CS 2742 (Logic in Computer Science) – Winter 2014
Lecture 19

Antonina Kolokolova

March 12, 2014

Recall the definition of the induction from last lecture:

$$\text{for all predicates } P, (P(a) \land \forall k(P(k) \to P(k+1))) \to \forall n \geq a, P(n)$$

Structure of a proof by induction:

1) **Predicate** State which $P(n)$ you are proving as a function of n.
2) **Base case:** Prove $P(a)$.
3) **Induction hypothesis:** State "Assume $P(k)$ holds" explicitly.
4) **Induction step:** Show how $P(k) \to P(k+1)$. That is, assuming $P(k)$ derive $P(k+1)$.

In this lecture, we will see several examples of using induction to prove various statements.

Example 1. Show that for all $n \geq 0$, $0 + 1 + \cdots + n = n(n+1)/2$. This is a classical example of application of math. induction.
Proof: Predicate: $P(n) = 0 + 1 + \cdots + n = n(n+1)/2$.
Base case: $n = 0$, then $0 = 0 \cdot (1/2)$. Let's also check $n = 1$: $0 + 1 = 1 = 1 \cdot (1+1)/2$
Induction hypothesis: Assume that for some $k \geq 0$ $0 + 1 + \cdots + k = k(k+1)/2$.
Induction step: Show that $P(k) \to P(k+1)$. Take $0 + 1 + \cdots + k + (k+1) = (0 + 1 + \cdots + k) + (k+1)$. By induction hypothesis, the sum in the first parentheses is $k(k+1)/2$. Now, $k(k+1)/2 + (k+1) = \frac{k(k+1)+2(k+1)}{2} = (k+2)(k+1)/2 = (k+1)(k+2)/2$, which is exactly the right hand side of $P(k+1)$.
Therefore, by induction, $\forall n \geq 0$, $0 + 1 + \cdots + n = n(n+1)/2$.

Note that in this case, the calculations would be slightly simpler if we would state the induction hypothesis and induction step as "Assume $P(k-1)$, prove $P(k)$". This is a valid argument, and is often used, as long as $k-1$ satisfies the restriction on n (in this case, $k-1 \geq 0$).

Example 2. Recall the following question: show that every amount of change ≥ 8 can be paid with only 3c and 5c coins. This time we will prove this using induction
Let $P(n) : \exists i, j \geq 0 \; n = 3i + 5j$
Base case: Let $n = 8$. Then $n = 3 + 5$, $i = j = 1$. For this method of solving the problem, it is also convenient to have a base case $n = 9 = 3 \cdot 3$, $i = 3$, $j = 0$.
Induction hypothesis: Assume that $\exists i, j \geq 0 \; k = 3i + 5j$. This assumption gives us the i and j which we will be using in the induction step.
Induction step: We want to show that $\exists i', j' \geq 0$ such that $k + 1 = 3i' + 5j'$. Look at i and j given to us by induction hypothesis, that is, i and j such that $k = 3i + 5j$. Consider the following two cases.
Case 1: $j > 0$. That is, at least one 5c coin was used to make k. Then we can replace this 5c coin with two 3c coins to get $k + 1$. That is, $i' = i + 2$ and $j' = j' - 1$, so $k + 1 = 3i' + 5j' = 3(i + 2) + 5(j - 1)$.
Case 2: $j = 0$. Suppose that there was no 5c coin used to make up k, that is, $k = 3i$ for some i. Since $k \geq 8$, $i \geq 3$. Now, to make $k + 1$ we can take three 3c coins out of i used to make up k and replace them by two 5c coins. That is, $i' = i - 3$ and $j' = 2$. Since $i \geq 3$, $i' \geq 0$, and $k + 1 = 3i' + 5j'$. This completes the proof.

Note how here we actively used the values i and j, existence of which was given to us by the induction hypothesis, to build our new i and j existence of which we were proving in the induction step. This is one reason why it is good to write out the induction hypothesis: to see the values that are available to be used in the induction step.

Example 3. Here is an example of proving an unequality by induction: $n^2 \leq 2^n$ for $n > 3$. You have seen this already in the assignment: this unequality says that for large enough numbers the size of a powerset 2^A is always larger than the size of a Cartesian product of a set with itself $A \times A$. Another way of looking at this unequality is from the algorithmic point of view: it says that for large enough input size n, an algorithm that runs in time $O(n^2)$ always runs faster than an algorithm that runs in time $O(2^n)$ and is not in $O(n^2)$.

We will prove this unequality by induction.
Predicate $P(n) : \; n^2 \leq 2^n$.
Base case: $P(4) : 4^2 = 16 \leq 2^4$
Induction hypothesis: assume that for $k > 3$, $k^2 \leq 2^k$.
Induction step: Assuming $P(k)$, prove that $(k+1)^2 \leq 2^{k+1}$.
First, $(k+1)^2 = k^2 + 2k + 1$ and $2^{k+1} = 2 \cdot 2^k = 2^k + 2^k$. By induction hypothesis, $k^2 + 2k + 1 \leq 2^k + 2k + 1$. It remains to show that $2k + 1 \leq 2^k$, where this is the second "copy" of 2^k in $2^k + 2^k$ expression. We could prove this by doing another induction proof, but in this case it can be done easier. Notice that it is sufficient to show that $2k + 1 \leq k^2$,

because then by induction hypothesis we will get $2k+1 \leq k^2 \leq 2^k$. To see that $2k+1 \leq k^2$, divide both sides of the unequality by k. Since k is positive, this preserves the unequality, resulting in $2 + 1/k \leq k$. But we do know that $k > 3$, so $2 + 1/k < 3 < k$. Therefore, $2k + 1 \leq k^2 \leq 2^k$, and so $k^2 + (2k + 1) \leq 2^k + 2^k$, completing the proof.

Puzzle 1. What is wrong with the following induction proof of: "All horses are white"?

1) Let $P(n)$ be: any n horses are white.
2) Base case: 0 horses are white.
3) Ind. hyp.: if any set of k horses are white, then any set of $k+1$ horses are white.

Let the proof of the induction step be as follows. Take a set of $k+1$ horses. Remove one horse; by induction hypothesis, all remaining horses are white. Now, put that horse back in and remove another horse. The remaining horses are again white by induction hypothesis, and so is the horse we took out the first time. Therefore, these $k+1$ horses are white, and as it was an arbitrary set of $k+1$ horses, induction step holds.

Therefore, all horses are white.