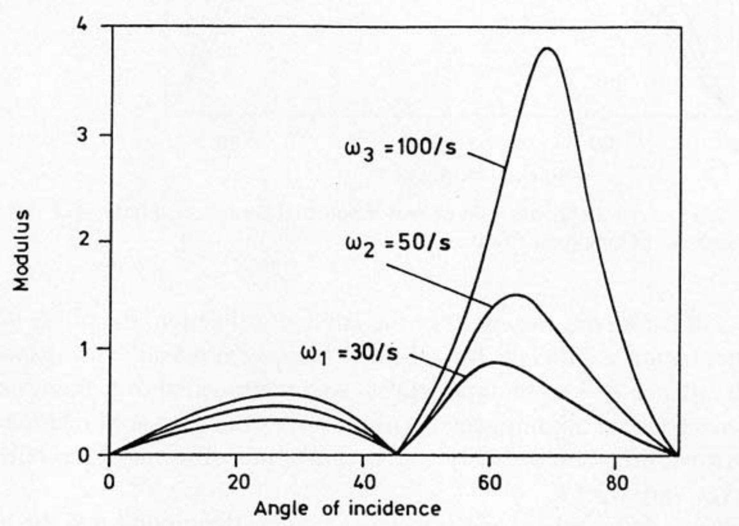

Fig. 8.5.11: Modulus of amplitude ratio of scalar potential for reflected longitudinal waves vs. angle of incidence of transverse waves.

$$D_{12} = 2\mu^S[(m_1n_2 + m_2n_1)\sin 2\beta_2 - 2n_1n_2\cos 2\beta_2] \ , \tag{8.304}$$

$$\begin{aligned} D_{13} \quad = \quad & 2\mu^S(m_1n_1 - m_2n_2)\sin 2\beta_2 + \\ & +[\lambda^S + 2\mu^S(n_1^2 - n_2^2)]\cos 2\beta_2 \ , \end{aligned} \tag{8.305}$$

$$D_{14} = 2\mu^S[(m_1n_2 + m_2n_1)\sin 2\beta_2 + 2n_1n_2\cos 2\beta_2] \ . \tag{8.306}$$

Once more, poreliquid pressure produced by the reflection of the longitudinal waves reads

$$p = \frac{n^S\rho^L\omega^2 + S_Li\omega}{(n^L)^2} r_1M_2[1 - \exp(ik_1nz)]\exp i(k_1mx - \omega t) \ , \tag{8.307}$$

where

$$n = n_1 + n_2i \ . \tag{8.308}$$

Finally, the numerical results will be discussed. The material properties for a saturated soil are given in Table I (see de Boer *et al.*, 1993). The phase

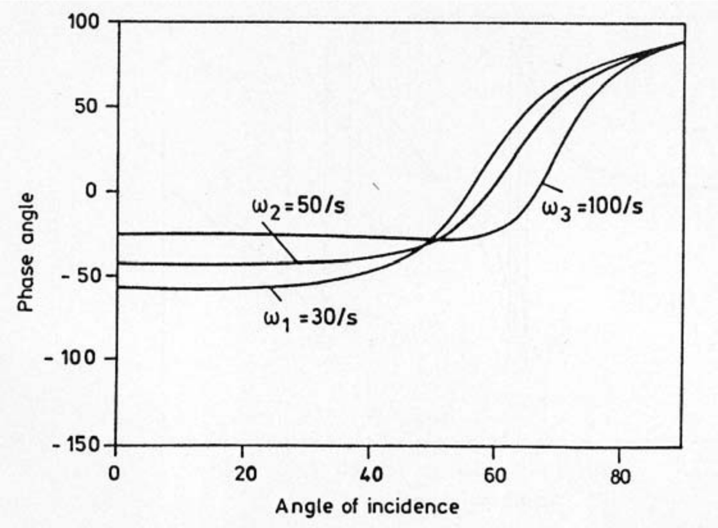

Fig. 8.5.12: Phase angle of amplitude ratio of scalar potential for reflected longitudinal waves vs. angle of incidence of transverse waves.

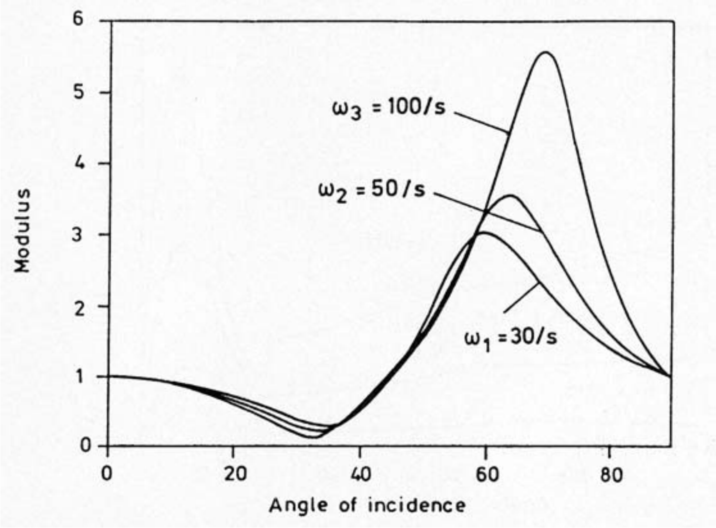

Fig. 8.5.13: Modulus of amplitude ratio of vector potential for reflected transverse waves vs. angle of incidence of transverse waves.

velocity and attenuation coefficient with frequency at different permeability for longitudinal and transverse waves are shown in Fig. 8.5.1 through 8.5.4.

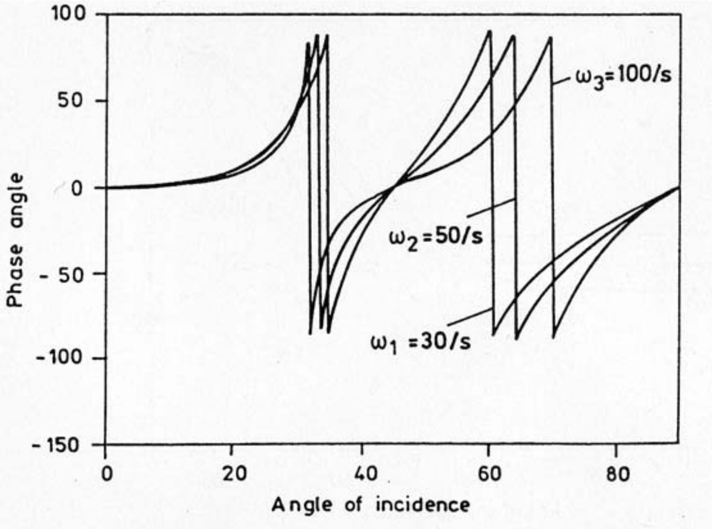

Fig. 8.5.14: Phase angle of amplitude ratio of vector potential for reflected transverse waves vs. angle of incidence of transverse waves.

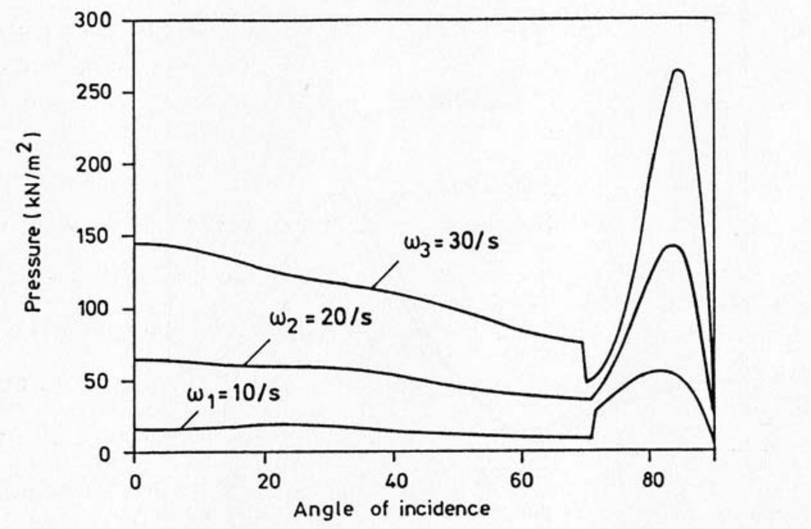

Fig. 8.5.15: Interface pressure (factor p) vs. angle of longitudinal waves at x = 0 m, z = 5 m and when t = 0.1 s.

From Figure 8.5.1 and 8.5.3 it is evident that these two coupled body waves propagating in the liquid saturated porous medium are endowed with viscous features, although the porous solid is assumed to be linear elastic. Thus, these

n^S	=	0.67	n^L	=	0.33
ρ^S	=	1.34 $\mathrm{Mg/m^3}$	ρ^L	=	0.33 $\mathrm{Mg/m^3}$
E	=	30 $\mathrm{MN/m^2}$	ν	=	0.20
λ^S	=	5.5833 $\mathrm{MN/m^2}$	μ^S	=	8.3750 $\mathrm{MN/m^2}$
qk^L	=	0.01 $\mathrm{m/s}$	γ^{LR}	=	10.00 $\mathrm{kN/m^3}$

Table I. Material properties

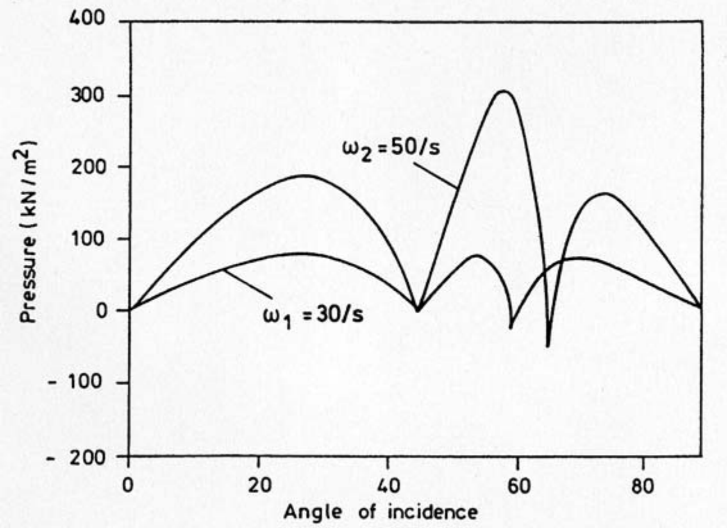

Fig. 8.5.16: Interface pressure (factor p) vs. angle of transverse waves at x = 0 m and z = 1 m, when t = 1 s.

two coupled waves are dissipative, as depicted in Figure 8.5.2 and 8.5.4, because of the interaction between the solid skeleton and liquid.

Figures 8.5.5 and 8.5.6 illustrate the attenuation coefficient for reflected transverse waves with respect to the angle of incidence of longitudinal waves. As mentioned above, the reflected transverse waves become inhomogeneous plane waves in the case of incidence of longitudinal waves; the phase velocity and attenuation coefficient of the reflected transverse waves vary with it.

For the reflected longitudinal and transverse waves, the moduli and the phase angles of the amplitude ratios with an angle of incidence of longitudinal waves are depicted in Figures 8.5.7 to 8.5.10. It is evident that the amplitude ratios of two reflected waves are dependent on the angle of incidence. A comparison between Figures 8.5.7 and 8.5.9 shows that the reflected waves are just the incident longitudinal waves themselves at zero angle of incidence in which the modulus of amplitude ratio for reflected longitudinal waves is 1.00 but that for reflected transverse waves is 0.00.

In the case of incidence of transverse waves, the moduli and the phase angle of the amplitude ratios of reflected longitudinal waves and reflected transverse waves against the angle of incidence are presented in Figures 8.5.11 to 8.5.14. It is clear that only transverse waves reflect themselves without reflected longitudinal waves at the angles of 0° and 45°, which is similar to results in Fig. 8.5.7 and 8.5.9.

Particularly, the results of the interface pressures with angles of incidence of longitudinal and transverse waves at different depths are depicted in Figures 8.5.15 and 8.5.16. It is shown that the interface pressure reaches maximum at a large angle of incidence.

b) Propagation of Acceleration Waves in Saturated Porous Solids

The present investigation is on the propagation of acceleration waves in porous media consisting of the incompressible solid and the incompressible liquid. Discussions on this problem can be found in the literature, e.g., Dziecielak (1986) investigated the behavior of acceleration waves in a liquid-filled porous medium. However, most of them were based on the motion equations proposed by Biot, which were not developed from the fundamental axioms and principles of mechanics and thermodynamics. Thus, the subsequent conclusions are unsatisfactory. In this section the propagation of acceleration waves in incompressible saturated porous solid will be studied. The theory of propagating singular surface, developed by Thomas (1957) as well as Truesdell and Toupin (1960) is utilized to examine the propagation of acceleration waves in the media.

The subject of our interest is a poroelastic solid filled with an ideal liquid, such that two constituents are assumed incompressible and the whole medium is statistically homogenous and isotropic. Possible thermal and chemical effects are excluded. After these preliminaries some general properties of acceleration waves will be discussed. Consider a singular curved surface $\sigma(t)$ with the curvilinear coordinates $\Theta^\beta(\beta = 1, 2)$ moving during the time t through the body B. By definition, the propagation of a singular surface is said to be acceleration waves if the motions, deformation gradients and velocities are continuous across $\sigma(t)$, but the accelerations, the second-order deformation gradient and their higher order derivatives suffer finite jumps across $\sigma(t)$. The geometrical and the kinematical conditions of compatibility for the first and the second derivatives of a function are presented in the reference, see Thomas (1957), Truesdell and Toupin (1960). However, the field variables across the surface $\sigma(t)$ are still restricted by the dynamic conditions of compatibility that are a part of the field equations when the discontinuity surface occurs.

For the present problem concerned, the dynamic conditions of compatibility are derived from the local mass, momentum and energy balances. It follows from Eringen (1975) that the jump field equations are:

Balance of mass

$$\Big[\!\Big|\rho^{\wp}(\mathbf{x}'_{\wp}\cdot\mathbf{n}-u_n)\Big|\!\Big]=0 \quad \text{or} \quad \Big[\!\Big|\rho^{\wp}U_{\wp}\Big|\!\Big]=0 \tag{8.309}$$

($\wp = S$, solid phase; $\wp = L$, liquid phase)[2].

Balance of momentum

$$\Big[\!\Big|\rho^{\wp}\mathbf{x}'_{\wp}(\mathbf{x}'_{\wp}\cdot\mathbf{n}-u_n)-\mathbf{T}^{\wp}\mathbf{n}\Big|\!\Big]=\mathbf{0}\ . \tag{8.310}$$

Balance of energy

$$\Big[\!\Big|\rho^{\wp}(\varepsilon^{\wp}+\frac{1}{2}\mathbf{x}'_{\wp}\cdot\mathbf{x}'_{\wp})(\mathbf{x}'_{\wp}\cdot\mathbf{n}-u_n)-\mathbf{T}^{\wp}\mathbf{x}'_{\wp}\cdot\mathbf{n}\Big|\!\Big]=0 \tag{8.311}$$

or

$$\Big[\!\Big|\rho^{\wp}(\varepsilon^{\wp}+\frac{1}{2}\mathbf{x}'_{\wp}\cdot\mathbf{x}'_{\wp})U_{\wp}\Big|\!\Big]+\Big[\!\Big|\mathbf{T}^{\wp}\mathbf{x}'_{\wp}\Big|\!\Big]\cdot\mathbf{n}=0\ . \tag{8.312}$$

In (8.309) and (8.312) the local speed of propagation for the constituent $\varphi^{\wp}$ is introduced by

$$U_{\wp}=u_n-\mathbf{x}'_{\wp}\cdot\mathbf{n}\ . \tag{8.313}$$

In the above expressions the unit normal vector $\mathbf{n}$ to the surfaces and the normal velocity u_n of the singular surface have been utilized. Note that temperature effects are neglected.

Now, let us examine the properties of acceleration waves propagating in a liquidsaturated porous medium. Since the velocity of each material particle of both constituents stays continuously across $\sigma(t)$, the local speeds of propagation in (8.313) are also continuous across the surface. Hence, from (8.309) we see that both densities are continous, i.e.

$$\Big[\!\Big|\rho^{\wp}\Big|\!\Big]=0\ . \tag{8.314}$$

---

[2] We replace the index α for the constituents used in the first part of this book with $\wp$ in order not to come into conflict with Greek letters in differential geometry.

Due to the constraint of incompressibility the volume fractions are continuous across the singular surface $\sigma(t)$:

$$\left[\!\left| n^{\wp} \right|\!\right] = 0 \,. \tag{8.315}$$

In turn, with the aid of (8.314) the jump condition (8.310) simplifies to

$$\left[\!\left| \mathbf{T}^{\wp} \right|\!\right] \mathbf{n} = \mathbf{0} \,, \tag{8.316}$$

which means that the stress tensor in both constituents are continuous across $\sigma(t)$. After the application of (8.316) to (8.311) or (8.312), it is found that the internal energy density of the mixture has no jump across the singular surface

$$\left[\!\left| \varepsilon^{\wp} \right|\!\right] = 0 \,. \tag{8.317}$$

Therefore, the dynamic conditions of compatibility are ensured automatically for acceleration waves in saturated porous solids.

With the use of well-known geometric and kinematic conditions of compatibility, the expressions of the jumps for the displacements $\mathbf{u}_{\wp}$ of the skeleton and the liquid may be written as

$$\left[\!\left| \operatorname{grad}\operatorname{grad}\mathbf{u}_S \right|\!\right] = \mathbf{a}_S \otimes \mathbf{n} \otimes \mathbf{n}, \quad \left[\!\left| \operatorname{grad}\operatorname{grad}\mathbf{u}_L \right|\!\right] = \mathbf{a}_L \otimes \mathbf{n} \otimes \mathbf{n} \,, \tag{8.318}$$

$$\left[\!\left| \frac{\partial^2 \mathbf{u}_S}{\partial t^2} \right|\!\right] = u_n^2 \mathbf{a}_S, \quad \left[\!\left| \frac{\partial^2 \mathbf{u}_L}{\partial t^2} \right|\!\right] = u_n^2 \mathbf{a}_L \,, \tag{8.319}$$

$$\left[\!\left| \operatorname{grad}\frac{\partial \mathbf{u}_S}{\partial t} \right|\!\right] = -u_n \mathbf{a}_S \otimes \mathbf{n}, \quad \left[\!\left| \operatorname{grad}\frac{\partial \mathbf{u}_L}{\partial t} \right|\!\right] = -u_n \mathbf{a}_L \otimes \mathbf{n} \,, \tag{8.320}$$

where the vectors

$$\mathbf{a}_S = \left[\!\left| \operatorname{grad}\operatorname{grad}\mathbf{u}_S \right|\!\right] \mathbf{n} \otimes \mathbf{n}, \quad \mathbf{a}_L = \left[\!\left| \operatorname{grad}\operatorname{grad}\mathbf{u}_L \right|\!\right] \mathbf{n} \otimes \mathbf{n} \tag{8.321}$$

are measures of the discontinuity strength along the direction of the acceleration jump of the solid and the liquid, respectively. If they are in the direction of $\mathbf{n}$ of the surface, we have longitudinal waves; otherwise, when they are normal to $\mathbf{n}$ we have transverse waves. Keeping in mind the assumption of small deformations we can replace the velocity $\mathbf{x}'_{\wp}$ with the time derivative of the displacement $\dfrac{\partial \mathbf{u}_{\wp}}{\partial t}$ for each constituent. With the aid of (8.320) it follows that

$$\left[\!\left| \operatorname{div} \mathbf{x}'_S \right|\!\right] = \left[\!\left| \operatorname{div} \frac{\partial \mathbf{u}_S}{\partial t} \right|\!\right] = -u_n \mathbf{a}_S \cdot \mathbf{n} \; , \tag{8.322}$$

$$\left[\!\left| \operatorname{div} \mathbf{x}'_L \right|\!\right] = \left[\!\left| \operatorname{div} \frac{\partial \mathbf{u}_L}{\partial t} \right|\!\right] = -u_n \mathbf{a}_L \cdot \mathbf{n} \; . \tag{8.323}$$

The balance equations of mass for the solid and liquid read:

$$(\rho^S)'_S = -\rho^S \operatorname{div} \mathbf{x}'_S, \;\; (\rho^L)'_L = -\rho^L \operatorname{div} \mathbf{x}'_L \; , \tag{8.324}$$

or considering the incompressibility conditions

$$(n^S)'_S = -n^S \operatorname{div} \mathbf{x}'_S, \;\; (n^L)'_L = -n^L \operatorname{div} \mathbf{x}'_L \; . \tag{8.325}$$

Thus, by use of (8.322) and (8.323) we obtain

$$\left[\!\left| (n^S)'_S \right|\!\right] = n^S u_n \mathbf{a}_S \cdot \mathbf{n}, \;\; \left[\!\left| (n^L)'_L \right|\!\right] = n^L u_n \mathbf{a}_L \cdot \mathbf{n} \; , \tag{8.326}$$

$$\left[\!\left| (\rho^S)'_S \right|\!\right] = \rho^S u_n \mathbf{a}_S \cdot \mathbf{n} \; , \;\; \left[\!\left| (\rho^L)'_L \right|\!\right] = \rho^L u_n \mathbf{a}_L \cdot \mathbf{n} \; . \tag{8.327}$$

Accordingly, from Eqs. (8.326) and (8.327) the time rate of the densities and the volume fractions of the constituents are continuous across the singular surface for the transverse waves. However, for the longitudinal waves, we have

$$\left[\!\left| (n^S)'_S \right|\!\right] = n^S u_n a^n_S \; , \;\; \left[\!\left| (n^L)'_L \right|\!\right] = n^L u_n a^n_L \; , \tag{8.328}$$

$$\left[\!\left| (\rho^S)'_S \right|\!\right] = \rho^S u_n a^n_S, \;\; \left[\!\left| (\rho^L)'_L \right|\!\right] = \rho^L u_n a^n_L \; , \tag{8.329}$$

which expresses the amplitudes of the solid skeleton and the liquid in the normal direction of the surface

$$a^n_S = \mathbf{a}_S \cdot \mathbf{n}, \;\; a^n_L = \mathbf{a}_L \cdot \mathbf{n} \; . \tag{8.330}$$

Therefore, by means of (8.322), (8.323) and (8.330) Eq. (8.185) becomes

$$n^S a^n_S + n^L a^n_L = 0 \; , \tag{8.331}$$

which indicates that the amplitudes in the solid skeleton and liquid are coupled.

Moreover, it is concluded from the constitutive equations for the stress in the solid that the pressure p is continuous across the surface $\sigma(t)$, i.e.

$$\left[\!\left| p \right|\!\right] = 0\ . \tag{8.332}$$

Henceforth, in the remainder of this investigation, we will assume that the material region ahead of the singular surface is initially at rest in a homogenous configuration, which is taken to be the reference configuration. It is clear that in this situation

$$u_n = U_{\wp}\ . \tag{8.333}$$

Finally, the propagation conditions of acceleration waves will be addressed. These can be obtained from the governing field equations (8.183) through (8.185). For this purpose we derive from (8.318)

$$\left[\!\left| \operatorname{grad}\operatorname{div}\mathbf{u}_S \right|\!\right] = (\mathbf{a}_S \cdot \mathbf{n})\mathbf{n}\ , \tag{8.334}$$

$$\left[\!\left| \operatorname{div}\operatorname{grad}\mathbf{u}_S \right|\!\right] = \mathbf{a}_S\ , \tag{8.335}$$

$$\left[\!\left| \operatorname{grad}\operatorname{div}\mathbf{u}_L \right|\!\right] = (\mathbf{a}_L \cdot \mathbf{n})\mathbf{n}\ . \tag{8.336}$$

With the help of the geometric condition of compatibility, see Trusdell and Toupin (1960), we obtain

$$\left[\!\left| \operatorname{grad} p \right|\!\right] = b\,\mathbf{n}\ , \tag{8.337}$$

where

$$b = \left[\!\left| \operatorname{grad} p \right|\!\right] \cdot \mathbf{n} = \left[\!\left| p_{,n} \right|\!\right] \tag{8.338}$$

is a measure of the pressure gradient discontinuity across the surface.

Since the field equations (8.183) and (8.184) hold in each of the regions divided by the singular surface $\sigma(t)$, bearing in mind that the velocities and the body forces of the solid and the liquid have no jump at the surface, we obtain with the aid of (8.319) and (8.334) to (8.337) across the surface

$$(\lambda^S + \mu^S)(\mathbf{a}_S \cdot \mathbf{n})\mathbf{n} + \mu^S \mathbf{a}_S - n^S b\mathbf{n} = \rho^S u_n^2 \mathbf{a}_S\ , \tag{8.339}$$

$$-\, n^L b\mathbf{n} = \rho^L u_n^2 \mathbf{a}_L\ . \tag{8.340}$$

Given the direction $\mathbf{n}$ and the possible velocity u_n the direction of the amplitudes of the solid and the liquid can be determined.

In order to get the possible propagation speeds, let us determine the projection of Eq. (8.339) and (8.340) in the direction on $\mathbf{n}$ and normal to $\mathbf{n}$. Upon elimination of the pressure gradient term, which is determined by (8.340), the projection of (8.339) and (8.340) in the direction of $\mathbf{n}$ and normal to $\mathbf{n}$ yields:

$$\begin{gathered}\{(n^L)^2(\lambda^S + 2\mu^S) - [(n^S)^2\rho^L + (n^L)^2\rho^S]u_n^2\}\mathbf{a}_S \cdot \mathbf{n} = 0 \ , \\ (\mu^S - \rho^S u_n^2)\mathbf{a}_S \times \mathbf{n} = \mathbf{0} \ , \\ \mathbf{a}_L \times \mathbf{n} = \mathbf{0} \ .\end{gathered} \tag{8.341}$$

These equations express that there are only two possible propagation speeds, which may be derived from the existence conditions of the nontrivial amplitudes. The results can be written as

$$u_{n1}^2 = \frac{(n^L)^2(\lambda^S + 2\mu^S)}{(n^L)^2\rho^S + (n^S)^2\rho^L} \ , \tag{8.342}$$

$$u_{n2}^2 = \frac{\mu^S}{\rho^S} \ . \tag{8.343}$$

Now, we replace u_n in $(8.341)_1$ with u_{n2}. As a result we obtain

$$\mathbf{a}_S \cdot \mathbf{n} = 0 \ , \tag{8.344}$$

and analogously, the replacement of u_n in $(8.341)_2$ with u_{n1} yields

$$\mathbf{a}_S \times \mathbf{n} = \mathbf{0} \ . \tag{8.345}$$

The above two relations demonstrate that the direction of the amplitudes in the waves with the speed u_{n1} and u_{n2} coincide with the directions of $\mathbf{n}$ and normal to $\mathbf{n}$. The waves with the speed u_{n1} are longitudinal waves in the two phases and the waves with u_{n2} are transverse waves only in the liquid. The amplitude of the longitudinal waves in the solid and liquid satisfy the relations (8.3.150). The interaction effect does not influence the propagation velocities but yields the attenuation acceleration waves.

Since the propagation speeds are constant and depend on the physical and mechanical properties of the medium, waves in the process of propagation will keep their initial shapes due to

$$\frac{\delta \mathbf{n}}{\delta t} = u_{n,\beta} \ \mathbf{a}^\beta = \mathbf{0} \ , \ \ \beta = 1, 2, \tag{8.346}$$

where $\dfrac{\delta \mathbf{n}}{\delta t}$ denotes the displacement derivative of the unit normal $\mathbf{n}$ with respect to surface $\sigma(t)$, and where $\mathbf{a}^{\beta}$ is the contravariant base vector on the surface $\sigma(t)$. Thus, the acceleration waves form a family of parallel surfaces.

It should be noted that for empty porous solids

$$\rho^L \to 0 \tag{8.347}$$

is valid. Then, it follows from (8.3.162) that the propagation speed of the longitudinal waves reduce to

$$u_{n1}^2 = \frac{\lambda^S + 2\mu^S}{\rho^S} \tag{8.348}$$

which is a counterpart of the classical elasticity.

Only a longitudinal and a transverse acceleration wave can propagate in incompressible porous media and the two propagation speeds and their directions of the amplitudes are provided by examining the existence of acceleration waves. The interaction between the phases does not influence the speeds of acceleration waves. Acceleration waves keep their initial shapes in the process of propagation.

c) Growth and Decay of Acceleration Waves

The propagation properties and the propagation conditions of acceleration waves in incompressible saturated porous solids have been discussed extensively in the preceding section. The existence of a longitudinal wave and a transverse wave in the incompressible two-phase porous medium has been revealed. The longitudinal disturbance in the liquid are carried by the wave motion in the solid skeleton. Their amplitudes have been indicated to obey a certain relation. In order to obtain a more complete information on the properties of acceleration waves, the growth and decay of acceleration waves in such media are investigated in the present study (see de Boer and Liu, 1995). The theory of propagating singular surfaces developed by Thomas (1957) is utilized to examine the growth and decay of acceleration waves in incompressible porous media.

Consider a moving surface of discontinuity $\sigma(t)$ in three-dimensional Euclidian space. The parametric equation of this surface is described by (see de Boer, 1982)

$$\mathbf{x} = \mathbf{x}(\Theta^\alpha, t), \quad \alpha = 1, 2\ . \tag{8.349}$$

Here Θ^α are the curvilinear coordinates of the surface $\sigma(t)$, t denotes the time, and $\mathbf{x}$ is the position vector for a reference point to the surface. For the oriented surfaces $\sigma(t)$ the tangent vectors read:

$$\mathbf{a}_\alpha = \mathbf{x}_{,\alpha} = \frac{\partial \mathbf{x}}{\partial \Theta^\alpha} \ , \tag{8.350}$$

where $\mathbf{a}_\alpha$ is the covariant base vector on the surface; the corresponding contravariant base vector is indicated by $\mathbf{a}^\alpha$. The surface metric tensor of the surface $\sigma(t)$ is given by

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \ . \tag{8.351}$$

The components of the second fundamental form of the surface satisfy

$$b_{\alpha\beta} = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta} = -\mathbf{n}_{,\alpha} \cdot \mathbf{a}_\beta \ , \tag{8.352}$$

where $\mathbf{n}$ is the unit normal vector of the surface $\sigma(t)$. The mean curvature Ω and the total or Gauss curvature K of the surfaces $\sigma(t)$ are the first and the second invariants of the second fundamental form $b^\alpha_{.\beta}$; these write:

$$\Omega = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = \frac{1}{2} b^\alpha_{.\,\alpha}, \quad K = \frac{1}{2} (b^\alpha_{.\,\alpha})^2 - \frac{1}{2} b^\alpha_{.\,\beta}\, b^\beta_{.\,\alpha} \ . \tag{8.353}$$

The normal velocity of the surface $\sigma(t)$ is specified by

$$u_n = n \cdot \dot{\mathbf{x}}, \quad \dot{\mathbf{x}} = \frac{\partial \mathbf{x}(\Theta^\alpha, t)}{\partial t} \ , \tag{8.354}$$

and the local speeds of propagation are defined by

$$U_\wp = u_n - \mathbf{x}'_\wp \cdot \mathbf{n}, \quad \wp = S, L \ , \tag{8.355}$$

which indicates a measure of the normal velocity of the surface $\sigma(t)$ with respect to the material particles. In our case, $\mathbf{x}'_\wp$ denotes the velocities of the particles of the two components in the two-phase porous media.

For an arbitrary time-dependent tensor valued field $\Phi(\mathbf{x}, t)$ (Φ may be components of a tensorial quantity) confined to the surfaces, the displacement derivative is introduced in the form

$$\frac{\delta \Phi}{\delta t} = \frac{\partial \Phi}{\partial t} + u_n \mathbf{n} \cdot \operatorname{grad} \Phi \ , \tag{8.356}$$

which is defined as the time derivative of Φ along the normal trajectory. A useful relation here should be mentioned as follows:

$$\frac{\delta \mathbf{n}}{\delta t} = -u_{n,\alpha} \mathbf{a}^\alpha = -u_{n,\alpha} a^{\alpha\beta} \mathbf{a}_\beta \ , \tag{8.357}$$

which relates the displacement of the unit normal vector $\mathbf{n}$ with the propagation speed u_n of the surface.

Being an arbitrary function, $\Phi(\mathbf{x}, t)$ has a jump across the surface $\sigma(t)$ given by

$$\Big[\!\Big|\Phi\Big|\!\Big] = \Phi^{+} - \Phi^{-} \neq 0 \,, \tag{8.358}$$

where Φ^{+} and Φ^{-} are the definite limits of the function Φ before and behind the surface $\sigma(t)$, respectively. In the following paragraph we introduce the denotations:

$$\begin{gathered}
A = \Big[\!\Big|\Phi\Big|\!\Big], \;\; B = \Big[\!\Big|\operatorname{grad}\Phi\Big|\!\Big]\cdot\mathbf{n}, \;\; C = \Big[\!\Big|\operatorname{grad}\operatorname{grad}\Phi\Big|\!\Big]\cdot\mathbf{n}\otimes\mathbf{n}\,, \\
\mathbf{A} = \Big[\!\Big|\operatorname{grad}\Phi\Big|\!\Big], \;\; \mathbf{B} = \Big[\!\Big|\operatorname{grad}\operatorname{grad}\Phi\Big|\!\Big]\mathbf{n}\,, \\
A' = \Big[\!\Big|\frac{\partial\Phi}{\partial t}\Big|\!\Big], \;\; B' = \Big[\!\Big|\operatorname{grad}\frac{\partial\Phi}{\partial t}\Big|\!\Big]\cdot\mathbf{n}\,.
\end{gathered} \tag{8.359}$$

With the aid of the above denotations (8.359) the geometrical conditions of compatibility have the followings forms (see Eringen and Suhubin, 1974):

$$\begin{aligned}
&\Big[\!\Big|\operatorname{grad}\Phi\Big|\!\Big] = B\mathbf{n} + a^{\alpha\beta}A\,|_{\alpha}\,\mathbf{a}_{\beta}\,, \\
&\Big[\!\Big|\operatorname{grad}\operatorname{grad}\Phi\Big|\!\Big] = C\mathbf{n}\otimes\mathbf{n} + \\
&+a^{\alpha\beta}(B\,|_{\alpha} + a^{\gamma\delta}b_{\alpha\gamma}A\,|_{\delta})(\mathbf{n}\otimes\mathbf{a}_{\beta} + \mathbf{a}_{\beta}\otimes\mathbf{n}) + \\
&+a^{\alpha\beta}a^{\gamma\delta}(A\,|_{\alpha\gamma} - Bb_{\alpha\gamma})\mathbf{a}_{\beta}\otimes\mathbf{a}_{\delta}
\end{aligned} \tag{8.360}$$

and the kinematical conditions of compatibility are as follows

$$\Big[\!\Big|\frac{\partial\Phi}{\partial t}\Big|\!\Big] = -u_n B + \frac{\delta A}{\delta t}\,, \tag{8.361}$$

$$\Big[\!\Big|\frac{\partial^2\Phi}{\partial t^2}\Big|\!\Big] = \Big(u_n C - \frac{\delta B}{\delta t} - a^{\alpha\beta}A\,|_{\alpha}\,u_{n,\beta}\Big)u_n + \frac{\delta A'}{\partial t}\,, \tag{8.362}$$

$$\Big[\!\Big|\operatorname{grad}\frac{\partial\Phi}{\partial t}\Big|\!\Big] = \Big(-u_n C + \frac{\delta B}{\delta t} + a^{\alpha\beta}A\,|_{\alpha}\,u_{n,\beta}\Big)\mathbf{n} + a^{\alpha\beta}A'\,|_{\alpha}\,\mathbf{a}_{\beta}\,, \tag{8.363}$$

where (...)| indicates “covariant derivatives”.

Henceforth, in the remainder of our investigation, we assume that the material region ahead of the singular surfaces is initially at rest in a homogeneous configuration, which is taken to be as the reference configuration. It is clear from (8.354) and (8.355) that in this case

$$u_n = U_{\wp}\,. \tag{8.364}$$

For the propagation of acceleration waves the following conditions are automatically satisfied

$$\llbracket \mathbf{u}_\wp \rrbracket = \mathbf{0}, \quad \llbracket \frac{\partial \mathbf{u}_\wp}{\partial t} \rrbracket = \mathbf{0}, \quad \llbracket n^\wp \rrbracket = 0, \quad \llbracket \rho^\wp \rrbracket = 0, \quad \llbracket p \rrbracket = 0 \,. \tag{8.365}$$

It follows from the discussion on the propagation conditions for the longitudinal and the transverse acceleration waves in incompressible saturated poroelastic solids that (see the preceding section)

$$\{(n^L)^2(\lambda^S + 2\mu^S) - [(n^S)^2\rho^L + (n^L)^2\rho^S]u_n^2\} \; \mathbf{a}^S \cdot \mathbf{n} = 0 \,, \tag{8.366}$$

$$n^S \mathbf{a}^S \cdot \mathbf{n} + n^L \mathbf{a}^L \cdot \mathbf{n} = 0 \,, \tag{8.367}$$

$$(\mu^S - \rho^S u_n^2)\mathbf{a}^S \times \mathbf{n} = \mathbf{0} \,, \tag{8.368}$$

$$\mathbf{a}^L \times \mathbf{n} = \mathbf{0} \,, \tag{8.369}$$

where the vectors

$$\mathbf{a}^S = \llbracket \operatorname{grad} \operatorname{grad} \mathbf{u}_S \rrbracket \mathbf{n} \otimes \mathbf{n}, \quad \mathbf{a}^L = \llbracket \operatorname{grad} \operatorname{grad} \mathbf{u}_L \rrbracket \mathbf{n} \otimes \mathbf{n} \tag{8.370}$$

are measures of the discontinuity strength along the direction of the acceleration jump of the solid and liquid, respectively. If they are in the direction of $\mathbf{n}$ to the surface, we have longitudinal waves; otherwise, when they are normal to $\mathbf{n}$ we have transverse waves.

It is indicated from (8.366) to (8.369) that two types of acceleration waves with two propagation speeds are realizable in incompressible porous media. The longitudinal waves can propagate either in solids or in liquids. Their amplitudes confirm to the relation (8.367). In other words, the waves in liquid are carried by the waves in the solid. These results originate from the incompressibility of the two constituents. The transverse waves propagate only in the solid skeleton. The two propagation velocities are derived from the conditions of the existence of non-trivial amplitudes; the results read:

$$u_{n1}^2 = \frac{(\mathbf{n}^L)^2(\lambda^S + 2\mu^S)}{(\mathbf{n}^L)^2\rho^S + (n^S)^2\rho^L}, \quad u_{n2}^2 = \frac{\mu^S}{\rho^S} \,. \tag{8.371}$$

Because the propagation velocities are constant we have with the aid of (8.357)

$$u_{n,\alpha} = 0, \quad \frac{\partial u_n}{\partial t} = 0, \quad \frac{\partial \mathbf{n}}{\partial t} = \mathbf{0} \,. \tag{8.372}$$

Since the propagation speeds are constant for acceleration waves, each wave front $\sigma(t)$ forms a family of parallel surfaces.

The measure of the pressure gradient discontinuity b, across the surfaces is obtained from the motion equation $(8.75)_2$, which takes the form

$$b = \llbracket \operatorname{grad} p \rrbracket \cdot \mathbf{n} = -\rho^L u_{n1}^2 \mathbf{a}^L \cdot \mathbf{n} \; ; \tag{8.373}$$

then, the pressure gradient discontinuity is determined by the amplitudes of the longitudinal waves.

In order to derive the differential equations governing the amplitudes of acceleration waves in the solid and liquid we may take the differentiation of the equations (8.75_1) to $(8.75)_3$ with respect to time. Consequently, we need higher order derivatives of the second order discontinuities. The iterated substitution of Φ with $\dfrac{\partial \mathbf{u}_\wp}{\partial t}$ into geometric conditions and kinematic conditions of compatibility of the second order (8.360) to (8.363) lead to the following jumps of the corresponding derivatives:

$$\left\llbracket \frac{\partial^2 \mathbf{u}_\wp}{\partial t^2} \right\rrbracket = u_n^2 \mathbf{a}^\wp, \quad \left\llbracket \frac{\partial^3 \mathbf{u}_\wp}{\partial t^3} \right\rrbracket = u_n^2 c^\wp + 2u_n^2 \frac{\delta \mathbf{a}^\wp}{\delta t} \; , \tag{8.374}$$

$$\begin{aligned} \left\llbracket \operatorname{grad} \operatorname{grad} \frac{\partial \mathbf{u}_\wp}{\partial t} \right\rrbracket \;&=\; \mathbf{c}^\wp \otimes \mathbf{n} \otimes \mathbf{n} - u_n(\mathbf{a}^\wp \,|_\alpha \otimes \mathbf{n} \otimes \mathbf{a}^\alpha + \\ &+\; \mathbf{a}^\wp \,|_\alpha \otimes \mathbf{a}^\alpha \otimes \mathbf{n}) + u_n \mathbf{a}^\wp \otimes \mathbf{a}^\alpha \otimes \mathbf{a}^\beta b_{\alpha\beta} \; , \end{aligned} \tag{8.375}$$

$$\left\llbracket \operatorname{grad} \frac{\partial^2 \mathbf{u}_\wp}{\partial t^2} \right\rrbracket = -u_n(\mathbf{c}^\wp + \frac{\delta \mathbf{a}^\wp}{\delta t}) \otimes \mathbf{n} + u_n^2 \mathbf{a}^\wp \,|_\alpha \otimes \mathbf{a}^\alpha \; , \tag{8.376}$$

$$\left\llbracket \operatorname{grad} \frac{\partial \mathbf{u}_\wp}{\partial t} \right\rrbracket = (-u_n c^\wp + \frac{\delta b^\wp}{\delta t})\mathbf{n} - u_n \mathbf{a}^{\alpha\beta} b_{,\beta} \mathbf{a}_\alpha \; , \tag{8.377}$$

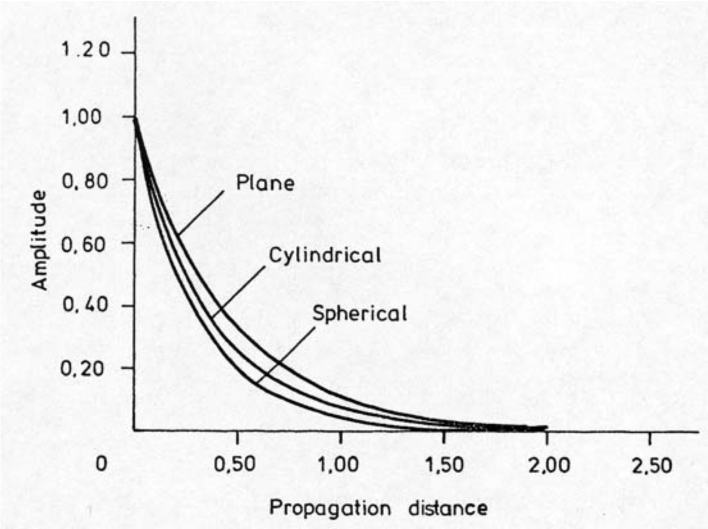

Fig. 8.5.17: Amplitude decay of longuitidinal waves in a solid skeleton (a_n^S versus r, $a_n^S(0) = 1.0$).

where

$$\mathbf{c}^{\wp} = \left[\!\!\left[\operatorname{grad}\operatorname{grad}\frac{\partial \mathbf{u}_{\wp}}{\partial t} \right]\!\!\right] \mathbf{n}\otimes\mathbf{n}, \quad c^{p} = \left[\!\!\left[\operatorname{grad}\operatorname{grad} p \right]\!\!\right] \cdot \mathbf{n}\otimes\mathbf{n} \tag{8.378}$$

are introduced as unknown quantities.

For simplicity we only examine the case of the so-called homogenous weak discontinuity. Namely, it is assumed that all amplitudes do not exchange their tangent to the wave fronts, viz.

$$a^{S}_{n,\alpha} = 0\,, \quad a^{S}_{\perp,\alpha} = 0\,, \tag{8.379}$$

where we denote

$$a^{S}_{n} = \mathbf{a}^{S}\cdot\mathbf{n}\,, \quad a^{S}_{\perp} = \mathbf{a}^{S}\cdot\boldsymbol{\gamma} \tag{8.380}$$

as the absolute amplitudes of the longitudinal and the transverse waves, $\boldsymbol{\gamma}$ is chosen as a unit vector perpendicular to the unit waves normal so that

$$\boldsymbol{\gamma}\cdot\mathbf{n} = 0\,. \tag{8.381}$$

Now, we take the differentiation of the equation (8.75) with respect to time, for which we further take the limit on both sides of the surfaces $\sigma(t)$. With the aid of (8.367) and with the insertion of the expression (8.376) we can obtain the following relation for the unknown quantities $\mathbf{c}^{S}$ and $\mathbf{c}^{L}$

$$\mathbf{c}^{L}\cdot\mathbf{n} = -\frac{n^{S}}{n^{L}}\mathbf{c}^{S}\cdot\mathbf{n}\,. \tag{8.382}$$

Next, we examine the amplitudes of the longitudinal waves in a well-known way. However, the treatment is lengthy and the details are neglected. With (8.374) to (8.377), (8.379), (8.382), and upon elimination of terms associated with $b^{\wp}$ and $c^{\wp}$, it follows from the equation $(8.75)_{1,2}$ that

$$\begin{aligned} &2[(n^{L})^{2}\rho^{S} + (n^{S})^{2}\rho^{L}]u^{2}_{n1}\frac{\delta a^{S}_{n}}{\delta t} - (\lambda^{S} + 2\mu^{S})(n^{L})^{2} + S_{L}u^{2}_{n1}a^{S}_{n} - \\ &-\{(\lambda^{S} + 2\mu^{S})(n^{L})^{2} - [(n^{L})^{2}\rho^{S} + (n^{S})^{2}\rho^{L}]u^{2}_{n1}\}\mathbf{c}^{S}\cdot\mathbf{n} = 0\,, \end{aligned} \tag{8.383}$$

where the relation

$$\mathbf{n}\cdot\mathbf{x}_{,\alpha} = 0 \tag{8.384}$$

and the relation (8.352) have been used. Utilizing the expression $(8.371)_{1}$, we can eliminate the unknown product $\mathbf{c}^{S}\cdot\mathbf{n}$ and obtain the differention equation governing the amplitude a^{S}_{n}:

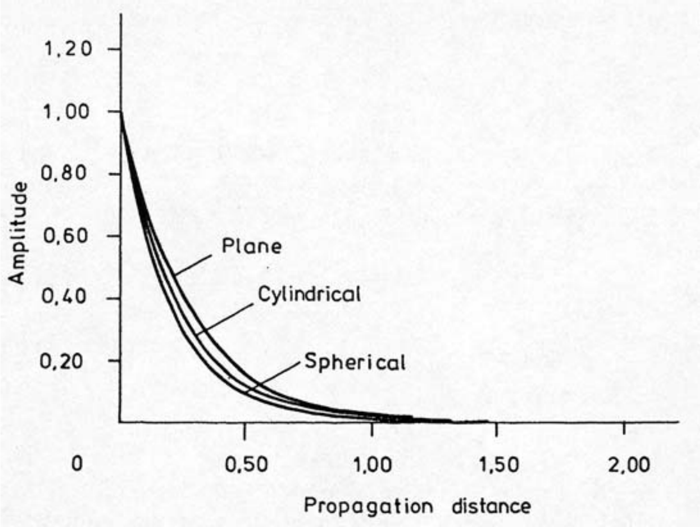

Fig. 8.5.18: Amplitude decay of transverse waves in a solid skeleton (a_T^S versus r, $a_\perp^S(0) = 1.0$).

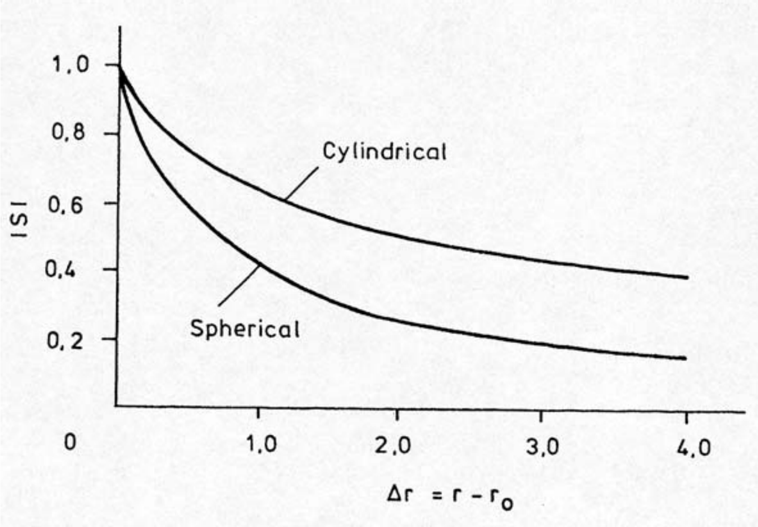

Fig. 8.5.19: Amplitude decay results based on linear elasticity theory.

$$\frac{\delta a_n^S}{\delta t} - \frac{1}{2}\left[b^{\alpha}_{\cdot\alpha} u_{n1} - \frac{S_L}{(n^L)^2\rho^S + (n^S)^2\rho^L} \right] a_n^S = 0 \; . \tag{8.385}$$

Analogously, we can derive the differential equations governing the amplitude $a_\perp^S$:

$$\begin{gathered} 2\rho^S u_{n2}^2 \frac{\delta a_\perp^S}{\delta t} - (\mu^S b_{\cdot\alpha}^\alpha u_{n2} - S_L u_{n2}^2) a_\perp^S - \\ -(\mu^S - \rho^S u_{n2}^2) \mathbf{c}^S \cdot \boldsymbol{\gamma} = 0 \ , \end{gathered} \tag{8.386}$$

where the identity

$$\mathbf{n} \cdot \boldsymbol{\gamma}_{,\alpha} = -\, \boldsymbol{\gamma} \cdot \mathbf{n}_{,\alpha} = b_{\alpha}^{\cdot\gamma} \mathbf{x}_{,\gamma} \cdot \boldsymbol{\gamma} \tag{8.387}$$

has been utilized. With the aid of the expression $(8.371)_2$, the amplitude for transverse waves is determined by

$$\frac{\delta a_\perp^S}{\delta t} - \left(\frac{1}{2} b_{\cdot\alpha}^\alpha u_{n2} - \frac{S_L}{2\rho^S} \right) a_\perp^S = 0 \ . \tag{8.388}$$

With the aid of (8.356) the following notation is introduced:

$$\frac{\delta}{\delta t} = u_n \left(\mathbf{n} \cdot \operatorname{grad} + \frac{1}{u_n} \frac{\partial}{\partial t} \right) = u_n \frac{d}{dr} \ , \tag{8.389}$$

where the total derivative in the directions of the normal to the singular surfaces $\sigma(t)$ is represented by d/dr with r as the distance of the wave fronts from their initial position. Then, the substitution by means of d/dr shows that the ordinary differential equations (8.385) and (8.388) take the forms

$$\frac{da_n^S}{dr} = (\Omega - M) a_n^S \ , \quad \frac{da_\perp^S}{dr} = (\Omega - N) a_\perp^S \ , \tag{8.390}$$

where the definition $(8.353)_1$ has been used. The coefficients M and N are defined by

$$M = \frac{S_L}{2[(n^L)^2 \rho^S + (n^S)^2 \rho^L] u_{n1}} \ , \quad N = \frac{S_L}{2\rho^S u_{n2}} \tag{8.391}$$

and Ω can be rewritten in term of a certain position, namely,

$$\Omega(r) = \frac{\Omega_0 - K_0 r}{1 - 2\Omega_0 r + K_0 r^2} \ . \tag{8.392}$$

Hence, the integration of $(8.390)_{1,2}$ together with (8.392) results in:

$$a_n^S(r) = a_n^S(0)(1 - 2\Omega_0 r + K_0 r^2)^{-1/2} \exp(-Mr) \ , \tag{8.393}$$

$$a_\perp^S(r) = a_\perp^S(0)(1 - 2\Omega_0 r + K_0 r^2)^{-1/2} \exp(-Nr) \ , \tag{8.394}$$

where $a_n^S(0)$ and $a_\perp^S(0)$ are the initial values of the absolute amplitudes of acceleration waves.

It is evident from $(8.391)_{1,2}$ as well as (8.393) and (8.394) that the growth and decay of acceleration waves depend on the initial shapes of the wave fronts and the diffusion effect amongst the phases. The diffusion effect produces only the attenuation of the amplitudes of acceleration waves but doesn't influence the propagation speeds.

In particular, the evolution of the amplitudes of acceleration waves may be reduced to the classical results in the case of non-dissipation where the amplitudes rely completely on the initial geometrical property of the waves fronts.

If the initial mean curvature Ω_0 is positive and the initial total curvature K_0 is negative, for instance in the case of the initial pseudosphere for the wave fronts, it follows from (8.393) and (8.394) that the amplitudes will reach an infinite value within a finite propagation distance. In other words, the acceleration waves in the porous medium may constitute a higher weak discontinuity. In the case of the negative initial mean curvature and the positive initial total curvature, the amplitudes of acceleration waves will decay till they vanish.

It is of interest to consider the plane ($\Omega_0 = 0,\ K_0 = 0$), the cylindrical ($\Omega_0 = \dfrac{-0.5}{r_0},\ K_0 = 0$) and the spherical ($\Omega_0 = -\dfrac{1}{r_0},\ K_0 = \dfrac{1}{r_0^2},$) acceleration waves for which the amplitudes are reduced into the following forms, respectively:

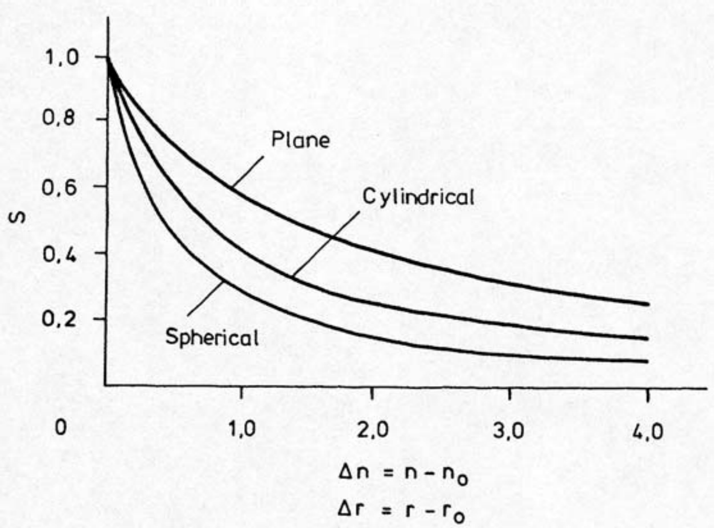

Fig. 8.5.20: Amplitude decay in finite time in isotropic elastic materials (after Chen, 1968).

for plane waves:

$$a_n^S(r) = a_n^S(0)\exp(-Mr)\ , \tag{8.395}$$

$$a_\perp^S(r) = a_\perp^S(0)\exp(-Nr)\ , \tag{8.396}$$

for cylindrical waves:

$$a_n^S(r) = a_n^S(0)\left(1+\frac{r}{r_0}\right)^{-1/2}\exp(-Mr)\ , \tag{8.397}$$

$$a_\perp^S(r) = a_\perp^S(0)\left(1+\frac{r}{r_0}\right)^{-1/2}\exp(-Nr)\ , \tag{8.398}$$

for spherical waves:

$$a_n^S(r) = a_n^S(0)\left(1+\frac{2}{r_0}r+\frac{1}{r_0}r^2\right)^{-1/2}\exp(-Mr)\ , \tag{8.399}$$

$$a_\perp^S(r) = a_\perp^S(0)\left(1+\frac{2}{r_0}r+\frac{1}{r_0}r^2\right)^{-1/2}\exp(-Nr)\ , \tag{8.400}$$

where r_0 is the initial semi-diameter of the wave fronts.

As a numerical illustration, it is assumed that $a_n^S(0) = a_\perp^S(0) = 1.0$, and $r_0 = 0.70$. The material constants are taken from Table I in the preceding Section 8.5 a). Fig. 8.5.17 and Fig. 8.5.18 show that the amplitudes in solids decay till they vanish in the finite propagation distance for the plane, the cylindrical, and the spherical acceleration waves. The corresponding results for the three cases of the waves in classical elasticity are depicted in Fig. 8.5.19 and Fig. 8.5.20, the latter is presented by Chen (1968). It is demonstrated that the evolution of the amplitudes of acceleration waves in porous media is similar to the characteristics in linear and isotropic elasticity.

d) Dispersion and Attenuation of Surface Waves in a Saturated Porous Medium

This section is aimed at a study of two types of surface waves (see Liu and de Boer, 1997): Rayleigh- and Love-type waves. The former is assumed to be bounded close to the surface of a semi-infinite saturated porous medium and the latter within a saturated porous medium layer. The discovery of these waves in single-phase (solid) half-space (Rayleigh, 1885; Love, 1911) was closely related to the seismic spectrum analysis. Unlike the other two types of body waves as P-type (longitudinal) and S-type (transverse) waves which stimulate relative minor disturbances close to the surface of the earth in an earthquake, for their energy is dispersed in a vast body of the earth's interior, the surface waves may bring about a significant damage-causing tremor to structures because these waves dissipate their energy less rapidly than P-type and S-type waves and they are essentially confined to the surface.

The concern of the surface waves in saturated porous media is of more practical significance in the field of seismic engineering because most of the geological materials can be grouped into a certain family of porous media. In particular, the portion of these media as structure foundation is mostly composed of saturated porous media such as watersaturated soil deposit. A study on Rayleigh waves in a porous medium half-space and Love waves in a porous medium layer has been presented by Deresiewicz (1961/62 a) who utilized Biot's (1956) theory to describe the porous media. However, despite remarkable success in the application of Biot's model to wave propagation and various dynamic responses, due to limited space we will mention here only a book by Bourbié *at al.* (1987), the model has been proven to remain in some respects unsatisfactory for the treatment of a saturated porous medium.

The basis of this section is the Theory of Porous Media (TPM). In order to make a comparison with the classical counterpart and the work by Deresiewicz (1961, 1962 b) possible, the assumption is made in this treatment that the porous medium consists of homogenous elastic solid skeleton filled with a liquid. The propagation of plane waves in a infinite body of a porous medium is briefly presented based on the contents of Section 8.5 a). Then, Rayleigh-type waves in a semi-infinite saturated porous medium and Love-type waves in a saturated porous layer are examined. Their frequency equations are derived relating the dependence if wave numbers, being complex, on frequency, which shows that both surface waves are dispersive and inhomogeneous. Their amplitudes damp off along the propagation surface of the porous medium whereas they decay exponentially with depth. It is also revealed that the classical results are the particular cases of the surface waves in the porous medium.

Rayleigh-type waves

The starting point for the treatment of Rayleigh and Love waves in the set of main equations of poroelasticity $(8.75)_{1,2,3}$. *Rayleigh waves* can propagate unperturbed along the surface of a semi-infinite isotropic solid in many respects like waves in the sea. Such a disturbance is especially significant in seismic wave propagation because, unlike bulk acoustic waves, they diffract on the surface but not into the volume of the earth. The most characteristic feature of Rayleigh waves is that the path of a particle in the medium is an ellipse with its major axis normal to the surface and that the shape of the ellipse changes as a function of depth.

Based on the study on plane waves in Section 8.5 a) we embark to examine Rayleigh-type waves on the free surface of a porous medium. The waves are assumed to be generated by a force source (e.g. the forces inside the earth that cause earthquake) at minus infinity so that the frequency is real and positive. A rectangular coordinate system is introduced such that the xy-plane coincides with the boundary of the half-space and the z-axis is directed into the medium. If a wave propagate along the coordinate plane (x, z) corresponding vector potential possesses only one nonzero component. Associated with this coordinate system, the monochromatic progressing waves in the xz-plane are considered in the usual manner by denoting

$$\overline{\Phi} = f(z)\exp(i\gamma_r x), \;\; \overline{h}_y = g(z)\exp(i\gamma_r x), \;\; \overline{h}_x = \overline{h}_z = 0 \; , \qquad (8.401)$$

where γ_r is the complex wave number of Rayleigh-type surface waves.

With the aid of (8.204) and (8.218) combined with (8.401) we have

$$f(z) = A_1\exp(-\bar{k}_1 z), \;\; g(z) = A_2\exp(-\bar{k}_2 z) \; , \qquad (8.402)$$

where A_1 and A_2 represent the amplitude constants and

$$\bar{k}_1^2 = \gamma_r^2 - k_1^2, \;\; \bar{k}_2^2 = \gamma_r^2 - k_2^2 \; . \qquad (8.403)$$

The scalar and vector potentials following from (8.200), (8.215) through (8.217), and (8.401) as well as from (8.402) and (8.403) are:

$$\begin{aligned} \Phi &= A_1\exp(-\bar{k}_1 z)\exp(i(\gamma_r x - \omega t)) \; , \\ h_y &= A_2\exp(-\bar{k}_2 z)\exp(i(\gamma_r x - \omega t)) \; , \\ h_x &= h_z = 0 \; . \end{aligned} \qquad (8.404)$$

Rayleigh-type waves are known to be a superposition of equivoluminal waves and irrotationally waves, as described in (8.404). However, much more attention should be paid to the case under study if a separation of variables representation of the potentials Φ and h_y in (8.404) is made into the products of an exponential decay function in the coordinate plane (x,z) and a function whose argument is phase; this gives (h_x and h_y are omitted)

$$\begin{aligned} \Phi = A_1 \quad & \exp(-[Re(\bar{k}_1)z + \mathrm{Im}(\gamma_r)x])\times \\ & \times\exp(i[Re(\gamma_r)x - \mathrm{Im}(\bar{k}_1)z - \omega t]) \; , \\ h_y = A_2 \quad & \exp(-[Re(\bar{k}_2)z + \mathrm{Im}(\gamma_r)x])\times \\ & \times\exp(i[Re(\gamma_r)x - \mathrm{Im}(\bar{k}_2)z - \omega t]) \; , \end{aligned} \qquad (8.405)$$

where $\mathrm{Re}(\bar{k}_1)$, $\mathrm{Re}(\bar{k}_2)$, and $\mathrm{Im}(\gamma_r)$ (Re and Im denote the real and imaginary parts, respectively) are required to be at least positive to ensure decay. If this adjustment holds, Rayleigh-type surface waves exist but those waves lose their conceptual generality, because the waves attenuate along the propagation direction (x-axis) also. Therefore, in a liquidsaturated porous medium these waves would not be called real surface waves. In the light of (8.405), both two phase planes are not perpendicular to the plane surface of the medium. Besides, these waves are inhomogeneous since the planes of constant phases are not parallel to the planes of constant amplitudes.

It follows from (8.188) and (8.189) as well as from (8.404) that the displacement components have the form

$$u_{sx} = \frac{\partial \Phi}{\partial x} - \frac{\partial h_y}{\partial z}, \quad u_{sy} = 0, \quad u_{sz} = \frac{\partial \Phi}{\partial z} + \frac{\partial h_y}{\partial x} \,. \tag{8.406}$$

The surface of the porous medium is assumed to be free of traction and of adequate permeability; thus, the boundary conditions are given by

$$(\mathbf{T}_E^S)_{zx}\Big|_{z=0} = \mu^S\Big(\frac{\partial u_{sx}}{\partial z} + \frac{\partial u_{sz}}{\partial x}\Big)\Big|_{z=0} = 0 \,, \tag{8.407}$$

$$(\mathbf{T}_E^S)_{zy}\Big|_{z=0} = \mu^S\Big(\frac{\partial u_{sy}}{\partial z} + \frac{\partial u_{sz}}{\partial y}\Big)\Big|_{z=0} = 0 \,, \tag{8.408}$$

$$(\mathbf{T}_E^S)_{zz}\Big|_{z=0} = \Big[2\mu^S\frac{\partial u_{sz}}{\partial z} + \lambda^S\Big(\frac{\partial u_{sx}}{\partial x} + \frac{\partial u_{sz}}{\partial z}\Big)\Big]_{z=0} = 0 \,, \tag{8.409}$$

$$p\Big|_{z=0} = 0 \,. \tag{8.410}$$

Condition (8.408) is identically satisfied due to the consideration of plane strain waves. Condition (8.410) is utilized to specify the undetermined poreliquid pressure. The residual two conditions give rise to the following two algebraic equations in terms of the amplitudes

$$2i\gamma_r\bar{k}_1A_1 + (\gamma_r^2 + \bar{k}_2^2)A_2 = 0 \,, \tag{8.411}$$

$$(2\mu^S\bar{k}_1^2 - \lambda^S k_1^2)A_1 - 2i\mu^S\gamma_r\bar{k}_2A_2 = 0 \,. \tag{8.412}$$

The requirement of nonzero solutions for the set of Equations (8.411) and (8.412) leads to vanishing of its determinant; in turn we obtain

$$4\mu^S\gamma_r^2\bar{k}_1\bar{k}_2 - [2\mu^S\gamma_r^2 - (2\mu^S + \lambda^S)k_1^2](2\gamma_r^2 - k_2^2) = 0 \,. \tag{8.413}$$

This is the dispersion relationship using which the complex number γ_r should be calculated with respect to frequency. Once γ_r is known, the phase velocity c_r being dependent on frequency in general, will be attained by

$$c_r = \mathrm{Re}\Big(\frac{\omega}{\gamma_r}\Big) \,, \tag{8.414}$$

while the attenuation coefficient along the surface propagation direction just takes the imaginary portion $\mathrm{Im}(\gamma_r)$ which is devoted to the internal interface dissipation in the porous medium.

When the interstitial liquid passed through the pores, an energy loss (dissipation) of a travelling wave will occur at the internal interface of the two phases due to the friction drag which is roughly expressed by the volume force $\hat{\mathbf{p}}^L$. This is why the saturated porous medium behaves like a viscoelastic material. The inherent internal irregularity in a porous medium makes a rigorous description hardly possible for this dissipation. For a surface wave over a long distance, the influence of the energy dissipation on the amplitude attenuation is apparent. We write out the time rate of the energy dissipation, which is raised by the friction force $\hat{\mathbf{p}}^L$, proportional to the velocity difference, by the formula

$$D = \int_v \hat{d}(\boldsymbol{x}, t) dv = \int_v \hat{\mathbf{p}}^L \cdot (\boldsymbol{x}'_L - \boldsymbol{x}'_S) dv \ , \tag{8.415}$$

where $\hat{d}(\boldsymbol{x}, t)$ is the time rate of the energy dissipation in a volume element. The aspects related with the energy flux and the energy loss in wave motions will be discussed in the next section.

In the classical Rayleigh equation, the wave velocity c_r is independent of frequency, i.e., the classical Rayleigh waves are dispersionless, and the wave velocity relies only on Poisson's ratio. However, the Rayleigh-type waves in a saturated porous medium are predicated to be dispersive and dissipative. Practical seismograph often exhibits waves similar in structure to classical Rayleigh waves. Simultaneously, seismographic records of distant earthquakes indicate dispersion and attenuation. These practical aspects would be an evidence for interpretation of this result.

To examine the dispersion relationship of Rayleigh-type wave numerically, the dependence of the phase velocity and the attenuation coefficient on the frequency ω in a saturated soil is plotted in Figures 8.5.21 and 8.5.22. The physical properties of the saturated soil are taken from Table I in Section 8.5 a). As shown in Figures 8.5.21 and 8.5.22, the phase velocity c_r of the Rayleigh-type wave depends on the frequency. For the higher permeability coefficient k^L the phase velocity c_r is also higher. But the attenuation coefficient is lower for the higher permeability coefficient because the attenuation coefficient characterizes the dissipative effect between the solid and the liquid phases.

Especially, the classical Rayleigh waves are a particular case of the Rayleigh-type waves in a porous medium, which can be seen to the next. If the saturated porous is suggested to be frictionless $(S_L = 0)$, from (8.206), (8.207), (8.221), and (8.222) we immediately have

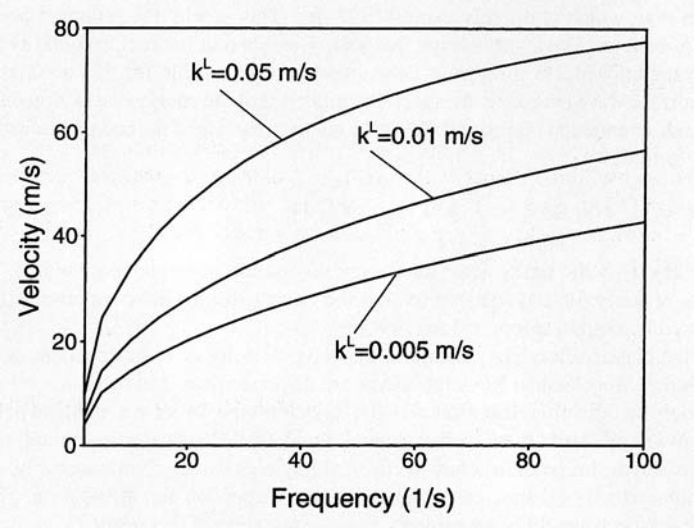

Fig. 8.5.21: Phase velocity of Rayleigh wave versus frequency at different permeability parameters.

$$\begin{aligned} a_1' &= \frac{\rho^S (n^L)^2 + \rho^L (n^S)^2}{(n^L)^2 \lambda^S + 2(n^L)^2 \mu^S} \omega^2 \ , \\ a_2' &= \frac{\rho^S \omega^2}{\mu^S}, \ \ b_1' = b_2' = 0 \ . \end{aligned} \tag{8.416}$$

In this case no dissipation is involved along the propagation surface and imiganary portions in (8.205) and (8.320) vanish. Morover, Eq. (8.403) takes the form

$$\begin{aligned} \bar{k}_1'^2 &= \gamma_r^2 - k_1'^2 = \gamma_r^2 (1 - \frac{c^2}{c_1^2}), \ \ k_1'^2 = \frac{\omega^2}{c_1^2} \ , \\ \bar{k}_2'^2 &= \gamma_r^2 - k_2'^2 = \gamma_r^2 (1 - \frac{c^2}{c_2^2}), \ \ k_2'^2 = \frac{\omega^2}{c_2^2} \ , \end{aligned} \tag{8.417}$$

where

$$\begin{aligned} c_1^2 &= \frac{(n^L)^2 \lambda^S + 2(n^L)^2 \mu^S}{\rho^S (n^L)^2 + \rho^L (n^S)^2} \ , \\ c_2^2 &= \frac{\mu^S}{\rho^S} \ , \end{aligned} \tag{8.418}$$

which are the propagation velocities of the two body waves in a two-phase saturated porous medium, respectively. In (8.418), $c_1 > c_r$ and $c_2 > c_r$ are

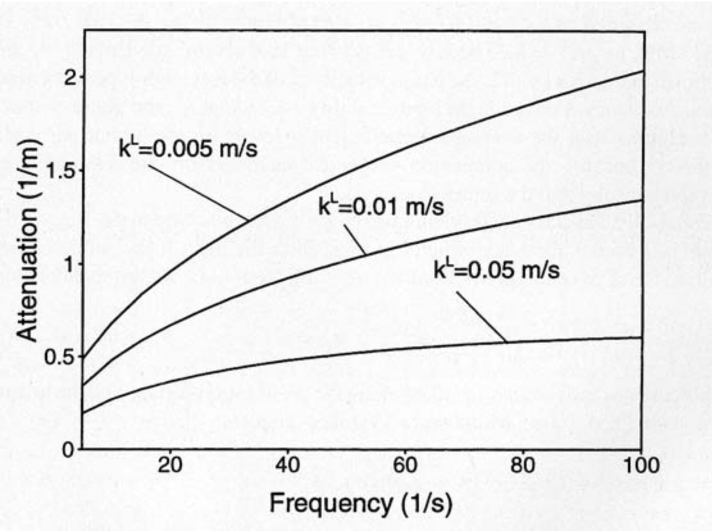

Fig. 8.5.22: Attenuation coeficient of Rayleigh wave versus frequency at different permeability parameters.

required that the propagation velocity of Rayleigh waves is less than the propagation velocities of the two body waves. Therefore, the frequency equation (8.413) takes the form

$$4\bar{k}_1'\bar{k}_2'\gamma_r^2 - (2\gamma_r^2 - \frac{\lambda^S + 2\mu^S}{\mu^S} k_1'^2)(2\gamma_r^2 - k_2'^2) = 0 \; . \tag{8.419}$$

If we further consider an empty porous solid skeleton without liquid ($\rho^L \to 0$), then the propagation velocity c_1 is reduced to

$$c_1'^2 = \frac{\lambda^S + 2\mu^S}{\rho^S} \tag{8.420}$$

with which the frequency equation (8.419) may further give

$$4\bar{k}_1'\bar{k}_2'\gamma_r^2 - (2\gamma_r^2 - k_2'^2)^2 = 0 \; , \tag{8.421}$$

or

$$16(1 - \frac{c_r^2}{c_1'^2})(1 - \frac{c_r^2}{c_2^2}) - (2 - \frac{c_r^2}{c_2^2})^4 = 0 \; . \tag{8.422}$$

The above Eqs. (8.421) and (8.422) are indeed the case in the classical Rayleigh waves; the propagation velocity c_r is independent of frequency and the wave shape is maintained.

Love-type waves

The original impetus to study Love-type waves may retrace its steps to the explanation of the presence of sizable transverse components of displacement in the main tremor of an earthquake. An intense horizontal component of motion is not a feature of Rayleigh waves because Rayleigh waves propagate only in the vertical plane. Meanwhile, such a disturbance is also impossible to be produced by a surface shear wave. This phenomenon was successfully described by Love (1911) who realized that the actual conditions in the earth must be different in some essential respects from those of a homogenous, isotropic half-space. He in turn conjectured that such a disturbance was a consequence of a layered construction of the earth. Love suggested an elastic thin layer which is added on elastic semi-infinite media with different material properties to the thin layer. This kind of horizontally polarized surface waves discovered by Love, later to bear his name, is constructed by shear waves localized in this superficial layer and retain its energy close to the surface. It is of particular interest to bear in mind that the phase velocity of shear waves in the elastic half-space exceeds that in the thin layer.

We now consider the propagation of Love-type waves within a thin saturated porous layer which is superimposed upon a homogenous, isotropic half-space. The Cartesian coordinate system is so adopted that the xy-plane coincides with the horizontal interface between the porous layer and the elastic substrate, and z-axis points vertically upward into the layer. Referring to these axes and recalling the shearing property of Love-type waves, we write the components of displacements of the two phases in x, y, z directions and the poreliquid pressure in the form

$$\begin{aligned} u_{Sx} = 0, \quad u_{Sy} = u_{Sy}(x,z,t), \quad u_{Sz} = 0\ , \\ u_{Lx} = 0, \quad u_{Ly} = u_{Ly}(x,z,t), \quad u_{Lz} = 0\ , \end{aligned} \tag{8.423}$$

$$p = p(x,z,t)\ . \tag{8.424}$$

It is evident that, unlike Rayleigh-type waves, Love-type waves are pure equivoluminal waves. Considering the fact that the shear disturbance does not produce variation of volume, the mass balance equation (8.185) is eliminated from the set of the governing equations of the porous medium.

Furthermore, effect of the external body force is neglected. Accordingly, with the help of (8.423) and (8.424) the equations of motion (8.75) are easily reduced to

$$\mu^S \triangle u_{Sy} = \rho^S \frac{\partial^2 u_{Sy}}{\partial t^2} - S_L\left(\frac{\partial u_{Ly}}{\partial t} - \frac{\partial u_{Sy}}{\partial t}\right)\ , \tag{8.425}$$

$$0 = \rho^L \frac{\partial^2 u_{Ly}}{\partial t^2} + S_L\left(\frac{\partial u_{Ly}}{\partial t} - \frac{\partial u_{Sy}}{\partial t}\right)\ , \tag{8.426}$$

where the Laplace operator is $\triangle = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}$. A time harmonic variation for each of the displacements of the solid skeleton and the liquid is assumed, i.e.

$$u_{Sy} = \overline{u}_{Sy}(x,z)\exp(i\omega t), \quad u_{Ly} = \overline{u}_{Ly}(x,z)\exp(i\omega t) \; . \tag{8.427}$$

Then, the substitution of (8.427) into (8.425) and (8.426) yields

$$(\triangle + k_3^2)\bar{u}_{Sy} = 0 \; , \tag{8.428}$$

$$\bar{u}_{Ly} = \frac{S_L i}{S_L i - \rho^L \omega}\, \bar{u}_{Sy} \; , \tag{8.429}$$

where

$$k_3^2 = a_3 - b_3 i \; , \tag{8.430}$$

$$a_3 = \frac{\rho^S S_L^2 \omega^2 + \rho^L S_L^2 \omega^2 + \rho^S (\rho^L)^2 \omega^4}{\mu^S [S_L^2 + (\rho^L)^2 \omega^2]} \; , \tag{8.431}$$

$$b_3 = \frac{\rho^L S_L^2 \omega^3}{\mu^S [S_L^2 + (\rho^L)^2 \omega^2]} \; . \tag{8.432}$$

Thus, the portion of spatially dependent skeleton displacement $\bar{u}_{Sy}$ is expressed by Eq. (8.428) with the complex number k_3, indicating the waves in the layer is dissipative on the one hand. On the other hand, Eq. (8.429) tells us that the wave in the liquid is coupled with that in the skeleton.

Suppose the resolution of (8.428) is

$$u_{Sy} = h(z)\exp(-i\gamma_l x) \; , \tag{8.433}$$

where γ_l is the wave number being a complex of Love-type waves. Then, the insertion of the resolution into (8.428) yields

$$h(z) = A_3 \cos \bar{k}_3 z + A_4 \sin \bar{k}_3 z \; , \tag{8.434}$$

where

$$\bar{k}_3^2 = k_2^2 - \gamma_l^2 \; . \tag{8.435}$$

The displacement of the solid skeleton in the porous layer is, by virtue of (8.427), (8.433) and (8.434), given by

$$u_{Sy} = (A_3 \cos \bar{k}_3 z + A_4 \sin \bar{k}_3 z)\exp[(\omega t - \gamma_l x) i] \; . \tag{8.436}$$

With respect to the half space of elastic materials, as in the case of the porous layer, denoting u_{Sb} to be the horizontal displacement component portion, being harmonic time dependent, the spatially dependent portion is governed by a Helmholtz equation, viz.

$$(\triangle + k_b^2)\bar{u}_{Sb} = 0 \ , \tag{8.437}$$

where

$$k_b^2 = \frac{\omega^2}{c_b^2}, \ \ c_b^2 = \frac{\mu}{\rho} \tag{8.438}$$

with μ as the shear modulus of the elastic base and ρ its density. Similarly, the solution of the displacement u_{Sb} is

$$u_{Sb} = A_b \exp(\bar{k}_b z) \exp[(\omega t - \gamma_l x)i] \ , \tag{8.439}$$

where

$$\bar{k}_b^2 = \gamma_l^2 - k_b^2, \ \ \mathrm{Re}(\bar{k}_b) > 0 \ . \tag{8.440}$$

At $z = h$, the layer should be free of traction and the transverse stress and displacement should be continuous across the interface at $z = 0$, thus

$$\mu^S \frac{\partial u_{Sy}}{\partial z}\bigg|_{z=h} = 0 \ , \tag{8.441}$$

$$\mu^S \frac{\partial u_{Sy}}{\partial z}\bigg|_{z=0} = \mu \frac{\partial u_{Sb}}{\partial z}\bigg|_{z=0} = 0 \ , \tag{8.442}$$

$$u_{Sy}\bigg|_{z=0} = u_{Sb}\bigg|_{z=0} \ . \tag{8.443}$$

The above expressions show that the solid skeleton constitutes a framework of porous bodies subjected to external traction at the free surface without considering the liquid in the pores. With the help of (8.436) and (8.439), the above described boundary conditions produce a set of algebraic equations in terms of amplitudes

$$\sin(\bar{k}_3 h)A_3 - \cos(\bar{k}_3 h)A_4 = 0 \ , \tag{8.444}$$

$$\mu^S \bar{k}_3 A_4 - \mu^S \bar{k}_b A_b = 0 \ , \tag{8.445}$$

$$A_3 - A_b = 0. \tag{8.446}$$

The unique condition of nonzero solutions for this set of algebraic equations requires that the determinant of coefficients of the amplitudes is zero, which leads to the frequency equation

$$\tan(\bar{k}_3 h) = \frac{\mu \bar{k}_b}{\mu^S \bar{k}_3} \ . \tag{8.447}$$

Since the wave number of Love-type wave γ_l is complex, the amplitude will suffer a frequency-dependent spatial attenuation characteristic. Thus, Love-type

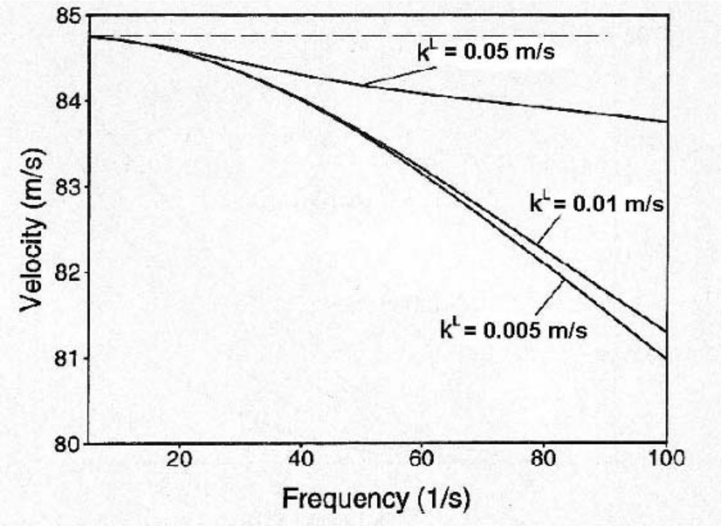

Fig. 8.5.23: Phase velocity of Love wave versus frequency at different permeability parameters. The dotted line indicates the propagation velocity of the shear wave in the substrate.

waves are also not called real surface waves. Furthermore, Love-type waves are also inhomogeneous waves. The expression (8.447) regulates the dependence of phase velocity c_l and attenuation coefficient of frequency ω, being a

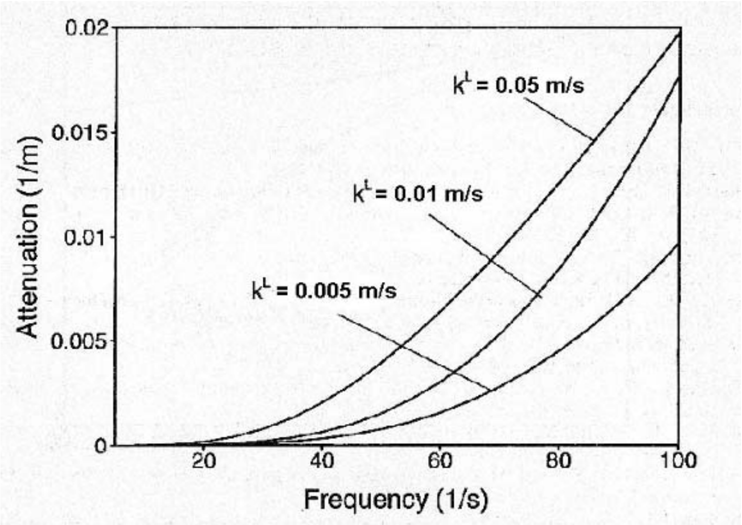

Fig. 8.5.24: Attenuation coeficient of Love wave versus frequency at different permeability parameters.

transcendental equation; a mathematical evaluation for (8.447) is furthermore required. The multiple branches of the tangent function in (8.447) imply that multiple roots will exist for any given frequency ω. Therefore, the dispersion curves and frequency spectrum should have multiple branches which correspond to various propagation modes. Despite the fact that (8.447) is identical in appearance with the result given by Deresiewicz (1961), it is fundamentally different if we detail the implication of the terms.

In order to study the dispersion property of Love-type wave numerically, we discuss the saturated soil superimposed on a soil base in which the bulk shear propagates at a velocity $84.76 m/s$. The physical property of the saturated soil is described by de Boer *et al.*(1993). In the calculation we choose the first root related with the multiple roots in (8.447). The propagation velocity and the attenuation coefficient of the first mode versus the frequency for Love-type wave in the soil are shown in Figures 8.5.23 and 8.5.24, respectively.

We note that under the condition of a frictionless saturated porous medium, $S_L = 0$, we have

$$a_3' = \frac{\rho^S \omega^2}{\mu^S}, \quad b_3' = 0 \; . \tag{8.448}$$

Furthermore, it follows from (8.435) and (8.440) that

$$\bar{k}_3'^2 = \gamma_l^2 \left(\frac{c_l^2}{c_2^2} - 1 \right) , \quad \bar{k}_b'^2 = \gamma_l^2 \left(1 - \frac{c_l^2}{c_b^2} \right) , \tag{8.449}$$

where $c_2 < c_l < c_b$ is required. With use of (8.447) we have

$$\mu \sqrt{1 - \frac{c_l^2}{c_b^2}} - \mu^S \sqrt{\frac{c_l^2}{c_2^2} - 1} \; \tan\left(\gamma_l h \sqrt{\frac{c_l^2}{c_2^2} - 1} \right) = 0 \; . \tag{8.450}$$

This is the same counterpart of elasticity given by Love (1911).

e) Inhomogeneous Plane Waves, Mechanical Energy Flux, and Energy Dissipation in a Two-Phase Porous Medium

We will deal with inhomogeneous plane waves, energy flux, and energy dissipation in a two-phase porous medium, consisting of an incompressible porous solid and an incompressible liquid (see Liu *et al.*, 1998).

Two types of coupled plane harmonic waves, namely P-type and S-type, are shown to travel in the two-phase porous medium. Each type of plane harmonic waves is described in terms of two complex-valued vectors: a complex-valued wave vector and a complex-valued amplitude vector. Each complex-valued wave vector is further decomposed into a real-valued propagation vector and a real-valued attenuation vector. The propagation vector is in general not parallel to the attenuation vector, but between them there exists an inhomogeneity angle for each type of waves, which implies that planes of constant phase do not coincide with planes of constant amplitude and thus both types of waves are inhomogeneous. As discussed intensively by Hayes (1980), associated with each complex-valued amplitude vector is an ellipse of the particle motions for each type of waves, and the major axis of the ellipse is neither parallel nor perpendicular to the propagation vector. These two types of waves are essentially not new, but they deserve more attention since they represent a fundamental characteristic of these waves in the two-phase porous medium.

Proceeding from the motion equations of the saturated porous medium an energy conservative equation for the whole medium is derived. It is shown that the whole energy rate plus the energy dissipation rate is equal to the energy transfer through the surface of the controlled volume, in another way, equal to a divergence of the energy flux vector. The remainder of this section is concerned with applications of the energy flux and the energy dissipation rate at the inhomogeneous plane waves. The mean energy flux vectors and the mean energy dissipation rates over a complete period are attained for each inhomogeneous plane wave.

The fundamental relations of the poroelasticity are laid down in Section 8.3; they are used in the next paragraphs.

Inhomogeneous plane waves

We consider a time-harmonic dependence in the form $\exp(-i\omega t)$ in which t is the time, ω the angular frequency, and i the imaginary unit. As pointed out by de Boer and Liu (1994), a coupled P-type wave and a coupled S-type wave may travel in the incompressible porous medium. Put in another way, a disturbance occurring in the skeleton is always accompanied by a disturbance in the liquid for each type of waves. Thus, the general solution of the field equations (8.183) to (8.185) in terms of the skeleton displacement $\mathbf{u}_S$ and the liquid displacement $\mathbf{u}_L$ can be written using Helmholtz's representation:

$$\mathbf{u}_S = \operatorname{grad}\Phi + \operatorname{curl}\mathbf{h}, \quad \mathbf{u}_L = \tau_1 \operatorname{grad}\Phi + \tau_2 \operatorname{curl}\mathbf{h} \ , \tag{8.451}$$

where τ_1 and τ_2 are real and complex valued coupling factors, respectively,

$$\tau_1 = -\frac{n^S}{n^L}, \quad \tau_2 = \frac{\rho^L S_L \omega i + S_L^2}{(\rho^L)^2\omega^2 + S_L^2} \ . \tag{8.452}$$

The scalar potential Φ with respect to P-type wave and vector potential $\mathbf{h}$ with respect to S-type wave are functions of $\boldsymbol{x}$ and t by satisfying the wave equations

$$(\triangle + \mathbf{k}^P \cdot \mathbf{k}^P)\Phi = 0 \ , \quad (\triangle + \mathbf{k}^T \cdot \mathbf{k}^T)\mathbf{h} = \mathbf{0} \ , \tag{8.453}$$

where $\triangle$ is the Laplace operator as well as $\mathbf{k}^P$ and $\mathbf{k}^T$ being complex-valued vectors, denote wave vectors. The subscripts *P* and *T* stand for a P-type (longitudinal) wave and an S-type (transverse) wave, respectively. The complex-valued wave vectors are identified by the relations

$$\mathbf{k}^P \cdot \mathbf{k}^P = a_1(\omega) + b_1(\omega)i \ , \quad \mathbf{k}^T \cdot \mathbf{k}^T = a_2(\omega) + b_2(\omega)i \ , \tag{8.454}$$

where

$$\begin{aligned}
a_1 &= \frac{\rho^S (n^L)^2 + \rho^L (n^S)^2}{(n^L)^2 \lambda^S + 2(n^L)^2 \mu^S}\, \omega^2 \ , \\
b_1 &= \frac{S_L \omega}{(n^L)^2 \lambda^S + 2(n^L)^2 \mu^S} , \\
a_2 &= \frac{\rho^S (\rho^L)^2 \omega^4 + \rho^S S_L^2 \omega^2 + \rho^L S_L^2 \omega^2}{\mu^S (\rho^L)^2 \omega^2 + \mu^S S_L^2} \ , \\
b_2 &= \frac{(\rho^L)^2 S_L \omega^3}{\mu^S (\rho^L)^2 \omega^2 + \mu^S S_L^2} \ .
\end{aligned} \tag{8.455}$$

The general plane wave solutions of Eq. (8.453) are

$$\Phi = \Phi_0 \exp[i(\mathbf{k}^P \cdot \boldsymbol{x} - \omega t)] \ , \quad \mathbf{h} = \mathbf{h}_0 \exp[i(\mathbf{k}^T \cdot \boldsymbol{x} - \omega t)] \ , \tag{8.456}$$

where Φ_0 is a complex-valued scalar and $\mathbf{h}_0$ a complex-valued vector. With the aid of (8.451) and (8.456) the general solutions in the terms of displacement field $\mathbf{u}_S$ for the P-type and S-type waves, respectively are given by

$$\begin{aligned}
\mathbf{u}_S^P &= i\mathbf{k}^P \Phi_0 \exp[i(\mathbf{k}^P \cdot \boldsymbol{x} - \omega t)] \qquad \text{for P} - \text{type wave} \ , \\
\mathbf{u}_S^T &= i\mathbf{k}^T \wedge \mathbf{h}_0 \exp[i(\mathbf{k}^P \cdot \boldsymbol{x} - \omega t)] \quad \text{for S} - \text{type wave} \ ,
\end{aligned} \tag{8.457}$$

where the symbol $\wedge$ stands for the cross product of two complex-valued vectors. In (8.457) $i\mathbf{k}^P\Phi_0$ and $i\mathbf{k}^T \wedge \mathbf{h}_0$ are the amplitude vectors for the two types of waves travelling in the skeleton, from which the amplitude vectors in the liquid can be obtained by multiplication with the coupling factors τ_1 and τ_2. We remember that the actual displacement vector is the real part of the corresponding complex-valued vector.

The variation of the poreliquid pressure p can be attained from the motion equations (8.65) or (8.66). It is to be noted that the variation of the poreliquid pressure p is related only to the scalar potential Φ and it follows that

$$p = \tau_3 \Phi \ , \quad \tau_3 = -\frac{n^S \rho^L \omega^3 + i\omega S_L}{(n^L)^2} \ , \tag{8.458}$$

where τ_3 is a frequency-dependent factor.

Now, we consider the essential features of the inhomogeneous plane waves in the porous medium. To this end, we express the complex-valued wave vectors in terms of the corresponding real and imaginary parts

$$\mathbf{k}^P = \mathbf{k}^{P+} + i\mathbf{k}^{P-} \ , \quad \mathbf{k}^T = \mathbf{k}^{T+} + i\mathbf{k}^{T-} \ , \tag{8.459}$$

where $\mathbf{k}^{P+}$ and $\mathbf{k}^{T+}$ are the propagation vectors and $\mathbf{k}^{P-}$ and $\mathbf{k}^{T-}$ the attenuation vectors for the P-type and S-type waves, respectively. The propagation vectors are perpendicular to planes of constant phase and the attenuation vectors are perpendicular to planes of constant amplitude for the two waves. Insertion of (8.459) into (8.456) leads to the representation

$$\begin{aligned} \Phi &= \Phi_0 \exp(-\mathbf{k}^{P-} \cdot \boldsymbol{x}) \exp([i(\mathbf{k}^{P+} \cdot \boldsymbol{x} - \omega t)] \ , \\ \mathbf{h} &= \mathbf{h}_0 \exp(-\mathbf{k}^{T-} \cdot \boldsymbol{x}) \exp([i(\mathbf{k}^{T+} \cdot \boldsymbol{x} - \omega t)] \ . \end{aligned} \tag{8.460}$$

In view of (8.456) and (8.457), $\mathbf{k}^P \cdot \boldsymbol{x} = \text{const.}$ and $\mathbf{k}^T \cdot \boldsymbol{x} = \text{const.}$ describe the planes of constant phase of the two types of waves while $\mathbf{k}^{P-} \cdot \boldsymbol{x} = \text{const.}$ and $\mathbf{k}^{T-} \cdot \boldsymbol{x} = \text{const.}$ turn out the planes of constant amplitude. The phase speeds are determined by $c^P = \dfrac{\omega}{|\mathbf{k}^{P+}|}$ for the P-type wave and $c^T = \dfrac{\omega}{|\mathbf{k}^{T+}|}$ for the S-type wave. As a matter of fact, $\mathbf{k}^{P-}$ and $\mathbf{k}^{T-}$ characterize an exponential decay of the amplitudes in the porous medium. The maximum decay rate does not take place in the direction of propagation but of $\mathbf{k}^{P-}$ and $\mathbf{k}^{T-}$. It is apparent that the planes of constant phase is in general not parallel to the planes of constant amplitude, see Fig. 8.5.25. Insertion of (8.459) into (8.454) yields

$$\begin{aligned} |\mathbf{k}^{P+}|^2 - |\mathbf{k}^{P-}|^2 = a_1, \quad |\mathbf{k}^{P+}||\mathbf{k}^{P-}| \cos\Theta_1 = \frac{1}{2} b_1 \ , \\ |\mathbf{k}^{T+}|^2 - |\mathbf{k}^{T-}|^2 = a_2, \quad |\mathbf{k}^{T+}||\mathbf{k}^{T-}| \cos\Theta_2 = \frac{1}{2} b_2 \ , \end{aligned} \tag{8.461}$$

where Θ_1 and Θ_2 are the inhomogeneity angles smaller than $\pi/2$ for the two types of waves, respectively, which are measured between the propagation vectors and the attenuation vectors. The equation system (8.461) is underdetermined unless further information is given as in the case of reflection-refraction problems.

In view of (8.456) and (8.457) each displacement field is described in terms of two complex-valued vectors: the amplitude vector and the wave vector. As a basic property, associated with each complex-valued amplitude vector is a directional ellipse of the particle motion for each type of waves. We now take the P-type wave in the skeleton as an example to gain an insight into the propagation of elliptically polarized inhomogeneous plane waves in the porous medium.

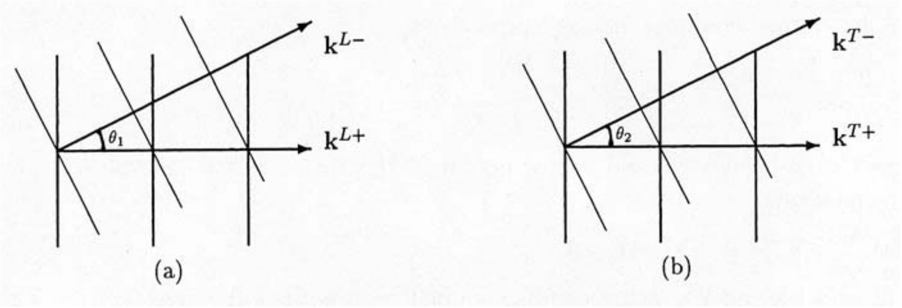

Fig. 8.5.25: The planes of constant phase (thick lines) are not parallel to the planes of constant amplitude (thin lines) (a) for P-type waves, (b) for S-type waves.

The complex-valued amplitude vector $i\mathbf{k}^P\Phi_0$, whereby it is assumed that the cross product of its real and imaginary parts is a non-zero vector, for the P-type wave may be written in the form

$$\mathbf{f}^P + i\mathbf{g}^P = \exp(-i\beta^P)(i\mathbf{k}^P\Phi_0), \quad \mathbf{f}^P \cdot \mathbf{g}^P = 0 \,, \tag{8.462}$$

where β^P is real determined by the orthogonality of $\mathbf{f}^P$ and $\mathbf{g}^P$. Moreover, $\mathbf{f}^P$ and $\mathbf{g}^P$ are real-valued coplanar with $\mathbf{k}^P$ determined by $i\mathbf{k}^P\Phi_0$. Consequently we rewrite the displacement field $\mathbf{u}_S^P$ for the P-type wave

$$\mathbf{u}_S^P = \exp(-\mathbf{k}^{P-} \cdot \boldsymbol{x})(\mathbf{f}^P + i\mathbf{g}^P)\exp[i(\mathbf{k}^{P+} \cdot \boldsymbol{x} - \omega t + \beta^P)] \,. \tag{8.463}$$

Recalling that a physical interpretation is ascribed to the real part and letting $\xi = \mathbf{k}^{P+} \cdot \boldsymbol{x} - \omega t + \beta^P$, immediately it follows from (8.463) that the actual displacement

$$\mathbf{u}_S^P = \exp(-\mathbf{k}^{P-} \cdot \boldsymbol{x})(\mathbf{f}^P \cos\xi - \mathbf{g}^P \sin\xi) \,. \tag{8.464}$$

Owing to $(8.462)_2$ the orthogonality of $\mathbf{f}^P$ and $\mathbf{g}^P$ allow us to obtain

$$\frac{(\mathbf{u}_S^P \cdot \mathbf{f}^P)^2}{[|\mathbf{f}^P|\exp(-\mathbf{k}^{P-} \cdot \boldsymbol{x})]^2} + \frac{(\mathbf{u}_S^P \cdot \mathbf{g}^P)^2}{[|\mathbf{g}^P|\exp(-\mathbf{k}^{P-} \cdot \boldsymbol{x})]^2} = 1 \,, \tag{8.465}$$

where $|\mathbf{f}^P|$, $|\mathbf{g}^P|$ are the moduli of $\mathbf{f}^P$, $\mathbf{g}^P$ and $\mathbf{u}_S^P \cdot \mathbf{f}^P$, $\mathbf{u}_S^P \cdot \mathbf{g}^P$ are the projections along $\mathbf{f}^P$, $\mathbf{g}^P$. This demonstrates that the solid displacement vector $\mathbf{u}_S^P$ for a P-type wave describes an ellipse in the plane of $\mathbf{k}^{P+}$ and $\mathbf{k}^{P-}$ with a pair of semidiameters of $|\mathbf{f}^P|\exp(-\mathbf{k}^{P-} \cdot \boldsymbol{x})$ and $|\mathbf{g}^P|\exp(-\mathbf{k}^{P-} \cdot \boldsymbol{x})$ being the major and minor axes of the ellipse, respectively. The direction of rotation is from $\mathbf{k}^{P+}$ to $\mathbf{k}^{P-}$. Due to the relation (8.462) the major axis or the minor axis of the polarized ellipse is neither in the direction of $\mathbf{k}^{P+}$ nor in the direction of $\mathbf{k}^{P-}$. Fig. 8.5.26 illustrates graphically the particle motion traces of the solid for the

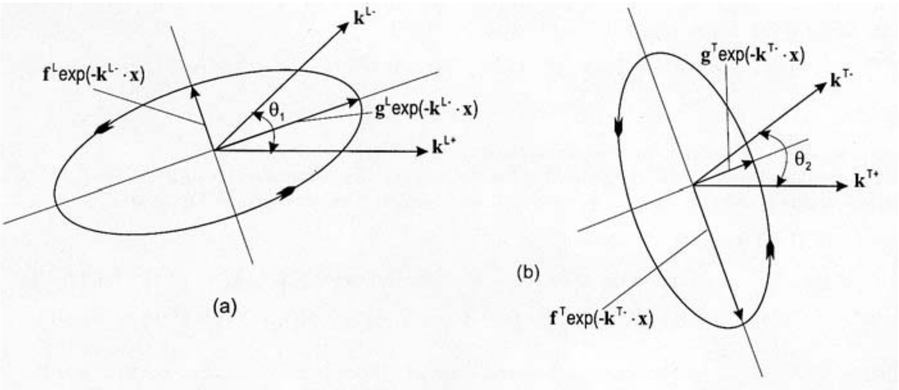

Fig. 8.5.26: The elliptical particle motion (a) for P-type wave in the skeleton, where planes of constant phase propagate in the direction of the vector $\mathbf{k}^{\mathrm{L}+}$, (b) for S-type wave in the fluid, where planes of constant phase propagate in the direction of the vector $\mathbf{k}^{\mathrm{T}+}$ (modification after Borcherdt, 1973.)

P-type wave. Accordingly, the P-type wave in the solid skeleton is elliptically polarized and the plane of polarization is determined by the complex-valued amplitude vector $\imath\mathbf{k}^P\Phi_0$.

The S-type wave in the solid skeleton must also be elliptically polarized because of its complex-valued amplitude vector $\imath\mathbf{k}^T \wedge \mathbf{H}_0$ from which, however, it can be anticipated that the polarization plane is neither orthogonal nor parallel, in general, to the complex-valued vector $\mathbf{k}^T$, since the polarization plane depends on $\imath\mathbf{k}^T \wedge \mathbf{H}_0$. The polarized ellipse is described by

$$\frac{(\mathbf{u}_S^T \cdot \mathbf{f}^T)^2}{[|\mathbf{f}^T| \exp(-\mathbf{k}^{T-} \cdot \boldsymbol{x})]^2} + \frac{(\mathbf{u}_S^T \cdot \mathbf{g}^T)^2}{[|\mathbf{g}^T| \exp(-\mathbf{k}^{T-} \cdot \boldsymbol{x})]^2} = 1 \, , \tag{8.466}$$

where $\mathbf{f}^T + \imath\mathbf{g}^T = \exp(-\imath\beta^T)(\imath\mathbf{k}^T \wedge \mathbf{H}_0), \quad \mathbf{f}^T \cdot \mathbf{g}^T = 0.$

The particle motion in the solid phase for the S-type waves is given graphically in Fig. 8.5.26. Note that the plane of polarization is not in the plane determined by $\mathbf{k}^T$.

Energy flux and energy dissipation rate

Now, we focus on the energy of a wave in the uniform porous medium. In the absence of body forces, we take the inner product of the motion equation (8.60), (8.61) with $\mathbf{v}_S$ and $\mathbf{v}_L$, respectively, and obtain

$$\rho^S(\mathbf{v}_S)'_S \cdot \mathbf{v}_S = \operatorname{div} \mathbf{T}^S \cdot \mathbf{v}_S - \hat{\mathbf{p}}^L \cdot \mathbf{v}_S \, , \tag{8.467}$$

$$\rho^L(\mathbf{v}_L)'_L \cdot \mathbf{v}_L = \operatorname{grad} p^L \cdot \mathbf{v}_L + \hat{\mathbf{p}}^L \cdot \mathbf{v}_L \, , \tag{8.468}$$

where we have set $p^L = -n^L p$. Recalling the symmetry of the stress tensor $\mathbf{T}^S$ and the following identities

$$\begin{aligned} \operatorname{div} \mathbf{T}^S \cdot \mathbf{v}_S &= \operatorname{div}(\mathbf{T}^S \mathbf{v}_S) - \mathbf{T}^S \cdot \mathbf{D}_S \,, & (8.469) \\ \operatorname{grad} p^L \cdot \mathbf{v}_L &= \operatorname{div}(p^L \mathbf{v}_L) - p^L(\mathbf{D}_L \cdot \mathbf{I}) \,. & (8.470) \end{aligned}$$

Eqs. (8.467) and (8.468) become

$$\begin{aligned} \rho^S(\mathbf{v}_S)'_S \cdot \mathbf{v}_S &= \operatorname{div}(\mathbf{T}^S \mathbf{v}_S) - \mathbf{T}^S \cdot \mathbf{D}_S - \hat{\mathbf{p}}^L \cdot \mathbf{v}_S \,, & (8.471) \\ \rho^L(\mathbf{v}_L)'_L \cdot \mathbf{v}_L &= \operatorname{div}(p^L \mathbf{v}_L) - p^L(\mathbf{D}_L \cdot \mathbf{I}) + \hat{\mathbf{p}}^L \cdot \mathbf{v}_L \,. & (8.472) \end{aligned}$$

The sum of both Eqs. (8.471) and (8.472) makes up the rate of mechanical work of the whole medium as

$$\begin{aligned} &\operatorname{div}(\mathbf{T}^S \mathbf{v}_S) + \operatorname{div}(p^L \mathbf{v}_L) - \mathbf{T}^S \cdot \mathbf{D}_S - p^L(\mathbf{D}_L \cdot \mathbf{I}) - \\ &-\rho^S(\mathbf{v}_S)'_S \cdot \mathbf{v}_S - \rho^L(\mathbf{v}_L)'_L \cdot \mathbf{v}_L + \hat{\mathbf{p}}^L(\mathbf{v}_L - \mathbf{v}_S) = 0 \,. \end{aligned} \qquad (8.473)$$

Integration over the control space B leads to

$$\begin{aligned} \int\limits_B \operatorname{div}(\mathbf{T}^S \mathbf{v}_S + (p^L \mathbf{v}_L) dv \;&= \int\limits_B [\mathbf{T}^S \cdot \mathbf{D}_S + p^L(\mathbf{D}_L \cdot \mathbf{I}) + \\ &\quad + \rho^S(\mathbf{v}_S)'_S \cdot \mathbf{v}_S + \rho^L(\mathbf{v}_L)'_L \cdot \mathbf{v}_L - \\ &\quad - \hat{\mathbf{p}}^L(\mathbf{v}_L - \mathbf{v}_S)] dv \,. \end{aligned} \qquad (8.474)$$

With the aid of Gauss' theorem the following relation

$$\begin{aligned} \int\limits_{\partial B} (\mathbf{T}^S \mathbf{v}_S + (p^L \mathbf{v}_L) \cdot \mathbf{n} da \;&= \int\limits_B [\mathbf{T}^S \cdot \mathbf{D}_S + p^L(\mathbf{D}_L \cdot \mathbf{I}) + \\ &\quad + \rho^S(\mathbf{v}_S)'_S \cdot \mathbf{v}_S + \rho^L(\mathbf{v}_L)'_L \cdot \mathbf{v}_L - \\ &\quad - \hat{\mathbf{p}}^L(\mathbf{v}_L - \mathbf{v}_S)] dv \end{aligned} \qquad (8.475)$$

is gained, where $\mathbf{n}$ is the unit normal vector at the integration surface ∂B.

The volume change rate of the liquid $\mathbf{D}_L \cdot \mathbf{I}$ is restrained by the saturation condition $n^S + n^L = 1$. Combining the saturation condition and the mass balance equation (8.52) in consideration of the incompressibility condition we may derive

$$n^L(\mathbf{D}_L \cdot \mathbf{I}) = -n^S(\mathbf{D}_S \cdot \mathbf{I}) - \operatorname{grad} n^L \cdot (\mathbf{v}_L - \mathbf{v}_S) \,. \qquad (8.476)$$

We use the above relation and the constitutive relations (8.62), (8.64) and rewrite Eq. (8.475) as

$$\begin{aligned}\int\limits_{\partial B}(\mathbf{T}^S\mathbf{v}_S + p^L\mathbf{v}_L)\cdot\mathbf{n}da &= \int\limits_{B}[\mathbf{T}^S_E\cdot\mathbf{D}_S - \hat{\mathbf{p}}^L_E\cdot(\mathbf{v}_L-\mathbf{v}_S)+ \\ &+ \rho^S(\mathbf{v}_S)'_S\cdot\mathbf{v}_S + \rho^L(\mathbf{v}_L)'_L\cdot\mathbf{v}_L]dv\ .\end{aligned} \tag{8.477}$$

Furthermore, because our study is confined to the scope of infinitesimal deformation we have linear geometrical and physical equations as in (8.68). Thus, we introduce the potential energy of the solid phase

$$W^S = \frac{1}{2}\int\limits_B \mathbf{T}^S_E\cdot\mathbf{E}_S dv \tag{8.478}$$

and the kinetic energies of the solid and liquid phases, respectively,

$$K^S = \frac{1}{2}\int\limits_B \rho^S\mathbf{v}_S\cdot\mathbf{v}_S dv,\quad K^L = \frac{1}{2}\int\limits_B \rho^L\mathbf{v}_L\cdot\mathbf{v}_L dv\ . \tag{8.479}$$

It follows from (8.477) to (8.479) that a balance of mechanical power of the whole porous medium yields

$$\begin{aligned}\int\limits_{\partial B}(\mathbf{T}^S\mathbf{v}_S + p^L\mathbf{v}_L)\cdot\mathbf{n}da &= (W^S)'_S + (K^S)'_S + (K^L)'_L - \\ &-\int\limits_B \hat{\mathbf{p}}^L_E\cdot(\mathbf{v}_L-\mathbf{v}_S)dv\ .\end{aligned} \tag{8.480}$$

It can be seen from (8.477) and (8.478) that the total potential energy of the entire medium is given by the effective stress acting upon the deformable solid skeleton while the poreliquid pressure makes no contribution to the potential energy for the whole medium. In other words the potential energy of the whole porous medium is stored in the solid skeleton even though the mechanical work of the external tractions includes the part by the poreliquid pressure as shown in the left-hand side of Eq. (8.477). The energy dissipation rate is reasonably defined by

$$\hat{d}(\mathbf{x},t) = -\hat{\mathbf{p}}^L_E\cdot(\mathbf{v}_L-\mathbf{v}_S)\ . \tag{8.481}$$

Analogous to the case of a one-component medium (Hayes, 1980), the energy flux vector $\mathbf{J}$ is defined by

$$\mathbf{J} = -\mathbf{T}^S\mathbf{v}_S - p^L\mathbf{v}_L\ . \tag{8.482}$$

In conclusion, the balance equation of mechanical power of the whole porous medium takes the form

$$(W^S + K^S)'_S + (K^L)'_L + \int\limits_B \hat{d}(\mathbf{x}, t) dv = \int\limits_{\partial B} \mathbf{J} \cdot \mathbf{n} da \; . \tag{8.483}$$

The above equation implies that the time rate of the total energy plus the energy dissipation rate is equal to the divergence of the energy flux vector. In addition, we recognize the energy flux intensity I as the energy transfer per unit area in the direction of every propagation

$$I = \mathbf{J} \cdot \mathbf{n} \; , \tag{8.484}$$

where $\mathbf{n}$ coincides with the direction of the propagation vector.

Energy flux and energy dissipation rate at inhomogeneous plane waves.

This section is concerned with the energy flux and the energy dissipation rate associated with the inhomogeneous plane waves. Owing to the oscillatory behavior, we are now interested in obtaining a mean energy flux vector $\langle \mathbf{J} \rangle$ over one complete period τ, which is defined as

$$\langle \mathbf{J} \rangle = \frac{1}{\tau} \int\limits_0^\tau \mathbf{J} dt \; . \tag{8.485}$$

We first take into account the representation of energy flux vectors $\mathbf{J}^P$ and $\mathbf{J}^T$ at the P-type and the S-type inhomogenous plane waves. For the P-type wave in the energy flux vector is attained from (8.482) and (8.1):

$$\mathbf{J}^P = -(\mathbf{T}^S_E)^P \mathbf{v}^P_S \; , \tag{8.486}$$

since the following relation holds for the P-type wave with the aid of (8.451) and (8.452):

$$n^S \mathbf{v}^P_S + n^L \mathbf{v}^P_L = n^S \mathbf{v}^P_S + n^L \tau_1 \mathbf{v}^P_S = \mathbf{0} \; , \tag{8.487}$$

where $\mathbf{v}^P_S$ and $\mathbf{v}^P_L$ are the particle velocities of the solid and the liquid for the longitudinal wave, respectively. Meanwhile, the poreliquid pressure is independent of the S-type inhomogeneous wave, we immediately write out from (8.482) and (8.1) the energy flux for the S-type wave

$$\mathbf{J}^T = -(\mathbf{T}^S_E)^T \mathbf{v}^T_S \; , \tag{8.488}$$

where $\mathbf{v}^T_S$ is the particle velocity of the solid for the transverse waves.

We have in mind that at the end of the calculations only the real part should be used. Before we calculate these energy flux vectors let us take into account two arbitrary quantities with time-harmonic variation as $\pi_1 = F \exp[i(\mathbf{k} \cdot \boldsymbol{x} - \omega t)]$

and $\pi_2 = G \exp[i(\mathbf{k} \cdot \boldsymbol{x} - \omega t)]$. Since the real part is indeed the physical part of the corresponding quantities, we multiply them as

$$\begin{aligned} \mathrm{Re}(\pi_1)\,\mathrm{Re}(\pi_2) &= \frac{1}{2}(\pi_1 + \pi_1^*)\frac{1}{2}(\pi_2 + \pi_2^*) \\ &= \frac{1}{2}\,\mathrm{Re}(\pi_1 \pi_2^*) + \frac{1}{4}(\pi_1 \pi_2 + \pi_1^* \pi_2^*) \ , \end{aligned} \tag{8.489}$$

where the index * denotes the complex conjugate. Remembering that the time average of harmonic functions of frequency 2ω vanishes and that $\pi_1^* \pi_2^*$ is not a function of time, we obtain an expression

$$\langle \mathrm{Re}(\pi_1)\,\mathrm{Re}(\pi_2) \rangle = \frac{1}{2}\,\mathrm{Re}(\pi_1^* \pi_2^*) \ . \tag{8.490}$$

At first, we calculate the energy flux vector at the P-type inhomogeneous wave and thus from (8.490) we write the mean energy flux vector

$$\langle \mathbf{J}^P \rangle = -\frac{1}{2}\,\mathrm{Re}[(\mathbf{T}_E^S)^P (\mathbf{v}_S^P)^*] \ . \tag{8.491}$$

Since $(\mathbf{v}_S^P)^* = i\omega(\mathbf{u}_S^P)^*$, (8.491) can be rewritten as

$$\langle \mathbf{J}^P \rangle = -\frac{1}{2}\omega\,\mathrm{Im}[(\mathbf{T}_E^S)^P (\mathbf{u}_S^P)^*] \ . \tag{8.492}$$

It follows from (8.68) and (8.457) that the stress tensor due to the P-type wave leads to

$$\begin{aligned} (\mathbf{T}_E^S)^P = \ & -\Phi_0[\ 2\mu^S \mathbf{k}^P \otimes \mathbf{k}^P + \\ & \qquad + \lambda^S (\mathbf{k}^P \cdot \mathbf{k}^P)\mathbf{I}] \exp[i(\mathbf{k}^P \cdot \boldsymbol{x} - \omega t)] \ . \end{aligned} \tag{8.493}$$

In order to apply (8.493) to the mean energy flux $\langle \mathbf{J}^P \rangle$ in (8.492) we have to evaluate $\mathbf{T}_E^S (\mathbf{u}_S^L)^*$. Substitution and some rearrangements yield

$$\begin{aligned} (\mathbf{T}_E^S)^P (\mathbf{u}_S^P)^* = i\Phi_0 \Phi_0^* \exp(-2\mathbf{k}^{P-} \cdot \mathbf{x})[2\mu^S (\mathbf{k}^P \cdot \mathbf{k}^{P*})\mathbf{k}^P + \\ + \lambda^S (\mathbf{k}^P \cdot \mathbf{k}^P)\mathbf{k}^{P*}] \ . \end{aligned} \tag{8.494}$$

Inserting (8.494) into (8.492) we immediately have

$$\begin{aligned} \langle \mathbf{J}^P \rangle = \frac{1}{2}\omega \exp(-2\mathbf{k}^{P-} \cdot \mathbf{x})|\Phi_0|^2\,\mathrm{Re}[(\lambda^S + 2\mu^S)(\mathbf{k}^P \cdot \mathbf{k}^{P*})\mathbf{k}^P + \\ + 2\mu^S \mathbf{k}^P \wedge (\mathbf{k}^P \wedge \mathbf{k}^{P*})] \ , \end{aligned} \tag{8.495}$$

where the relation

$$(\mathbf{k}^P \cdot \mathbf{k}^{P*})\mathbf{k}^P = (\mathbf{k}^P \cdot \mathbf{k}^P)\mathbf{k}^{P*} + \mathbf{k}^P \wedge (\mathbf{k}^P \wedge \mathbf{k}^{P*}) \tag{8.496}$$

has been used.

Analogously, the energy flux at the S-type wave is written as

$$\langle \mathbf{J}^T \rangle = \frac{1}{2}\omega \,\mathrm{Im}[(\mathbf{T}_E^S)^T (\mathbf{u}_S^T)^*] \,, \tag{8.497}$$

where the stress tensor due to the S-type wave propagation is given by

$$\begin{aligned} (\mathbf{T}_E^S)^T = \; & -\mu^S[(\mathbf{k}^T \wedge \mathbf{h}_0) \otimes \mathbf{k}^T + \\ & +\mathbf{k}^T \otimes (\mathbf{k}^T \wedge \mathbf{h}_0)] \exp[i(\mathbf{k}^T \cdot \mathbf{x} - \omega t)] \,. \end{aligned} \tag{8.498}$$

Moreover, a simple calculation turns out

$$\begin{aligned} (\mathbf{T}_E^S)^T (\mathbf{u}_S^T)^* = \; & \mu^S \exp(-2\mathbf{k}^{T-} \cdot \boldsymbol{x})[(\mathbf{k}^T \wedge \mathbf{h}_0)\mathbf{k}^T \cdot \mathbf{k}^{T*} \wedge \mathbf{h}_0^* + \\ & +\mathbf{k}^T(\mathbf{k}^T \wedge \mathbf{h}_0) \cdot (\mathbf{k}^{T*} \wedge \mathbf{h}_0^*)] \,. \end{aligned} \tag{8.499}$$

Insertion of (8.499) and (8.497) yields

$$\begin{aligned} \langle \mathbf{J} \rangle^T = \; & \frac{1}{2}\omega\mu^S \exp(-2\mathbf{k}^{T-} \cdot \boldsymbol{x}) \,\mathrm{Re}[(\mathbf{k}^T \wedge \mathbf{h}_0)\mathbf{k}^T \cdot \mathbf{k}^{T*} \wedge \mathbf{h}_0^* + \\ & + \mathbf{k}^T(\mathbf{k}^T \wedge \mathbf{h}_0) \cdot (\mathbf{k}^{T*} \wedge \mathbf{h}_0^*)] \,. \end{aligned} \tag{8.500}$$

In the sequel we investigate the energy dissipation rate at each type of waves. With the aid of (8.70) and (8.481) as well as (8.490) the time-averaged energy dissipation rate becomes:

$$\langle \hat{d}(\mathbf{x}, t) \rangle = -\frac{1}{2} S_L \,\mathrm{Re}[(\mathbf{v}_L - \mathbf{v}_S) \cdot (\mathbf{v}_L - \mathbf{v}_S)^*] \,. \tag{8.501}$$

For the P-type wave we write the mean energy dissipation rate as

$$\langle \hat{d}^P(\mathbf{x}, t) \rangle = -\frac{1}{2} S_L \,\mathrm{Re}[(\mathbf{v}_L^P - \mathbf{v}_S^P) \cdot (\mathbf{v}_L^P - \mathbf{v}_S^P)^*] \tag{8.502}$$

and with the aid of (8.457) and (8.487) we write further

$$\langle \hat{d}^P(\mathbf{x},t)\rangle = -\frac{S_L\omega^2}{2(n^L)^2}\,\mathrm{Re}[\mathbf{u}_S^P \cdot (\mathbf{u}_S^P)^*]\ . \tag{8.503}$$

Insertion of (8.461) in (8.502) gives

$$\begin{aligned}\langle \hat{d}^P(\mathbf{x},t)\rangle &= -\frac{S_L\omega^2}{2(n^L)^2}\,\mathrm{Re}[\exp(-2\mathbf{k}^{P-}\cdot\boldsymbol{x})\Phi_0\Phi_0^*\mathbf{k}^P\cdot\mathbf{k}^{P*}]\\ &= -\frac{S_L\omega^2}{2(n^L)^2}\exp(-2\mathbf{k}^{P-}\cdot\boldsymbol{x})|\Phi_0|^2|\mathbf{k}^P|^2\ .\end{aligned} \tag{8.504}$$

The time-averaged energy dissipation rate at the S-type wave can be written as

$$\langle \hat{d}^T(\mathbf{x},t)\rangle = -\frac{1}{2}S_L\,\mathrm{Re}[(\mathbf{v}_L^T-\mathbf{v}_S^T)\cdot(\mathbf{v}_L^T-\mathbf{v}_S^T)^*]\ . \tag{8.505}$$

With the aid of (8.451) the above expression becomes

$$\langle \hat{d}^T(\mathbf{x},t)\rangle = -\frac{1}{2}S_L\omega^2\,\mathrm{Re}[(\tau_2-1)(\tau_2-1)^*\mathbf{u}_S^T\cdot(\mathbf{u}_S^T)^*]\ . \tag{8.506}$$

We insert (8.457) into (8.506) and arrive at

$$\begin{aligned}\langle \hat{d}^T(\mathbf{x},t)\rangle = \ & -\frac{1}{2}S_L\omega^2[(\tau_2-1)^2\exp(-2\mathbf{k}^{T-}\cdot\boldsymbol{x})\times\\ & \times\mathrm{Re}[(\mathbf{k}^T\wedge\mathbf{h}_0)\cdot(\mathbf{k}^{T*}\wedge\mathbf{h}_0^*)]\ ,\end{aligned} \tag{8.507}$$

where we further evaluate

$$\begin{aligned}\mathrm{Re}[(\mathbf{k}^T\wedge\mathbf{h}_0)\cdot(\mathbf{k}^{T*}\wedge\mathbf{h}_0^*)] &= \mathrm{Re}[(\mathbf{k}^T\cdot\mathbf{k}^{T*})(\mathbf{h}_0\cdot\mathbf{h}_0^*)] -\\ &\quad -(\mathbf{k}^T\cdot\mathbf{h}_0^*)(\mathbf{k}^{T*}\cdot\mathbf{h}_0)]\\ &= |\mathbf{k}^T|^2|\mathbf{h}_0|^2 - \mathrm{Re}[(\mathbf{k}^T\cdot\mathbf{h}_0^*)(\mathbf{k}^T\cdot\mathbf{h}_0^*)^*]\\ &= |\mathbf{k}^T|^2|\mathbf{h}_0|^2 - (\mathbf{k}^T\cdot\mathbf{h}_0^*)^2\ .\end{aligned} \tag{8.508}$$

Consequently, the mean energy dissipation rate for the S-type wave is evaluated as

$$\begin{aligned}\langle \hat{d}^T(\mathbf{x},t)\rangle = \; & -\frac{1}{2}S_L\omega^2|(\tau_2-1)^2|\exp(-2\mathbf{k}^{T-}\cdot\boldsymbol{x})\times \\ & \times[|\mathbf{k}^T|^2|\mathbf{h}_0|^2-|\mathbf{k}^T\cdot\mathbf{h}_0^*|^2]\;.\end{aligned} \tag{8.509}$$

The incompressible porous media model is utilized to describe the two-phase porous medium. A coupled P-type wave and a coupled S-type wave may propagate in the porous media. The displacement field for each type of waves is characterized in terms of a complex-valued amplitude vector and a complex-valued wave vector which includes a real-valued propagation vector and real-valued attenuation vector. It is verified that each coupled wave is inhomogeneous since the planes of constant phase is in general not parallel to the planes of constant amplitude due to the dissipative property of the porous medium. The trace of the particle motion of each type of wave is of elliptical polarization associated with the complex-valued amplitude vector.

The explicit expression of the mean energy flux vector and the mean energy dissipation rate are given over a complete period for each type of inhomogeneous waves.

f) Propagation and Evolution of Wave Fronts in Saturated Porous Solids

We investigate the evolution equations governing the amplitudes of discontinuity waves in porous media. Because our study is confined to infinitesimal deformations and to a linear poroelastic solid filled with an ideal liquid, it follows, if we neglect convective terms, from (8.73) that the balance equations of momentum may be rewritten in the form

$$\begin{aligned}&(\lambda^S+\mu^S)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}_S+\mu^S\,\mathrm{div}\,\mathrm{grad}\,\mathbf{u}_S-n^S\,\mathrm{grad}\,p\,+ \\ &\quad +\rho^S(\mathbf{b}_S-\frac{\partial^2\mathbf{u}_S}{\partial t^2})+S_L(\frac{\partial\mathbf{u}_L}{\partial t}-\frac{\partial\mathbf{u}_S}{\partial t})=\mathbf{0}\;,\end{aligned} \tag{8.510}$$

$$-n^L\,\mathrm{grad}\,p+\rho^L(\mathbf{b}_L-\frac{\partial^2\mathbf{u}_L}{\partial t^2})-S_L(\frac{\partial\mathbf{u}_L}{\partial t}-\frac{\partial\mathbf{u}_S}{\partial t})=\mathbf{0}\;. \tag{8.511}$$

In order to derive the differential equations of the amplitudes, we need to evaluate the field equations (8.510) and (8.511) on the wave fronts. To this end, from the iterated substitution of Φ in $(8.360)_2$ and (8.362) by $\mathbf{u}_\wp$ into the geometrical and kinematical conditions of compatibility of the second order we have the following jumps:

$$\begin{aligned}[\![\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}_\wp]\!] = \; & (\mathbf{c}^\wp\cdot\mathbf{n})\mathbf{n}+a^{\alpha\beta}[\![(\mathbf{a}^\wp_{,\alpha}\cdot\mathbf{n})\mathbf{x}_{,\beta}\,+ \\ & +(\mathbf{a}^\wp_{,\alpha}\cdot\mathbf{x}_{,\beta})\mathbf{n}]\!]-b^{\alpha\beta}(\mathbf{a}^\wp\cdot\mathbf{x}_{,\alpha})\mathbf{x}_{,\beta}\;,\end{aligned} \tag{8.512}$$

$$\llbracket \operatorname{div}\operatorname{grad}\mathbf{u}_S \rrbracket = \mathbf{c}^S - b^{\alpha}_{.\alpha}\mathbf{a}^S \ , \tag{8.513}$$

$$\left\llbracket \frac{\partial^2 \mathbf{u}_{\wp}}{\partial t^2} \right\rrbracket = u_n^2 \mathbf{c}^{\wp} - 2\mathbf{u}_n \frac{\delta \mathbf{a}^{\wp}}{\delta t} \ , \tag{8.514}$$

where

$$\mathbf{a}^{\wp} = \llbracket \operatorname{grad}\mathbf{u}_{\wp} \rrbracket \mathbf{n} \tag{8.515}$$

is defined as the amplitude vectors in the solid skeleton and in the liquid. Moreover, the following relation

$$\mathbf{n}\cdot\mathbf{x}_{,\alpha} = 0 \tag{8.516}$$

has been used, and $b^{\alpha}_{.\alpha}$ is the trace of the surface tensor $b^{\alpha}_{.\beta}$ and

$$\mathbf{c}^{\wp} = \llbracket \operatorname{grad}\operatorname{grad}\mathbf{u}_{\wp} \rrbracket \mathbf{n}\otimes\mathbf{n} \tag{8.517}$$

are introduced as unknown vectors.

Next, we first examine the amplitude evaluation of the longitudinal waves in a well-known way. However, the procedure is lengthy and the details are therefore purposely neglected. After the field equation (8.510) and (8.5.11) are written on the wave fronts and the relations (8.512) to (8.516) are substituted into them. Upon elimination of the porewater pressure gradient $[\operatorname{grad} p]$, we have

$$\begin{aligned}
& n^L(\lambda^S + \mu^S)[(\mathbf{c}^S\cdot\mathbf{n}) \ + a^{\alpha\beta}(\mathbf{a}^S_{,\alpha}\cdot\mathbf{x}_{,\beta})] + \\
& + n^L\mu^S(\mathbf{c}^S\cdot\mathbf{n} - b^{\alpha}_{.\alpha}\mathbf{a}^S\cdot\mathbf{n}) - \\
& - n^L\rho^S\Big(u_n^2\mathbf{c}^S\cdot\mathbf{n} - 2u_n\frac{\delta\mathbf{a}^S}{\delta t}\cdot\mathbf{n}\Big) + \\
& + n^S\rho^L\Big(u_n^2\mathbf{c}^L\cdot\mathbf{n} - 2u_n\frac{\delta\mathbf{a}^L}{\delta t}\cdot\mathbf{n}\Big) + \\
& + S_L u_n(\mathbf{a}^S\cdot\mathbf{n} - \mathbf{a}^L\cdot\mathbf{n}) = 0 \ .
\end{aligned} \tag{8.518}$$

For simplicity we examine only the case of the so-called homogenous weak discontinuity in our following analysis. Namely, all amplitudes tangent to the wave fronts are assumed as unchangeable quantities, i.e,

$$a^S_{n,\alpha} = 0, \quad a^S_{e,\alpha} = 0 \ , \tag{8.519}$$

where we denote

$$a_n^S = \mathbf{a}^S \cdot \mathbf{n}, \quad a_e^S = \mathbf{a}^S \cdot \mathbf{e} \tag{8.520}$$

as the amplitudes of the longitudinal and the transverse waves; $\mathbf{e}$ is chosen as a unit vector perpendicular to the unit wave normal $\mathbf{n}$ such that

$$\mathbf{e} \cdot \mathbf{n} = 0 \,. \tag{8.521}$$

Moreover, in the scope of infinitesimal deformations, the equation for the volume fractions $(8.111)_3$ are simplified into

$$\operatorname{div}\Big(n^S \frac{\partial \mathbf{u}_S}{\partial t} + n^L \frac{\partial \mathbf{u}_L}{\partial t} \Big) = 0 \,. \tag{8.522}$$

In view of (8.363) it is easy to have

$$\left[\!\left[\operatorname{div} \frac{\partial \mathbf{u}_\wp}{\partial t} \right]\!\right] = -u_n \mathbf{c}^\wp \cdot \mathbf{n} + \frac{\partial \mathbf{a}^\wp}{\partial t} \cdot \mathbf{n} - a^{\alpha\beta} u_n \mathbf{a}^\wp_{,\alpha} \cdot \mathbf{x}_{,\beta} \,. \tag{8.523}$$

Thus, recalling $(8.341)_3$ and the saturation condition in the reference placement and upon substitution of the above expression (8.523) into (8.522) we obtain the restriction for the unknown vector $\mathbf{c}^\wp$ such that

$$n^S \mathbf{c}^S \cdot \mathbf{n} + n^L \mathbf{c}^L \cdot \mathbf{n} = 0 \,, \tag{8.524}$$

where (8.519) and (8.520) have been utilized. It should be mentioned here that the constraint (8.524) is valid only for the P-type coupled waves.

With the aid of (8.524) a rearrangement of (8.518) immediately gives rise to

$$\begin{aligned} & 2[(n^L)^2 \rho^S + (n^S)^2 \rho^L] u_{n1} \frac{\delta a_n^S}{\delta t} + S_L u_{n1} a_n^S - \\ & -(n^L)^2 (\lambda^S + 2\mu^S) b^{\alpha}_{.\alpha} a_n^S + \{(n^L)^2 (\lambda^S + 2\mu^S) - \\ & -[(n^L)^2 \rho^S + (n^S)^2 \rho^L] u_{n1}\} \mathbf{c}^S \cdot \mathbf{n} = 0 \,. \end{aligned} \tag{8.525}$$

Utilizing the expression (8.342) we can eliminate the unknown dot product $\mathbf{c}^S \cdot \mathbf{n}$ and obtain the differential equation governing the amplitude a_n^S

$$\frac{\delta a_n^S}{\delta t} - \frac{1}{2} \left[\!\left[b^{\alpha}_{.\alpha} u_{n1} - \frac{S_L}{(n^L)^2 \rho^S + (n^S)^2 \rho^L} \right]\!\right] a_n^S = 0 \,. \tag{8.526}$$

Analogously, we take the projection of the field equations (8.510) and (8.511) across the wave fronts in the direction of $\mathbf{e}$ and bear in mind that only the transverse waves are realizable in the solid skeleton in this case, the result is arranged in the form

$$\begin{aligned}
&(\lambda^S+\mu^S)[a^{\alpha\beta}(\mathbf{a}^S_{,\alpha}\cdot\mathbf{n})(\mathbf{x}_{,\beta}\cdot\mathbf{e})-b^{\alpha\beta}(\mathbf{a}^S\cdot\mathbf{x}_{,\alpha})(\mathbf{x}_{,\beta}\cdot\mathbf{e})+\\
&+\mu^S(\mathbf{c}^S\cdot\mathbf{e}-b^{\alpha}_{.\alpha}\mathbf{a}^S\cdot\mathbf{e})\;-\rho^S\Big(u_n^2\mathbf{c}^S\cdot\mathbf{e}-2u_n\frac{\delta\mathbf{a}^S}{\delta t}\cdot\mathbf{e}\Big)+\\
&+S_L u_n\mathbf{a}^S\cdot\mathbf{e}=0\;.
\end{aligned}\tag{8.527}$$

In the case of the homogenous weak discontinuity, Eq. (8.518) becomes

$$\begin{aligned}
&2\rho^S u_{n2}\frac{\delta\mathbf{a}^S}{\delta t}\cdot\mathbf{e}-(\mu^S b^{\beta}_{.\alpha}-S_L u_{n2})\mathbf{a}^S\cdot\mathbf{e}+\\
&+(\mu^S-\rho^S u_{n2}^2)\mathbf{c}^S\cdot\mathbf{e}=0\;,
\end{aligned}\tag{8.528}$$

where the identity

$$\mathbf{e}_{,\alpha}\cdot\mathbf{n}=-\,\mathbf{e}\cdot\mathbf{n}_{,\alpha}=b^{\cdot\gamma}_{\alpha}\mathbf{x}_{,\gamma}\cdot\mathbf{e}\tag{8.529}$$

has been utilized. With the aid of the expression (8.343), the amplitude for the transverse waves is determined by

$$\frac{\delta a^S_e}{\delta t}-\Big(\frac{1}{2}b^{\alpha}_{.\alpha}u_{n2}-\frac{S_L}{2\rho^S}\Big)a^S_e=0\;.\tag{8.530}$$

With (8.356), the following notation may be introduced as

$$\frac{\delta}{\delta t}=u_n\Big(\mathbf{n}\cdot\mathrm{grad}+\frac{1}{u_n}\frac{\delta}{\delta t}\Big)=u_n\frac{d}{dr}\;,\tag{8.531}$$

where the total derivative in the direction of the normal $\mathbf{n}$ to the singular surfaces $\sigma(t)$ is represented by $\frac{d}{dr}$. Then, the substitution by means of $\frac{d}{dr}$ shows that the ordinary differential equations (8.526) and (8.530) take the forms

$$\frac{da^S_n}{dr}=(\Omega-\mathrm{M})a^S_n\;,\tag{8.532}$$

$$\frac{da^S_e}{dr}=(\Omega-\mathrm{N})a^S_e\;,\tag{8.533}$$

where definition $(8.353)_1$ has been used, and the coefficients M and N are defined by

$$\mathrm{M}=\frac{S_L}{2[(n^L)^2\rho^S+(n^S)^2\rho^L]u_{n1}}\;,\tag{8.534}$$

$$\mathrm{N} = \frac{S_L}{2\rho^S u_{n2}} \ . \tag{8.535}$$

Furthermore, Ω can be rewritten in terms of a certain position; namely

$$\Omega(r) = \frac{\Omega_0 - \mathrm{K}_0 r}{1 - 2\Omega_0 r + \mathrm{K}_0 r^2} \ , \tag{8.536}$$

where r is the distance of the wave fronts from their initial position. Hence the integration of (8.532) and (8.533) together with (8.536) results in:

$$a_n^S(r) = a_n^S(0)(1 - 2\Omega_0 r + \mathrm{K}_0 r^2)^{-1/2} \exp(-\mathrm{M}r) \ , \tag{8.537}$$

$$a_e^S(r) = a_e^S(0)(1 - 2\Omega_0 r + \mathrm{K}_0 r^2)^{-1/2} \exp(-\mathrm{N}r) \ , \tag{8.538}$$

where $a_n^S(0)$ and $a_e^S(0)$ are the initial values of the absolute amplitudes of waves.

It is evident from (8.524) through (8.538) that growth and decay of the waves depend on the initial geometrical shapes of the waves fronts and the diffusion effects between the phases. The diffusion effects only produces the attenuation of the amplitudes of the waves but does not influence the propagation speeds.

In particular, the evolution of the amplitudes of wave fronts may reduce to the classical result in the case of non-dissipation where the amplitudes rely completely on the initial geometrical property of the wave fronts.

If the initial mean curvature Ω_0 is positive and the initial total curvature K_0 is negative, for instance in the case of the initial pseudosphere for the wave fronts, it follows from (8.537) and (8.538) that the amplitudes will reach an infinite value within a finite propagation distance. In other words, the discontinuity waves in the porous media may constitute a higher weak discontinuity. In the case of the negative initial mean curvature and the positive initial total curvature, the amplitudes of wave fronts will decay to vanish. It is of interest to consider the plane $(\Omega_0 = 0,\ \mathrm{K}_0 = 0)$, the cylindrical $\Big(\Omega_0 = -\dfrac{0.5}{r_0},\ \mathrm{K}_0 = 0\Big)$ and the spherical $\Big(\Omega_0 = -\dfrac{1}{r_0},\ \mathrm{K}_0 = -\dfrac{1}{r_0^2}\Big)$ waves for which the amplitudes are reduced into the following forms, respectively:

for plane waves:

$$a_n^S(r) = a_n^S(0) \exp(-\mathrm{M}r) \ , \tag{8.539}$$

$$a_e^S(r) = a_e^S(0) \exp(-\mathrm{N}r) \ ; \tag{8.540}$$

for cylindrical waves:

$$a_n^S(r) = a_n^S(0)\Big(1 + \frac{1}{r_0} r\Big)^{-1/2} \exp(-\mathrm{M}r) \; , \tag{8.541}$$

$$a_e^S(r) = a_e^S(0)\Big(1 + \frac{1}{r_0} r\Big)^{-1/2} \exp(-\mathrm{N}r) \; ; \tag{8.542}$$

for spherical waves:

$$a_n^S(r) = a_n^S(0)\Big(1 + \frac{2}{r_0} r + \frac{1}{r_0^2} r^2\Big)^{-1/2} \exp(-\mathrm{M}r) \; , \tag{8.543}$$

$$a_e^S(r) = a_e^S(0)\Big(1 + \frac{2}{r_0} r + \frac{1}{r_0^2} r^2\Big)^{-1/2} \exp(-\mathrm{N}r) \; , \tag{8.544}$$

where r_0^2 is the initial semi-diameter of the wave fronts.

In conclusion, it is revealed that a coupled longitudinal wave and a pure transverse wave are realizable in two-phase incompressible porous media. The longitudinal waves propagate either in the solid skeleton or in the liquid with the same propagation speed. In this coupled wave mode a certain restriction between the amplitudes is required by the saturation condition together with the incompressibility of porous media. The jump of the poreliquid pressure may be determined by the amplitude of the coupled waves. The transverse waves propagate only in the solid phase. The two propagation velocities are determined by the material constants and the volume fractions but do not depend on the diffusion coefficient S_L. In the case of homogenous weak discontinuities, the differential equation governing the amplitudes of the waves of two types are then derived and the explicit solutions for the two cases are obtained. These results indicate that the amplitudes of wave fronts may either decay to vanish or grow to infinity in finite time, which depends on the geometric property of the initial shapes of the wave fronts and the diffusion effect between the two phases.

Chapter 9

POROPLASTICITY FOR METALLIC POROUS SOLIDS

A general poroplasticity was already developed in Section 6.6 including porous solids consisting of frictional or metallic material. Of special interest are metallic porous solids because their plastic material behavior can be relatively easily described by a von-Mises-like yield condition and an associated flow rule (Green, 1972). The corresponding theory represents an extension of the plasticity theory by von Mises (1913), in the sense that in the extended theory not only the deviatoric stress but also the hydrostatic stress state is bounded. Moreover, the associated flow rule allows the descriptions of volume changes, a special feature of porous solids owing to the considerable influence of the pores on the stress and deformations states. The influence of the volume change of the real material in the elastic-plastic region can be completely neglected, because the plastic volume changes of the real material is zero and the elastic volume strains are very small within the frameworks of the geometrically linear theory in comparison with the volume changes of the pores. Therefore, we will treat an incompressible porous solid, filled with an incompressible liquid. This model covers a wide range of applications in the plasticity theory of metallic porous solids.

Fortunately, the yield condition and the associated flow rule are contained in the general theory (Section 6.6). We only have to neglect several material dependent parameters in the yield condition (6.93) and have to change the meaning of the parameter in the flow rule.

With these simplified basic constitutive equations we will developed a plasticity theory for ideal-plastic behavior and for hardening material with work and kinematic hardening, at first. We will manipulate the basic constitutive equations and will show that the "elastic-plastic-tangent" is symmetric, invertible and positive-definite.

These properties are very useful for the construction of general theorems in Section 9.2. There are many general theorems possible as it is known from the von Mises theory. Due to space limitation we will consider explicitly only two of them, namely the unique theorem and the minimum-maximum principle for ideal-plastic behavior.

9.1 Stress-Strain Relations

It has already been mentioned that the basic constitutive equations for the description of plastic and elastic plastic behavior namely yield condition and flow rule, are contained in the corresponding relations (6.93) and (6.109). In order be able to derive the yield condition and flow rule for metallic materials we have to set

$$\beta = 0, \quad \gamma = 0, \quad \varepsilon = 0 \, , \tag{9.1}$$

and

$$\mu = \alpha^2 \, . \tag{9.2}$$

Then, the yield condition (6.93) reads as:

$$F = \frac{1}{\sqrt{2}} \sqrt{\bar{\mathbf{S}}^D \cdot \bar{\mathbf{S}}^D + \alpha^2 (\bar{\mathbf{S}} \cdot \mathbf{I})^2} - \kappa = 0 \tag{9.3}$$

and the flow rule as

$$\mathrm{D}\mathbf{E}_{SP} = \mathrm{D}\lambda \frac{\partial F}{\partial \bar{\mathbf{S}}} \, . \tag{9.4}$$

We will treat the rigid ideal-plastic behavior of the solid material, at first.

a) Rigid Ideal-Plastic Behavior

The beginning of plastic deformations of a liquid-saturated porous body is bound to the yield condition. Bearing in mind that for rigid-ideal-plastic behavior κ and α^2 are constants and $\bar{\mathbf{S}}_E^S = \mathbf{S}_E^S$ the flow rule (9.4) yields with (9.3)

$$\mathrm{D}\mathbf{E}_{SP} = \mathrm{D}\lambda \frac{1}{2\kappa} [\mathbf{S}_E^{SD} + \alpha^2 (\mathbf{S}_E^S \cdot \mathbf{I}) \mathbf{I}] \, . \tag{9.5}$$

The criterion for the persistence of plastic deformations reads as

$$F = 0 \;\; \text{and} \;\; \frac{\partial F}{\partial \mathbf{S}_E^S} \cdot \mathrm{D}\mathbf{S}_E^S = 0 \, , \tag{9.6}$$

whereas, based on $F = 0$,

$$\frac{\partial F}{\partial \mathbf{S}_E^S} \cdot \mathrm{D}\mathbf{S}_E^S < 0 \,, \tag{9.7}$$

yields the condition for unloading. The scalar factor $\mathrm{D}\lambda$ in (9.5) can be expressed by the plastic strains. For this purpose we proceed from (9.5) and form

$$\mathrm{D}\mathbf{E}_{SP}^D \cdot \mathrm{D}\mathbf{E}_{SP}^D = (\mathrm{D}\lambda)^2 \frac{1}{4\kappa^2} \mathbf{S}_E^{SD} \cdot \mathbf{S}_E^{SD} \tag{9.8}$$

and

$$\left(\frac{1}{3\alpha} \mathrm{D}\mathbf{E}_{SP} \cdot \mathbf{I}\right)^2 = (\mathrm{D}\lambda)^2 \frac{1}{4\kappa^2} \alpha^2 (\mathbf{S}_E^S \cdot \mathbf{I})^2 \tag{9.9}$$

The summation of (9.8) and (9.9) leads to

$$\begin{aligned} &\mathrm{D}\mathbf{E}_{SP}^D \cdot \mathrm{D}\mathbf{E}_{SP}^D + \left(\frac{1}{3\alpha} \mathrm{D}\mathbf{E}_{SP} \cdot \mathbf{I}\right)^2 \\ &= \ (\mathrm{D}\lambda)^2 \frac{1}{4\kappa^2} [\mathbf{S}_E^D \cdot \mathbf{S}_E^D + \alpha^2 (\mathbf{S}_E^S \cdot \mathbf{I})^2] \end{aligned} \tag{9.10}$$

Considering (9.5) Eq. (9.10) can be simplified to

$$\mathrm{D}\mathbf{E}_{SP}^D \cdot \mathrm{D}\mathbf{E}_{SP}^D + \left(\frac{1}{3\alpha} \mathrm{D}\mathbf{E}_{SP} \cdot \mathbf{I}\right)^2 = \frac{1}{2} (\mathrm{D}\lambda)^2 \,, \tag{9.11}$$

which yields

$$\mathrm{D}\lambda = \sqrt{2} \sqrt{\mathrm{D}\mathbf{E}_{SP}^D \cdot \mathrm{D}\mathbf{E}_{SP}^D + \left(\frac{1}{3\alpha} \mathrm{D}\mathbf{E}_{SP} \cdot \mathbf{I}\right)^2} \tag{9.12}$$

or if we bring the time t into consideration

$$\dot{\lambda} = \sqrt{2} \sqrt{\dot{\mathbf{E}}_{SP}^D \cdot \dot{\mathbf{E}}_{SP}^D + \left(\frac{1}{3\alpha} \dot{\mathbf{E}}_{SP} \cdot \mathbf{I}\right)^2} \,, \tag{9.13}$$

where the superscript dot notes the material time derivative, whereby we have neglected the convective terms. This form (9.13) is important for the development of minimum-maximum principles in Section 9.2.

b) Elastic-Plastic Behavior with Hardening

Now, we will treat an elastic-plastic model with work and kinematic hardening. The governing constitutive equations for this model are contained in

(6.93),(6.94), (9.5) and (9.12). In consideration of the restrictions introduced at the beginning of this section we have after (6.123) and (6.124)

$$\mathrm{D}\mathbf{E}_S = (\overset{4}{\mathbf{P}}{}^S)^{-1}\mathrm{D}\mathbf{S}_E^S \tag{9.14}$$

with

$$(\overset{4}{\mathbf{P}}{}^S)^{-1} = (\overset{4}{\mathbf{K}}{}^S)^{-1} + \frac{1}{h}\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \frac{\partial F}{\partial \bar{\mathbf{S}}}\right) \tag{9.15}$$

as well as after (6.133) and (6.134)

$$\mathrm{D}\mathbf{S}_E^S = \overset{4}{\mathbf{P}}{}^S\,\mathrm{D}\mathbf{E}_S \tag{9.16}$$

with

$$\overset{4}{\mathbf{P}}{}^S = \overset{4}{\mathbf{K}}{}^S - \frac{\overset{4}{\mathbf{K}}{}^S\,\dfrac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \overset{4}{\mathbf{K}}{}^S\,\dfrac{\partial F}{\partial \bar{\mathbf{S}}}}{h + \overset{4}{\mathbf{K}}{}^S \cdot \left(\dfrac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \dfrac{\partial F}{\partial \bar{\mathbf{S}}}\right)}\,. \tag{9.17}$$

It can be easily proven that the fourth-order tensor $(\overset{4}{\mathbf{P}}{}^S)^{-1}$ in (9.15) is the inverse tensor to $\overset{4}{\mathbf{P}}{}^S$. The tensor product of $\overset{4}{\mathbf{P}}{}^S$ and $(\overset{4}{\mathbf{P}}{}^S)^{-1}$ yields with (9.15) and (9.17):

$$\begin{aligned}
\overset{4}{\mathbf{P}}{}^S\,(\overset{4}{\mathbf{P}}{}^S)^{-1} &= \overset{4}{\mathbf{K}}{}^S\,(\overset{4}{\mathbf{K}}{}^S)^{-1} + \frac{1}{h}\,\overset{4}{\mathbf{K}}{}^S\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \frac{\partial F}{\partial \bar{\mathbf{S}}}\right) - \\
&- \frac{\overset{4}{\mathbf{K}}{}^S\,\dfrac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \overset{4}{\mathbf{K}}{}^S\,\dfrac{\partial F}{\partial \bar{\mathbf{S}}}}{h + \overset{4}{\mathbf{K}}{}^S \cdot \left(\dfrac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \dfrac{\partial F}{\partial \bar{\mathbf{S}}}\right)}(\overset{4}{\mathbf{K}}{}^S)^{-1} - \\
&- \frac{1}{h}\frac{\overset{4}{\mathbf{K}}{}^S\,\dfrac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \overset{4}{\mathbf{K}}{}^S\,\dfrac{\partial F}{\partial \bar{\mathbf{S}}}}{h + \overset{4}{\mathbf{K}}{}^S \cdot \left(\dfrac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \dfrac{\partial F}{\partial \bar{\mathbf{S}}}\right)}\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \frac{\partial F}{\partial \bar{\mathbf{S}}}\right)\,.
\end{aligned} \tag{9.18}$$

Considering the symmetry of the elastic material tensor $\mathbf{K}^S$ and the tensor with the derivatives of the yield condition the above relation (9.18) can be reformulated:

$$\begin{aligned}
\overset{4}{\mathbf{P}}{}^S\,(\overset{4}{\mathbf{P}}{}^S)^{-1} &= \overset{4}{\mathbf{K}}{}^S\,(\overset{4}{\mathbf{K}}{}^S)^{-1} + \frac{1}{h}\left(\frac{\partial F}{\partial \bar{\mathbf{S}}}\otimes \overset{4}{\mathbf{K}}{}^S\,\frac{\partial F}{\partial \bar{\mathbf{S}}}\right) - \\
&- \frac{1}{h + \dfrac{\partial F}{\partial \bar{\mathbf{S}}}\cdot\left(\overset{4}{\mathbf{K}}{}^S\, \dfrac{\partial F}{\partial \bar{\mathbf{S}}}\right)}\left(\frac{\partial F}{\partial \bar{\mathbf{S}}}\otimes \overset{4}{\mathbf{K}}{}^S\,\frac{\partial F}{\partial \bar{\mathbf{S}}}\right) - \\
&- \frac{1}{h}\,\frac{\overset{4}{\mathbf{K}}{}^S\, \dfrac{\partial F}{\partial \bar{\mathbf{S}}}\otimes \overset{4}{\mathbf{K}}{}^S\,\dfrac{\partial F}{\partial \bar{\mathbf{S}}}}{\dfrac{\partial F}{\partial \bar{\mathbf{S}}}\cdot\left(\overset{4}{\mathbf{K}}{}^S\,\dfrac{\partial F}{\partial \bar{\mathbf{S}}}\right)+h}\left(\frac{\partial F}{\partial \bar{\mathbf{S}}}\otimes \overset{4}{\mathbf{K}}{}^S\,\frac{\partial F}{\partial \bar{\mathbf{S}}}\right).
\end{aligned} \tag{9.19}$$

The last three terms in (9.19) can be summed up. The result is that

$$\overset{4}{\mathbf{P}}{}^S\,(\overset{4}{\mathbf{P}}{}^S)^{-1} = \overset{4}{\mathbf{K}}{}^S\,(\overset{4}{\mathbf{K}}{}^S)^{-1} = \overset{4}{\mathbf{I}} \tag{9.20}$$

and that $(\overset{4}{\mathbf{P}}{}^S)^{-1}$ is indeed the fourth-order reverse tensor of $\overset{4}{\mathbf{P}}{}^S$.

Another import feature of $\overset{4}{\mathbf{P}}{}^S$, which is useful for the development of general theorems, will now be proved, namely the positive definiteness of $\overset{4}{\mathbf{P}}{}^S$. The proof of the definiteness of $\overset{4}{\mathbf{P}}{}^S$ is based on the requirement

$$\omega = \overset{4}{\mathbf{P}}{}^S \cdot(\mathbf{A}\otimes\mathbf{A}) > 0, \tag{9.21}$$

whereby $\mathbf{A}$ denotes an arbitrary tensor of second order. With the abriviation

$$L = \frac{\partial F}{\partial \bar{\mathbf{S}}}\cdot\left(\overset{4}{\mathbf{K}}{}^S\,\frac{\partial F}{\partial \bar{\mathbf{S}}}\right) + h \tag{9.22}$$

we can write

$$\omega = \overset{4}{\mathbf{K}}{}^S\cdot(\mathbf{A}\otimes\mathbf{A}) - \frac{1}{L}\left(\overset{4}{\mathbf{K}}{}^S\,\frac{\partial F}{\partial \bar{\mathbf{S}}}\otimes\overset{4}{\mathbf{K}}{}^S\,\frac{\partial F}{\partial \bar{\mathbf{S}}}\right)\cdot(\mathbf{A}\otimes\mathbf{A})\,. \tag{9.23}$$

The quantity L is always positive, if

$$\frac{\partial F}{\partial \bar{\mathbf{S}}} \neq 0 \tag{9.24}$$

and if Poisson's ratio ν^S of the solid fulfils the condition

$$\nu^S \leq \frac{1}{2}\,. \tag{9.25}$$

From (9.23) we obtain

$$\mathrm{L}\omega = \mathrm{L}\,\overset{4}{\mathbf{K}}{}^S \cdot (\mathbf{A} \otimes \mathbf{A}) - \left(\overset{4}{\mathbf{K}}{}^S \, \frac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \overset{4}{\mathbf{K}}{}^S \, \frac{\partial F}{\partial \bar{\mathbf{S}}} \right) \cdot (\mathbf{A} \otimes \mathbf{A}) \tag{9.26}$$

and with (9.22)

$$\begin{aligned} \mathrm{L}\omega &= h\,\overset{4}{\mathbf{K}}{}^S \cdot (\mathbf{A} \otimes \mathbf{A}) - \left(\overset{4}{\mathbf{K}}{}^S \, \frac{\partial F}{\partial \bar{\mathbf{S}}} \otimes \overset{4}{\mathbf{K}}{}^S \, \frac{\partial F}{\partial \bar{\mathbf{S}}} \right) \cdot (\mathbf{A} \otimes \mathbf{A}) + \\ &+ \left[\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \left(\overset{4}{\mathbf{K}}{}^S \, \frac{\partial F}{\partial \bar{\mathbf{S}}} \right) \right] [\overset{4}{\mathbf{K}}{}^S \cdot (\mathbf{A} \otimes \mathbf{A})] \, . \end{aligned} \tag{9.27}$$

We reformulate (9.27)

$$\begin{aligned} \mathrm{L}\omega &= h\,\mathbf{A} \cdot \overset{4}{\mathbf{K}}{}^S\,\mathbf{A} - \left[\left(\overset{4}{\mathbf{K}}{}^S \, \frac{\partial F}{\partial \bar{\mathbf{S}}} \right) \cdot \mathbf{A} \right] \left[\left(\overset{4}{\mathbf{K}}{}^S \, \frac{\partial F}{\partial \bar{\mathbf{S}}} \right) \cdot \mathbf{A} \right] + \\ &+ \left[\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \left(\overset{4}{\mathbf{K}}{}^S \, \frac{\partial F}{\partial \bar{\mathbf{S}}} \right) \right] [(\overset{4}{\mathbf{K}}{}^S\,\mathbf{A}) \cdot \mathbf{A})] \, . \end{aligned} \tag{9.28}$$

Because of the special properties of the parameters h and the elastic material tensor, the first term in (9.28) is always positive. The second and third terms of (9.28) can be reformulated, if one represents the elastic material tensor with the Lamé constant

$$\lambda^S = \frac{2G^S \nu^S}{1 - 2\nu^S} \tag{9.29}$$

in dependence of G^S and ν^S

$$\begin{aligned} \xi = 4{G^S}^2 \Bigg\{ &\left[\left(\overset{4}{\mathbf{I}} + \frac{\nu^S}{1-2\nu^S} \, \overset{4}{\bar{\mathbf{I}}} \right) \frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \frac{\partial F}{\partial \bar{\mathbf{S}}} \right] \left(\overset{4}{\mathbf{I}} + \frac{\nu^S}{1-2\nu^S} \, \overset{4}{\bar{\mathbf{I}}} \right) \mathbf{A} \cdot \mathbf{A} \Bigg\} - \\ &- \left[\left(\overset{4}{\mathbf{I}} + \frac{\nu^S}{1-2\nu^S} \, \overset{4}{\bar{\mathbf{I}}} \right) \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A} \right) \right] \left[\left(\overset{4}{\mathbf{I}} + \frac{\nu^S}{1-2\nu^S} \, \overset{4}{\bar{\mathbf{I}}} \right) \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A} \right) \right] \end{aligned} \tag{9.30}$$

or

$$\xi = 4{G^S}^2 \zeta \, , \tag{9.31}$$

where

$$\begin{aligned} \zeta = \Bigg\{ &\left[\left(\overset{4}{\mathbf{I}} + \frac{\nu^S}{1-2\nu^S} \, \overset{4}{\bar{\mathbf{I}}} \right) \frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \frac{\partial F}{\partial \bar{\mathbf{S}}} \right] \left(\overset{4}{\mathbf{I}} + \frac{\nu^S}{1-2\nu^S} \, \overset{4}{\bar{\mathbf{I}}} \right) \mathbf{A} \cdot \mathbf{A} \Bigg\} - \\ &- \left[\left(\overset{4}{\mathbf{I}} + \frac{\nu^S}{1-2\nu^S} \, \overset{4}{\bar{\mathbf{I}}} \right) \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A} \right) \right] \left[\left(\overset{4}{\mathbf{I}} + \frac{\nu^S}{1-2\nu^S} \, \overset{4}{\bar{\mathbf{I}}} \right) \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A} \right) \right] . \end{aligned} \tag{9.32}$$

If we apply calculation rules for the four-order identity tensors $\overset{4}{\mathbf{I}}$ and $\overset{4}{\bar{\mathbf{I}}} = (\mathbf{I} \otimes \mathbf{I})$ by de Boer (1982) we obtain:

$$\begin{aligned}
\zeta \;=\;& \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \frac{\partial F}{\partial \bar{\mathbf{S}}}\right)(\mathbf{A} \cdot \mathbf{A}) + \\
+\;& \frac{\nu^S}{1 - 2\nu^S}\left[\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \frac{\partial F}{\partial \bar{\mathbf{S}}}\right)(\mathbf{A} \cdot \mathbf{I})(\mathbf{A} \cdot \mathbf{I}) + \right. \\
+\;& \left. \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)(\mathbf{A}) \cdot \mathbf{A})\right] + \\
+\;& \left(\frac{\nu^S}{1 - 2\nu^S}\right)^2 \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)(\mathbf{A} \cdot \mathbf{I}) - \\
-\;& \left\{\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A}\right)\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A}\right) + \right. \\
+\;& \left(\frac{\nu^S}{1 - 2\nu^S}\right)\left[\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A}\right)\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)(\mathbf{A} \cdot \mathbf{I}) + \right. \\
+\;& \left. \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)(\mathbf{A} \cdot \mathbf{I})\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A}\right)\right] + \\
+\;& \left. \left(\frac{\nu^S}{1 - 2\nu^S}\right)^2 \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)(\mathbf{A} \cdot \mathbf{I})\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)(\mathbf{A} \cdot \mathbf{I})\right\} .
\end{aligned} \tag{9.33}$$

This relation (9.33) can be simplified:

$$\begin{aligned}
\zeta \;=\;& \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \frac{\partial F}{\partial \bar{\mathbf{S}}}\right)(\mathbf{A} \cdot \mathbf{A}) + \frac{\nu^S}{1 - 2\nu^S}\left[(\mathbf{A} \cdot \mathbf{I})\frac{\partial F}{\partial \bar{\mathbf{S}}} - \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)\mathbf{A}\right] \times \\
& \times \left[(\mathbf{A} \cdot \mathbf{I})\frac{\partial F}{\partial \bar{\mathbf{S}}} - \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{I}\right)\mathbf{A}\right] - \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A}\right)\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A}\right) .
\end{aligned} \tag{9.34}$$

Since the second term in (9.34) is a quadratic expression, it is always larger or equal to zero, if Ineq. (9.21) is assumed. In consideration of Schwartz's inequality for tensors of second order

$$\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A}\right)\left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \mathbf{A}\right) \leq \left(\frac{\partial F}{\partial \bar{\mathbf{S}}} \cdot \frac{\partial F}{\partial \bar{\mathbf{S}}}\right)(\mathbf{A} \cdot \mathbf{A}) , \tag{9.35}$$

it follows that the scalar quantity ζ as well as $L\omega$ is larger or equal to zero, i. e.:

$$\nu^S \leq \frac{1}{2}, \;\; \mathbf{A} \neq \mathbf{O} : \; \overset{4}{\mathbf{P}^S} \cdot (\mathbf{A} \otimes \mathbf{A}) > 0 . \tag{9.36}$$

In consideration of the stated assumptions in Eq. (9.21) the elasto-plastic material tensor is positive definit and thus, also the inverse tensor $(\overset{4}{\mathbf{P}}{}^{S})^{-1}$.

9.2 General Theorems for Saturated Porous Solids in the Rigid Ideal-Plastic Range

It is well-known that the constitutive equations in the classical linear elasticity fulfils important principles, e. g. the uniqueness theorem of solutions and minimum-maximum principles. In classical plasticity of one-component material on the basis of von Mises fundamental works one treated general principles relatively late. However, today there also exists a well-rounded theory with all of the principles and theorems known from elasticity and even extended principles characteristic for plastic behavior, e. g. limit design theorems.

In the Theory of Porous Media there are only a few contributions which are concerned with the fulfillment of general principles by the constitutive equations of the porous solid. The main reason for this deficit is without doubt the complex structure of the constitutive theory in the plastic range. One has to differentiate at least between two distinct different behaviors of the material, used for porous solid, namely frictional (granular and brittle) and metallic materials. Owing to the complicated constitutive equations for frictional materials in the plastic range (see Section 6.6) there have not been any substantial contribution to the fulfillment of general theorems. Fortunately it could be shown that the plastic behavior of metallic porous bodies can sufficiently be describe by an extended form of von Mises basic constitutive equations (see this Chapter), which are relatively simple and which allow the proof of general theorems. This will be worked out for a special material behavior in this and next section, namely for saturated porous solids in the rigid ideal-plastic state.

a) Preliminaries

The uniqueness theorem and minimum-maximum principles for solutions of boundary value problems considered in this section concerns a rigid ideal-plastic material with its non-capillary intercommunicating void spaces filled with an incompressible liquid. It is assumed that the porous material is isotropic and the pores statistically distributed over the whole space occupied by the medium in question with an isotropic distribution function. We assume the compressibility of the peculiar constituents to be much smaller then the compressibility of the medium as a whole.

The final result is the proof of the uniqueness theorem and minimum-maximum principles.

For the clearness of our investigations we summarize the basic equations needed for the proof.

In the following paragraphs the investigations are restricted to a binary non-capillary model, namely an incompressible porous solid filled with an incompressible liquid, within the geometrically linear theory. From the equations of motion (4.15) and the constraint $(4.17)_4$, neglecting the inertia forces the dynamical equations for the solid and fluid phases are obtained as:

$$\begin{aligned} \operatorname{div} \mathbf{S}^S + \rho^S \mathbf{b} - \hat{\mathbf{p}}^L = \mathbf{0} \ , \\ \operatorname{grad} p^L + \rho^L \mathbf{b} + \hat{\mathbf{p}}^L = \mathbf{0} \ , \end{aligned} \tag{9.37}$$

where

$$p^L = -\, n^L p \tag{9.38}$$

denotes the partial hydrostatic stress state in the liquid, see de Boer (1996) and where for convenience Cauchy's stress tensor is replaced with the symmetric Piola-Kirchhoff tensor..

With the dynamical equations (9.37) we are now able to develop the principle of virtual velocities and of virtual stresses. These principles are useful for the development of minimum and maximum principles (see e.g., de Boer, 1974, de Boer and Ehlers, 1980, de Boer and Kowalski, 1985) and for the proof of general theorems, as, e.g., the uniqueness theorem (see, e.g. de Boer and Kowalski 1986).

In order to derive the principle of virtual velocities it is necessary to introduce the virtual velocities $\bar{\mathbf{v}}_S$ (for the partial solid constituent) and $\bar{\mathbf{v}}_L$ (for the partial fluid constituent). These virtual velocities must be *kinematically admissible* which is defined as follows:

A virtual velocity state is kinematically admissible if it is smooth and satisfies internal and external kinematic constraints.

Note in passing that by internal kinematic constraints such constraints as rigidity, incompressibility or saturation are denoted and by external kinematic constraints the geometric boundary conditions on $\partial \mathcal{B}_v$, where the velocities are prescribed.

Multiplying the dynamic equations (9.37) with the virtual velocities $\bar{\mathbf{v}}_S$ and $\bar{\mathbf{v}}_L$ and integrating over the whole control space $\mathcal{B}$, which is shaped by the skeleton, we obtain for the mixture body

$$\begin{aligned} \int_{\mathcal{B}} [\, (\operatorname{div} \mathbf{S}^S + \rho^S \mathbf{b} - \hat{\mathbf{p}}^L) \cdot \bar{\mathbf{v}}_S + \\ + (\operatorname{grad} p^L + \rho^L \mathbf{b} + \hat{\mathbf{p}}^L) \cdot \bar{\mathbf{v}}_L] dv = 0 \ . \end{aligned} \tag{9.39}$$

Considering the calculating rules

$$\begin{aligned} \operatorname{div} \mathbf{S}^S \cdot \bar{\mathbf{v}}_S &= \operatorname{div}(\mathbf{S}^S \bar{\mathbf{v}}_S) - \mathbf{S}^S \cdot \bar{\dot{\mathbf{E}}}_{SP} \ , \\ \operatorname{grad} p^L \cdot \bar{\mathbf{v}}_L &= \operatorname{div}(p^L \bar{\mathbf{v}}_L) - p^L (\bar{\mathbf{D}}_L \cdot \mathbf{I}) \ , \end{aligned} \tag{9.40}$$

where the superscript dot $\dot{\bar{\mathbf{E}}}_{SP}$ is the material time derivative with respect to the partial solid body. Bearing in mind that the virtual velocities fulfill the geometrical boundary conditions, we obtain with the help of Gauss' theorem

$$\begin{aligned}&\int\limits_{\partial\mathcal{B}_t} (\,\mathbf{s}^S\cdot\bar{\mathbf{v}}_S+\mathbf{t}^L\cdot\bar{\mathbf{v}}_L)da+\int\limits_{\mathcal{B}}(\rho^L\bar{\mathbf{v}}_L+\rho^S\bar{\mathbf{v}}_S)\cdot\mathbf{b}\,dv\\ &=\int\limits_{\mathcal{B}}[\mathbf{S}^S\cdot\dot{\bar{\mathbf{E}}}_{SP}+p^L(\bar{\mathbf{D}}_L\cdot\mathbf{I})-\hat{\mathbf{p}}^L\cdot\bar{\mathbf{v}}_{LS}]dv\;,\end{aligned}\tag{9.41}$$

with

$$\bar{\mathbf{v}}_{LS}=\bar{\mathbf{v}}_L-\bar{\mathbf{v}}_S\;.\tag{9.42}$$

The surface integral in (9.41) covers the area $\partial\mathcal{B}_t$ with the normal unit vector $\mathbf{n}$, where the surface forces

$$\mathbf{s}^S=\mathbf{S}^S\mathbf{n}\;,\quad \mathbf{t}^L=p^L\mathbf{n}\tag{9.43}$$

are prescribed. The tensors $\bar{\mathbf{D}}_L$ and $\dot{\bar{\mathbf{E}}}_{SP}$ stand for the symmetrical parts of the velocity gradients of the virtual velocities $\bar{\mathbf{v}}_L$ and $\bar{\mathbf{v}}_S$.

According to the derivation of the principle of virtual velocities (9.41) the principle of virtual stresses (forces) can be derived. Hereby we are relying on a virtual state of stresses and forces $\hat{\mathbf{S}}^S$, $\hat{p}^L$, $\hat{\mathbf{b}}$ and $\hat{\hat{\mathbf{p}}}^L$, which must be *dynamically admissible*:

A virtual stress state is dynamically admissible if it is smooth and satisfies the internal and external dynamic constraints (dynamic equations, yield functions) and the dynamic boundary conditions on $\partial\mathcal{B}_t$.

We commence from the dynamical equations of motion (9.37) which must be fulfilled by the virtual stress states

$$\begin{aligned}\operatorname{div}\hat{\mathbf{S}}^S+\rho^S\hat{\mathbf{b}}-\hat{\hat{\mathbf{p}}}^L &= \mathbf{0}\;,\\ \operatorname{grad}\hat{\mathbf{p}}^L+\rho^L\hat{\mathbf{b}}+\hat{\hat{\mathbf{p}}}^L &= \mathbf{0}\;.\end{aligned}\tag{9.44}$$

We multiply the equations of motion for the virtual stresses (9.44) with the real partial velocities $\mathbf{v}_S$ and $\mathbf{v}_L$ and integrate over the whole control space. Applying the calculating rules (9.40) and manipulating the virtual work similarly to the aforementioned method we arrive at

$$\begin{aligned}&\int\limits_{\partial\mathcal{B}_v} (\hat{\mathbf{s}}^S\cdot\mathbf{v}_S+\hat{\mathbf{t}}^L\cdot\mathbf{v}_L)da+\int\limits_{\mathcal{B}}(\rho^L\mathbf{v}_L+\rho^S\mathbf{v}_S)\cdot\hat{\mathbf{b}}\,dv\\ &=\int\limits_{\mathcal{B}}[\hat{\mathbf{S}}^S\cdot\dot{\mathbf{E}}_{SP}+\hat{p}^L(\mathbf{D}_L\cdot\mathbf{I})-\hat{\hat{\mathbf{p}}}^L\cdot\mathbf{v}_{LS}]dv\;,\end{aligned}\tag{9.45}$$

whereby $\partial\mathcal{B}_v$ is the part of the boundary, where the velocities are prescribed and where $\mathbf{D}_L$ and $\dot{\mathbf{E}}_{SP}$ are the symmetric parts of the velocity gradients of $\mathbf{v}_\alpha$. The velocity difference $\mathbf{v}_{LS}$ is determined by (9.42) with the real velocities $\mathbf{v}_\alpha$.

Next, we reformulate the principles of virtual work (9.41) and (9.45) with the help of Fig. 9.2.1. From a stress-free placement proceed by, applying external forces to the actual placement. By superposition with a virtual velocity state we arrive at the actual plus virtual state which is marked by an asterix. Now, we substitute the virtual velocities $\bar{\mathbf{v}}_S$ and $\bar{\mathbf{v}}_L$ by the differences of the velocities $\overset{*}{\mathbf{v}}_S - \mathbf{v}_S$ and $\mathbf{v}_L^* - \mathbf{v}_L$, which is immediately recognized from Fig. 9.2.1. With this substitution we obtain a convenient form of the principle of virtual work (9.41) suitable for the derivation of minimum and maximum principles.

$$\begin{aligned}
&\int\limits_{\partial\mathcal{B}_t} [\mathbf{s}^S \cdot (\overset{*}{\mathbf{v}}_S - \mathbf{v}_S) + \mathbf{t}^L \cdot (\overset{*}{\mathbf{v}}_L - \mathbf{v}_L)]da + \\
&+ \int\limits_{\mathcal{B}} [\rho^L(\overset{*}{\mathbf{v}}_L - \mathbf{v}_L) + \rho^S(\overset{*}{\mathbf{v}}_S - \mathbf{v}_S)] \cdot \mathbf{b}\, dv \\
&= \int\limits_{\mathcal{B}} [\mathbf{S}^S \cdot (\dot{\mathbf{E}}_{SP}^* - \dot{\mathbf{E}}_{SP}) + p^L(\overset{*}{\mathbf{D}}_L - \mathbf{D}_L) \cdot \mathbf{I} - \\
&- \hat{\mathbf{p}}^L \cdot (\overset{*}{\mathbf{v}}_{LS} - \mathbf{v}_{LS})]dv\,.
\end{aligned} \tag{9.46}$$

In a similar way, Eq. (9.45) will be reformulated. We substitute the virtual boundary forces and the stresses by the differences of the real and virtual plus actual states. This can be seen in Fig. 9.2.1. By application of a virtual state of forces we obtain the virtual state.

Next, by applying the real force and velocity states, we arrive at the virtual plus actual state which we will mark by a superscript zero. The splitting of the virtual state in a virtual state plus actual state minus actual state in Eq. (9.45) yields a slightly changed form of the principle of virtual forces (see Fig. 9.2.1):

$$\begin{aligned}
&\int\limits_{\partial\mathcal{B}_v} [(\overset{\circ}{\mathbf{s}}{}^S - \mathbf{s}^S) \cdot \mathbf{v}_S \; + \; (\overset{\circ}{\mathbf{t}}{}^L - \mathbf{t}^L) \cdot \mathbf{v}_L]da \\
&= \int\limits_{\mathcal{B}} [(\overset{\circ}{\mathbf{S}}{}^S - \mathbf{S}^S) \cdot \dot{\mathbf{E}}_{SP} \; + \; (\overset{\circ}{p}{}^L - p^L)(\mathbf{D}_L \cdot \mathbf{I}) - \\
&- \; (\overset{\circ}{\hat{\mathbf{p}}}{}^L - \hat{\mathbf{p}}^L) \cdot \mathbf{v}_{LS}]dv\,.
\end{aligned} \tag{9.47}$$

In obtaining (9.47) it is assumed that the body force densities are not changed in the two stress states.

Both versions of the principle of virtual work (9.46) and (9.47) will serve to develop minimum and maximum principles and to prove the uniqueness of the

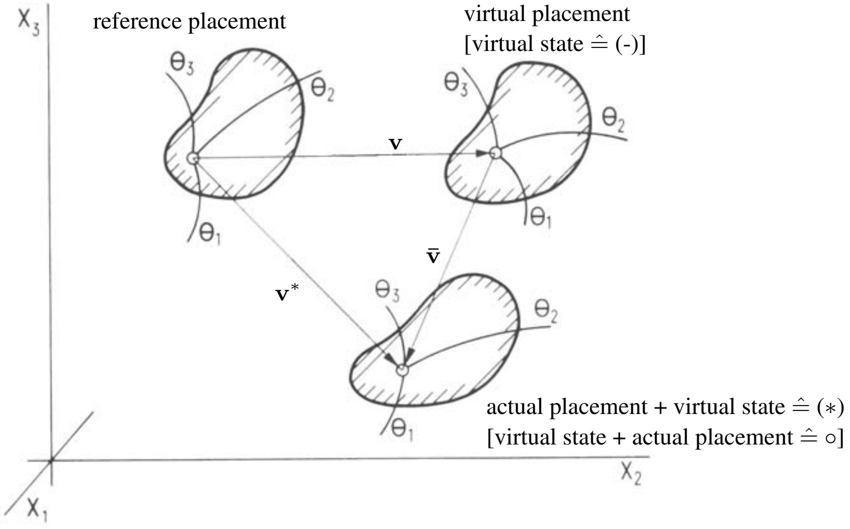

Fig. 9.2.1: The deformation states

solutions of boundary and initial value problems for a certain class of material behavior.

The structure of the constitutive equations is as follows: The constitutive relations for the liquid or gas constituents are governed by weighted (by the volume fraction) porefluid pressures whereas the constitutive equation for the stress tensor of the porous solid constituent consists of two parts, namely of the weighted (by the volume fraction of the solid phase) porefluid pressure and of "effective stresses" determined by the motion of the partial solid phase. The porefluid pressure is indeterminable by constitutive relations if the fluid is incompressible. In the case of a compressible fluid the porefluid pressure is related to the real density if the fluid behaves elastically (see de Boer, 1996). In the sequel the main results of the constitutive theory will be reviewed.

$$\begin{aligned} \mathbf{S}^S &= -n^S p\mathbf{I} + \mathbf{S}^S_E \,, \\ \mathbf{T}^L &= -n^L p\mathbf{I} \,, \\ \hat{\mathbf{p}}^L &= p\,\mathrm{grad}\, n^L + \hat{\mathbf{p}}^L_E \,, \quad \hat{\mathbf{p}}^L_E = -S_L \mathbf{v}_{LS} \,. \end{aligned} \tag{9.48}$$

The interaction force $\hat{\mathbf{p}}^L$ consists also of two parts, the weighted porefluid pressure and an "effective"interaction force which is, close to the mixture equilibrium state, proportional to the relative velocity of both phases, see $(9.48)_4$, where S_L is a positive response constant.

With the constitutive relations (9.48) the principles of virtual work will be revisited. We reformulate the principle of virtual work (virtual velocities) (9.41) with the saturation condition (2.2). After differentiating this condition and applying some manipulations (see de Boer, 2000 a) we attain a virtual velocity state at

$$\bar{\mathbf{D}}_L \cdot \mathbf{I} = -\frac{n^S}{n^L}\,\dot{\bar{\mathbf{E}}}_{SP} \cdot \mathbf{I} - \frac{1}{n^L}\ \operatorname{grad} n^L \cdot \bar{\mathbf{v}}_{LS} \tag{9.49}$$

or by splitting the virtual velocities

$$\begin{aligned}(\overset{*}{\mathbf{D}}_L - \mathbf{D}_L) \cdot \mathbf{I} = \quad & - \quad \frac{n^S}{n^L}(\dot{\mathbf{E}}^*_{SP} - \dot{\mathbf{E}}_{SP}) \cdot \mathbf{I} - \\ & - \quad \frac{1}{n^L}\operatorname{grad} n^L \cdot (\overset{*}{\mathbf{v}}_{LS} - \mathbf{v}_{LS})\ .\end{aligned} \tag{9.50}$$

Eq. (9.49) clearly shows that $\mathbf{D}_L \cdot \mathbf{I}$ or in this case $\bar{\mathbf{D}}_L \cdot \mathbf{I}$ depends on some other kinematic quantities.

Now we substitute in (9.41) the virtual volume change of the partial liquid. This procedure considering $(9.48)_2$ leads to

$$\begin{aligned}&\int_{\mathcal{B}}\{(\mathbf{S}^S\ +\ n^S p\mathbf{I}) \cdot (\dot{\mathbf{E}}^*_{SP} - \dot{\mathbf{E}}_{SP}) + \\ &+ p \operatorname{grad} n^L \cdot (\overset{*}{\mathbf{v}}_{LS} - \mathbf{v}_{LS}) - \hat{\mathbf{p}}^L \cdot (\overset{*}{\mathbf{v}}_{LS} - \mathbf{v}_{LS})\}dv \\ &= \int_{\partial\mathcal{B}_t}[\mathbf{s}^S \cdot (\overset{*}{\mathbf{v}}_S - \mathbf{v}_S) + \mathbf{t}^L \cdot (\overset{*}{\mathbf{v}}_L - \mathbf{v}_L)]da + \\ &+ \int_{\mathcal{B}}[\rho^S(\overset{*}{\mathbf{v}}_S - \mathbf{v}_S) + \rho^L(\overset{*}{\mathbf{v}}_L - \mathbf{v}_L)] \cdot \mathbf{b}\ dv\ .\end{aligned} \tag{9.51}$$

Considering the constitutive equations (9.48) the principle of virtual work (9.51) yields

$$\begin{aligned}&\int_{\mathcal{B}}[\mathbf{S}^S_E \cdot (\dot{\mathbf{E}}^*_{SP} - \dot{\mathbf{E}}_{SP}) -\ \hat{\mathbf{p}}^L_E \cdot (\overset{*}{\mathbf{v}}_{LS} - \mathbf{v}_{LS})]dv \\ &= \int_{\partial\mathcal{B}_t}[\mathbf{s}^S \cdot (\overset{*}{\mathbf{v}}_S - \mathbf{v}_S) + \mathbf{t}^L \cdot (\overset{*}{\mathbf{v}}_L - \mathbf{v}_L)da + \\ &+ \int_{\mathcal{B}}[\rho^S(\overset{*}{\mathbf{v}}_S - \mathbf{v}_S) + \rho^L(\overset{*}{\mathbf{v}}_L - \mathbf{v}_L)] \cdot \mathbf{b}\ dv\ .\end{aligned} \tag{9.52}$$

Relations (9.51) and (9.52) clearly show that the structure of the stress tensor of the solid skeleton contains the effective stress principle of Fillunger and von Terzaghi. It should be emphasized that in order to prove general theorems with the principle of virtual work only such a structure is possible. This is completely in accordance with the results of thermodynamic investigations (see de Boer, 2000 a) and with experimental observations (see Lade and de Boer, 1997). Moreover, the porefluid pressure is related to the change of the real density of the fluid which is proportional to the volume change of the real fluid material. Thus, these investigations confirm the results, recently derived in several contributions (see de Boer, 2000 a).

b) The Uniqueness Theorem for Solutions of Boundary Value Problems

In the sequel the uniqueness of solutions of boundary-value problems will be investigated. We state that the constitutive equations presented at the beginning of Section 9.1 are unique except possibly in the common non-deformed regions of the body. The proof is performed, being based on the virtual work equation and Schwarz' inequality.

We assume that a boundary value problem has two possible stress and velocity states under the same load, namely

$$\overset{1}{\mathbf{S}}{}^S_E \ , \overset{1}{\hat{\mathbf{p}}}{}^L_E, \overset{1}{\mathbf{v}}_S, \overset{1}{\mathbf{v}}_L \quad \text{and} \quad \overset{2}{\mathbf{S}}{}^S_E \ , \overset{2}{\hat{\mathbf{p}}}{}^L_E, \overset{2}{\mathbf{v}}_S, \overset{2}{\mathbf{v}}_L \ . \tag{9.53}$$

From the virtual work equation being expressed for the two possible stress and velocity states and after subtracting one from the other, we obtain, see the derivation of Eq. (9.52):

$$\begin{aligned} \int_{\mathcal{B}} [(\overset{1}{\mathbf{S}}{}^S_E - \overset{2}{\mathbf{S}}{}^S_E) \cdot (\overset{1}{\dot{\mathbf{E}}}_{SP} - \overset{2}{\dot{\mathbf{E}}}_{SP}) - \\ - (\overset{1}{\hat{\mathbf{p}}}{}^L_E - \overset{2}{\hat{\mathbf{p}}}{}^L_E) \cdot (\overset{1}{\mathbf{v}}_{LS} - \overset{2}{\mathbf{v}}_{LS})] dv = 0 \ . \end{aligned} \tag{9.54}$$

If the solutions of the boundary-value problem are not unique, the integrand in (9.54) must vanish. Otherwise, if this is not the case, the uniqueness of the solutions of boundary value problems in the rigid ideal plastic state is proved.

With the constitutive equation for $\hat{\mathbf{p}}^L_E$ $(9.48)_4$ it is immediately recognized that

$$-(\overset{1}{\hat{\mathbf{p}}}{}^L_E - \overset{2}{\hat{\mathbf{p}}}{}^L_E) \cdot (\overset{1}{\mathbf{v}}_{LS} - \overset{2}{\mathbf{v}}_{LS}) \geq 0 \ . \tag{9.55}$$

The equality sign is only valid if $\overset{1}{\mathbf{v}}_{LS} = \overset{2}{\mathbf{v}}_{LS}$, i.e. if the velocity states are unique.

We now show that the first expression in (9.54) is also non-negative, i.e.

$$I = (\overset{1}{\mathbf{S}}{}^S_E - \overset{2}{\mathbf{S}}{}^S_E) \cdot (\overset{1}{\dot{\mathbf{E}}}_{SP} - \overset{2}{\dot{\mathbf{E}}}_{SP}) \geq 0 \ . \tag{9.56}$$

In order to prove (9.56), it is convenient to use the inverse form of the constitutive equation (9.5). After some manipulations

$$\mathbf{S}_E^S = \frac{2\kappa}{\dot{\lambda}}[\dot{\mathbf{E}}_{SP}^D + (\dot{\mathbf{E}}_{SP} \cdot \mathbf{I}/9\alpha^2)\mathbf{I}] \tag{9.57}$$

is obtained. With the constitutive relation (9.57) the expression I in (9.56) takes the form whereby we have put the factor 2κ into $\dot{\lambda}$ and $\overset{1}{\dot{\lambda}}$ and $\overset{2}{\dot{\lambda}}$ respectively:

$$\begin{aligned} I = \; & \Big\{ \frac{1}{\overset{1}{\dot{\lambda}}} \Big[\overset{1}{\dot{\mathbf{E}}}{}_{SP}^D + (\overset{1}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}/9\alpha^2)\mathbf{I} \Big] - \\ & - \frac{1}{\overset{2}{\dot{\lambda}}} \Big[\overset{2}{\dot{\mathbf{E}}}{}_{SP}^D + \Big(\overset{2}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}/9\alpha^2 \Big)\mathbf{I} \Big] \Big\} \cdot (\overset{1}{\dot{\mathbf{E}}}_{SP} - \overset{2}{\dot{\mathbf{E}}}_{SP}) \end{aligned} \tag{9.58}$$

or

$$\begin{aligned} I = \; & \frac{1}{\overset{1}{\dot{\lambda}}} \Big[\overset{1}{\dot{\mathbf{E}}}{}_{SP}^D \cdot \overset{1}{\dot{\mathbf{E}}}{}_{SP}^D + (\overset{1}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}/3\alpha)^2] + \\ + \quad & \frac{1}{\overset{2}{\dot{\lambda}}} [\overset{2}{\dot{\mathbf{E}}}{}_{SP}^D \cdot \overset{2}{\dot{\mathbf{E}}}{}_{SP}^D + (\overset{2}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}/3\alpha)^2 \Big] - \\ - \quad & \Big[\overset{1}{\dot{\mathbf{E}}}{}_{SP}^D \cdot \overset{2}{\dot{\mathbf{E}}}{}_{SP}^D + (\overset{1}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}/9\alpha^2)(\overset{2}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}) \Big] \Big(\frac{1}{\overset{1}{\dot{\lambda}}} + \frac{1}{\overset{2}{\dot{\lambda}}} \Big) \, . \end{aligned} \tag{9.59}$$

The last term in (9.59) can be written as

$$\begin{aligned} A \; & = \Big[\overset{1}{\dot{\mathbf{E}}}{}_{SP}^D \cdot \overset{2}{\dot{\mathbf{E}}}{}_{SP}^D + (\overset{1}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}/9\alpha^2)(\overset{2}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}) \Big] \left(\frac{1}{\overset{1}{\dot{\lambda}}} + \frac{1}{\overset{2}{\dot{\lambda}}} \right) = \\ & = \Big[\overset{1}{\dot{\mathbf{E}}}{}_{SP}^D + \frac{1}{\sqrt{3}} (\overset{1}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}/3\alpha)\mathbf{I} \Big] \cdot \Big[\overset{2}{\dot{\mathbf{E}}}{}_{SP}^D + \frac{1}{\sqrt{3}} (\overset{2}{\dot{\mathbf{E}}}_{SP} \cdot \mathbf{I}/3\alpha)\mathbf{I} \Big] \times \\ & \quad \times \left(\frac{1}{\overset{1}{\dot{\lambda}}} + \frac{1}{\overset{2}{\dot{\lambda}}} \right) . \end{aligned} \tag{9.60}$$

Using Schwarz' inequality and considering (9.13) we can state that

$$A < \frac{1}{2} (\overset{1}{\dot{\lambda}} + \overset{2}{\dot{\lambda}}) \, . \tag{9.61}$$

Moreover, the expression I (9.59) turns out in consideration of (9.60) and (9.13) to be

$$I = \frac{1}{2}(\overset{1}{\dot{\lambda}} + \overset{2}{\dot{\lambda}}) - A \tag{9.62}$$

and with the result (9.61)

$$I \geq 0 \, . \tag{9.63}$$

Thus the statement (9.56) is proven, i.e. the integral (9.54) vanishes only if the uniqueness of the boundary-value problem is fulfilled.

We now assume that the stress state $\overset{2}{\mathbf{S}}{}^S_E$ does not plastify the solid matrix, however, the stress state $\overset{1}{\mathbf{S}}{}^S_E$ does. Then we have

$$\overset{1}{\dot{\mathbf{E}}}_{SP} \neq \mathbf{O} \, , \quad \overset{2}{\dot{\mathbf{E}}}_{SP} = \mathbf{O} \, , \tag{9.64}$$

and

$$\begin{aligned} \overset{1}{\mathbf{S}}{}^{SD}_E \cdot \overset{1}{\mathbf{S}}{}^{SD}_E \quad &+ \quad \alpha^2(\overset{1}{\mathbf{S}}{}^S_E \cdot \mathbf{I})^2 = 2\kappa^2 \, , \\ \overset{2}{\mathbf{S}}{}^{SD}_E \cdot \overset{2}{\mathbf{S}}{}^{SD}_E \quad &+ \quad \alpha^2(\overset{2}{\mathbf{S}}{}^S_E \cdot \mathbf{I})^2 \, \leq \, 2\kappa^2 \, . \end{aligned} \tag{9.65}$$

In the sequel it will be shown that such two states of stress and velocity for given boundary conditions are impossible. For this purpose we prove that the integrand in (9.54) is different from zero, namely always positive

$$(\overset{1}{\mathbf{S}}{}^S_E - \overset{2}{\mathbf{S}}{}^S_E) \cdot \overset{1}{\dot{\mathbf{E}}}_{SP} \geq \, 0 \, , \tag{9.66}$$

if $\overset{1}{\mathbf{S}}{}^S_E \neq \overset{2}{\mathbf{S}}{}^S_E$.

With the flow rule for $\dot{\mathbf{E}}_{SP}$, namely (9.5) we have

$$\begin{aligned} & \frac{1}{2\kappa}(\overset{1}{\mathbf{S}}{}^S_E - \overset{2}{\mathbf{S}}{}^S_E) \cdot \overset{1}{\dot{\lambda}} \, [\overset{1}{\mathbf{S}}{}^S_E + \alpha^2(\overset{1}{\mathbf{S}}{}^S_E \cdot \mathbf{I})\mathbf{I}] \\ = \quad & \frac{\overset{1}{\dot{\lambda}}}{2\kappa}\{2\kappa^2 - [\overset{1}{\mathbf{S}}{}^{SD}_E \cdot \overset{2}{\mathbf{S}}{}^{SD}_E + \alpha^2(\overset{1}{\mathbf{S}}{}^S_E \cdot \mathbf{I})(\overset{2}{\mathbf{S}}{}^S_E \cdot \mathbf{I})]\} \, , \end{aligned} \tag{9.67}$$

where the yield condition (9.3) has been considered in (9.67) neglecting all hardening effects.

Again, using Schwarz' inequality and considering the conditions $(9.65)_{1,2}$ it can be shown that the last term in (9.67) is always less than $2\kappa^2$:

$$\overset{1}{\mathbf{S}}{}^S_E \cdot \overset{2}{\mathbf{S}}{}^{SD}_E + \alpha^2(\overset{1}{\mathbf{S}}{}^{SD}_E \cdot \mathbf{I})(\overset{2}{\mathbf{S}}{}^S_E \cdot \mathbf{I}) < 2\kappa^2 \, . \tag{9.68}$$

The equality sign in (9.66) is only given if

$$\overset{1}{\mathbf{S}}{}^S_E = \overset{2}{\mathbf{S}}{}^S_E \quad . \tag{9.69}$$

For two different stress states $\overset{1}{\mathbf{S}}{}^S_E$ and $\overset{2}{\mathbf{S}}{}^S_E$ such as

$$\begin{aligned} \overset{1}{\mathbf{S}}{}^S_E \cdot \overset{1}{\mathbf{S}}{}^S_E \quad &+ \quad \alpha^2(\overset{1}{\mathbf{S}}{}^S_E \cdot \mathbf{I})^2 < 2\kappa^2 \ , \\ \overset{2}{\mathbf{S}}{}^{SD}_E \cdot \overset{2}{\mathbf{S}}{}^{SD}_E \quad &+ \quad \alpha^2(\overset{2}{\mathbf{S}}{}^S_E \cdot \mathbf{I})^2 < 2\kappa^2 \end{aligned} \tag{9.70}$$

we have $\overset{1}{\dot{\mathbf{E}}}_{SP} = \mathbf{O}$ and $\overset{2}{\dot{\mathbf{E}}}_{SP} = \mathbf{O}$. The Eq. (9.54) is satisfied by $\overset{1}{\mathbf{v}}_{LS} = \overset{2}{\mathbf{v}}_{LS}$ even if $\overset{1}{\mathbf{S}}{}^S_E \neq \overset{2}{\mathbf{S}}{}^S_E$. Thus, in rigid regions the uniqueness of the solutions of boundary-value problems can be violated.

The uniqueness theorem for solutions of boundary-value problems for a rigid ideal-plastic porous solid filled with a fluid can thus be stated as follows:

If two or more complete solutions of a boundary-value problem can be determined, then the stress fields of the solutions are identical except possibly in the common non-deformed regions.

c) Minimum and Maximum Principles for Rigid Ideal-Plastic Behavior

Exact solutions of boundary and initial value problems are very difficult to obtain in the porous media theory. It is therefore desirable to succeed in developing minimum and maximum principles, which permit the boundary or initial value problems to be solved approximately. In the sequel this will be performed for rigid ideal-plastic behavior of the solid phase and elastic compressible and incompressible behavior of the fluid phase in a binary porous media model.

The minimum principle for velocities

To begin with, we develop the minimum principle for velocities. As usual in the plasticity theory we define a kinematically admissible smooth velocity field:

A smooth velocity field is denoted as kinematically admissible, if it fulfills the internal and external constraints.

We consider the term:

$$\begin{aligned} \overset{*}{H} = &\int_{\mathcal{B}} \overset{*}{\mathbf{S}}{}^S_E \cdot \dot{\mathbf{E}}^*_{SP} \, dv - \frac{1}{2} \int_{\mathcal{B}} \overset{*}{\hat{\mathbf{p}}}{}^L_E \cdot \overset{*}{\mathbf{v}}_{LS} \, dv - \\ - &\int_{\partial \mathcal{B}_t} (\mathbf{s}^S \cdot \overset{*}{\mathbf{v}}_S + \mathbf{t}^L \cdot \overset{*}{\mathbf{v}}_L) da - \int_{\mathcal{B}} (\rho^S \overset{*}{\mathbf{v}}_S + \rho^L \overset{*}{\mathbf{v}}_L) \cdot \mathbf{b} \, dv \ , \end{aligned} \tag{9.71}$$

where $\overset{*}{\mathbf{S}}{}_E^S$ fulfils (9.3) in connection with (9.5) and $\overset{*}{\hat{\mathbf{p}}}{}_E^L$ $(9.48)_4$ and state that *Out of all kinematically admissible velocity fields the really existent field makes the term (9.71) a minimum.*

In order to prove the above theorem we consider the term $\overset{*}{H} - H$, whereby H represents (9.71) which includes the real velocities. If the theorem is correct, $\overset{*}{H} - H$ must always be positive, i.e.

$$\begin{aligned} \overset{*}{H} - H = & \int_{\mathcal{B}} (\overset{*}{\mathbf{S}}{}_E^S \cdot \dot{\mathbf{E}}_{SP}^* - \mathbf{S}_E^S \cdot \dot{\mathbf{E}}_{SP}) dv - \\ & - \frac{1}{2} \int_{\mathcal{B}} (\overset{*}{\hat{\mathbf{p}}}{}_E^L \cdot \overset{*}{\mathbf{v}}_{LS} - \hat{\mathbf{p}}_E^L \cdot \mathbf{v}_{LS})\, dv - \\ & - \int_{\partial\mathcal{B}_t} [\mathbf{s}^S \cdot (\overset{*}{\mathbf{v}}_S - \mathbf{v}_S) + \mathbf{t}^L \cdot (\overset{*}{\mathbf{v}}_L - \mathbf{v}_L) da - \\ & - \int_{\mathcal{B}} [\rho^S (\overset{*}{\mathbf{v}}_S - \mathbf{v}_S) + \rho^L (\overset{*}{\mathbf{v}}_L - \mathbf{v}_L) \cdot \mathbf{b}\, dv \geq 0 \,. \end{aligned} \tag{9.72}$$

We replace in (9.72) the integrals over the surface and body forces by the principle of virtual velocities (9.52) and obtain

$$\begin{aligned} \overset{*}{H} - H = & \int_{\mathcal{B}} \{ \overset{*}{\mathbf{S}}{}_E^S \cdot \dot{\mathbf{E}}_{SP}^* - \mathbf{S}_E^S \cdot \dot{\mathbf{E}}_{SP} - \mathbf{S}_E^S \cdot (\dot{\mathbf{E}}_{SP}^* - \dot{\mathbf{E}}_{SP}) - \\ & - [\frac{1}{2} \overset{*}{\hat{\mathbf{p}}}{}_E^L \cdot \overset{*}{\mathbf{v}}_{LS} - \frac{1}{2} \hat{\mathbf{p}}_E^L \cdot \mathbf{v}_{LS} - \hat{\mathbf{p}}_E^L \cdot (\overset{*}{\mathbf{v}}_{LS} - \mathbf{v}_{LS})]\} dv \geq 0 \,. \end{aligned} \tag{9.73}$$

In order to prove the validity of inequality (9.73) it is necessary to calculate the following stress powers first, considering the constitutive equations (9.3) – (9.5) as well as $(9.48)_4$. It is

$$\begin{aligned} \mathbf{S}_E^S \cdot \dot{\mathbf{E}}_{SP} &= \sqrt{2}\kappa \sqrt{\dot{\mathbf{E}}_{SP}^D \cdot \dot{\mathbf{E}}_{SP}^D + (\dot{\mathbf{E}}_{SP} \cdot \mathbf{I}/3\alpha)^2} \,, \\ \overset{*}{\mathbf{S}}{}_E^S \cdot \dot{\mathbf{E}}_{SP}^* &= \sqrt{2}\kappa \sqrt{\dot{\mathbf{E}}_{SP}^{D*} \cdot \dot{\mathbf{E}}_{SP}^{D*} + \left(\dot{\mathbf{E}}_{SP}^* \cdot \mathbf{I}/3\alpha\right)^2} \,, \end{aligned} \tag{9.74}$$

$$\begin{aligned} \hat{\mathbf{p}}_E^L \cdot \mathbf{v}_{LS} &= - S_L \mathbf{v}_{LS} \cdot \mathbf{v}_{LS} \,, \\ \overset{*}{\hat{\mathbf{p}}}{}_E^L \cdot \overset{*}{\mathbf{v}}_{LS} &= - S_L \overset{*}{\mathbf{v}}_{LS} \cdot \overset{*}{\mathbf{v}}_{LS} \,. \end{aligned} \tag{9.75}$$

Moreover, it can easily be shown with Schwartz' inequality (see de Boer and Kowalski, 1985) that

$$\begin{aligned}\mathbf{S}^S_E \cdot (\dot{\mathbf{E}}^*_{SP} - \dot{\mathbf{E}}_{SP}) \quad &\leq \sqrt{2}\kappa\Bigg[\sqrt{\dot{\mathbf{E}}^{D*}_{SP} \cdot \dot{\mathbf{E}}^{D*}_{SP} + (\dot{\mathbf{E}}^*_{SP} \cdot \mathbf{I}/3\alpha)^2} - \\ &\qquad - \sqrt{\dot{\mathbf{E}}^{D}_{SP} \cdot \dot{\mathbf{E}}^{D}_{SP} + (\dot{\mathbf{E}}_{SP} \cdot \mathbf{I}/\mathbf{3}\alpha)^2}\,\Bigg]\end{aligned} \tag{9.76}$$

and

$$\hat{\mathbf{p}}^L_E \cdot (\overset{*}{\mathbf{v}}_{LS} - \mathbf{v}_{LS}) > \frac{1}{2} S_L \, \overset{*}{\mathbf{v}}_{LS} \cdot \overset{*}{\mathbf{v}}_{LS} - \frac{1}{2} S_L \mathbf{v}_{LS} \cdot \mathbf{v}_{LS} \; . \tag{9.77}$$

Thus, with (9.76) and (9.77) it has been proven that $\overset{*}{H} - H$ in (9.73) is always positive, provided that $\dot{\mathbf{E}}_{SP}$ is different from zero.

If in a part of the continuum $\dot{\mathbf{E}}_{SP}$ is equal to zero the relation for $\overset{*}{H} - H$ in (9.73) reduces to

$$\begin{aligned}\overset{*}{H} - H \quad &= \int_{\mathcal{B}} \{\overset{*}{\mathbf{S}}{}^S_E \cdot \dot{\mathbf{E}}^*_{SP} - \mathbf{S}^S_E \cdot \dot{\mathbf{E}}^*_{SP} - [\frac{1}{2}\, \overset{*}{\hat{\mathbf{p}}}{}^L_E \cdot \overset{*}{\mathbf{v}}_{LS} \; - \\ &\quad - \frac{1}{2} \hat{\mathbf{p}}^L_E \cdot \mathbf{v}_{LS} - \hat{\mathbf{p}}^L_E \cdot (\overset{*}{\mathbf{v}}_{LS} - \mathbf{v}_{LS})]\} dv \; .\end{aligned} \tag{9.78}$$

Now, the second term in (9.78) will be investigated. Applying (9.5) to $\dot{\mathbf{E}}^*_{SP}$ we obtain

$$\mathbf{S}^S_E \cdot \dot{\mathbf{E}}^*_{SP} = \overset{*}{\dot{\lambda}} \frac{1}{2\kappa} [\mathbf{S}^{SD}_E + \frac{1}{3} (\mathbf{S}^S_E \cdot \; \mathbf{I})\mathbf{I}] \cdot [\overset{*}{\mathbf{S}}{}^{SD}_E + \alpha^2 (\overset{*}{\mathbf{S}}{}^S_E \; \cdot \mathbf{I})\mathbf{I}] \tag{9.79}$$

or with Schwarz' inequality and (9.3) as well as (9.13)

$$\mathbf{S}^S_E \cdot \dot{\mathbf{E}}^*_{SP} \leq \sqrt{2}\kappa \sqrt{\dot{\mathbf{E}}^{D*}_{SP} \cdot \dot{\mathbf{E}}^{D*}_{SP} + (\dot{\mathbf{E}}^*_{SP} \cdot \mathbf{I}/3\alpha)^2} \; . \tag{9.80}$$

Considering (9.77) and (9.80) it is easily recognized that always

$$\overset{*}{H} - H \geq 0 \; . \tag{9.81}$$

Thus, we have proved the statement that the real velocity field makes the term $\overset{*}{H}$ in (9.71) a minimum.

The expression for H can be slightly reformulated if we consider the rate of mechanical work of the real state

$$\begin{aligned}&\int_{\mathcal{B}} (\mathbf{S}^S_E \cdot \dot{\mathbf{E}}_{SP} - \hat{\mathbf{p}}^L_E \cdot \mathbf{v}_{LS}) dv - \int_{\partial\mathcal{B}_t} (\mathbf{s}^S \cdot \mathbf{v}_S + \mathbf{t}^L \cdot \mathbf{v}_L) da - \\ &- \int_{\partial\mathcal{B}_v} (\mathbf{s}^S \cdot \mathbf{v}_S + \mathbf{t}^L \cdot \mathbf{v}_L) da - \int_{\mathcal{B}} (\rho^S \mathbf{v}_S + \rho^L \mathbf{v}_L) \cdot \mathbf{b} \; dv = 0 \; .\end{aligned} \tag{9.82}$$

If we insert Eq. (9.82) into H (9.71) we can express H in the following way:

$$H = \frac{1}{2} \int\limits_{\mathcal{B}} \hat{\mathbf{p}}^L_E \cdot \mathbf{v}_{LS}\, dv + \int\limits_{\partial \mathcal{B}_v} (\mathbf{s}^S \cdot \mathbf{v}_S + \mathbf{t}^L \cdot \mathbf{v}_L) da \; . \tag{9.83}$$

With (9.83) we can construct a modified minimum problem

$$\begin{aligned}
&\int\limits_{\mathcal{B}} (\overset{*}{\mathbf{S}}{}^S_E \cdot \dot{\mathbf{E}}^*_{SP}\, dv - \frac{1}{2} \int\limits_{\mathcal{B}} \overset{*}{\hat{\mathbf{p}}}{}^L_E \cdot \overset{*}{\mathbf{v}}_{LS}\, dv - \\
&- \int\limits_{\partial \mathcal{B}_t} (\mathbf{s}^S \cdot \overset{*}{\mathbf{v}}_S + \mathbf{t}^L \cdot \overset{*}{\mathbf{v}}_L) da - \int\limits_{\mathcal{B}} (\rho^S \mathbf{v}_S + \rho^L \mathbf{v}_L) \cdot \mathbf{b}\, dv \geq \\
&\geq \frac{1}{2} \int\limits_{\mathcal{B}} \hat{\mathbf{p}}^L_E \cdot \mathbf{v}_{LS} dv + \int\limits_{\partial \mathcal{B}_v} (\mathbf{s}^S \cdot \mathbf{v}_S + \mathbf{t}^L \cdot \mathbf{v}_L) da \; .
\end{aligned} \tag{9.84}$$

This form of the minimum problem will be used to derive a restriction theorem in connection with the maximum problem which will be developed in the next section.

The maximum principle for stresses

At first we define a *dynamically admissible stress field*:

A smooth stress field is denoted as dynamically admissible (marked by a superscript o) if it fulfils the boundary and dynamic conditions and does not violate the yield condition (9.3).

We commence from the expression:

$$\overset{\circ}{L} = \int\limits_{\partial \mathcal{B}_v} (\overset{\circ}{\mathbf{s}}{}^S_E \cdot \mathbf{v}_S + \overset{\circ}{\mathbf{t}}{}^L \cdot \mathbf{v}_L) da + \frac{1}{2} \int\limits_{\mathcal{B}} \overset{\circ}{\hat{\mathbf{p}}}{}^L_E \cdot \overset{\circ}{\mathbf{v}}_{LS}\, dv \quad , \tag{9.85}$$

whereby $\overset{\circ}{\mathbf{v}}_{LS}$ obeys the constitutive equation $(9.48)_4$, and prove the theorem:

Out of all possible dynamically admissible stress fields the really existent one makes the expression (9.85) a maximum.

In this case, the difference $L - \overset{\circ}{L}$ (where L represents (9.85) with the real stress field) must always be positive, i.e.

$$\begin{aligned}
L - \overset{\circ}{L} = &\int\limits_{\partial \mathcal{B}_v} [(\mathbf{s}^S - \overset{\circ}{\mathbf{s}}{}^S) \cdot \mathbf{v}_S + (\overset{\circ}{\mathbf{t}}{}^L - \mathbf{t}^L) \cdot \mathbf{v}_L] da + \\
&+ \frac{1}{2} \int\limits_{\mathcal{B}} (\hat{\mathbf{p}}^L_E \cdot \mathbf{v}_{LS} - \overset{\circ}{\hat{\mathbf{p}}}{}^L_E \cdot \overset{\circ}{\mathbf{v}}_{LS}) dv \geq 0 \; .
\end{aligned} \tag{9.86}$$

The difference $L-\overset{\circ}{L}$ in (9.86) can be easily reformulated with the aid of the principle of virtual stresses (forces) (9.47):

$$\begin{aligned} L-\overset{\circ}{L}=\int_{\mathcal{B}}[(\mathbf{S}_E^S-\overset{\circ}{\mathbf{S}}{}_E^S)\cdot\dot{\mathbf{E}}_{SP}+\frac{1}{2}(\hat{\mathbf{p}}_E^L\cdot\mathbf{v}_{LS}-\overset{\circ}{\hat{\mathbf{p}}}{}_E^L\cdot\overset{\circ}{\mathbf{v}}_{LS})-\\ -(\hat{\mathbf{p}}_E^L-\overset{\circ}{\hat{\mathbf{p}}}{}_E^L)\cdot\mathbf{v}_{LS}]dv \end{aligned} \tag{9.87}$$

or considering the constitutive equation for $\hat{\mathbf{p}}_E^L$ and $\overset{\circ}{\hat{\mathbf{p}}}{}_E^L$, respectively, in (9.87)

$$L-\overset{\circ}{L}=\int_{\mathcal{B}}[(\mathbf{S}_E^S-\overset{\circ}{\mathbf{S}}{}_E^S)\cdot\dot{\mathbf{E}}_{SP}-\frac{1}{2}(\hat{\mathbf{p}}_E^L-\overset{\circ}{\hat{\mathbf{p}}}{}_E^L)\cdot(\mathbf{v}_{LS}-\overset{\circ}{\mathbf{v}}_{LS})]dv \tag{9.88}$$

is obtained. The second term on the right-hand side of (9.88) is always positive as it can easily be proven considering $(9.48)_4$. In order to show that also the first term on the right-hand side of (9.88) is always positive we reformulate this term using the constitutive equation (9.5)

$$\begin{aligned} (\mathbf{S}_E^S-\overset{\circ}{\mathbf{S}}{}_E^S)\cdot\dot{\mathbf{E}}_{SP} &= \dot{\lambda}\frac{1}{2\kappa}\{\mathbf{S}_E^S\cdot[\mathbf{S}_E^{S^D}+\alpha^2(\mathbf{S}_E^S\cdot\mathbf{I})\mathbf{I}]-\\ &\quad -\overset{\circ}{\mathbf{S}}{}_E^S\cdot[\mathbf{S}_E^{S^D}+\alpha^2(\mathbf{S}_E^S\cdot\mathbf{I})\mathbf{I}]\} \end{aligned} \tag{9.89}$$

or considering the yield condition (9.3) and after some manipulations we get:

$$\begin{aligned} (\mathbf{S}_E^S-\overset{\circ}{\mathbf{S}}{}_E^S)\cdot\dot{\mathbf{E}}_{SP} &= \dot{\lambda}\{2\kappa^2-[\overset{\circ}{\mathbf{S}}{}_E^{S^D}+\\ &+\alpha\frac{1}{\sqrt{3}}(\overset{\circ}{\mathbf{S}}{}_E^S\cdot\mathbf{I})\mathbf{I}]\cdot[\mathbf{S}_E^{S^D}+\alpha\frac{1}{\sqrt{3}}(\mathbf{S}_E^S\cdot\mathbf{I})\mathbf{I}]\}\,. \end{aligned} \tag{9.90}$$

As both stress states in (9.90) do not exceed the yield limit one recognizes using Schwarz' inequality that always

$$[\overset{\circ}{\mathbf{S}}{}_E^{S^D}+\alpha\frac{1}{\sqrt{3}}(\overset{\circ}{\mathbf{S}}{}_E^S\cdot\mathbf{I})\mathbf{I}]\cdot[\mathbf{S}_E^{S^D}+\alpha\frac{1}{\sqrt{3}}(\mathbf{S}_E^S\cdot\mathbf{I})\mathbf{I}]\le 2\kappa^2\,. \tag{9.91}$$

Therefore, the integrand in (9.88) is positive because $\dot{\lambda}$ is assumed to be positive. Thus, the theorem (9.86) has been proved. Hence, because $\overset{\circ}{L}\le L$, the inequality

$$\begin{aligned} \int_{\partial\mathcal{B}_v}(\overset{\circ}{\mathbf{s}}{}^S\cdot\mathbf{v}_S &+ \overset{\circ}{\mathbf{t}}{}^L\cdot\mathbf{v}_L)da+\frac{1}{2}\int_{\mathcal{B}}\overset{\circ}{\hat{\mathbf{p}}}{}_E^L\cdot\overset{\circ}{\mathbf{v}}_{LS}\,dv\le\\ &\le \int_{\partial B_v}(\mathbf{s}^S\cdot\mathbf{v}_S+\mathbf{t}^L\cdot\mathbf{v}_L)da+\frac{1}{2}\int_{\mathcal{B}}\hat{\mathbf{p}}_E^L\cdot\mathbf{v}_{LS}\,dv \end{aligned} \tag{9.92}$$

is valid.

Restriction statement

With the result gained in the last two sections the real stress and velocity states can be restricted. Considering (9.81), (9.92) and (9.84) we have

$$\overset{*}{H} \geq H = L > \overset{\circ}{L}\ , \quad \text{i.e.} \tag{9.93}$$

$$\begin{aligned}
&\int\limits_{\mathcal{B}} \overset{*}{\mathbf{S}}{}^{S}_{E} \cdot \dot{\mathbf{E}}^{*}_{SP}\, dv - \frac{1}{2} \int\limits_{\mathcal{B}} \overset{*}{\hat{\mathbf{p}}}{}^{L}_{E} \cdot \overset{*}{\mathbf{v}}_{LS}\, dv - \int\limits_{\partial\mathcal{B}_t} (\mathbf{s}^S \cdot \overset{*}{\mathbf{v}}_S + \mathbf{t}^L \cdot \overset{*}{\mathbf{v}}_L) da - \\
&- \int\limits_{\mathcal{B}} (\rho^S\, \overset{*}{\mathbf{v}}_S + \rho^L\, \overset{*}{\mathbf{v}}_L) \cdot \mathbf{b}\, dv \\
&\geq \int\limits_{\mathcal{B}} \mathbf{S}^S_E \cdot \dot{\mathbf{E}}_{SP}\, dv - \frac{1}{2} \int\limits_{\mathcal{B}} \hat{\mathbf{p}}^L_E \cdot \mathbf{v}_{LS}\, dv - \\
&- \int\limits_{\partial\mathcal{B}_t} (\mathbf{s}^S \cdot \mathbf{v}_S + \mathbf{t}^L \cdot \mathbf{v}_L) da - \int\limits_{\mathcal{B}} (\rho^S \mathbf{v}_S + \rho^L \mathbf{v}_L) \cdot \mathbf{b}\, dv \\
&= \frac{1}{2} \int\limits_{\mathcal{B}} \hat{\mathbf{p}}^L_E \cdot \mathbf{v}_{LS}\, dv + \int\limits_{\partial\mathcal{B}_v} (\mathbf{s}^S \cdot \mathbf{v}_S + \mathbf{t}^L \cdot \mathbf{v}_L) da \\
&\geq \frac{1}{2} \int\limits_{\mathcal{B}} \overset{\circ}{\hat{\mathbf{p}}}{}^{L}_{E} \cdot \overset{\circ}{\mathbf{v}}_{LS}\, dv + \int\limits_{\partial\mathcal{B}_v} (\overset{\circ}{\mathbf{s}}{}^{S} \cdot \mathbf{v}_S + \overset{\circ}{\mathbf{t}}{}^{L} \cdot \mathbf{v}_L) da\ .
\end{aligned} \tag{9.94}$$

If the interacting forces $\hat{\mathbf{p}}^L_E$ vanish and if incompressibility is assumed, the well-known theorems of classical plasticity theory are recognized (see Prager and Hodge, 1954). Again, it should be emphasized that (9.94) can serve to restrict the real stress and velocity states in the case of rigid-ideal plastic behavior. However, there is less experience in this field.

For *elastic-plastic hardening behavior* corresponding principles can be proved. In the elastic hardening range we can describe the material behavior with the constitutive equations (9.14) and (9.16) in consideration of (9.15) and (9.17). Since these constitutive equations are formulated in the rates it is advisable to develop the principles of virtual work in the rates. Then, it is possible to treat general principles, such as minimum-maximum problems and uniqueness theorems as has, e. g., been done in classical elasto-plasticity (see, e. g., de Boer and Ehlers, 1980).

We have shown that the fourth order material dependent tensor and its reverse, respectively, is positive definite. Thus, all general principles can be proved to be valid according to the method of de Boer and Ehlers (1980). However, owing to space limitations we will renounce the representation of these principles, valid for elastic-plastic deformations with hardening.

Chapter 10

APPLICATIONS IN ENGINEERING AND BIOMECHANICS

With the development of kinematics, balance equations, and the constitutive theory for saturated and empty porous solids in the last years, an ensured base has been provided for the investigation of special problems in different fields of engineering and biomechanics. Great progress has been achieved in such differing domains as soil mechanics, chemical engineering, material science, and environmental engineering, as well as in biomechanics. In this section some problems in the various fields will be discussed and the current state of the treatment of these problems will be addressed.

10.1 Soil Mechanics

The vast domain of soil mechanics is a preferred field for applying porous media theory. Saturated and empty sandy bodies or clay fulfill, to a high extent, the basic assumption of the porous media theory regarding statistically distributed pores. Therefore, it is not surprising that the porous media theory has its roots in soil mechanics (see the extensive review by de Boer, 2000 a). Based on experimental observations – in particular, in soil mechanics – fundamental findings, e.g., the effective stress concept, have been recognized. In the meantime, modern porous media theory has been successfully applied to static and dynamic problems in soil mechanics.

a) Consolidation Problem and Localization Phenomena

Whereas, in the past the consolidation problem (the time-dependent settlement of soils under load) was investigated with elasticity laws of the Hookean and non-Hookean types (see, e.g., Lancellotta and Preziosi, 1997, and the review by de Boer, 2000 a, as well as Arnod *et al.*, 1998, 1999), in recent times elastic-plastic relations for the solid skeleton have been used in order to investigate the settlement and other properties of loaded saturated porous solids. Skolnik

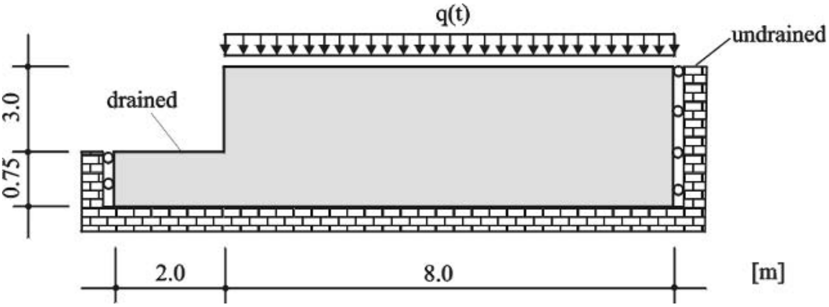

Fig. 10.1.1: Loaded slope.

(1998) proceeded consequently from the yield function (6.92) along with the flow rule (6.109) and the constitutive equation for the elastic-plastic behavior of the solid skeleton (6.133) and (6.134) to formulate the numerical solution strategy. This strategy was applied by Skolnik (1999) to different problems in soil mechanics (see, e.g., Figs. 10.1.1 and 10.1.2). In several papers, Ehlers and Volk (1997 a, b, 1998, 1999) took polar and non-polar granular elastic-plastic solid matrices, made of frictional material, into consideration. In the case of a saturated porous solid, they assumed that the pore spaces are filled with an incompressible viscous poreliquid. For the investigation of the mechanical behavior of the solid skeleton, they used kinematics, balance equations, and the constitutive theory described in Sections 3, 4, and 6, as a base. One of the goals of their papers is the calculation of localization phenomena for selected problems in soil mechanics, as for example, the classical base failure (Figs. 10.1.3 and 10.1.4) and slope failure problems. In order to improve their numerical results (which were obtained with the finite element method) they considered

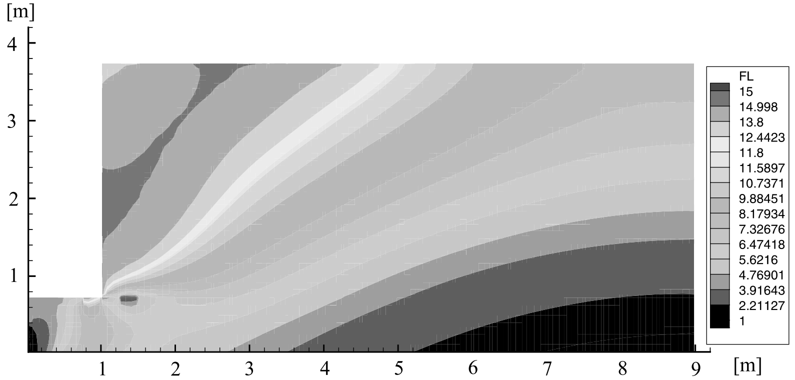

Fig. 10.1.2: Equivalent stresses for an increasing load ($q = 47.75 kN/m^2$).

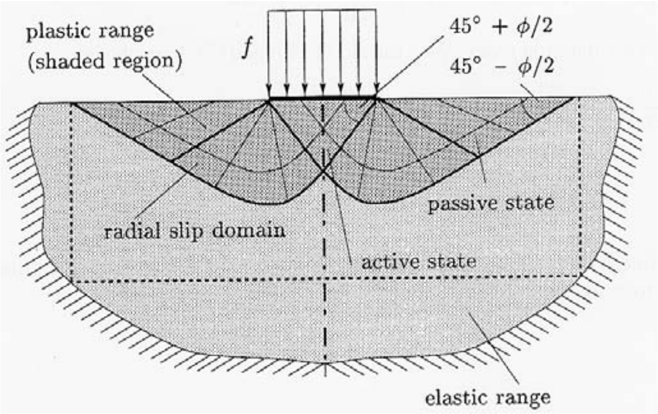

Fig. 10.1.3: The classical base failure problem (ϕ is the angle of internal friction).

the fluid-viscosity and the micropolar grain rotations as a regularization instrument. Ehlers and Volk (1998) wrote: "In the present article, it is shown that the inclusion of fluid-viscosity in the saturated case and the inclusion of micropolar grain rotations both in the saturated and in the non-saturated case leads to a regularization of the shear band problem." The results are encouraging and correspond to the classical solutions in soil mechanics. The procedure – which Ehlers and Volk described above – is, however, in some respect questionable. Does it mean that only the consideration of the fluid-viscosity and micropolar approach gives such excellent results?

In a recent paper Ehlers (2000) discussed, proceeding from an elasto-viscoplastic solid skeleton, dilatant and compressive shear bands in saturated geomaterials.

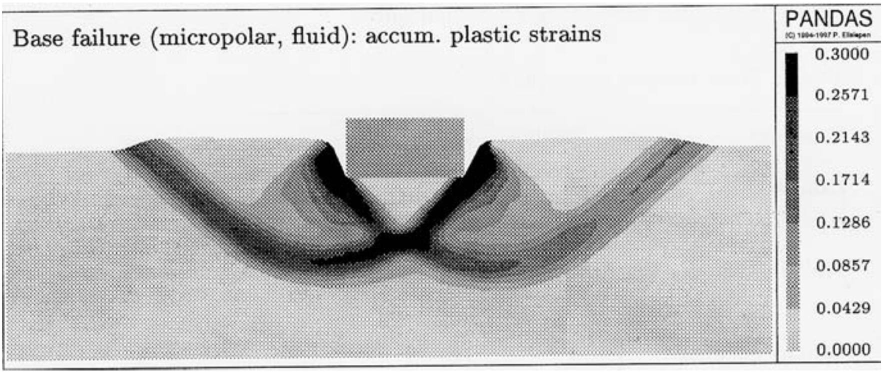

Fig. 10.1.4: Accumulated plastic strain of a liquid-saturated skeleton material (micropolar formulation) 20 days after having applied the external load.

b) Phase Transitions

Two important phase transitions occur in soil mechanics (also in some other fields of engineering), namely in the drying process and in the freezing process. Although the processes show similar characteristics, they are very different, see Chapter 7.7. It seems that, in engineering, drying processes have attracted much more interest than freezing and defrosting processes (possibly due to their importance in chemical engineering).

Kowalski (1987) was the first scientist to develop a theory for *drying processes* which was based on the fundamentals of continuum mechanics considering the coupling of the heat and moisture transport with the deformations of the solid skeleton. Kowalski (1990) later turned again to the drying process problem and extended his earlier treatise by considering the propagation of the evaporation surface (Stefan problem). The main differences between the papers of 1990 and 1987 consist of treating a ternary model instead of a binary model. Kowalski's investigations (also his subsequent papers) will be reviewed in Section 8.2 b).

Freezing and defrosting behavior in saturated brittle and granular media are treated less than drying processes in the literature. This is valid, in particular, for saturated granular materials within the porous media theory, although freezing processes, e.g., to improve the strength of soils, are important. As the frost-resistance of concrete is of great importance in the practice, earlier contributions in this field had the goal of investigating the freezing processes of porous materials from the engineering point of view. Schäfer (1964), as well as Manns and Hartmann (1977), performed a large amount of experimental investigations. These yielded the result that the frost resistance of concrete is influenced to a great extent by the amount and the kind of its pores. The analysis of damaged concrete had shown that this is caused by the frozen water in the capillary pores where the frozen water expands up to 9 % and builds up a considerable hydrostatic pressure. In this context, we refer to the papers of Zech (1981) and Setzer *et al.* (1994). The investigations were, however, performed with complete detachment from fundamental mechanical theories.

In recent times, Fasano *et al.* (1993) have discussed the problem of defrosting for a simplified mechanical model, though mainly from the mathematical point of view concerning the existence and uniqueness of the solution. Selvadurai and Shinde (1993) modelled the influence of frozen soils on pipes laid in soil, using a very simple theory and the finite element method. Moreover, they gave a comprehensive overview of the existing literature related to this topic.

The study of the literature has led to the conclusion that freezing and defrosting processes have not been treated within the framework of the theory of porous media at all, whereas the continuum-mechanical treatment of drying (vapourization) processes in porous media has attracted much more interest. This is a challenge for scientists working in the field of porous media to investigate the thermomechanic response of the freezing processes of saturated solid skeletons and to implement the findings gained from the phase change of liquid to ice in the modelling of freezing and defrosting processes in porous materials, with the help of continuum mechanics, in a mathematically as well as physically consistent manner.

A ternary model has also been proposed by de Boer (1993, 1994). The components of this model are a compressible porous solid and two immiscible compressible fluids, each with its own temperature fields. The goal of these papers, and subsequent contributions, has been to develop a general thermodynamic frame for phase transitions. The thermodynamic investigations concern the fact that the chemical potentials, as well as the temperatures of all components, must be different if mass exchanges between the constituents should occur. Musielak *et al.* (1994) formulated the boundary and initial value problem of a drying problem in the one-dimensional case for a ternary model, and performed, with the help of the finite element method, the numerical simulation of the commencement of the drying process.

The thermodynamic foundation of phase transitions in porous media was further developed by de Boer (1995 a) for a ternary model consisting of compressible constituents. However, de Boer's (1995 a) investigations are only valid for a simplified model, because, in the ternary model, the influence of the configuration pressure on the hydrostatic pressure of the solid material was neglected.

In de Boer and Kowalski (1995), thermodynamic restrictions for phase changes were developed for a ternary model with incompressible solid and liquid constituents and a compressible gas. In the appendix of this paper, the explicit evaluation of the entropy inequality is represented in detail.

The contributions of de Boer were improved due to new findings in the kinematics of the porous media theory and were brought to a primary close, as far as the thermodynamic frame is concerned, by de Boer and Bluhm (1999) and by de Boer *et al.* (2003).

c) Dynamics

Although dynamic problems play a role in many branches of engineering, it is in soil mechanics where these problems have the greatest influence (see, e.g., earthquakes and densification of sand). Therefore, we will review the domain of dynamics in this section.

In a series of papers, dynamic problems have been analytically treated with incompressible models at the University of Essen in Germany, see Section 8.5. The commonly accepted opinion, that there are two dilatational waves and one rotational wave, concluded on the basis of Biot's (1956) theory and from Plona's (1982) experimental work, could not be confirmed. It was able to predict the existence of only one dilatational wave as it should be for incompressible saturated porous solids. The result of Biot (1956) is not surprising, since his model is based on two compressible constituents.

An extended review on shock wave propagation in porous media was given by Sorek *et al.* (1999). They reviewed about 53 papers concerned with theoretical developments and experimental observations.

Dynamic problems have also been investigated by Gubaidullin and Kuchugurina (1999), though, in part, however, for simplified porous media models.

Parallel to the analytical investigations of the dynamic behavior of saturated porous solids, the numerical solution of initial- and boundary-value problems in the dynamic range has made great progress (see Breuer, 1997 a – c, 1999 a,b). However, in this field, there are still many problems left to investigate.

Readers interested in some aspects of modern approaches to dynamic problems of saturated porous media are referred to the special issues of the Journal *Transport in Porous Media* (Kowalski and Kubik (eds.), 1992, de Boer (ed.), 1999 b).

10.2 Chemical Engineering

In chemical engineering, many applications of porous media theory are known. In the following paragraphs we will discuss, however, only two problems.

a) Powder Compaction

For aerospace and other high-tech applications, many powder-metallurgy alloys have been developed in order to achieve better combinations of strength, toughness, fatigue resistance, and resistance to stress-corrosion cracking than those found in alloys produced by ingot metallurgy (see Doraivelu *et al.*, 1984). The success of powder-metallurgy processing depends heavily upon the ability to produce a near-net-shape economically. The forming of this shape is dependent on the success of the die compaction process in delivering defect-free, uniform-density green parts (Lewis *et al.*, 1993). The compaction process for compressing the powder is, without any doubt, the main process in manufacturing engineered products in powder metallurgy. Therefore, this process should be clearly understood from the mechanical and thermodynamic point of view, more so as there are many difficulties which exist in the compaction process for powders. This concerns, for example, the non-homogeneous density distribution and considerably large residual stresses in the green end product. Hence, a need exists to develop a mathematical model which can predict mechanical phenomena for the compaction process.

This mathematical model must be based on an appropriate plasticity theory, because the compression of the metallic powder during the compaction process is caused by the plastic deformation of the powder. Due to the metallic properties of the powder or the green, a von-Mises-type of plasticity theory seems to be

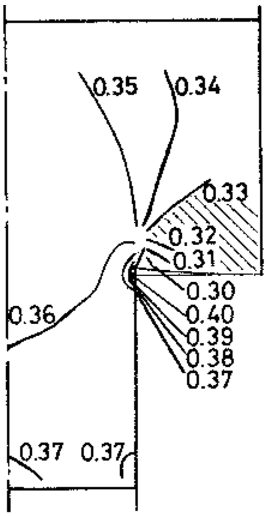

Fig. 10.2.1: Test results for the relative density

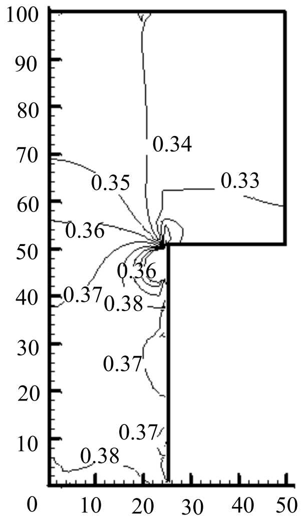

Fig. 10.2.2: Numerical results for the relative density

appropriate. However, the yield function of von Mises (1913) has to be extended in order to include the considerably large volume changes of the powder during the compaction process due to porosity. This effect is unknown in the classical plasticity of metals, where the incompressibility of the material in the plastic range is assumed (see von Mises, 1913).

Such a theory which meets the above statements has been developed in Section 6.6 and Chapter 9. The incorporation of the elastic deformation of the powder is justified by the numerical results represented at the end of this section.

In the following, we will study the compaction of copper powder in a T-shaped die (see Jägering, 1998). The die is filled with a copper powder, which is compacted up to a relative density of 0.332 and thickness of 20 mm before the real experiment starts. The relative density is defined as the quotient of the momentary (partial) density and the theoretically-possible maximum (real) density. The quotient corresponds to the volume fraction n^S:

$$\rho^{rel} = \frac{\rho^S}{\rho^{SR}} = n^S \, . \tag{10.1}$$

The compaction of the copper powder in the T-shaped die will be simulated on the basis of the theory of elastic-plastically deformed gas-filled porous solids. However, we will neglect the physical properties of the gas in the pores. For this reason, the partial density ρ^G of the gas is introduced with 10^{-5} kg/m^3.

The calculation shows that the poregas pressure is, in this case, equal to zero. Thus, the simulation can also be interpreted as the description of the mechanical behavior of an empty porous solid. The Lamé response parameters in the elastic range are given by $\mu^S = 5555.55\ kN/m^2$ and $\lambda^S = 1058.20\ kN/m^2$. The friction coefficient between powder and stamp has the value of 0.2 (see Morimoto *et al.*, 1982). For the description of the friction phenomena, we use Coulomb's friction law, where the friction force is equal to the product of the force normal to the wall of the die and the friction coefficient. The finite element calculation has been performed with 120 elements. In order to control the convergence, a second calculation with 480 elements has been simulated. Moreover, the plastic response parameters α and κ are assumed to be 0.2 and 0.6, respectively. For more information, see Jägering (1998). In Figure 10.2.1, the course of the relative density of the experiment is represented. Figure 10.2.2 shows the corresponding values obtained by the finite element method. From the comparison of the numerically gained results and the test results, it becomes apparent that porous media theory can sufficiently predict the test results correctly. Thus, the porous media theory seems to be excellently suited to describe the mechanical behavior of the compaction of metallic powders in a die.

b) Drying Processes

Drying processes (phase transitions which are accompanied by a liquid transport) occur in many fields of engineering, e.g., in soil mechanics, agriculture and, in particular, in chemical engineering. Thus, the problem of liquid transport in porous solids, and its removal from them, has an important practical feature. As indicated, the drying process consists of a change of fluid into moisture and the transport of the liquid in capillary-porous solids.

The drying process is very complex and not all problems have been solved so far. The thermodynamics of phase transition of water into gas (steam) has only recently come to a primary close (see de Boer and Bluhm, 1999) and the capillary rise of fluids in capillary-porous bodies is still under study concerning the continuum mechanical description of this phenomenon.

It seems that Kowalski (1987) was the first scientist who tried a continuum mechanical approach to the complex drying process. Kowalski (1987) pointed out: "This paper corresponds to the review article by Luikov (1975), and stands for the generalization of the concept presented there by taking into consideration the deformability of the capillary-porous body.

The basic aim of this paper is to perform a system of differential equations describing the distribution of strains (and, indirectly, stresses), temperature and concentration in a fluid-saturated capillary-porous body in the simultaneous presence of mechanical, thermal and concentration fields. Such a system of equations can be used to describe the thermomechanical changes in fluid-saturated porous bodies during the constant drying rate period.

In this period the evaporation surface covers the boundary surface of the medium. The thermodynamics of the nonconstant drying rate period and, in particular, the problem of receding of the evaporation surface will be published in a separate paper."

This was done by Kowalski (1990) in an extended article *Thermodynamics of dried mechanics*. In this paper, the author generalized and continued his earlier contribution from 1987. "The generalization consists in considering the phase transitions not only on the evaporation surface but also inside the whole body, what actually takes place in real processes. The continuation means here an extension of the previous theory to the so-called second period of drying."

Kowalski's (1990) treatise is based on the balance equations of porous media theory and restrictions implied on the constitutive relations by the second law of thermodynamics (entropy inequality). The consideration of the fundamentals of porous media theory leads to a clear concept of heat and mass transfer.

The mathematical model consists of the following four constituents:

"(i) an elastic porous matrix (skeleton of the medium),

(ii) a liquid bounded physically and chemically with the skeleton (cellular liquid); the kinematics of the constituent is the same as that of the skeleton,

(iii) a liquid bounded physically and mechanically with the skeleton (capillary liquid) having its own kinematics,

(iv) humid air contained in the medium in the form of bubbles having also its own kinematics."

Kowalski (1990) divided the whole drying process into two periods: the constant drying rate period (period I) and the decreasing drying rate period (period II). Then he developed the balance equations for the two periods and derived thermodynamic restrictions for the constitutive equations. Moreover, Kowalski (1990) summarized his investigations and laid down the final set of equations for moist colloidal (elastic gels) and capillary-porous bodies. Finally, he wrote down a linearized set of differential equations describing the heat and mass transfer and the deformations of dried material coupled with the transfer. In addition, he addressed some considerations regarding the moving evaporation surface which separates the wet and dry regions (Stefan problem). Whereas the transfer of heat and mass as well as elastic behavior was extensively treated by Kowalski (1990), the phase transition of fluid into gas was obviously not discussed further.

Kowalski (1994) later returned to this point and stated that the mass supply $\hat{\rho}^G$ of the gas phase due to phase transition from the liquid phase φ^L was proportional to the difference between the chemical potentials of liquid μ^L and gas μ^G related to the respective temperature Θ^L and Θ^G (see also de Boer, 1993, 2000 a).

The goals of Kowalski's further contributions (see Kowalski, 1996 a,b, Kowalski and Rybicki, 1996, Kowalski *et al.*, 1997) to the drying problem were the modelling of shrinkage (linear-elastic and isotropic material), the estimation of the influence of the moisture concentration and the temperature on the formation of drying-induced stresses, and the study of the response of the dried materials to a variation in drying conditions.

Kowalski (2000) arrived at a temporary conclusion in his paper *Toward a Thermodynamics and Mechanics of Drying Processes*, where he provided "a wider presentation of mechanics of intensively dried moistened capillary-porous materials." The extension of his previous works consisted of the development of a non-linear theory for the heat and mass transfer and the mechanical phenomena in moist materials manifested in a coupled system of differential equations. Finally, he presented a simple numerical example.

Concerning additional references, in particular older papers, the reader is referred to Kowalski (2003).

Although one cannot agree with all simplifying assumptions and, in parts, to some mathematical representations, one can state that the work involved in the contribution of Kowalski was considerable. For the first time, a uniform theory of the complex drying process of moist, capillary-porous materials based on the fundamentals of porous media theory had been developed.

10.3 Building Physics

In building physics, the phenomena of coupled matter (moisture) and heat transport in capillary-porous media are of special interest. For example, a lot of damages to buildings results from the influence of moisture in concrete and brick walls. Moreover, the control of heat conduction is absolutely necessary in order to save energy.

Due to the eminent importance of these phenomena there have been many attempts to create mathematical models in order to describe thermomechanical effects.

The first approaches obviously go back to the 1940s when Krischer (1940, 1942) developed the balance equations for the moisture and heat transport. In this context, Luikov (1966, 1975) must be named. His balance equations are nearly identical with those of Krischer (1940, 1942). However, the ansätze for the transport of liquid and steam differ in some points. In three papers de Vries contributed to the theory of moisture transport (Philip and de Vries, 1957, de Vries, 1958, 1962). In his works de Vries referred, for the first time, to results of soil physics.

In his dissertation, Kießl (1983) further developed the description of the coupled moisture and heat transport, based on the results of Krischer, Luikov and de Vries. Neiß (1982) introduced a new element into the discussion of the coupled moisture and heat transport, namely the description of transport processes under frost conditions. Further contributions are due to Garrecht (1992), Künzel (1994), Häupl and Stopp (1987, 1988). These treatises contain, apart from the modelling, some numerical algorithms to solve the coupled system of differential equations.

Most of the descriptive equations in the papers mentioned above are obtained purely intuitively (in parts, connected with inadequate mathematics), not founded with the fundamentals of porous media theory. A first approach to develop a theory for diffusive and convective matter, as well as energy transport in capillary-porous building materials, on the base of a multi-component model is due to Grunewald (1997). Unfortunately, his dissertation remains in some parts a purely formalistic treatment of the aforementioned problem.

Thus, one can state that little has been done to derive the description of some physical phenomena from the fundamentals of the porous media theory. Most of the describing equations for, e.g., moisture transport and heat conduction in saturated porous solids, have been obtained purely intuitively. In the following two sections, it will be shown how usefully porous media theory can be applied to describe the moisture transport and the heat conduction in saturated rigid porous solids. However, the treatment of the above-mentioned physical phenomena in the next two sections should be considered only as a first approach, because there is no experience with the governing equations developed for moisture transport and heat conduction from the fundamental relations of porous media theory, see de Boer, 1999 a, de Boer and Didwania, 2000, 2002, 2004.

a) Transport of Moisture

In building physics, the phenomena of coupled matter (moisture) and heat transport in capillary-porous media are of special interest. For example, a lot of damages to buildings results from the influence of moisture in concrete and brick walls. Moreover, the control of heat conduction is absolutely necessary in order to save energy.

In the following section, we will show how the porous media theory can also be applied to transport phenomena in building physics, namely the transport of moisture in a capillary-porous solid.

The ternary model consists of a rigid capillary-porous solid φ^S at repose, of a free incompressible liquid φ^L, and compressible gas (humid air) φ^G. In building physics the moisture c is defined by the quotient of the mass of free liquid dm^L per volume element dv (see Hohmann and Setzer, 1997)

$$c = \frac{dm^L}{dv} \, . \tag{10.2}$$

It would not be difficult to also include the liquid content in the humid air in the definition (10.2). However, this is not usual in building physics. Eq. (10.2) can be further evaluated. With

$$dm^L = \rho^L dv \tag{10.3}$$

we have

$$c = \rho^L \ . \tag{10.4}$$

Other definitions of the moisture c are possible; the interested reader is referred to the literature on building physics, see, e.g., Lutz *et al.* (1985). However, the basic equation for describing the transport of moisture is not changed. Thus, for simplicity we will use the definition (10.2) and the consequence (10.4).

From Section 4.1, we obtain the balance equations of mass for the free liquid and gas phases, whereby we assume that the mass exchange with the solid constituent can be neglected:

$$\frac{\partial \rho^L}{\partial t} + \operatorname{div}(\rho^L \mathbf{v}_L) = \hat{\rho}^L \ , \tag{10.5}$$

$$\frac{\partial \rho^G}{\partial t} + \operatorname{div}(\rho^G \mathbf{v}_G) = \hat{\rho}^G \tag{10.6}$$

with

$$\hat{\rho}^L + \hat{\rho}^G = 0 \ . \tag{10.7}$$

The equations of motion can be formulated as follows:

$$\operatorname{div} \mathbf{T}^S + \rho^S \mathbf{g} + \hat{\mathbf{p}}^S = \mathbf{0} \ , \tag{10.8}$$

$$\operatorname{div} \mathbf{T}^L + \rho^L \mathbf{g} + \hat{\mathbf{p}}^L = \hat{\rho}^L \mathbf{v}_L \ , \tag{10.9}$$

$$\operatorname{div} \mathbf{T}^G + \rho^G \mathbf{g} + \hat{\mathbf{p}}^G = \hat{\rho}^G \mathbf{v}_G \ , \tag{10.10}$$

where $\mathbf{g}$ is the acceleration due to gravity. As the transportation process occurs slowly, we have neglected the inertia forces in (10.8) through (10.10). The interaction forces $\hat{\mathbf{p}}^\alpha$ (where α stands for S = solid, L = free liquid, G = gas) are restricted by

$$\hat{\mathbf{p}}^S + \hat{\mathbf{p}}^L + \hat{\mathbf{p}}^G = \mathbf{0} \ . \tag{10.11}$$

From the constitutive theory in de Boer (2000 a), we can read the following constitutive equations:

$$\begin{aligned}
&\mathbf{T}^L = n^L p^t \mathbf{I}\,, \quad \mathbf{T}^G = n^G p^t\,\mathbf{I} + p_E^G \mathbf{I}\,,\\
&p_E^G = n^G \rho^G \frac{\partial \psi^G}{\partial n^G}\,,\\
&\mathbf{T}^G = p^G \mathbf{I} = n^G p^{GR} \mathbf{I}\,,
\end{aligned} \tag{10.12}$$

where p^t defines the effective liquid suction in the capillary zone,

$$\begin{aligned}
\hat{\mathbf{p}}^L = \; & - \; p^t \,\mathrm{grad}\, n^L - (\rho^S \frac{\partial \psi^S}{\partial \rho^L} + \rho^G \frac{\partial \psi^G}{\partial \rho^L})\,\mathrm{grad}\,\rho^L - \\
& - \; (\beta_S + \beta_L + \beta_G)\,\mathrm{grad}\,\Theta \; - \beta_V^L \mathbf{v}_L - \beta_V^G \mathbf{v}_G\,,\\
\hat{\mathbf{p}}^G = \; & - \; p^t \,\mathrm{grad}\, n^G - \rho^G \frac{\partial \psi^G}{\partial \rho^L}\,\mathrm{grad}\,\rho^L - \\
& - \; (\gamma_S + \gamma_L + \gamma_G)\,\mathrm{grad}\,\Theta - \gamma_V^L \mathbf{v}_L - \gamma_V^G \mathbf{v}_G\,,
\end{aligned} \tag{10.13}$$

where $\mathbf{v}_S = \mathbf{0}$ has been used and where $\beta_S, \beta_L, \beta_G$ and β_V^L, β_V^G as well as $\gamma_S, \gamma_L, \gamma_G$ and γ_V^L, γ_V^G are response parameters depending on the temperature Θ.

The constitutive relations (10.12) are gained from the dissipation inequality by an evaluation process, see the recent papers by de Boer and Didwania (2000, 2002, 2004).

These constitutive equations for the interaction volume forces $\hat{\mathbf{p}}^L$ and $\hat{\mathbf{p}}^G$ depend on the temperature gradient, on the velocities $\mathbf{v}_L$ and $\mathbf{v}_G$ and on the free Helmholtz energy functions of the solid and gas phases connected with the gradient of the density of the liquid. It has been shown in de Boer and Didwania (2000, 2002, 2004) that capillary effects can be described with the constitutive equations for interaction forces (10.13). Moreover, the classical results in capillarity theory by Young and Laplace, which are well-known in Physics and Chemical Engineering, are contained in the approach by de Boer and Didwania (2000, 2002, 2004).

With the mass and momentum balance equations (10.5) through (10.7) and (10.8) through (10.10), as well as the constitutive relations (10.12) and (10.13), the basis for the development of the describing differential equation for the transport moisture c is known. From the mass balance equation for the free liquid (10.5), in combination with the definition (10.2), we obtain

$$\frac{\partial c}{\partial t} + \mathrm{div}(c\mathbf{v}_L) = \hat{\rho}^L\,. \tag{10.14}$$

The equations of motion (10.8) and (10.9), together with the constitutive equations (10.12) and (10.13), yield:

$$\begin{aligned} & n^L \text{ grad } p^t + \rho^L \mathbf{g} - (\rho^S \frac{\partial \psi^S}{\partial \rho^L} + \rho^G \frac{\partial \psi^G}{\partial \rho^L}) \text{ grad } \rho^L - \\ & - \beta_\Theta \text{ grad } \Theta = \beta_v^L \mathbf{v}_L + \beta_v^G \mathbf{v}_G + \hat{\rho}^L \mathbf{v}_L \ , \end{aligned} \tag{10.15}$$

$$\begin{aligned} & n^G \text{ grad } p^{GR} + \rho^G \mathbf{g} - (p^t - \ p^{GR}) \text{ grad } n^G - \\ & - \rho^G \frac{\partial \psi^G}{\partial \rho^F} \text{ grad } \rho^F - \gamma_\Theta \text{ grad } \Theta \\ & = \gamma_v^L \mathbf{v}_L + \gamma_v^G \mathbf{v}_G + \hat{\rho}^G \mathbf{v}_G \ , \end{aligned} \tag{10.16}$$

where the abbreviations

$$\begin{aligned} \beta_\Theta &= \beta_S + \beta_L + \beta_G \ , \\ \gamma_\Theta &= \gamma_S + \gamma_L + \gamma_G \end{aligned} \tag{10.17}$$

have been introduced.

The equations of motion (10.15) and (10.16) are coupled in the velocities $\mathbf{v}_L$ and $\mathbf{v}_G$. From (10.16), we obtain

$$\begin{aligned} \mathbf{v}_G \ = \ & \frac{1}{\hat{\rho}^G + \gamma_v^G} [n^G \text{ grad } p^{GR} + \rho^G \mathbf{g} - \\ & - (p^t - p^{GR}) \text{ grad } n^G - \rho^G \frac{\partial \psi^G}{\partial \rho^L} \text{ grad } \rho^L - \\ & - \gamma_\Theta \text{ grad } \Theta] - \frac{\gamma_v^L}{\hat{\rho}^G + \gamma_v^G} \mathbf{v}_L \ . \end{aligned} \tag{10.18}$$

The insertion of $\mathbf{v}_G$ into (10.15) leads to

$$\begin{aligned} \mathbf{v}_L \ = \ & \frac{1}{A} \{ n^L \text{ grad } p^t + \rho^L \mathbf{g} - \\ & - (\rho^S \frac{\partial \psi^S}{\partial \rho^L} + \ \rho^G \frac{\partial \psi^G}{\partial \rho^L}) \text{ grad } \rho^F - \beta_\Theta \text{ grad } \Theta + \\ & + B[n^G \text{ grad } p^{GR} + \rho^G \mathbf{g} - (p^t - p^{GR}) \text{ grad } n^G - \\ & - \rho^G \frac{\partial \psi^G}{\partial \rho^L} \text{ grad } \rho^L - \ \gamma_\Theta \text{ grad } \Theta] \} \end{aligned} \tag{10.19}$$

with

$$\begin{aligned} A &= \frac{(\hat{\rho}^L + \beta_V^L)(\hat{\rho}^G + \gamma_V^G) - \beta_V^G \gamma_V^L}{\hat{\rho}^G + \gamma_V^G} \, , \\ B &= - \frac{\beta_V^G}{\hat{\rho}^G + \gamma_V^G} \, . \end{aligned} \tag{10.20}$$

Introducing the pressure heads

$$h^L = - \frac{p^t}{\gamma^{LR}} + \frac{U}{\mid \mathbf{g} \mid} \, , \quad h^G = - \frac{p^{GR}}{\gamma^{GR}} + \frac{U}{\mid \mathbf{g} \mid} \tag{10.21}$$

and considering (10.19), from (10.14) the differential equation for the moisture transport is obtained, where U is the gravity potential and $\rho^L = c$ (see 10.4):

$$\begin{aligned} \frac{\partial c}{\partial t} &= -\mathrm{div}\{ \frac{c}{A} [- \, n^L \gamma^{LR} \, \mathrm{grad} \, h^L - \\ &- B n^G \gamma^{GR} \, \mathrm{grad} \, h^G - B(p^t - p^{GR}) \, \mathrm{grad} \, n^G - \\ &- [\rho^S \frac{\partial \psi^S}{\partial \rho^L} + \rho^G \frac{\partial \psi^G}{\partial \rho^L} (1 - B)] \, \mathrm{grad} \, c - \\ &- (\beta_\Theta + B \gamma_\Theta) \, \mathrm{grad} \, \Theta] \} + \hat{\rho}^L \, . \end{aligned} \tag{10.22}$$

The difference $p^t - p^{GR}$ in (10.22) is known in the theory of capillary-porous materials as the capillary pressure.

A similar differential equation is known in building physics, see, e.g., Garrecht (1992). However, the differential equation in Garrecht (1992), and also other approaches in building physics have been developed purely intuitively. They are not based on the principles of mechanics and thermodynamics of porous media theory. Therefore, the identification of various terms in the corresponding relation to (10.22) in Garrecht (1992) is cumbersome.

It should be mentioned that there has been no experience with the evaluation of (10.22) until now. The identification of the parameters A and B in (10.20) is still awaiting research. It seems, however, that A and B are reasonable coefficients, because they contain the mass supply terms which influence, without any doubt, the permeability of the mixture body. Moreover, the constitutive equations for the momentum supplies in (10.13) are, in no way, evaluated. This will be done in further publications.

b) Heat Conduction in a Fluid-Saturated Capillary-Porous Solid

This section is based on a report by de Boer and Ehlers (1986 b) and the considerations of de Boer (2000 a), de Boer and Didwania (2000, 2002, 2004). For the investigation of the heat conduction in a rigid solid skeleton saturated with an incompressible fluid and a compressible gas, we shall use the constitutive relations derived in de Boer and Didwania (2000, 2002, 2004). We investigate only thermodynamic processes with small temperature gradients and small velocities:

$$\begin{aligned}
\psi^S &= \psi^S(\Theta, \rho^L)\ , \ \ \psi^L = \psi^L(\Theta)\ , \ \ \psi^G = \psi^G(n^G, \Theta, \rho^L)\ , \\
\eta^\alpha &= -\frac{\partial \psi^\alpha}{\partial \Theta}\ , \\
\hat{\mathbf{p}}^L &= -\, p^t \operatorname{grad} n^L - (\rho^S \frac{\partial \psi^S}{\partial \rho^L} + \rho^G \frac{\partial \psi^G}{\partial \rho^L}) \operatorname{grad} \rho^L - \\
&\quad -\, \beta_\Theta(\Theta) \operatorname{grad} \Theta - \beta_v^L(\Theta) \mathbf{v}_L - \beta_v^G(\Theta) \mathbf{v}_G\ , \\
\hat{\mathbf{p}}^G &= -\, p^t \operatorname{grad} n^G - \rho^G \frac{\partial \psi^G}{\partial \rho^L} \operatorname{grad} \rho^L - \\
&\quad -\, \gamma_\Theta(\Theta) \operatorname{grad} \Theta - \gamma_v^L(\Theta) \mathbf{v}_L - \gamma_v^G(\Theta) \mathbf{v}_G\ , \\
\mathbf{q}^S &+ \mathbf{q}^L + \mathbf{q}^G = -\, \alpha_\Theta(\Theta) \operatorname{grad} \Theta + \ \Theta \beta_\Theta(\Theta) \mathbf{v}_L + \\
&\quad +\, \Theta \gamma_\Theta(\Theta) \mathbf{v}_G\ , \\
\mathbf{T}^L &= n^L p^t \mathbf{I}\ , \quad \mathbf{T}^G = n^G p^t \mathbf{I} + p_E^G \mathbf{I}\ , \\
p_E^G &= n^G \rho^G \frac{\partial \psi^G}{\partial n^G}\ ,
\end{aligned} \tag{10.23}$$

where $\alpha_\Theta, \beta_\Theta, \gamma_\Theta, \beta_v^L, \beta_v^G$, and γ_v^L, γ_v^G are response parameters (see de Boer, 2000 a) and where the dependence of ψ^S and ψ^G on the density ρ^L is limited to a thin film between the solid and liquid and the gas and liquid constituents, respectively.

Considering (10.23), the saturation constraint, and replacing the internal energy ε with the free Helmholtz energy function (5.5), from (4.34) the balance of energy for the ternary model under study is gained:

$$\begin{aligned}
- &\quad \rho^S (\psi^S)'_S - \rho^L (\psi^L)'_L - \rho^G (\psi^G)'_G - \\
- &\quad \rho^S \eta^S \Theta'_S - \rho^L \eta^L \Theta'_L - \rho^G \eta^G \Theta'_G - \\
- &\quad \rho^S \Theta (\eta^S)'_S - \rho^L \Theta (\eta^L)'_L - \rho^G \Theta (\eta^G)'_G +
\end{aligned} \tag{10.24}$$

$$
\begin{aligned}
+ \quad & (\mathbf{T}^L - n^L p^t \mathbf{I}) \cdot \mathbf{D}_L + (\mathbf{T}^G - n^G p^t \mathbf{I}) \cdot \mathbf{D}_G + \\
+ \quad & \rho^S r^S + \rho^L r^L + \rho^G r^G - \operatorname{div}(\mathbf{q}^S + \mathbf{q}^L + \mathbf{q}^G) - \\
- \quad & [\hat{\mathbf{p}}^L + p^t \operatorname{grad} n^L + \ (\rho^S \frac{\partial \psi^S}{\partial \rho^L} + \rho^G \frac{\partial \psi^G}{\partial \rho^L}) \operatorname{grad} \rho^L] \cdot \mathbf{v}_L - \\
- \quad & (\hat{\mathbf{p}}^G + p^t \operatorname{grad} n^G + \rho^G \frac{\partial \psi^G}{\partial \rho^F} \operatorname{grad} \rho^L) \cdot \mathbf{v}_G = 0 \; .
\end{aligned}
$$

The mass supply $\hat{\rho}^L$ is neglected in the balance equation (10.24). With (10.23) and (10.24), we have

$$
\begin{aligned}
- \quad & \rho^S \ \frac{\partial \psi^S}{\partial \Theta} \Theta'_S - \rho^L \ \frac{\partial \psi^L}{\partial \Theta} \Theta'_L - \rho^G \ \frac{\partial \psi^G}{\partial \Theta} \Theta'_G + \\
+ \quad & \rho^S \ \frac{\partial \psi^S}{\partial \Theta} \Theta'_S + \rho^L \ \frac{\partial \psi^L}{\partial \Theta} \Theta'_L + \rho^G \ \frac{\partial \psi^G}{\partial \Theta} \Theta'_G + \\
+ \quad & \rho^S \Theta \ \frac{\partial^2 \psi^S}{\partial \Theta^2} \ \Theta'_S + \rho^F \Theta \ \frac{\partial^2 \psi^F}{\partial \Theta^2} \Theta'_L + \rho^G \ \frac{\partial^2 \psi^G}{\partial \Theta^2} \Theta'_G + \\
+ \quad & p^G_E (\mathbf{D}_G \cdot \mathbf{I}) + \rho^S r^S + \rho^L r^L + \rho^G r^G + \qquad (10.25) \\
+ \quad & \operatorname{div}(\alpha_\Theta \operatorname{grad} \Theta - \Theta \beta_\Theta \mathbf{v}_L - \Theta \gamma_\Theta \mathbf{v}_G) + \\
+ \quad & \beta_\Theta \operatorname{grad} \Theta \cdot \mathbf{v}_L + \beta^L_V \mathbf{v}_L \cdot \mathbf{v}_L + \\
+ \quad & \gamma_\Theta \operatorname{grad} \Theta \cdot \mathbf{v}_G + \gamma^G_V \mathbf{v}_G \cdot \mathbf{v}_G \ = 0 \; .
\end{aligned}
$$

where use has been made of the fact that $\beta^G_V + \gamma^L_V = 0$ due to thermodynamic restrictions.

With the assumption that the distribution of the volume fraction in the reference placement is homogeneous ($n^S_{0S} = \text{const.}$, $n^L_{0L} = \text{const.}$, and $n^G_{0G} = \text{const.}$), and that the response parameters β_Θ and γ_Θ are spatially independent, relation (10.25) reduces to:

$$
\begin{aligned}
& \rho^S \Theta \ \frac{\partial^2 \psi^S}{\partial \Theta^2} \Theta'_S + \rho^L \Theta \frac{\partial^2 \psi^L}{\partial \Theta^2} \Theta'_L + \rho^G \Theta \frac{\partial^2 \psi^G}{\partial \Theta^2} \Theta'_G + \\
& + \rho^S r^S + \rho^L r^L + \rho^G r^G + \operatorname{div}(\alpha_\Theta \operatorname{grad} \Theta) + \qquad (10.26) \\
& + p^G_E (\mathbf{D}_G \cdot \mathbf{I}) + \phi_D = 0 \; .
\end{aligned}
$$

The quantity

$$\begin{aligned}\phi_D \;=\;& -\,\Theta\beta_\Theta(\mathbf{D}_L\cdot\mathbf{I}) -\,\Theta\gamma_\Theta(\mathbf{D}_G\cdot\mathbf{I}) + \\ & +\,\beta_V^L\mathbf{v}_L\cdot\mathbf{v}_L + \gamma_V^G\mathbf{v}_G\cdot\mathbf{v}_G \geq\; 0\end{aligned} \tag{10.27}$$

denotes the dissipation function caused by the flow of the incompressible fluid and the compressible gas. The condition $\phi_D \geq 0$ results from the positive response parameters $-\,\Theta\beta_\Theta, -\,\Theta\gamma_\Theta,\ \beta_V^L$, and γ_V^G.

In analogy to classical continuum mechanics of one-component continua, the notion specific heat for constant volume c_V^α can be introduced:

$$c_V^\alpha = \Theta\frac{\partial\eta^\alpha}{\partial\Theta}\;. \tag{10.28}$$

With $(10.23)_4$ and (10.28), relation (10.26) simplifies to

$$\begin{aligned}-\quad & \rho^S c_V^S\Theta_S' - \rho^L c_V^L\Theta_L' - \rho^G c_V^G\Theta_G' + \\ +\quad & \rho^S r^S + \rho^L r^L + \rho^G r^G + \\ +\quad & \operatorname{div}(\alpha_\Theta \ \mathrm{grad}\ \Theta) + p_E^G(\mathbf{D}_G\cdot\mathbf{I}) + \phi_D \;=\; 0\;.\end{aligned} \tag{10.29}$$

Equation (10.29) can be further reduced, since the velocity of the rigid solid phase $\mathbf{v}_S$ is equal to zero. Therefore,

$$\begin{aligned}-\quad & \rho^S c_V^S\frac{\partial\Theta}{\partial t} - \rho^L c_V^L\Theta_L' - \rho^G c_V^G\Theta_G' + \\ +\quad & \rho^S r^S + \rho^L r^L + \rho^G r^G + \operatorname{div}(\alpha_\Theta \ \mathrm{grad}\ \Theta) + \\ +\quad & p_E^G(\mathbf{D}_G\cdot\mathbf{I}) + \phi_D \;=\; 0\;,\end{aligned} \tag{10.30}$$

where ϕ_D is expressed by (10.27).

Furthermore, it may happen that the sum of the external heat supplies $\rho^S r^S + \rho^L r^L + \rho^G r^G$, as well as the sum of the external heat flux $\mathbf{q}^S + \mathbf{q}^L + \mathbf{q}^G$, vanish. However, the condition $\mathbf{q}^S + \mathbf{q}^L + \mathbf{q}^G = \mathbf{0}$ is only possible with $\mathrm{grad}\Theta = \mathbf{0}$ (homogeneous distribution of the temperature), compare $(10.23)_5$. With these assumptions, (10.30) takes a very convenient form:

$$-\,(\rho^S c_V^S + \rho^L c_V^L + \rho^G c_V^G)\frac{\partial\Theta}{\partial t} + p_E^G(\mathbf{D}_G\cdot\mathbf{I}) + \phi_D \;=\; 0\;. \tag{10.31}$$

This relation shows that an increase in temperature is possible even if the external heat supply and flux are absent. It is recognized that the increase in temperature is caused by the change of the volumetric strain of the gas.

Finally, we will compare the equation of heat conduction (10.29) with that of one-component continua – rigid solid and incompressible inviscid fluid – in classical continuum mechanics neglecting the properties of the gas phase.

In the case of a *rigid* solid $(n^S = 1)$, we have

$$\rho^L c_V^L = 0, \quad \rho^G c_V^G = 0, \quad \rho^L r^L + \rho^G r^G = 0, \quad \alpha_\Theta = \alpha_\Theta^S \, . \tag{10.32}$$

Moreover,

$$\phi_D \;=\; 0 \tag{10.33}$$

and the interaction force $\hat{\mathbf{p}}^L$ is identical to zero. With (10.32) and (10.33), we obtain the equation of heat conduction in a rigid solid with $\alpha_\Theta = \alpha_\Theta^S$:

$$-\,\rho^S c_V^S \Theta_S' + \operatorname{div}(\alpha_\Theta^S \operatorname{grad}\Theta) + \rho^S r^S \;=\; 0 \, , \tag{10.34}$$

see, e.g., Parkus (1976).

In the case of an *incompressible fluid*,

$$\begin{aligned} &\rho^S c_V^S = 0 \, , \quad \rho^G c_V^G = 0 \, , \\ &\rho^S r^S + \rho^G r^G = 0 \, , \quad \text{and} \quad \alpha_\Theta = \alpha_\Theta^L \end{aligned} \tag{10.35}$$

are valid and the interaction force $\hat{\mathbf{p}}^L$ is identical to zero. The dissipation function simplifies to

$$\phi_D = 0 \, . \tag{10.36}$$

With (10.36) and (10.31), we obtain the equation of heat conduction for an incompressible, inviscid fluid:

$$-\,\rho^L c_V^L \Theta_L' + \operatorname{div}(\alpha_\Theta^L \operatorname{grad}\Theta) + \rho^L r^L = 0 \, . \tag{10.37}$$

This result can be proven by comparison with the corresponding considerations of either Serrin (1959) or Truckenbrodt (1968).

10.4 Biomechanics

Porous media theory is an essential part of biomechanics. Since Mow's *et al.* (1980) introduction of a binary model (similar to Bowen's, 1980, approach) with incompressible constituents to investigate creep and stress relaxation of articular cartilage, a vast amount of contributions in biomechanics, based on the fundamentals of porous media theory, has been published. Within the framework of this review, it is impossible to mention and discuss all of these valuable papers. We will restrict our review, therefore, to some selected papers and will discuss only the mechanical aspect. During the 1980s, much effort had been made to investigate single problems in biomechanics on the basis of the so-called linear Kuei, Lai, and Mow (KLM) biphasic theory for cartilage (see Armstrong *et al.*, 1984). In this paper, analytical solutions for the deformations, flux of the fluid, creep, and stress relaxations of fluid-filled elastic porous solids (e.g., cartilage) were developed. In the contribution of Holmes *et al.* (1985), viscoelastic behavior of the solid phase was taken into account.

An excellent review article *Cartilage and diarthrodial joints as paradigms for hierarchical materials and structures* was written by Mow and Ratcliffe (1992). This article gives a clear overview on the most essential mechanical properties of cartilage and diarthrodial joints. In particular, Mow and Ratcliff (1992) explained many notions of biomechanics and classical mechanics. Thus, this review article is excellently suited to give an introduction to that part of biomechanics concerned with the description of the behavior of loaded cartilage.

A decisive step towards a consistent theory for describing the mechanical behavior of articular cartilage was made by Lai *et al.* (1991). They developed a ternary model for the swelling and deformation behaviors of articular cartilage. In the introduction, they stated: “Many specialized theories have ... been developed, in particular, the theory for a mixture of one incompressible solid phase and $N-1$ incompressible fluid phases was developed in 1980 by Bowen ... This provides the basic framework for our development of our triphasic theory. In this theory, the Helmholtz energy functions for the phases depend on the deformation of the solid phase, temperature, and the densities of the $N-1$ fluid phases. In the present study, this theoretical framework is specialized to our study of articular cartilage which contains an elastic solid matrix, a fluid phase (water) and a third phase for the ionic species of the salt. However, to include the fixed charge groups of the solid matrix of cartilage, we specifically introduce into the Helmholtz energy functions a dependence on c^L (c^L is the fixed charge density, Eq/unit tissue water volume, the author). In addition, for the ion phase and the fluid phase, we incorporate the traditional chemical potentials for electrolyte and polyelectrolyte solutions into the theory, thus providing a theoretical bridge between the continuum mixture theory applied to soft tissues ... and the physico-chemical theory.”

Lai *et al.* (1991) adopted Bowen’s (1980) kinematics, balance equations, and the constitutive theory, however, with further extension of the set of process variables. They also adopted Bowen’s (1980) introduction of the chemical potential tensor (which is not contained in classical thermodynamics). Moreover, they repeated the interpretation of Bowen (1980) concerning the incompressibility constraint, which does not conform to modern standards in porous media theory (see Bluhm *et al.*, 1995, de Boer, 2000 a).

Through evaluation of the entropy inequality, Lai *et al.* (1991) gained restrictions for Cauchy's stress tensor of the various constituents and for the momentum supplies. The remaining part of their considerable and valuable paper is concerned with some special problems and the discussion of boundary conditions.

The subsequent papers of the school of biomechanics at the Columbia University, New York, have been concerned, e.g., with the incorporation of the viscoelastic behavior of the solid phase in their models (see Section 6.7) and the application of triphasic mechano-electrochemical theory (Lai *et al.*, 1991) to the transport of water and ions through a finite-thickness layer of charged, hydrated soft tissue (Gu *et al.*, 1993, 1997), to the one-dimensional confined swelling problem, and the modelling of an isolated cell as a spherical mixture. Furthermore, Gu *et al.* (1999) turned back to the transport problem in tissues in their paper: *Transport of Multi-Electrolytes in Charged Hydrated Biological Soft Tissues*. This paper is based on the triphasic theory extended by Gu *et al.* (1994, 1998). The goal of Gu *et al.* (1999) was to "briefly review our multiphasic (component) mixture theory, and present applications for it to investigate the role of the fixed charges in mechano-electrokinetic transduction effects."

Contributions to the mechanical behavior of the fluid flow in bones have been published by Cowin *et al.* (1995), Weinbaum *et al.* (1994) and Zeng *et al.* (1994).

Some important papers have been published by biomechanicians from The Netherlands. Mixture models for soft tissues and skin were used by Snijders (1986), and Oomens and van Campen (1987). However, their constitutive theory was not adequately developed. This is also valid for the contribution of Huyghe *et al.* (1989). Snijders *et al.* (1992) discussed a mathematical model for the intervertebral disk. Oomens *et al.* (1996) remarked: "Snijders *et al.* (1992) developed a triphasic mixture model for these tissues, a numerical solution procedure based on the finite-element method, and implemented it in the commercial software code DIANA... This theory strongly resembles the triphasic mixture theory developed by Lai *et al.* (1991). One of the major differences is that Snijders does not use the concept of a chemical expansion stress."

Snijders *et al.* (1995) developed a triphasic finite element model for swelling porous media on the base of Lai's *et al.* (1991) approach. van Kemenade *et al.* (1997) adopted Snijders *et al.* (1995) simplified version of Lai's *et al.* (1991) theory to model intervertebral disc tissues.

An informative introduction to swelling and compression of intervertebral disc tissue is contained in the dissertation thesis by Houben (1996).

Huyghe and Janssen (1997) "have developed a finite deformation quadriphasic theory which includes both electric current and electric potential, and which has turned out to fit experimental data of confined swelling and compression reasonably well, using material parameters consistent with other experimental data". The mathematical model consists of a charged porous solid, a free fluid, cations, and anions.

The contribution of Frijns *et al.* (1997) is concerned with the experimental validation of the quadriphasic model of Huyghe and Janssen (1997).

Recently in her doctoral thesis van Kemenade (1998) investigated the water and ion transport through intact and damaged skin, in particular, the loss of water from the body to a dry environment. Her investigations are based on the triphasic model of Snijders *et al.* (1995).

In their contribution *Thermo-Chemo-Electro-Mechanical Formulation of Saturated Charged Porous Solids*, Huyghe and Janssen (1999) presented a theory of swelling incompressible charged porous media. The base of their work is Bowen's (1980) theory which they have adjusted to their special problem. In the evaluation of the entropy inequality, they correctly introduced the saturation condition as a constraint provided with a Lagrange multiplier which they identified as a pressure.

10.5 Some Other Fields of Application

Due to space limitations, not all fields of application can be reviewed in length as has been done in Sections 10.1 through 10.4. Therefore, only several additional applications will be described briefly.

A field of growing interest is *environmental mechanics*. Typical problems in this domain are transport of contaminants in clayey soils, debris flow, and flow of an ice-till mixture. We will restrict the short review to these examples.

In a recent paper, Kaczmarek and Hueckel (1997) reported on *Transport of contaminants of clayey soils and hydraulic barrier*. In the introduction, they stated: "The most significant parameters which determine the contaminant transport across a barrier are: hydraulic conductivity, diffusivity, and sorptive capability of clayey materials as well as the dependence of these parameters on porefluid contents. Due to the densely packed microstructure, and significant amount of immobile water adsorbed on minerals the size of pores of the clayey materials available for transport is of the order of micrometers. As a result the coefficient of water hydraulic conductivity is usually less then $10^{-6} cm/s \ldots$ Since the transport of contaminants in such materials is driven by moderate hydraulic gradients diffusion plays at least comparable role in transport as advection. At variance with the transport in groundwater mechanical dispersion is the secondary factor. As a result the time scale of the processes of migration of chemicals in the materials is long and range few hundred years." The theoretical treatment of the transport of contaminants and the consolidation problem is a little short.

Another important problem is the debris flow. Hutter *et al.* (1996) devoted an extended review article to this subject. They explained: "Broadly speaking, a debris flow represents the gravity driven flow of a mixture of various sizes of sediment (from clay to boulders), water and air, down a steep slope, often initiated by heavy rainfall and/or landslides." Hutter *et al.* (1996) first gave a survey of the existing literature and then discussed some aspects of the phenomenology of debris flow. In particular, they addressed dilatancy, internal friction and cohesion, fluidization, and particle segregation. The main section of the valuable paper is concerned with the mathematical modelling of debris flow on the basis of porous media theory. Finally, the numerical implementation of their model was discussed.

Gravity-driven shear flow of an ice-till mixture has been treated by Wu and Hutter (1999). Their model is again based on the fundamentals of the porous media theory. In this paper hints to other references, describing the respective problem, can be found.

Another field of applications is *soil physics (agriculture)*, see, e.g., Raats *et al.* (1996).

In this survey the soil structure and transport processes (implications for water, gases, nutrients, pesticides, and contaminants in soil) are investigated. In the first treatise of this contribution, Raats *et al.* stated "Transport processes in soil are of great importance to the environment of plant roots and all other types of soil-life. Transport processes in soils also determine the exchange of water, heat, gases, nutrients and contaminants between the soil and the other compartments of the environment (plants, atmosphere, groundwater and surface water). Knowledge of transport processes offers insight into the evolution of the effects of soil pollution and provides clues for prevention and rehabilitation."

Moreover, concerning the formulation of models, Raats remarked: "Several models describing transport processes are already available. In many of these models, it is assumed that soils can be considered to be rigid, unstructured, homogeneous, isotropic ... and isothermal.

In the protection of soils, the need often exists for models which also take into consideration the swelling and shrinkage, heterogeneity and anisotropy ... of the solid phase on various spatial scales, and temperature gradients. Descriptions of the exchange, between the soil and the compartments plant, atmosphere, groundwater, and surface water, are also important points for special attention. In the formulation of new models, initial inspiration is usually derived from various basic disciplines (continuum mechanics, especially of mixtures, physical chemistry and statistical physics), but also from experience accumulated with transport processes in soil, plant and atmosphere." In addition, the volume (Raats *et al.*, 1996) contains the discription of 11 projects of basic research concerning transport processes with many valuable references.

Well-known transport phenomena occur also in the field of the *petroleum industry*. Oil and natural gas, driven by, in parts, high pressure are gained from porous reservoirs. Recently, attention has been focused on foamy oil flow consisting of two phases: oil and gas. There is a special issue of the journal "Transport in Porous Media" (Maini and Hayes (eds.), 1999) devoted exclusively to these problems. In the introduction the editors stated: "The petroleum industry is keenly interested in learning more about this phenomena, as it may hold the key to the profitable exploitation of many otherwise unattractive heavy oil reservoirs." An introductory critical review by Sheng *et al.* (1999) gives an excellent survey of the problem under study with about 80 references.

Moreover, the field of *material science* should be mentioned. The powder compaction described in Section 10.2 a) can be considered as a part of material science. Other materials, which can be treated with the methods of porous media theory, are metallic foams (see, e.g., Ehlers and Eipper, 1999) and ceramic composites (see, e.g., Besmann *et al.*, 1991). All mentioned materials are used for high-tech applications and it is exactly in this field that it can be expected that other porous materials with special properties will be developed.

The porous media theory is also used in the field of *local water supplies*. Zimmer (1997) applied the Richards equations, see Richards (1931) (a combination of the partial balance equations of mass and momentum), to the retention and filtration problem of rainwater.

Finally, the mechanical behavior of plants, in particular, the analysis of *plant growth* in the continuum mechanical context, should be mentioned. First approaches have already been developed, see, e.g., Karalis (ed.) (1992) with a chapter devoted to plant growth. In this chapter Silk (1992) remarked, however: "Much work remains to understand the dynamic of growth." Indeed, in this field and also in other fields of engineering and biomechanics many problems remain unsolved and require further investigations.

Chapter 11

CONCLUSIONS AND OUTLOOK

In this book the current state of the macroscopic porous media theory and trends have been set forth and developed and numerous contributions have been cited and reviewed. The investigations concerning the fundamentals of the theory of porous media have revealed that in the last decade a consistent theory has been derived, consistent with the basic principles of continuum mechanics, in particular, the objectivity and the dissipation principles. In the elastic and the elastic-plastic ranges of the porous solids, e.g., mathematical models have been developed which are comparable to those developed in the field of the classical theory of one-component materials within the framework of the geometrically linear theory.

Moreover, based on the fundamentals of porous media theory, several applications in various fields have been successfully treated, as shown in Chapter 10, in soil mechanics, chemical engineering, biomechanics, building physics and several other fields.

Whereas the material independent relations seem to have reached their final state, the constitutive theory is still under study. For example, the viscoelastic and the viscoplastic behavior of the porous solid should, due to their importance, be investigated extensively. Moreover, the constitutive theory of the pore-fluids also needs, in some parts, improvements. This statement is also valid for phase transitions in saturated porous solids and some transport mechanisms, e.g., the capillary phenomenon. For capillary-porous media there is a lot of research work needed. First, appropriate ansätze of the free Helmholtz energy functions of the solid and gas phases must be chosen; second, with these ansätze initial and boundary value problems must be calculated in order to prove the correctness of the recently developed constitutive restrictions for the volume forces in comparison with test observations.

Moreover, in the course of completing the theory, new problems have arisen. From this point onwards only two of these will be discussed. The first problem concerns the stability of a water/steam mixture during heating and evaporation. At this point water transforms into steam. In the first step water is the carrier of steam bubbles. In the further process the number of bubbles increase until at a certain point the mixture changes considerable, namely to a mixture, where the steam is the carrier of droplets. In continuum mechanics of phase transitions in porous media it is not known, when the transformation occurs. The development of stability criteria for the onset of the transformations is a challenging task for scientists working in the field of the Theory of Porous Media.

The second problem arises when the pore size (hydraulic radius) approaches the nano ranges. It has been discussed in Section 7.5 that the capillary rise in porous bodies in continuum mechanics of porous solids has been successfully described. This new capillarity theory is valid in a wide range of the pore sizes, namely for the hydraulic radius from 1mm to approximately 30nm (1nm = 10^{-6}mm). When the hydraulic radius is smaller then 30nm, the intermolecular forces are so large that the mechanical properties of the pore content are changed considerably, for e.g., water is transformed to prestructured condensate and structured water. These types of water do not freeze under normal conditions.

As to whether the Theory of Porous Media can be applied to porous solids in the small nano range, is currently under study at the University of Essen.

There are many other issues in the Theory of Porous Media. However, with the remarks on the above two topics I will close the outlook.

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