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On a Generalization of Groups with All Subgroups Subnormal.

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ABSTRACT - In this article we prove the following result: Let G be a Fitting p-group. If for every proper subgroup H of G, $H^G \neq G$ and $H^{(n)}$ is hypercentral for a non-negative integer n, then $G' \neq G$.

1. Introduction.

We first consider the following question:

Let G be a group with $H^G \neq G$ for every proper subgroup H of G. Is $G' \neq G$?

Obviously groups having the property in the question are a generalization of groups with all subgroups subnormal. The class of all groups in which every proper subgroup has a proper normal closure is considered in [7]. In [4] Möhres proved that if G is a group with all subgroups subnormal then G is soluble. We mainly exploit Möhres' ideas in the proofs of our results. Many remarkable results can be found in the Literature related to groups with all subgroups subnormal which we do not include here.

In this article we prove the following result:

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THEOREM. *Let G be a Fitting p-group. If for every proper subgroup H of G, $H^G \neq G$ and $H^{(n)}$ is hypercentral for a non-negative integer n, then $G' \neq G$.*

2. Proper normal closures.

Let $\phi_0(x) = x$. Define

$$\phi_i(x_1, \dots, x_{2^i}) = [\phi_{i-1}(x_1, \dots, x_{2^{i-1}}), \phi_{i-1}(x_{2^{i-1}+1}, \dots, x_{2^i})].$$

Now a group G is soluble if and only if there is a positive integer n such that $\phi_n(G) = 1$.

Our results are closely related to the class of groups with the following property: for every given sequence $x_1, x_2, \dots$ of elements in the group there exists a natural number d such that

$$\phi_d(x_1, \dots, x_{2^d}) = 1\,.$$

Let denote the class of groups having the above property by $\wp$. Now we give some properties of this class of groups.

LEMMA 2.1. *Let G be a group. If every given sequence $x_1, x_2, \dots$ of elements in G there exists a natural number d such that*

$$\phi_d(x_1, \dots, x_{2^d}) = 1\,,$$

then G is hyperabelian and $G' \neq G$.

PROOF. It is enough to prove that G contains a nontrivial normal abelian subgroup for the first part of the assertion. Suppose that this is not the case. Clearly there exist elements $x_1, x_2 \in G$ such that $\phi_1(x_1, x_2) = [x_1, x_2] \neq 1$. Assume that we have found elements $x_1, x_2, \dots, x_{2^i}$ in G such that

$$y = \phi_i(x_1, \dots, x_{2^i}) \neq 1\,.$$

Now by the assumption $\langle y \rangle^G$ cannot be soluble. This implies that

$$[\langle y \rangle, (\langle y \rangle^G)^{(i)}] \neq 1\,.$$

Hence there exist elements $x_{2^i+1}, \dots, x_{2^{i+1}}$ in $\langle y \rangle^G$ such that

$$\phi_{i+1}(x_1, \dots, x_{2^{i+1}}) = [\phi_i(x_1, \dots, x_{2^i}), \phi_i(x_{2^i+1}, \dots, x_{2^{i+1}})] \neq 1\,.$$

This means that there exists a sequence of elements $x_1, x_2, \dots$ in G such that $\phi_i(x_1, \dots, x_{2^i}) \neq 1$ for every natural number i. But this is a contradiction.

Now assume that $G' = G$, then $Z_2(G) = Z_1(G)$ by Grün's Lemma, i.e., $Z(G/Z(G)) = 1$. Since the property given in the hypothesis is inherited to homomorphic images we may assume that $Z(G) = 1$. Let x_1, x_2 be elements in G such that $\phi_1(x_1, x_2) = [x_1, x_2] \neq 1$, as above. Assume that we have found elements $x_1, x_2, \dots, x_{2^i}$ in G such that

$$y = \phi_i(x_1, \dots, x_{2^i}) \neq 1 .$$

Now we can find an element x in G such that $[x, y] \neq 1$. Since G is perfect, $x \in G^{(i+1)}$ and hence we can find elements $x_{2^i+1}, \dots, x_{2^{i+1}}$ in G such that

$$\phi_{i+1}(x_1, \dots, x_{2^{i+1}}) =$$

$$= [y, \phi_i(x_{2^i+1}, \dots, x_{2^{i+1}})] = [\phi_i(x_1, \dots, x_{2^i}), \phi_i(x_{2^i+1}, \dots, x_{2^{i+1}})] \neq 1 .$$

Thus we see that there exist elements $x_1, x_2, \dots$ in G such that $\phi_i(x_1, \dots, x_{2^i}) \neq 1$ for every natural number i, a contradiction.

Let G be the direct product of a non-soluble hypercentral p-group (Example 6.12 of [8] for example) and a non-hypercentral soluble p-group of derived length n (see Example 6.10 of [8]). Then $G^{(n)}$ is hypercentral and proof of the theorem shows that G has the above property. This example shows that the class of all soluble groups and the class of all hypercentral groups are proper subclasses of $\wp$ and $\wp$ is a proper subclass of the class of all hyperabelian groups, since McLain's group $M(\mathbb{Q}, F)$ where F is a field of characteristic $p > 0$ (see 12.1.9 of [6]) is an example of a perfect hyperabelian group.

Now we give some auxiliary lemmas to prove the theorem.

LEMMA 2.2. *Let G be a locally nilpotent perfect p-group. Suppose that for every proper subgroup H of G and for every given sequence of elements $x_1, x_2, \dots$ in H there exists a natural number d such that*

$$\phi_d(x_1, \dots, x_{2^d}) = 1$$

and $H^G \neq G$. Then there exist a proper normal subgroup N, a finite subgroup U of G such that

$$\bigcap_{y \in G \setminus N} \langle U, y \rangle \neq U$$

and $Z(G/N) = 1$.

PROOF. Assume that the assertion is false. By the first part of the proof of Lemma 4 of [4], for every finite subgroup U of G, element $a \in G \setminus U$, proper subgroup T of G and outer commutator word $\phi(x_1, \dots, x_n)$ (see [4] for the definition) there exist $y_1, \dots, y_n$ in G such that $\phi(y_1, \dots, y_n) \notin T$ and $a \notin \langle U, y_1, \dots, y_n \rangle$. Let K be a proper subgroup of G. If we put $Z(G/K^G) = Z/K^G$ then $Z \neq G$ and $Z(G/Z) = 1$, since G is perfect. Now

$$\bigcap_{z \in G \setminus Z} \langle z \rangle = 1$$

by hypothesis. If a is a non-trivial element in G then there exists an element z_1 in $G \setminus Z$ such that $a \notin \langle z_1 \rangle$. Assume that we have found elements $z_2, \dots, z_{2^n}$ in G such that $\phi_n(z_1, \dots, z_{2^n}) \notin Z$ and $a \notin \langle z_1, \dots, z_{2^n} \rangle$. Now there exist elements $z_{2^n+1}, \dots, z_{2^{n+1}}$ in G such that

$$\phi_n(z_{2^n+1}, \dots, z_{2^{n+1}}) \notin C_G(\phi_n(z_1, \dots, z_{2^n})\, Z)$$

and $a \notin \langle z_1, \dots, z_{2^n}, z_{2^n+1}, \dots, z_{2^{n+1}} \rangle$, since $Z(G/Z) = 1$. This implies that

$$\phi_{n+1}(z_1, \dots, z_{2^n}, z_{2^n+1}, \dots, z_{2^{n+1}}) = \\ = [\phi_n(z_1, \dots, z_{2^n}), \phi_n(z_{2^n+1}, \dots, z_{2^{n+1}})] \notin Z\,.$$

Put $X = \langle z_i \colon i = 1, 2, \dots \rangle$. Then X is a proper subgroup of G, since $a \notin X$ and $\phi_d(z_1, \dots, z_{2^d}) \neq 1$ for all natural number d. But this is a contradiction.

The following lemma is a version of (5) Lemma of [4].

LEMMA 2.3. *Let G be a Fitting p-group. If $H^G \neq G$ and for every given sequence of elements $x_1, x_2, \dots$ in H and for every proper subgroup H of G there exists a natural number d such that*

$$\phi_d(x_1, \dots, x_{2^d}) = 1\,,$$

then $G' \neq G$.

PROOF. Assume that $G' = G$. By Lemma 2.2

$$\bigcap_{y \in G \setminus N} \langle U, y \rangle \neq U$$

for a finite subgroup U of G and for a proper normal subgroup N of G

such that $Z(G/N) = 1$. Let

$$a \in \Big(\bigcap_{y \in G \setminus N} \langle U, y \rangle \Big) \setminus U$$

and put $M = \langle U, a \rangle^G N$ and $Z(G/M) = Z/M$. Now $Z(G/Z) = 1$. Since G/Z is hyperabelian there exists an element y in G such that $\langle y \rangle^G Z/Z$ is an infinite elementary abelian p-group. Now $\langle y, a, U \rangle^G$ is nilpotent and $\langle y, a, U \rangle^G / (\langle y, a, U \rangle^G \cap Z)$ is an infinite elementary abelian p-group. Put $R = \langle y, a, U \rangle^G \cap Z$. By (6) Satz of [3] there exists a subgroup V of $\langle y, a, U \rangle^G$ such that $U < V$, $a \notin V$ and VR/R is infinite. But then there exists an element $v \in V \setminus N$ such that $a \notin \langle U, v \rangle$, a contradiction.

PROOF OF THE THEOREM. Since $H^{(n)}$ is hypercentral, there exists an ordinal β such that

$$1 = Z_0(H^n) \lhd Z_1(H^{(n)}) \lhd \dots Z_\beta(H^{(n)}) = H^{(n)} \lhd H .$$

Put $Z_\gamma(H^{(n)}) = L_\gamma$ for all ordinals $\gamma \leqslant \beta$ and $L_{\beta+1} = H$. Let $x_1, x_2, \dots$ be a given sequence of elements in H. Assume that $\phi_i(x_1, \dots, x_{2^i}) \neq 1$ for all i. Put $y_1 = \phi_n(x_1, \dots, x_{2^n})$. Then there exists an ordinal α_1 such that $y_1 \in$ $\in L_{\alpha_1+1} \setminus L_{\alpha_1}$. Now we have

$$y_2 = \phi_{n+1}(x_1, \dots, x_{2^{n+1}}) = [y_1, \phi_n(x_{2^n+1}, \dots, x_{2^{n+1}})] \in L_{\alpha_1}$$

and thus there exists an ordinal α_2 such that $\alpha_1 > \alpha_2$. By continuing in this way we see that $\alpha_1 > \alpha_2 > \dots$ is an infinite descending chain of ordinals, a contradiction. Now we have that $\phi_d(x_1, \dots, x_{2^d}) = 1$ for a natural number d. By Lemma 2.3 $G' \neq G$.

COROLLARY. *Let G be a Fitting p-group and let for every proper subgroup H of G,* $H^G \neq G$. *Then,*

(i) *if every proper subgroup of G is soluble then G is soluble.*

(ii) *if every proper subgroup of G is hypercentral then* $G' \neq G$.

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REFERENCES

[1] A. O. ASAR, *Locally nilpotent p-groups whose proper subgroups are hypercentral or nilpotent-by-Chernikov*, J. London Math. Soc., **2**, 61 (2001), pp. 412-422.

[2] W. MÖHRES, *Gruppen deren Untergruppen alle subnormal sind*, Würzburg Ph.D. thesis Aus Karlstadt (1988).

[3] W. MÖHRES, *Torsionsgruppen, deren Untergruppen alle subnormal sind*, Geometriae Dedicata, **31** (1989), pp. 237-244.
[4] W. MÖHRES, *Auflösbarkeit von Gruppen deren untergruppen alle subnormal sind*, Arch. Math., **54** (1990), pp. 232-235.
[5] D. J. S. ROBINSON, *Finiteness Conditions and Generalized Soluble Groups*, Vols. **1** and **2**, (Springer-Verlag 1972).
[6] D. J. S. ROBINSON, *A course in the theory of groups*, (Springer-Verlag, Heidelberg-Berlin-Newyork 1982).
[7] M. J. TOMKINSON, *A Frattini-like subgroup*, Math. Proc. Camb. Phil. Soc., **77** (1975), pp. 247-257.
[8] M. WEINSTEIN, *Examples of groups* (Polygonal Publishing USA 1977).

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