Computational Continuum Mechanics
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Introduction to Tensors - 1
Lecture - 03
Tensor and Tensor Algebra - 1

Coming to the next topic which is, Permutation or the Alternative Symbol.

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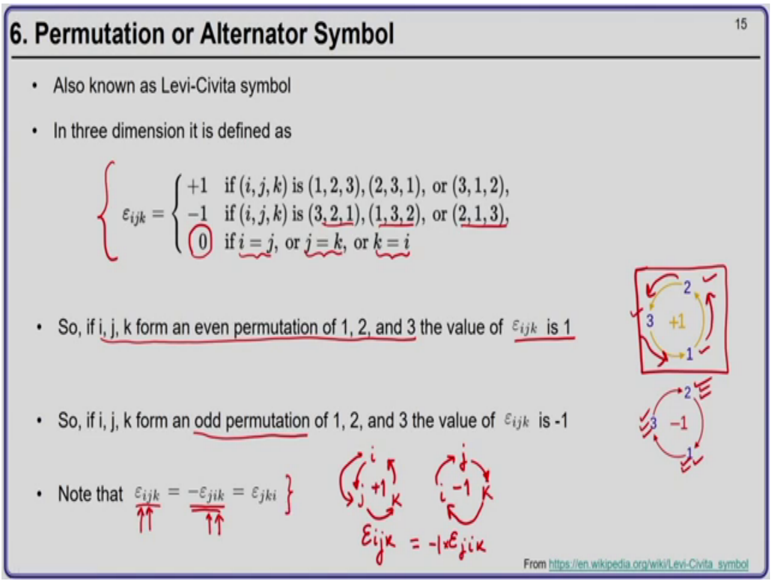

So, this is also called the Levi- Civita symbol ok. And in three dimensions it is defined as following expression which is given over here. It is represented by the symbol epsilon and it has three subscript; i j and k ok. In some of the literature you can find that instead of epsilon they also use e ok. So, it is not a standard symbol epsilon, but for this course we will stick to this symbol.

So, this permutation or the alternator symbol has 3 values ok. It is plus 1 if i j k is in this order 1 2 3 or 2 3 1 or 3 1 2 ok. It has a value of minus 1 if i j k are in this particular order 3 2 1 1 3 2 or 2 1 3. For all other values of i j k if i is equal to j or j is equal to k or if k is equal to i. If any of the index is same as the other index, then the value of this permutation symbol is 0 ok.

If i j k form an even permutation of 1 2 3 which is shown here in this particular picture ok. So, you can see we have written 1 2 3 in a particular order. And if i j k is 1 2 or 3 they follow this order or their 2 3 1 or 3 1 2, in that case epsilon i j k will take the value of plus 1; so this is called the even permutation. In the odd permutation this order gets reversed ok. So, you can see here initially it was anticlockwise direction, now the direction has become clockwise.

So, if i j k take the values 1 3 1 3 2 or 3 2 1 or 2 1 3 in that case the value of the permutation symbol will be equal to minus 1 ok. Another important property that you can yourself figure it out is that if you have epsilon i j k and if you reverse any 2 indices ok. Say for example, here if i interchange i with j in that case ok, so you have i ok, so you have j you have k ok. So, you have this particular order ok. So, the value is plus 1 ok.

Now, if you are interchange so if you are interchange i and j ok. So, what will happen? You will have j i and k ok. So, what has happened to the order? It has reverse so now, you have minus 1 ok. So, this was epsilon i j k and this is epsilon j i k ok. So, the value of epsilon i j k is minus 1, epsilon value i j k is plus 1. So, this will be equal to minus 1 times epsilon j i k and that is what you have it here ok.

And if you reverse one more time if you are interchange i and k here ok. So, then one more minus sign will come and these 2 minus signs will become plus. So, epsilon i j k is same as epsilon j k i ok. So, this property is very useful when you are deriving expressions ok and when you are going through your equations in indicial notation. So, this comes very handy.

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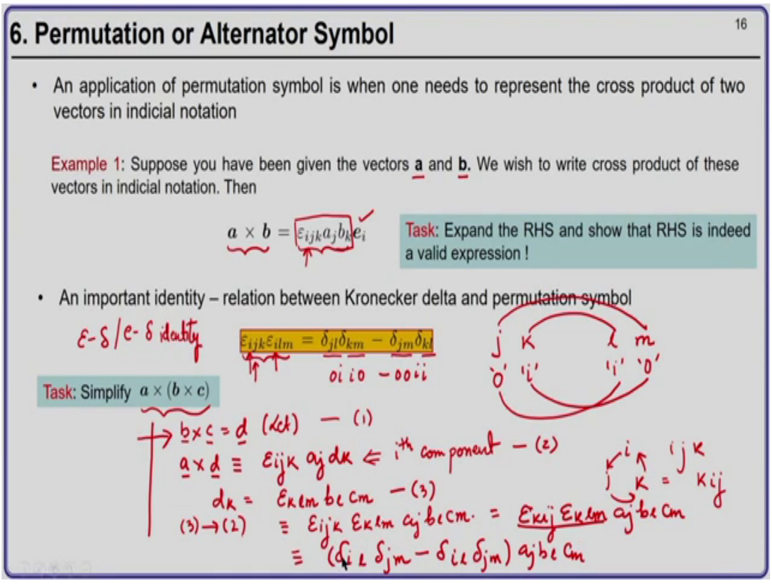

So, one of the application of permutation symbol is an one needs to express the cross product of two vectors in indicial notation ok. So, take for example, you are given two vectors a and b so a and b are two vectors and now we wish to write the cross product of these two vectors in indicial notation ok.

So, the cross product of two vectors is a cross b and you know the cross product of two vectors also gives you another vector ok. So, the way the cross product is written is epsilon i j k, b j, a j, b k, e i ok. So, the coefficient of this base vector e i basis vector e i is epsilon i j k, a j, b k ok. So, that is how the permutation symbol enters the enables you to write the cross product of two vectors in indicial notation ok.

So, now to verify that this is indeed the case what I suggest you do is; you expand the right hand side ok. So, the right hand side you can see both i I mean all 3 indices i j and k are repeated index ok. So, which means a summation is implied over these indices. So, what you can do is you can expand this in using the concepts that we discussed in the last slides. And then you can indeed show that this single expression on the right hand side is indeed what is given by a cross b ok.

So, that is a task for you. You expand the right hand side and show that the right hand side is indeed a valid expression ok. So, if you do this you will come to note that and you will appreciate that how indicial notation helps you to or write concisely very long expressions ok. So, another very important relation is the relation between the permutation symbol and the Kronecker delta ok. So, this relation is shown here and this is also called as the epsilon delta or the e delta identity ok.

So, what this identity says? This identity says that if you have a product of two alternative symbols in which at least one of the indices is common ok. So, if you look here the first permutation symbol has i and also the second permutation symbol has i at the same place and the other two indices in the first alternative symbol are j and k while in the second alternative symbol it is l and m ok.

So, using this you can convert; using this identity you can convert the expression containing two alternative symbols with at least one repeated index in terms of the Kronecker delta ok. So, that is how you do it epsilon i j k epsilon i l m is delta j l, delta k m, delta j m, delta k l ok. So, the proof of this identity we are not discussing in this course ok, but you can very well fine and if you have any question on how to do it you can always approach me ok.

So, the way to remember this identity and this identity comes very handy ok. So, the way to remember this identity is see; j is what? Ok. So, we write j k and we write l m ok. So, this we call outer, this is inner, this is inner, this is outer. So, on the right hand side if you see: delta j l which is outer inner and delta k m which is inner outer, minus delta j m which is outer outer and delta k l which is inner inner ok.

So, this is like outer inner, inner outer minus outer outer inner. So, that is one way you can remember this identity of course, when you repeat I mean when you do a lot of examples this becomes very easy to remember ok, but initially you have to remember in this way; this way it helps ok.

So, now let us look into one of the applications of the permutation symbol and how it helps us to prove certain vector identities ok. So, suppose you have been asked to simplify this expression; a cross b cross c ok. Now, you can do this very well using the concepts that you have discussed I mean that you know from your school days, but here we will like to address the simplification through indicial notation ok. Why we like to do this is because we want to practice our indicial notation ok.

So, let us start first we notice that b cross c is also a vector d ok. So, let us say let b cross c be a vector d ok. So, then we can have a cross d ok. Now, this can be written as in indicial notation as epsilon i j k, a j, d k and for this particular proof we will not write the base vector e i ok. So, because whatever is true for one component of a vector of or this quantity will be true for all other component. So, we will just take one i th component. So, this is what is the i th component ok. So, this is the i th component ok. Now, let us say the first is equation 1 the this is equation 2 ok.

Now, what will be d k? Ok. So, from equation 1 d k will be epsilon k because this correspond to the k th component epsilon k and because in expression 2 i is the free index I cannot use i here and j in expression 2 j is already occurring twice. So, I cannot use j also. So, what I will do? I will use l and m ok. So, this becomes epsilon k l m and b l c m. So, using expression 1 so now, I can put 3 in expression 2 ok. So, what I get? I get epsilon i j k epsilon k l m, a j b l cm.

Now, there are two permutation symbols and I see there is a common index k ok. So, this I can write epsilon i j k epsilon and now you have k l m you have k l m. I can write this as l m k; l m k a j b l c m. And now I can apply let us say I will just rub this, let us change the index of the first permutation symbol ok; i j k will be same as k i j k i j ok. So, I will write epsilon k i j epsilon k l m a j b l c m ok. Now, I can apply the e delta identity or the epsilon delta identity that we just discussed ok.

So, why I have done this is; is because I want to match the repeat an index in my expression with 1 which I mentioned above in the e delta identity ok. So, the two permutation symbols can be written in terms of the Kronecker delta as epsilon i k outer inner ok. No, sorry so let me rub this. Yeah epsilon i l outer inner inner outer which is inner is j outer is m for delta j m minus delta inner inner i l outer outer j m a j b l c m ok.

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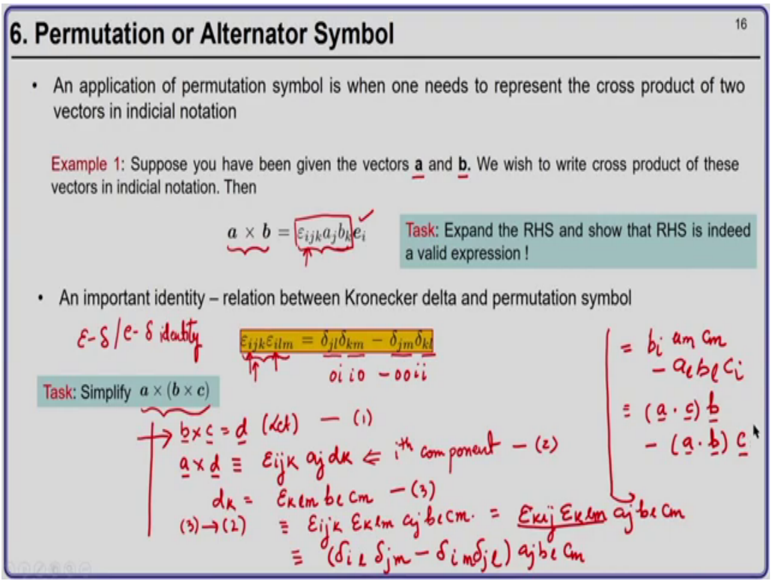

So, now let me make some space here I will rub this and I will continue from the top ok. So, now, I have the Kronecker deltas and I can open up the brackets and use the substitution property of the Kronecker delta. So, this I will leave it to you. I will write directly the final expression, you expand the brackets open up the bracket and use the with substitution property.

So, what you will have is; b i a m, c m minus you will have sorry this is i l i m ok. Sorry this is ok, let me rub this has become the same; a a l, b l, and then c i. So, what is a m, c m? This is now I will write this so I have simplified now, but this is an indicial notation. So, I have to now write in direct notation.

So, I will write a m, c m is a dot c into vector b minus a dot b into vector c ok. So, that is what this expression a cross b cross c simplifies to ok. So, following so you can see how very nicely I can use the concept of indicial notation and permutation symbol to verify certain vector identities.

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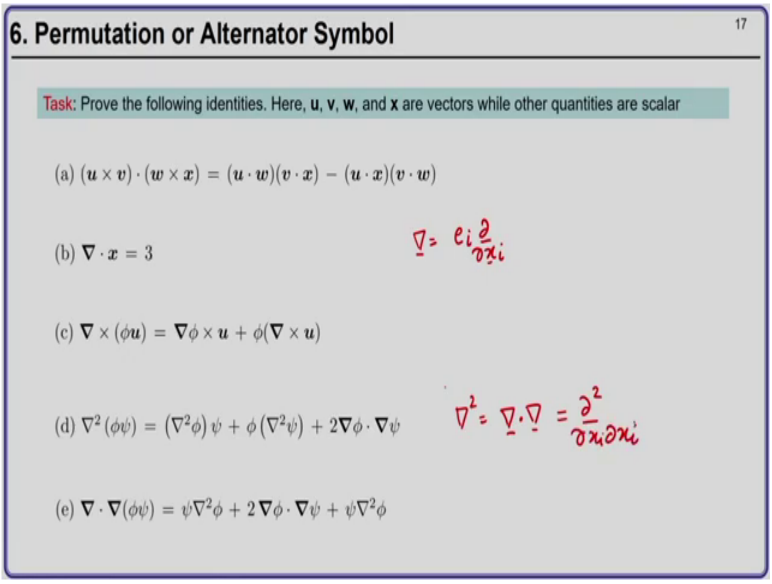

So, next you can try to prove some of the following identities here which I have given. So, there are five identities that you have to prove ok. So, you can try it yourself and if you have any doubt you can always contact me ok. Just remember delta here is ei del by del x i and because i is repeated so there is a summation which is involved ok.

And del square the fourth one del square is nothing, but del dot del which is nothing, but del square by del x i del x i ok. So, using these concepts you should be able to prove this five identities ok.

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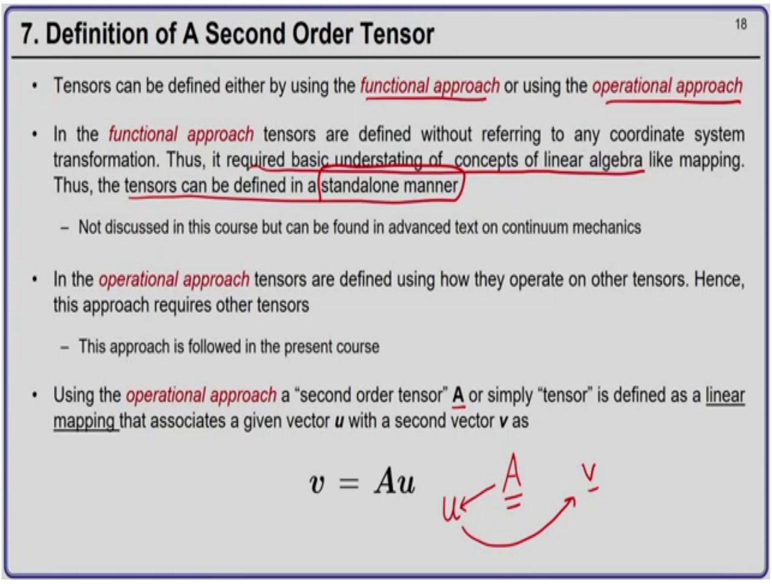

So, next we move to our main topic which is; what is meant by a second order tensor? ok. So, before defining a second order tensor you need to notice that there are two approaches to define tensors one is called the functional approach and the other one is called the operational approach ok. So, in functional approach the tensors are defined without referring to any coordinate system ok.

So, that is it requires basic understanding of the concepts of linear algebra like you need to know mapping ok. And the advantage of this approach is that the tensors can be defined in a standalone manner ok. So, tensors can be defined in a standalone manner you do not need to take any other tensors or vectors help to define the tensors ok.

But this approach is not discussed in this course this is not approached discuss in this course because it is a little more involved and because of time constraint we like to go for the second approach which is called the operational approach. So, in operational approach as the name suggests we go for defining tensors per how they operate on other tensors ok.

So, in this approach we require some other tensors to define tensors other tensors ok. So, this is the approach that we follow in this course. So, using this operational approach a second order tensor a ok. And, when we say a second order tensor sometime we interchangeably use the word tensor ok. So, and when we say tensor many time people mean only second order tensor ok. So, a second order tensor a or simply tensor is defined as a linear mapping that associates a given vector u with a second vector v ok.

So, what happens? A second order tensor it operates on this vector u and it gives you another vector v ok. So, it is operating on another vector is also a tensor a 1st order tensor because it has 1 index. So, this is called the operational approach of defining a second order tensor it operates on a vector to give you another vector ok.

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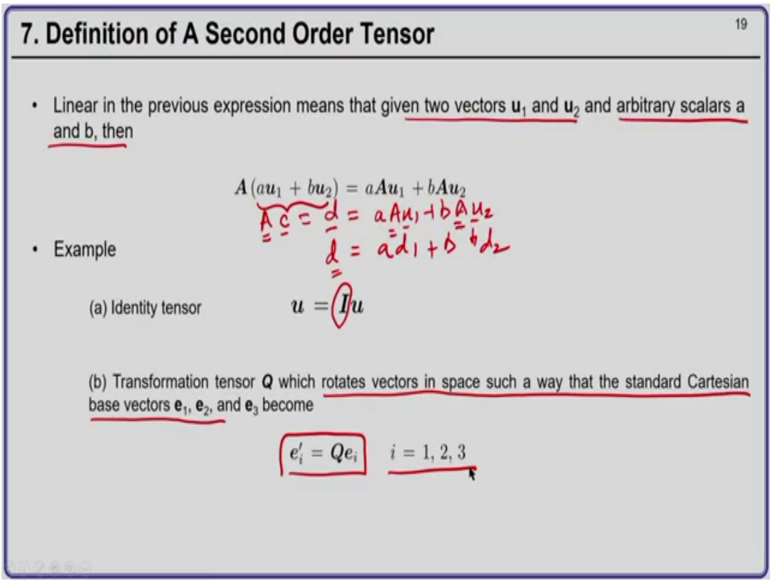

Now, let us see what is meant by the linear mapping ok. So, we saw in our introductory lectures what is meant by linear functions ok. So, similarly a linear mapping is a mapping such that given two vectors u 1 and u 2 and arbitrary scalars a and b ok.

The resulting vector obtained by taking the linear combination of u 1 and u 2 given by a u 1 plus b u 2 ok. And when this vector let us say c; when this vector c is mapped through this tensor A you get another vector d and this vector d will be same as A times a u 1 which is the first plus b times A u 2 ok.

So, when that second order tensor a maps the vector u 1 and let us say that vector is d 1. And when a u 2 a tensor a maps u 2 let us say we get vector d 2. So, d will be equal to a times d 1 plus b times d 2. So, if this property holds then the mapping is called a linear mapping. And some of the examples of second order tensor are the identity tensor for example; I is denoted by symbol I and this is a tensor second order tensor which maps the vector back to it is itself ok. That can always happen you take a vector u and the tensor maps it to itself that tensor second order tensor is called a identity tensor.

Another tensor we have already looked into is called the transformation tensor which rotates vectors in space such that the standard cartesian base vectors e 1, e 2, e 3 become e i dash equal to Q e i where i goes from 1, 2, 3 ok. So, the Q is called the transformation tensors ok, transformation tensor.

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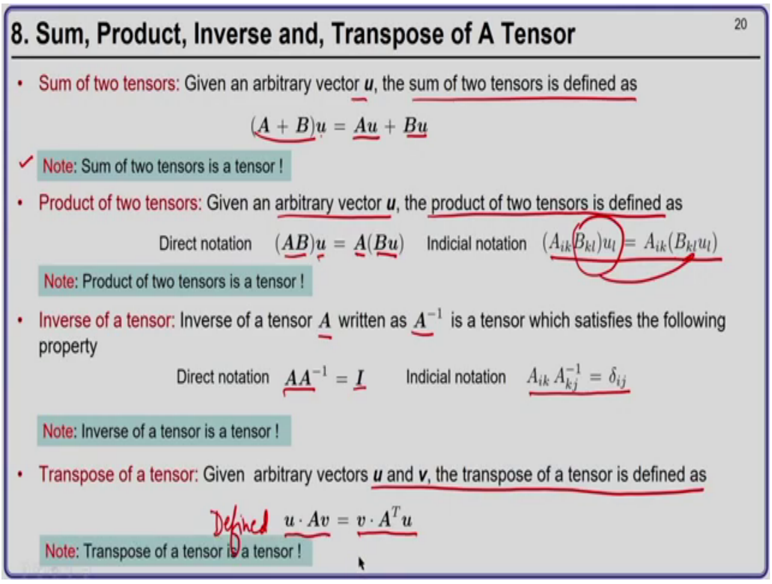

Now, there are some operations on tensors the first one is the sum of two tensors. So, thus because we are following the operational approach we will define these operations through how they operate on vectors ok. So, given an arbitrary vector u the sum of two tensors A and

B is defined as A plus B into u is same as A u plus B u ok. You remember some of two tensors is also a tensor product of two tensors.

So, given an arbitrary vector u the product of two tensors A and B is defined as A into B u is same as A into B u ok. So, this can be written in indicial notation which is given over here ok. So, A into B can be written as A i k, B k l and then you have u l. So, you can take these two terms in the bracket and you can write A i k B k l u l which is nothing, but B u.

So, will show what is the direct notation and what is the indicial notation for operations ok. So, remember product of two tensors is also a tensor. What is the inverse of a tensor? So, inverse of a tensor is usually written as A inverse. So, you have given A tensor A.

So, it is inverse is written as A inverse and this is a tensor which satisfies following property ok. So, A A inverse gives you a second order identity tensor I that is the indicial I mean direct notation. And the indicial notation is A i k A k j is delta i j. The delta i j is your Kronecker delta ok. So, inverse of a second order tensor is also a tensor.

Now, another important concept is transpose of a tensor ok. So, given arbitrary vectors u and v the transpose of a tensor is defined as u dot A v just look closely you have u dot A v that is how it is defined that is how we say the transpose of a second order tensor A is defined. So, u dot A v is v dot a transpose u. So, if a tensor for any arbitrary vectors u and v satisfy this property then this is called the transpose of the tensor A ok.

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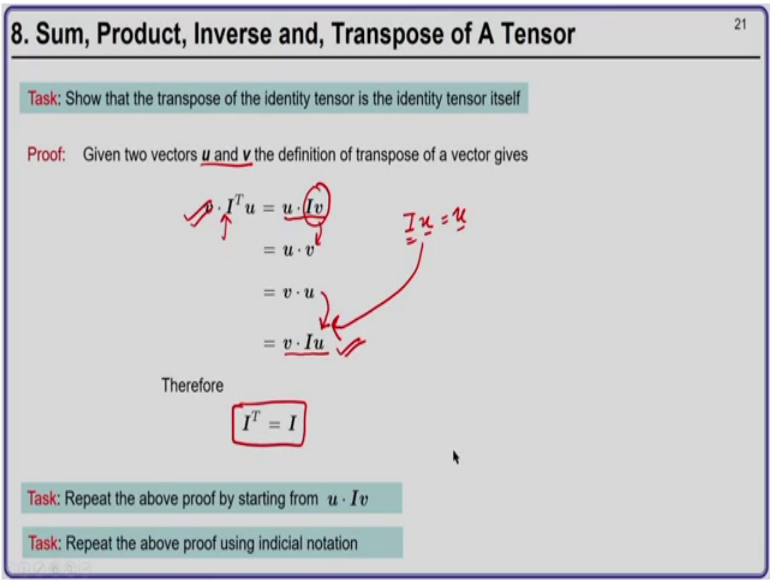

So, transpose of a tensor is also a tensor now we look into one proof ok. We will try to prove that the transpose of identity tensor is the identity tensor itself ok. So, let us start from the right hand side or the previous expression; take two arbitrary vectors u and v and we write let I transpose let I transpose be the transpose of the identity tensor ok. So, I can write v dot I transposed u. And now using the definition of transpose of a tensor I can write this expression as u dot I v ok. So, this is using the definition of the identity I mean transpose of a second order tensor.

Now, using the definition of identity tensor I can simplify I v ok. So, identity tensor is a tensor which maps a vector to itself so I v will be same as v so. Now, you have u dot v now I can interchange ok, so now, I can use the property of dot product of two vectors. So, a dot b same as b dot a so I can write u dot v as v dot u. And now I can write u as I u I can write because I u will be equal to u. So, I can use this expression and right v dot u as v dot I u ok.

So, we started with v dot I transposed u and we have ended it, ended at v dot I u. So, if we compare both the expression on the left hand side and one on the right hand side we can see that I transpose is same as I ok. So, this proves that the transpose of the identity tensor is the identity tensor itself ok.

Now, the task for you is again you start with the following expression u dot I v and try to prove that I is equal to I transpose, I have proven I transpose is equal to I you prove that I is equal to I transpose ok. And another thing is we have done this using direct notation try to repeat the same proof using indicial notation. Again, if you have any question query or doubt you can always get in touch with me ok.

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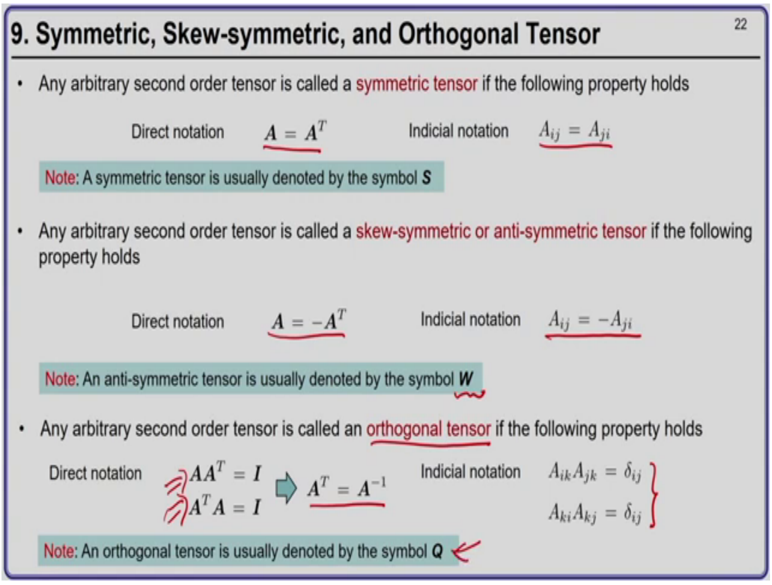

So, some other important class of tensors are called symmetric tensor. So, what is a symmetric tensor? A symmetric tensor A is 1 where, A is equal to a transpose ok. So, indicial notation we can write A i j is A j i ok. A symmetric tensor usually is denoted by symbol S ok. What is meant by a skew symmetric or anti-symmetric tensor?

So, if A is a anti symmetric tensor then A is equal to minus of a transpose or in indicial notation we can write A i j equal to minus A j i ok. So, usually anti-symmetric tensor is denoted by symbol W ok. And any arbitrary second order tensor is called an orthogonal tensor is the following property holds ok.

So, if a is a tensor which is a orthogonal tensor, then A A transpose will be equal to identity tensor or A transpose A will be equal to identity tensor ok, which means that A transpose is same as A inverse because A A inverse is identity. So, this is the indicial notation for the direct notation that we have written ok. So, an orthogonal tensor is usually denoted by symbol Q and that is how you will find in many books that symbol Q is used for orthogonal tensor ok.

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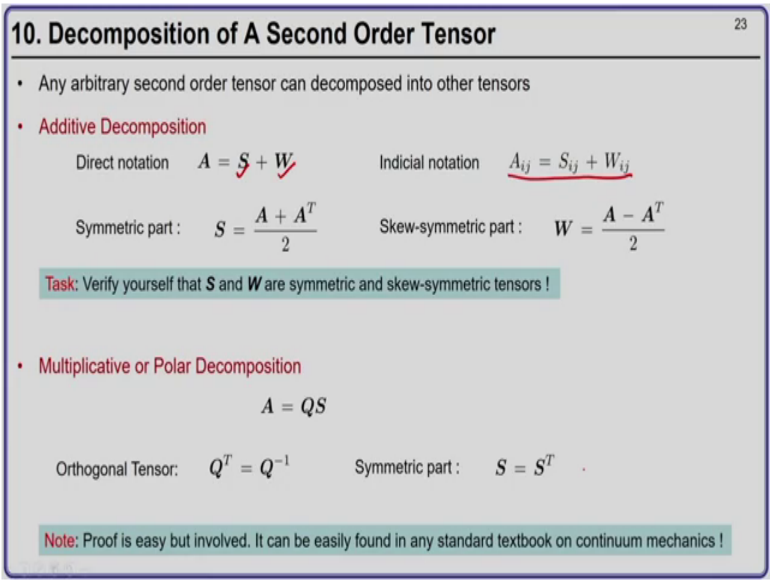

Now, a second order tensor can be any arbitrary second order tensor can be decomposed into other form of tensors ok. So, one side decomposition is called the additive decomposition, where if you are given when arbitrary tensor A it can be decomposed into what is called a symmetric tensor S and an anti symmetric tensor W ok.

In indicial notation A i j is S i j plus W i j; where S is given by A plus A transpose by 2 and W is given by A minus A transpose by 2 ok. So, you can check for yourself that S and W are indeed symmetric and anti symmetric tensor. If S is symmetric for example, then S transpose should be equal to S. So, you can take the transpose of the symmetric part and try to verify yourself ok.

Another decomposition which is a common and will come to it later in kinematics is the multiplicative or the polar decomposition. So, you have given an arbitrary tensor A. So, you can decompose that tensor into what is called an orthogonal tensor Q and a symmetric tensor S. So, A can be written multiplicatively as Q into S ok. So, Q is a orthogonal tensor, so Q transpose is equal to minus Q Q minus 1 inverse of Q. And symmetric part S is equal to S transpose ok.

So, this proof is not very difficult ok, but it is little involved and it can be found in standard texts on continuum mechanics which is skipped in this course ok. So, with this we will end and we will move to next topic ok.

Thank you.