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NERON–SEVERI GROUP FOR NONALGEBRAIC ELLIPTIC SURFACES III

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ABSTRACT

We give an Appell–Humbert type theorem for the Neron–Severi group $NS(X)/Tors\ NS(X)$ in the case of an elliptic bundle surface $X \to B$, extending a result from [4] in the case of primary Kodaira surfaces.

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1. INTRODUCTION

This paper is a natural continuation of our previous papers [2], [3]. Let $X \to B$ be a non–kählerian elliptic surface with the general fibre an elliptic curve E. It is known that every non–kählerian elliptic surface is a quasi–bundle (see, for example [3]). In this paper, by using the results from [4], [6], we precise the torsion of the Neron–Severi group of X, $Tors\ NS(X)$, and the description of the group $NS(X)/Tors\ NS(X)$ given in [3]. By using similar arguments as in [4] we give an Appell–Humbert theorem for the group $NS(X)/Tors\ NS(X)$ in the case of an elliptic bundle surface $X \to B$, extending a result from [4] in the case of primary Kodaira surfaces.

2. NERON–SEVERI GROUP FOR NON–KÄHLERIAN SURFACES

All varieties will be defined over the field $\mathbb{C}$ of complex numbers. An *elliptic surface* $\varphi : X \to B$ is a proper, connected, holomorphic map from a (compact, connected, smooth) surface X to a (compact, connected, smooth) curve B, such that the general fibre X_b ($b \in B$) is a non–singular elliptic curve (the holomorphic structure may depend on b). We shall always assume that φ is *relatively minimal*, i.e. all the fibres are free from (-1)–curves.

Let $F = \sum n_i D_i$ be a singular fibre of φ, where D_i's are the irreducible reduced components and the n_i's are their multiplicities. Let m denote the greatest common divisor of n_i's. If $m \geq 2$, then the fibre F is called *multiple fibre of multiplicity* m and we will write $F = mD$, where $D = \sum (n_i/m) D_i$.

An elliptic surface $\varphi : X \to B$ is called a *quasi–bundle* if all smooth fibres are pairwise isomorphic, and the only singular fibres are multiples of smooth (elliptic) curves. If moreover φ has no singular fibres then $\varphi : X \to B$ is said to be a *fibre bundle*.

Let E be an elliptic curve and let us consider its universal covering sequence:

$$0 \to \Gamma \to \mathbb{C} \to E \to 0, \quad \Gamma \cong \mathbb{Z}^2. \tag{1}$$

An *elliptic bundle* $\varphi : X \to B$ is a principal fibre bundle whose typical fibre and structure group are the elliptic curve E. These holomorphic fibre bundles are classified by the cohomology set $H^1(B, \mathcal{E}_B)$, where $\mathcal{E}_B$ is the sheaf of germs of local holomorphic maps from B to E. To describe $H^1(B, \mathcal{E}_B)$, one uses the exact cohomology sequence

$$H^1(B, \Gamma) \to H^1(B, \mathcal{O}_B) \to H^1(B, \mathcal{E}_B) \xrightarrow{c} H^2(B, \Gamma) \to 0\ , \tag{2}$$

induced by (1); see, for example, [1], Chapter V. 5.

It is known that every non–kählerian elliptic surface $\varphi : X \to B$ is a quasi–bundle; see, for example [3], Lemma 1. Let $m_1 D_1, ..., m_l D_l$ be all the multiple fibres of φ, where D_i is an elliptic curve and $m_i \geq 2$ for all $i = 1, ..., l$. By [3] Theorem 5, we know that the group $NS(X)\otimes\mathbb{Q}$ is isomorphic to the group $Hom(J_B,E)\otimes\mathbb{Q}$, where J_B is the Jacobian variety of the curve B and $Hom(J_B,E)$ is the group of the homomorphisms of abelian varieties. A more precise statement can be given:

THEOREM 1. *Let $\varphi : X \to B$ be a non–kählerian elliptic surface. Then the torsion group of the Neron–Severi group is generated by $c_1(D_i)$, $i = 1,...,l$, and we have an isomorphism*

$$NS(X)/Tors\ NS(X) \cong Hom(J_B, E^\vee),$$

where J_B is the Jacobian variety of the curve B, $E^\vee$ is the dual curve of E (isomorphic to E) and $Hom(J_B,E^\vee)$ is the group of homomorphisms of abelian varieties.

Proof. Let a_i be an element of $\frac{1}{m_i}\Gamma$ such that the order of the point $[a_i]$ of the torus E corresponding to a_i is precisely $m_i \geq 2$, the multiplicity of the singular fibre m_iD_i, $i = 1,...,l$. Let Γ_0 be a lattice in $\mathbb{C}$ generated by Γ and a_i, $i = 1,...,l$ and put $E_0 = \mathbb{C}/\Gamma_0$. Then, the elliptic curve E_0 is isomorphic to E/H, where H is a subgoup of E generated by $[a_1],...,[a_l]$. Let $h : E \to E_0$ be the canonical surjection. Then, by [6], Lemma 12 there exists an elliptic bundle $Y \to B$ with the fibre the elliptic curve E_0 and a holomorphic mapping $f : X \to Y$ such that the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ B & \xrightarrow{id_B} & B \end{array}$$

is commutative and f is unramified outside the multiple fibres. By [6], Theorem 17, we have an exact sequence:

$$0 \to Hom(J_B,E^\vee) \to NS(X)/\tilde{F}_2 \to \tilde{N}(X) \to 0, \qquad (3)$$

where $\tilde{F}_2$ is a finite subgroup (X is non–kähler) of $H^2(X,\mathbb{Z})$ generated by $c_1(D_i)$, $i = 1,...,l$ and $\tilde{N}(X)$ is a subgroup of $NS(E)$. Moreover, $\tilde{N}(X) \cong h^*\tilde{N}(Y)$, where $\tilde{N}(Y)$ is the corresponding subgroup of $NS(E_0)$. By Theorem 3.1 in [2] and Theorem 5 in [6], we get $\tilde{N}(Y) = 0$ and, therefore, $\tilde{N}(X) = 0$. It follows that the subgroup $Tors\ NS(X)$ of the Neron–Severi group $NS(X)$ is generated by $c_1(D_i)$, $i = 1,...,l$ and we have an isomorphism $NS(X)/Tors\ NS(X) \cong Hom(J_B,E^\vee)$, where J_B is the Jacobian variety of the curve B, $E^\vee$ is the dual torus of E (isomorphic to E) and $Hom(J_B,E^\vee)$ is the group of homomorphisms of abelian varieties.

3. APPELL–HUMBERT THEOREM FOR ELLIPTIC BUNDLES

Let $\varphi : X \to B$ be an elliptic bundle over the curve B with the fibre an elliptic curve E defined by $\xi \in H^1(B, \mathcal{E}_B)$ with $c(\xi) \neq 0$ (see the exact sequence (2)). By [1], Chapter V, Proposition 5.3, we know that X is non–kählerian. If B is an elliptic curve, X is a primary Kodaira surface. In [4], Lemma 5, we gave a description of the group $NS(X)/Tors\ NS(X)$ for $X \to B$ a primary Kodaira surface, similar to the Appell–Humbert Theorem for complex tori (see for example [10] or [5]). Here, we shall extend this result for any elliptic bundle over a curve B. So, let us suppose that the genus of B, g is greater than 2.

The fundamental group of the fibre E is isomorphic to the lattice $\Gamma \subset \mathbb{C}$ generated by $\{\beta_1,\ \beta_2\}$, the fundamental group Λ of the base B is generated by $\{\mu_1, ..., \mu_g, \lambda_1, ..., \lambda_g\}$ with one relation $\prod_{i=1}^{g}[\mu_i,\ \lambda_i] = 1$, and we have the following central extension:

$$0 \to \Gamma \xrightarrow{j} G \xrightarrow{\pi} \Lambda \to 0, \tag{4}$$

where G is the fundamental group $\pi_1(X)$ of the elliptic bundle $X \to B$ (see, for example [7], Chapter II, Lemma 7.3). In fact, the group G has a presentation of the following form:

$$G =< \tilde{\mu}_1, ..., \tilde{\mu}_g, \tilde{\lambda}_1, ..., \tilde{\lambda}_g, \tilde{\beta}_1, \tilde{\beta}_2 | \prod_{i=1}^{g}[\mu_i,\ \lambda_i] = \tilde{\beta}_1^m, \tilde{\beta}_1, \tilde{\beta}_2 \text{ central} > . \tag{5}$$

Of course, $j(\beta_i) = \tilde{\beta}_i,\ i = 1, 2$ and $\Gamma \cong j(\Gamma)$ is in the center of G. We shall identify Γ with its image $j(\Gamma)$ in G and let $s : \Lambda \to G$ be a *cross–section*, i.e. $\pi \circ s = 1_\Lambda$ and the group Λ is identified with G/Γ (we can choose the cross–section such that $s(\mu_i) = \tilde{\mu}_i$ and $s(\lambda_i) = \tilde{\lambda}_i,\ i = 1, ..., g$). Then the elements of G can be uniquely written in the form $\gamma s(\lambda)$ where $\gamma \in \Gamma$ and $\lambda \in \Lambda$. The sum $s(\lambda)s(\lambda')$ must lie in the same coset as $s(\lambda\lambda')$, so there are unique elements $h_0(\lambda, \lambda') \in \Gamma$ such that always

$$s(\lambda)s(\lambda') = h_0(\lambda, \lambda')s(\lambda\lambda').$$

Now, consider $\mathbb{Z}$ as a trivial G–module and let

$$res : H^2(G, \mathbb{Z}) \longrightarrow H^2(\Gamma, \mathbb{Z}) \tag{6}$$

be the restriction homomorphism. Because $\mathbb{Z}$ is a trivial G–module, the inflation homomorphism has the form:

$$inf : H^2(\Lambda, \mathbb{Z}) \longrightarrow H^2(G, \mathbb{Z}). \tag{7}$$

Let $\Pi_{J_B} = (\Pi, I_g) \in \mathcal{M}_{g,2g}(\mathbb{C})$ be a normalised period matrix for the Jacobian variety of the curve B. We know from [2], Theorem 3.1 that the group $NS(X)/Tors\ NS(X)$

is isomorphic to the group $Hom(J_B, E)$ of homomorphisms of abelian varieties. Now, by using some results in cohomology of groups (see [4]), we obtain the following Appell–Humbert type result:

THEOREM 2. *Let $X \to B$ be an elliptic bundle over the curve B with the fibre an elliptic curve E, defined by $\xi \in H^1(B, \mathcal{E}_B)$ with $c(\xi) \neq 0$. Then the group $NS(X)/\, Tors NS(X)$ is isomorphic to the subgroup of $\mathcal{M}_{2,2g}(\mathbb{C})$*

$$\mathcal{NS} := \left\{ \mathcal{A} = \begin{pmatrix} A_1 & \dots & A_g & C_1 & \dots & C_g \\ B_1 & \dots & B_g & D_1 & \dots & D_g \end{pmatrix} = (\alpha \mid \theta) \in \mathcal{M}_{2,2g}(\mathbb{Z}) : \right.$$

$$\left. (\Pi\, {}^t\theta - {}^t\alpha) \begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix} = 0 \right\}.$$

Proof. By using the Lyndon spectral sequence

$$E_2^{pq} = H^p(\Lambda, H^q(\Gamma, \mathbb{Z})) \Rightarrow H^{p+q}(G, \mathbb{Z}),$$

defined by the exact sequence (4) one obtains as in Lemma 3 in [4] that the restriction map (6) is zero. From Lemma 2 in [4], we get a homomorphism v such that the sequence

$$H^2(\Lambda, \mathbb{Z}) \stackrel{inf}{\to} H^2(G, \mathbb{Z}) \stackrel{v}{\to} H^1(\Lambda, H^1(\Gamma, \mathbb{Z})) \tag{8}$$

is exact. As in Lemma 4 in [4] one obtains that

$$NS(X)/Tors\ NS(X) \hookrightarrow H^1(\Lambda, H^1(\Gamma, \mathbb{Z}))$$

and the canonical surjection $NS(X) \to NS(X)/Tors\ NS(X)$ can be identified with the restriction of the homomorphism v to $NS(X)$. Thus we get the isomorphism

$$NS(X)/Tors\ NS(X) \cong Hom(J_B, E^\vee), \tag{9}$$

where $E^\vee$ is the dual of E, $E^\vee = Pic^0(E) = \mathbb{C}'/\Gamma'$ and $\Gamma' = Hom_{\mathbb{Z}}(\Gamma, \mathbb{Z}) \cong H^1(\Gamma, \mathbb{Z})$ is the dual lattice in the "complex space" $\mathbb{C}' = Hom_{\mathbb{Z}}(\Gamma, \mathbb{R})$ (see, for example [8], 1.4). Because Γ is a lattice in $\mathbb{C}$, we can extend uniquely any $f \in Hom_{\mathbb{Z}}(\Gamma, \mathbb{R})$ to a real linear map $\tilde{f} : \mathbb{C} \to \mathbb{R}$. Thus

$$Hom_{\mathbb{Z}}(\Gamma, \mathbb{R}) \cong Hom_{\mathbb{R}}(\mathbb{C}, \mathbb{R}) \cong \mathbb{C}'$$

and we put a complex structure on this real vector space defining $if(\gamma) := -\tilde{f}(i\gamma)$, $\gamma \in \Gamma$. Let Λ_1 be the lattice in $\mathbb{C}^g$ defined by Λ ($\Lambda_1 = H_1(B, \mathbb{Z})$). Then, $J_B \cong \mathbb{C}^g/\Lambda_1$. By [10], p. 175, we have the isomorphism

$$NS(X)/Tors\ NS(X) \cong \{h : \mathbb{C}^g \to \mathbb{C}' : h \text{ is } \mathbb{C}-\text{linear}, h(\Lambda_1) \subset \Gamma'\}.$$

Let $\{\beta'_1, \beta'_2\} \subset \Gamma'$ be the dual basis of $\{\beta_1, \beta_2\}$ (i.e. $\beta'_i(\beta_j) = \delta_{ij},\ i,j = 1,2$). We have chosen a normalised period matrix $\Pi_{J_B} = (\Pi \mid I_g)$ for the Jacobian J_B, where $\Pi,\ I_g \in \mathcal{M}_g(\mathbb{C})$ and the period matrix for $E^\vee$ is $\Pi' = (\beta'_1, \beta'_2) \in \mathcal{M}_{1,2}(\mathbb{C}')$. The rational representation $\rho_r(h)$ is given by a matrix

$$\mathcal{A} = (\alpha \mid \theta) = \begin{pmatrix} A_1 & ... & A_g & C_1 & ... & C_g \\ B_1 & ... & B_g & D_1 & ... & D_g \end{pmatrix} \in \mathcal{M}_{2,2g}(\mathbb{Z})$$

and the analytic representation $\rho_a(h)$ is given by a matrix $A = (a_1 a_2 ... a_g) \in \mathcal{M}_{1,g}(\mathbb{C}')$. Then we have the equality:

$$A\Pi_{J_B} = \Pi'\mathcal{A} \tag{10}$$

(see, for example, [9], p. 10).

Let $a_j = t_{j1}\beta'_1 + t_{j2}\beta'_2$ with $t_{j1},\ t_{j2} \in \mathbb{R}$ and put $T_j = t_{j2}\beta_1 - t_{j1}\beta_2,\ j = 1,2,...,g$. Let $T = {}^t(T_1 T_2 ... T_g) \in \mathcal{M}_{g,1}(\mathbb{C})$. By computation, we get from (10) the equations:

$$\Pi T = {}^t\alpha \begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix},\ I_g T = {}^t\theta \begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix}. \tag{11}$$

Finally, one obtains the relation

$$\left(\Pi\, {}^t\theta - {}^t\alpha\right) \begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix} = 0, \tag{12}$$

i.e. g relations for the elements of the period matrices. Thus we get the desired isomorphism

$$NS(X)/\, Tors\ NS(X) \cong \mathcal{NS}.$$

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