B.Sc. IInd Year
(Maths Hons.)
Advanced Calculus
(Paper IIIrd)

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Multiple points & Curve Tracing :-

Lecture-1

Singular points :- Those points on a curve at which the curve behaves in an unusual manner are called singular points. Such points can be divided into two main heads;
(i) Point of inflexion (ii) Multiple points

Concavity and Convexity at a point of a curve :-
Let P be a point on the curve and AB the given straight line not passing through P. Then the curve is said to be concave or convex at P w.r.to AB, according as a sufficient small arc containing P lies entirely within or without the acute angle PAB; where AP is tangent at P.

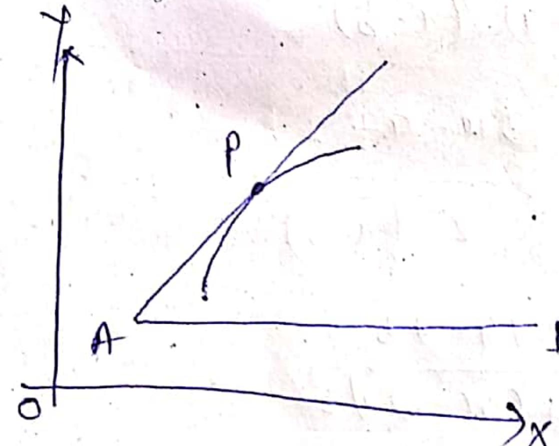


Figure: (i)

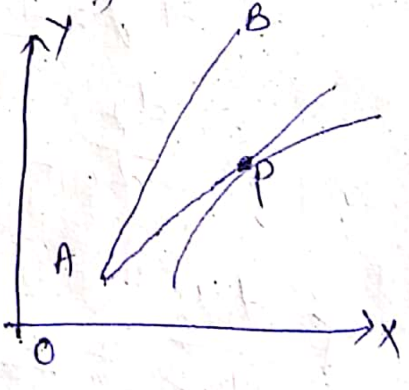
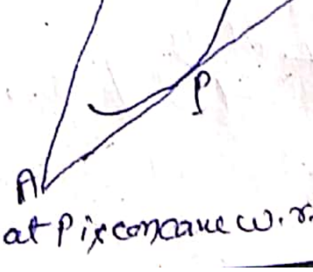


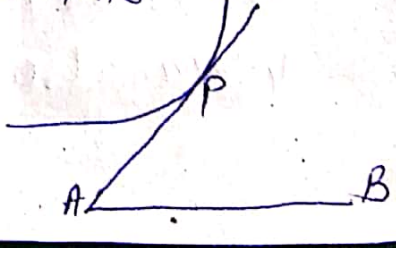
Figure: (ii)

Thus in figure (i) the curve at P is concave w.r.to AB; and in figure (ii) it is convex w.r.to AB.



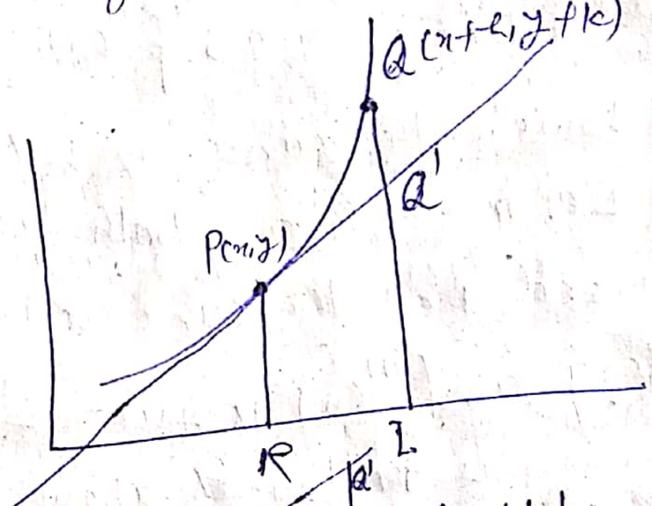
The curve at P is concave w.r.t. AB.

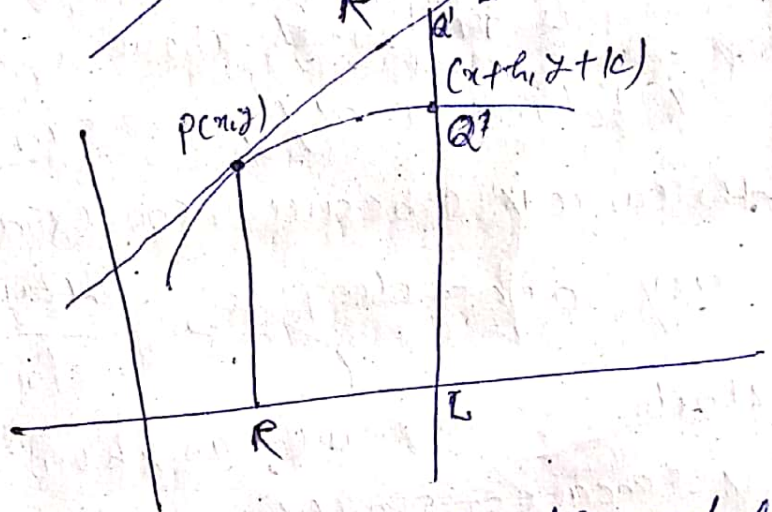
The curve at P is convex w.r.t. AB



A test of concavity or convexity :

Let $P(x, y)$ be a point on the curve $y = f(x)$ and $Q(x+h, y+k)$ be another neighbouring point to it.





Let PR, QL be the ordinates of P and Q, and let the tangent at P cuts QL at Q'.

The eqn of tangent at P is

$$Y - y = f'(x)\,(X - x)$$

Putting $X = x + h$, we obtained

$$LQ' = y + f'(x)\,h$$

Again by Taylor's theorem, we have

$$LQ = f(x+h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \cdots$$

By subtraction, we have $LQ - LQ' = \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \cdots$ (1)

Now, if h is small enough the sign of right hand side of (1) will depend on the first term i.e. $f''(x)$ as h^2 is always +ve.

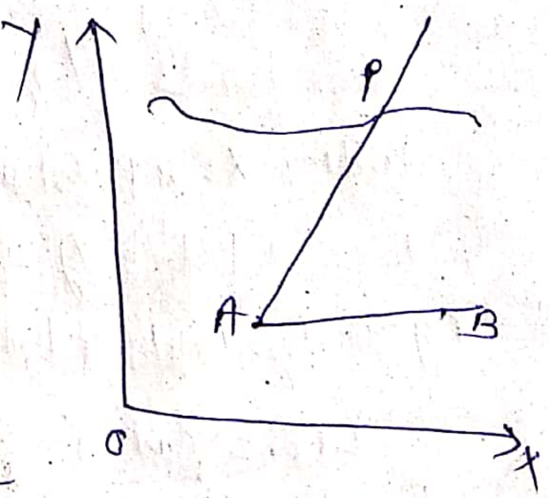
Hence the curve is concave or convex at P to the axis of x according as $f''(x)$ is −ve or +ve.

If however, the curve is below x-axis the sign of LQ and LQ' are both −ve and the case is reversed i.e. the curve is concave or convex according as $f''(x)$ is +ve or −ve. Also remembering that y is +ve for points above x-axis and −ve for points below x-axis; we can state the rule that the curve is concave or convex at P w.r.t to the x-axis according as $y\frac{d^2y}{dx^2}$ is −ve or +ve.

In a similar manner, we can show that the curve is concave or convex at P w.r.t the axis of y according as $x\frac{d^2x}{dy^2}$ is −ve or +ve.

Points of inflexion :-

A point on a curve is said to be a point of inflexion if the curve is concave on one side and convex on the other side of the point P w.r.to the line AB. At such a point curve crosses its tangent.



Test for point of inflexion :— If $f''(x) = 0$ at the point P we have.

$$LQ - LQ' = \frac{-h^3}{L^3} f'''(x) + \frac{h^4}{L^4} f^{iv}(x) + \text{———}$$

If h is small enough the sign of the R.H.S. will depend on $h^3 f'''(x)$; which changes sign with h. Hence, the curve is convex to the axis of x on one side of P and concave on the other side of P, so there is a point of inflexion at P; if $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$.

Example :— Show that the points of inflexion of the curve $y^2 = (x-a)^2(x-b)$ lie on the line $3x + a = 4b$.

Solution :— The curve is

$$y^2 = (x-a)^2(x-b)$$

$$\Rightarrow y = \pm (x-a)\sqrt{(x-b)}$$

$$\therefore \frac{dy}{dx} = \pm \frac{3x - a - 2b}{2\sqrt{(x-b)}}$$

$$\& \frac{d^2y}{dx^2} = \pm \frac{3x + a - 4b}{4(x-b)^{3/2}}$$

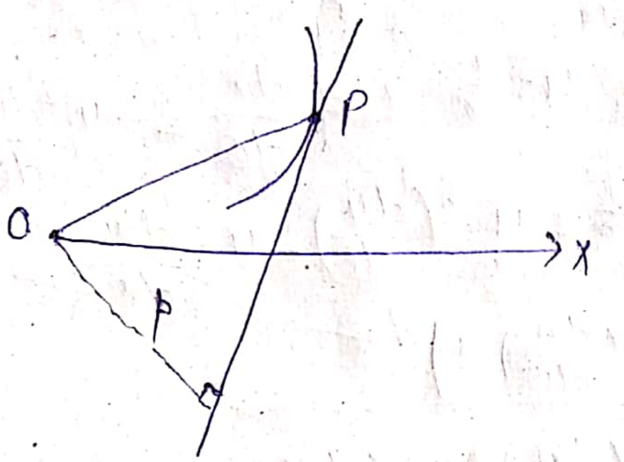
For points of inflexion $\frac{d^2y}{dx^2} = 0$, so we have

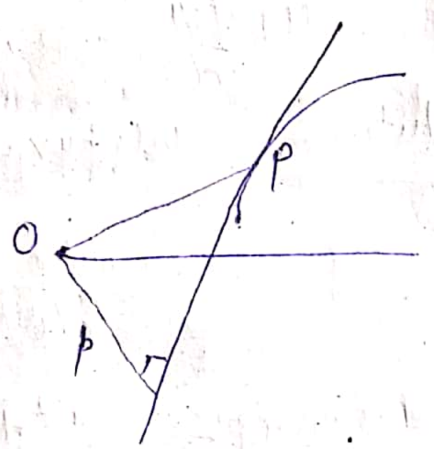
$$3x + a - 4b = 0$$

i.e. the points of inflexion lie on the straight line. $3x + a = 4b$.

We can also show that $\frac{d^3y}{dx^3} \neq 0$, at $x = \frac{4b - a}{3}$.

Concavity, convexity and points of inflexion for polar curves :-



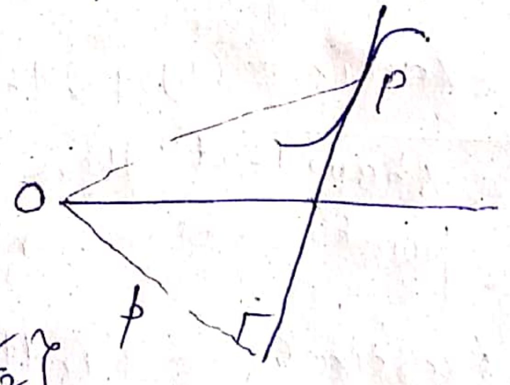


From the above figure, it is clear that as p, the $\perp^{ar}$ from the pole O on the tangent at P (any point on the curve), increases as r increases, then the curve is concave at P to the pole.

i.e. the curve is concave at P to the pole if $\frac{dp}{dr}$ is +ve there.

Similarly, the curve is convex at P to the pole if $\frac{dp}{dr}$ is -ve there.

If $\frac{dp}{dr} = 0$ at P, +ve for points lying one side of P & -ve for points lying on the other side of P; there is a point of inflexion at P.



Also $\rho = r\frac{dr}{dp} = \dfrac{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$

$$\Rightarrow \frac{dp}{dr} = \frac{r\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}}{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}$$

Hence, there is a point of inflexion at P; if

$$r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right) = 0.$$

Example ①: Show that the point of inflexion on the curve $r = b\theta^n$ are given by $r = b[-n(n+1)]^{n/2}$.

Solution: Given that $r = b\theta^n$

$$\Rightarrow \frac{dr}{d\theta} = bn.\theta^{n-1}$$

and $$\frac{d^2r}{d\theta^2} = bn(n-1)\theta^{n-2}$$

for the point of inflexion,

$$\frac{d^2r}{d\theta^2} = bn(n-1)\theta^{n-2}; \quad \frac{dr}{d\theta} = bn.\theta^{n-1}$$

using in $r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = 0$

$$\Rightarrow b^2\theta^{2n} + 2b^2n^2\theta^{2n-2} - b\theta^n bn(n-1)\theta^{n-2} = 0$$

$$\Rightarrow b^2\theta^{2n} + 2b^2n^2\theta^{2n-2} - b^2.n(n-1)\theta^{2n-2} = 0$$

$$\Rightarrow b^2\theta^{2n} + b^2n^2\theta^{2n-2} + b^2n.\theta^{2n-2} = 0$$

$$\Rightarrow [\theta^2 + n^2 + n]\, b^2\theta^{2n-2} = 0$$

$$\Rightarrow \theta^2 + n(n+1) = 0$$

$$\Rightarrow \theta^2 = -n(n+1)$$

$$\Rightarrow \theta = [-n(n+1)]^{1/2}$$

$$\Rightarrow \theta^n = [-n(n+1)]^{n/2}$$

$$\Rightarrow r = b\theta^n = b[-n(n+1)]^{n/2}$$

(Hence proved).

Example ②:— For a curve given by its polar equation show that the points of inflexion are given by

$$u + \frac{d^2u}{d\theta^2} = 0 \text{ ; where } u = \frac{1}{r}.$$

Solution:— We have $u = \frac{1}{r}$; $\Rightarrow \frac{dr}{d\theta} = -u^{-2}\frac{du}{d\theta}$

$$\Rightarrow \frac{d^2r}{d\theta^2} = 2u^{-3}\left(\frac{du}{d\theta}\right) - u^{-2}\frac{d^2u}{d\theta^2}$$

Then $$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)}$$

$$= \frac{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]^{3/2}}{u^3\left[u + \frac{d^2u}{d\theta^2}\right]}$$

At the point of inflexion, ρ is infinite

$$u^3\left[u + \frac{d^2u}{d\theta^2}\right] = 0$$

$$\Rightarrow u + \frac{d^2u}{d\theta^2} = 0.$$

Exercise:— ① For the curve $(\theta^2 - 1) r = a\theta^2$, show that there is a point of inflexion at the point where $r = \frac{3a}{5}$.

Exercise ②:— Find the point of inflexion of the curve $y(a^2 + x^2) = x^3$.

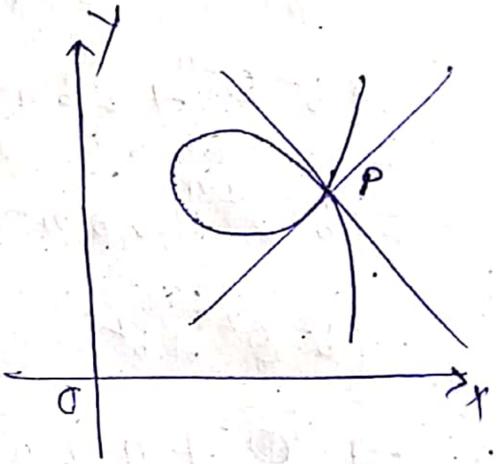
Multiple points:— A point is called a multiple point if more than one branches of the curve pass trough it. A point is called a double point if two branches of the curve pass through it and a triple point if three branches pass through it. In general, a point is called a multiple point of the order n if n branches of the curve pass through it

Classification of double points:—

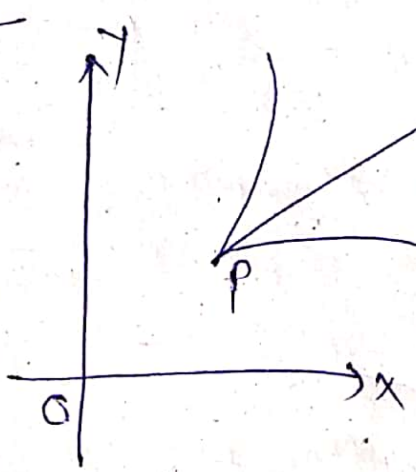
The double points are of three types:

(i) Node (ii) Cusp (iii) Conjugate point.

Node:— If the two branches through a double point are real and tangents to them are not coincident, then the point is called a node. As shown in side figure.

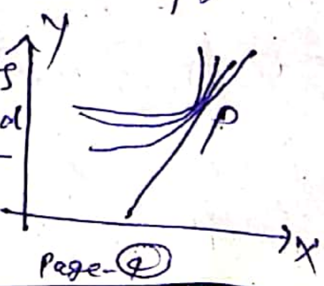


Cusp:—



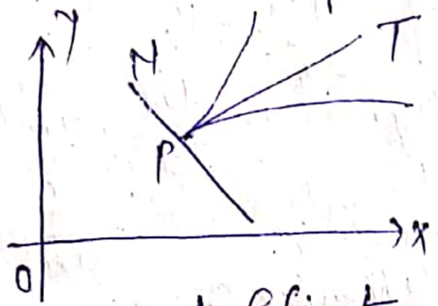
If the two branches through a double point are real and tangents to them are coincident, then the point is called a cusp. As shown in side figure.

Conjugate points:— If there are no real points on the curve in the neighbourhood of the point P then P is called a conjugate point or isolated point. Usually, the tangent at such a point are imaginary.

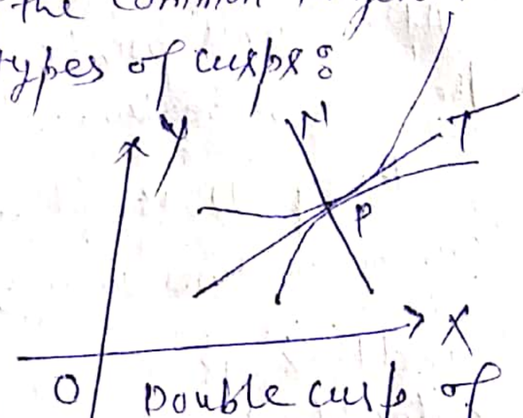


Lecture-3

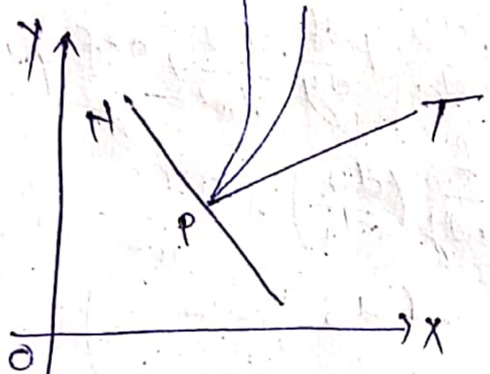
Species of Cusps: A cusp is single or double according as the two branches lie on the same or different sides of the common normal. Also, a cusp is of the first or second species according as both the branches lie on the different or same side of the common tangent.
Hence, we have the following types of cusps:



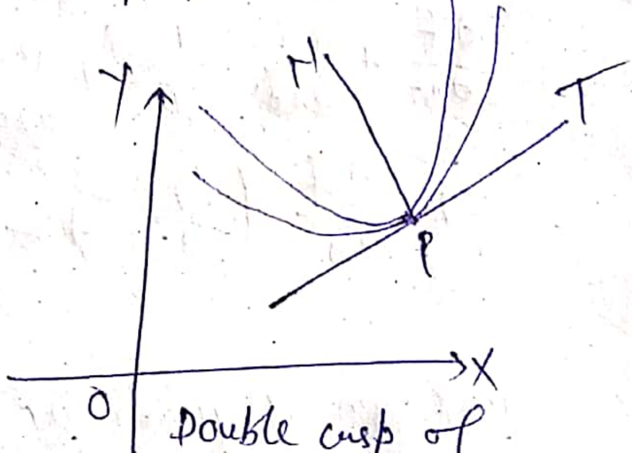
Single cusp of first species



Double cusp of first species.



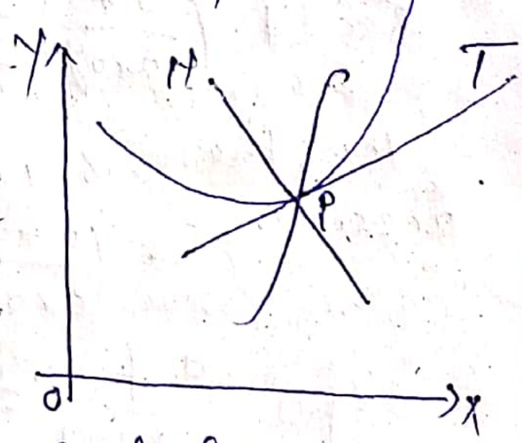
Single cusp of second species



Double cusp of second species.

Point of osculinflexion:

In the side figure, the two branches the cusp are on the same side on the tangent, while the two branches below the cusp are; there is change of species.
Such a point is called the point of osculinflexion.



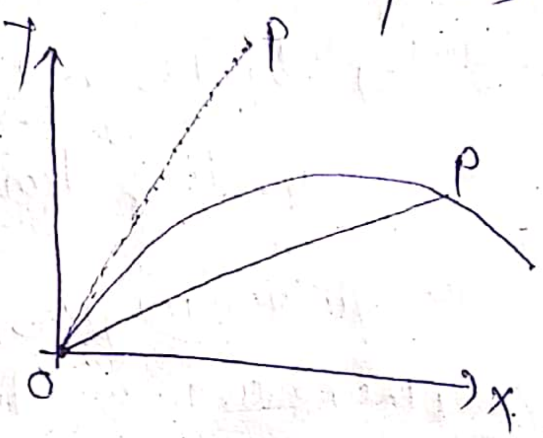
Point of osculinflexion.

Tangents at the origin :— If a curve passes through the origin and is given by a rational, integral, algebraic equation, the equation of tangent or tangent at the origin is obtained by equating to zero the terms of the lowest degree in the equation to the curve.

Let the given curve be

$$a_1x + a_2y + b_1x^2 + b_2xy + b_3y^2 + c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3 + \cdots = 0 \quad \text{——— (I)}$$

There is no constant term as the curve passes through the origin.



Let $P(x,y)$ be any point on the given curve. Then the equation of straight line OP is

$$Y = \left(\frac{y}{x}\right) X$$

The eqn of the tangent at the origin O is,

$$Y = \lim_{x \to 0} \left(\frac{y}{x}\right) X \quad \text{——— (II)}$$

For the time being, exclude the case, when the y-axis is tangent i.e. when $\lim_{x\to 0} \frac{y}{x} = \pm\infty$.

Case I :- Let $a_2 \neq 0$, dividing (I) by x and taking limits as $x \to 0$; we have

$$a_1 + a_2\left[\lim_{x\to 0} \frac{y}{x}\right] = 0 \quad \text{——— (III)}$$

Eliminating $\lim_{x\to 0} \frac{y}{x}$ between (II) & (III); we get

$$a_1X + a_2Y = 0$$

Replacing X and Y by x and y, the tangent to (I) at the origin is: $a_1x + a_2y = 0$, and this result could be obtained by equating to zero the terms of the lowest degree in (I).

Case II : If $a_2 = 0$; then by (ii); $a_1 = 0$

so, let $a_1 = a_2 = 0$, and b_2, b_3 are not both zero.

Dividing (I) by x^2, and taking limits as $x \to 0$, we get

$$b_1 + b_2 \lim_{x\to 0} \frac{y}{x} + b_3 \lim_{x\to 0} \frac{y^2}{x^2} = 0$$

$$\Rightarrow b_1 + b_2 m + b_3 m^2 = 0 \quad \text{———— (iv)}$$

denoting $\lim_{x\to 0} \frac{y}{x}$ by m.

The equation (iv) gives, in general, two values of m & so, there are two tangents at the origin.

Eliminating m between (iv) & (II), the equation of the tangents is

$$b_1x^2 + b_2xy + b_3y^2 = 0$$

This eqn can be obtained from (I) by equating to zero the lowest degree terms when $a_1 = 0, a_2 = 0$.

Case III : Let $a_1 = a_2 = b_1 = b_2 = b_3 = 0$, then we can show that the above rule is still true; and so on.

If the tangents at the origin is the axes of y, by interchanging the axes of x and y; we easily see that the rule still holds.

Thus, we see that all the tangents including y-axis can be obtained by equating to zero the lowest degree terms.

Position and Character of double points:-

Let the equation of the curve be $f(x,y)=0$, and $P(x,y)$ be any point on it.

Differentiating the eqn of the curve, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0 \quad \text{—— (I)}$$

At a double point, there are two tangents and so there must be two values of $\frac{dy}{dx}$ at P.

The eqn (I) is of the first degree in $\frac{dy}{dx}$, and so it can have two values of $\frac{dy}{dx}$ only when $\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$

and this is necessary condition for $f(x,y)=0$ to have a double point.

Differentiating (I) w.r.t. x, we have

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{dy}{dx}+\frac{\partial^2 f}{\partial y\partial x}\cdot\frac{dy}{dx}+\frac{\partial f}{\partial y}\cdot\frac{d^2y}{dx^2}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2=0$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2+2\frac{\partial^2 f}{\partial x\partial y}\left(\frac{dy}{dx}\right)+\frac{\partial^2 f}{\partial x^2}=0 \quad \text{—— (II)}$$

$$\left(\text{as } \frac{\partial f}{\partial y}=0\right)$$

This eqn is quadratic in $\left(\frac{dy}{dx}\right)$. The two tangents will be different, coincident or imaginary according as

$$4\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2-4\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 f}{\partial x^2} >, =, \text{ or } < 0.$$

Thus there is a node, cusp or conjugate points according as $\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 >, = \text{ or } < \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 f}{\partial x^2}$

If $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial y^2}=0$

then the point $P(x,y)$ will not be a double point.

Lecture - 4

Nature of a Cusp :- Let $(-h, k)$ be the cusp.
Transfer the origin to the point $(-h, k)$ and then find the equation of tangent by equating to zero lowest degree terms. Let $(ax+by)^2 = 0$ be the equation of the coincident tangents. Then the length of the $\perp^{ar}$ from (x, y) on the tangent is $\frac{ax+by}{\sqrt{a^2+b^2}}$.

which is proportional to $ax+by$. Let us suppose

$$P = ax+by$$

Eliminating y between this and the equation of the given curve; we shall get a relation between P & x. Neglecting powers of P higher than two, we get a quadratic equation in P. The sign and reality of the values of P for +ve and -ve values of x will decide the nature of the cusp. Thus, if for a +ve small x, the two values of P are of opposite signs, we get a first species on the right, and so on.

Example ① :- Show that the origin is a conjugate point on the curve $x^4 - ax^2y + axy^2 + a^2y^2 = 0$

Solution :- We have given curve

$$x^4 - ax^2y + axy^2 + a^2y^2 = 0 \quad \text{——— (1)}$$

For tangents at origin, equating to zero the lowest degree terms $y^2 = 0$ i.e. there are two coincident tangents. Hence, there can be either a cusp or conjugate points at the origin.

The eqn of curve is written as

$$a(x+a)y^2 - ax^2y + x^4 = 0$$

$$\Rightarrow y = \frac{ax^2 \pm \sqrt{(a^2x^4 - 4a^2x^4 - 4ax^5)}}{2a(x+a)}$$

$$= \frac{ax^2 \pm x^2\sqrt{(-4ax - 3a^2)}}{2a(x+a)}$$

If x is small, the sign of $-4ax - 3a^2$ will depend on $-3a^2$ which is $-ve$. Hence y is imaginary near the origin, so origin is a conjugate point.

Example ②:- Determine the ranges of values of x in which the curve $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is concave upwards or downwards. Find also the point of inflexion.

Solution:- Given that

$$y = x^4 - 6x^3 + 12x^2 + 5x + 7 \quad \text{---} (I)$$

$$\Rightarrow \frac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5 \quad \text{———} (II)$$

$$\& \ \frac{d^2y}{dx^2} = 12x^2 - 36x + 24 \quad \text{———} (III)$$

We see that, when $x < 1$; then $\frac{d^2y}{dx^2} > 0$;

When $x = 1$; $\frac{d^2y}{dx^2} = 0$; When $1 < x < 2$; $\frac{d^2y}{dx^2} < 0$;

When $x=2$; $\frac{d^2y}{dx^2}=0$; When $x>2$, then $\frac{d^2y}{dx^2}>0$

Hence the curve is concave upwards in the intervals $(-\infty,1) \cup (2,\infty)$ and concave downwards in the interval $(1,2)$.

Again $\frac{d^3y}{dx^3}=24x-36$

Therefore at $x=1$, and 2; $\frac{d^3y}{dx^3}\neq 0$

Hence $x=1$ and $x=2$ are points of inflexion.

Example (3):- Determine the nature and character of the double points on the curve $x^3+y^3-3axy=0$.

Solution:- The curve passes through the origin. Tangents at the origin, obtained by equating the lowest degree terms to zero, are $xy=0$ i.e. $x=0$ & $y=0$.

Since the tangents are real & distinct, the origin is a node.

Example (4):- Examine the nature of double points on the curve $(x+y)^3=\sqrt{2}(y-x+2)^2$

Solution:- Let $f(x,y)=(x+y)^3-\sqrt{2}(y-x+2)^2=0$ — (I)

$\Rightarrow \frac{\partial f}{\partial x}=0 \Rightarrow 3(x+y)^2+2\sqrt{2}(y-x+2)=0$ — (II)

& $\frac{\partial f}{\partial y}=0 \Rightarrow 3(x+y)^2-2\sqrt{2}(y-x+2)=0$ — (III)

Adding (II) & (III), we have $6(x+y)^2=0$

$\Rightarrow x+y=0$ — (iv)

Subtracting (III) from (II), we have

$y-x+2=0$ — (v)

Solving (iv) & (v); we have

$x = 1\ ;\ y = -1$

These values also satisfy the equation of curve.

Hence $(1, -1)$ is a double point.

Transfering the origin to $(1, -1)$; we have

$$(x+1+y-1)^3 = \sqrt{2}\left[(y-1)-(x+1)+2\right]^2$$

$$\Rightarrow (x+y)^3 = \sqrt{2}\,(y-x)^2 \quad \text{—— (vi)}$$

Hence the tangents at new origin are

$$(y-x)^2 = 0$$

Put $y - x = p$, i.e. $y = x + p$ in (vi), we have

$$(p+2x)^3 = \sqrt{2}\,p^2$$

$$\Rightarrow p^3 + 3p^2 2x + 3p(2x)^2 + 8x^3 - \sqrt{2}\,p^2 = 0$$

Neglecting p^3 which is small, we have

$$(6x - \sqrt{2})p^2 + 12x^2 p + 8x^3 = 0$$

$$\Rightarrow p = \frac{-12x^2 \pm \sqrt{\{144x^4 - 32x^3(6x - \sqrt{2})}}{2(6x - \sqrt{2})}$$

$$= \frac{-6x^2 \pm \sqrt{\{8(\sqrt{2}x^3)}}{(6x - \sqrt{2})}$$

$$= \frac{-6x^2 \pm \sqrt{8(\sqrt{2}x^3)}}{(6x - \sqrt{2})} \text{ on neglecting } +12x^4$$

p is real when $x > 0$ and two values of p are of opposite signs as $x^{3/2}$ is greater than x^2 when x is small.

Hence there is a single cusp of the first species at $(1, -1)$.

Lecture-5

Tracing of Cartesian curves:-

The following procedure adopt for tracing of cartesian curves:- **Step I: Symmetry:-** First observe whether the curve is symmetrical about any line by applying the following rules:

(i) If all the powers of y which occur in the equation of the curve are even, the curve is symmetrical about x-axis.

(ii) If all the powers of x which occur in the equation of the curve are even, the curve is symmetrical about y-axis.

(iii) If all the powers of both x and y which occur in the equation of the curve are even, the curve is symmetrical about both the axes.

(iv) If on interchanging x and y both the equation of the curve remains unaltered, the curve is symmetrical about the line $y=x$.

(v) If on changing the signs of x and y both, the equation of the curve is unaltered, there is symmetry in the opposite quadrants.

Step II: At the origin:- observe whether the curve passes through the origin. If it passes through the origin, find the equation of tangent or tangents there. If origin is a singular point, find its nature.

Step III: Other points on the curve: (i) Find out the points where the curve crosses the coordinate axes. Also find out the tangents at these points.

(ii) Find out other suitable points on the curve whose presence can be detected easily.

Step IV :- Asymptotes :- First find the asymptotes parallel to the coordinate axes. The curve will not go beyond these asymptotes. Then, find out all other asymptotes and examine on which side of the asymptote the curve lies.

Step V :- Points where tangents parallel to Axes :- Find $\frac{dy}{dx}$, and see for what values of x it is zero or infinite. Thus, we will know that points at which the tangent is parallel to x-axis or y-axis. At such points, the ordinate or abscissa generally changes its character from increasing to decreasing or vice versa.

Step VI :- Singular points :- Find out the singular points of curve and its nature.

Step VII :- Solve for y and/x :- Solve the equation of the curve for y or x, whichever is convenient. Note how y varies for different suitable values of x both +ve and -ve.

If, however, there is a symmetry about x-axis or in opposite quadrants, +ve values of y need be considered. The curve for -ve values of y can be drawn from symmetry.

Similarly; if the curve is symmetrical about y-axis the curve for -ve values of x can be drawn from symmetry.

Step VIII :- Imaginary values of y :- The equation has been already solved for y. If y is founded imaginary when x lies between a and b, then the curve does not lie in the region bounded by the lines

$$x = a \text{ and } x = b.$$

Examples :- ① Trace the curve $y^2(2a-x) = x^3$.

Solution :- We have given curve

$$y^2(2a-x) = x^3$$

(I) The power of y is even, so there is symmetry about x-axis.

(II) The curve passes through the origin. the tangents at origin are $y^2=0$ i.e: two coincident tangents. Hence the origin is a cusp.

(III) The curve does not crosses the axes anywhere except at origin.

(IV) Equating to zero the coefficient of highest power of y; we see that the asymptote parallel to y-axis is $x=2a$.

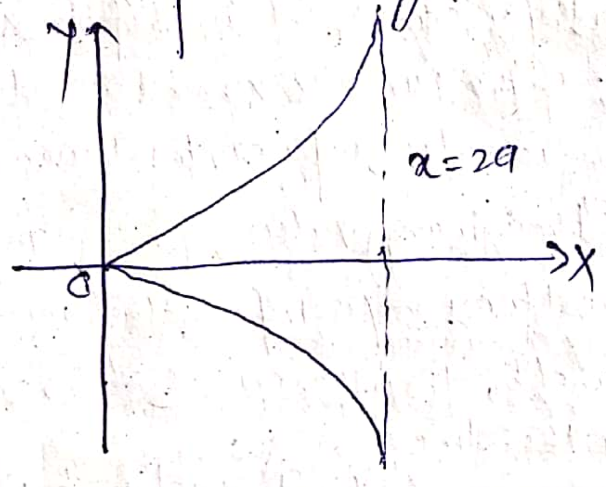
The remaining asymptotes are imaginary.

(V) Solving for y; we have $y=\pm\frac{x^{3/2}}{\sqrt{2a-x}}$

When $x>2a$; y is imaginary, so the curve does not lies beyond $x=2a$.

Also when x is $-ve$; y is imaginary, so the curve does not lie on the $-ve$ side of x axis.

Again when $0<x<2a$, y is real. Hence the curve is shown as following.



Example (2): — Trace the curve $y^2(a^2+x^2)=x^2(a^2-x^2)$.

Solution: Given the curve is

$$y^2(a^2+x^2)=x^2(a^2-x^2)$$

(I) All the powers of both x and y are even, hence the curve is symmetrical about the both the axes.

(II) The curve passes through the origin. Tangents at the origin are. $y^2 - x^2 = 0$ i.e. $y = \pm x$. Tangents are real and non-coincident, hence origin is node.

(III) Putting $y=0$; we get $x = 0; \pm a$, So the curve cuts the x-axis at the points $(0,0)$, $(\pm a, 0)$. The curve cuts y-axis only at the point $(0,0)$.

(IV) Solving for y, the equation of curve is $y^2 = \dfrac{x^2(a^2-x^2)}{a^2+x^2}$

y^2 is +ve i.e. y is real; when $x^2 < a^2$ i.e. $-a < x < a$.
Hence the curves lies between $x = a$ and $x = -a$.

(V) On differentiating; we have

$$2y\frac{dy}{dx} = \frac{(2a^2x - 4x^3)(a^2+x^2) - 2x\,x^2(a^2-x^2)}{(a^2+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^4 - 2a^2x^2 - x^4}{(a^2+x^2)^{3/2}(a^2-x^2)^{1/2}}$$

At $x = \pm a$; $\dfrac{dy}{dx} = \infty$.

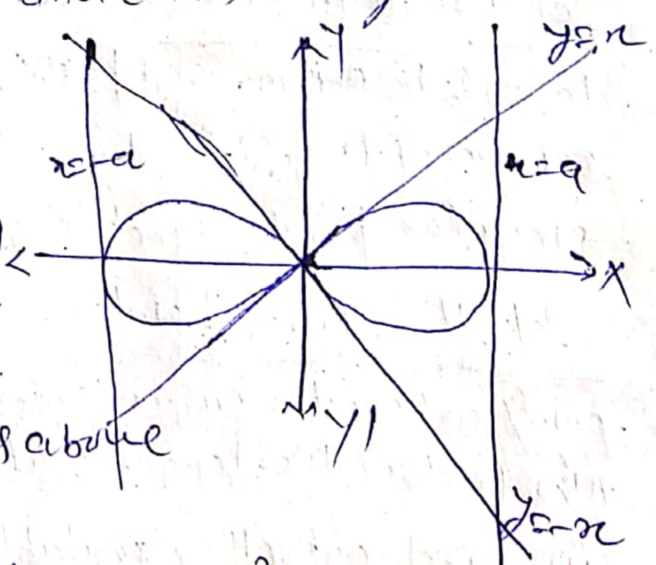
Hence at the points $(a,0)$ and $(-a,0)$ tangents are $\perp^{r}$ to the x-axis.

Again $\dfrac{dy}{dx} = 0$

if $x^2 = a^2(\sqrt{2}-1)$

So, there is a maximum at

$x = \pm a(\sqrt{2}-1)^{1/2}$



Hence the curve is shown as above

Exercise:- Trace the curve $y^2 = \dfrac{x^2(1+x^2)}{1-x^2}$

Lecture – 6

Tracing of polar curves:–

Step I: Symmetry: (i) If on changing θ to $-\theta$ the equation of curves remains unchanged, there is symmetry about the initial line. Ex. $r = a(1-\cos\theta)$.

Step II: (ii) If the equation of curve remains unchanged when r is changed into $-r$, there is symmetry about the pole. $r^2 = 2a\cos\theta$ (example).

(iii) If the eqn of curve remains unchanged when θ is changed into $\pi-\theta$, or when θ is changed to $-\theta$ and r into $-r$, then the curve is symmetrical about the line $\theta = \frac{\pi}{2}$. Example: $r = a(1-\sin\theta)$

(iv) If the equation of the curve remains unchanged when θ changed into $\frac{\pi}{2}-\theta$; there is symmetry about the line $\theta = \frac{\pi}{4}$. Example; $r = a\sin 2\theta$.

And similar procedure applying, we get symmetry of the curve.

Step II: Pass through the pole:– The curve will pass through the pole, if r is zero, for some values of θ, say $\theta = \alpha$. In that case $\theta = \alpha$ is tangent at the pole.

Step III: Solve for r or θ:– Solve the eqn of curve for r or θ whichever is convenient. If the equation is solved for r consider how r varies as θ increases from 0 to $+\infty$, and again as θ diminishes from 0 to $-\infty$. If convenient, we can form a table of corresponding values of θ and r.

If the eqn of curve contains only periodic functions the value of r repeats after some values of θ, then we need not give further values

Step IV: Imaginary values of r: — If on solving the equation of curve as above, r is found to be imaginary for every value of θ between $\theta=\alpha$ and $\theta=\beta$, then the curve does not exist in the region bounded by $\theta=\alpha$ & $\theta=\beta$.

Step V: Asymptotes: — If the curve possesses an infinite branch we find its asymptote.

Step VI: Direction of the tangent: — Find $\tan\phi = r\dfrac{d\theta}{dr}$

This will indicate the direction of the tangent at any point on the curve.

Example ①: — Trace the cardioid $r = a(1+\cos\theta)$.

Solution: (i) When θ is changed to $-\theta$, the eqn of curve is unaltered. Hence the curve is symmetrical about initial line.

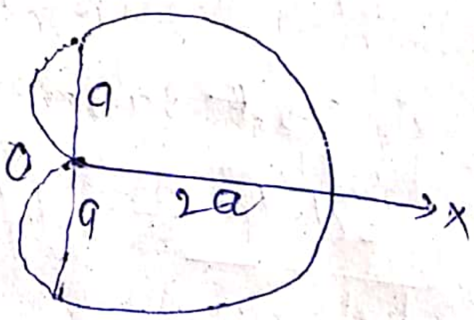
(ii) If $\theta=\pi$, $r=0$, hence the curve passes through the pole and $\theta=\pi$ is tangent at the pole.

(iii) We now consider the values of r corresponding to some values of θ

$\theta=0$, $r=2a$; $\theta=\frac{\pi}{2}$, $r=a$

$\theta=2\pi/3$; $r=a/2$

$\theta=\pi$; $r=0$



As θ increases from 0 to π, r is +ve and decreases from $2a$ to 0. When θ increases from π to 2π, r is +ve and increases from 0 to $2a$.

Exercise: Trace the curve $r^2 = a^2\cos 2\theta$

Tracing of parametric curve:-

If the equation of curve is given in a parametric form $x = \phi(t)$; $y = \psi(t)$, the parameter t can be easily eliminated in some cases. The resulting equation is in cartesian co-ordinates and can be traced easily.

The alternative method is as follows:

(I) Find $dy/dx = \dfrac{dy/dt}{dx/dt}$

(II) Give the parameter t a series of values & plot the corresponding values of x & y, noting that the slope at the point (x, y) is given by dy/dx

Example: Trace the curve

$$x = a(t + \sin t)\,;\; y = a(1 - \cos t)$$

as t varies in the interval $(-\pi, \pi)$.

Solution: We have $\dfrac{dx}{dt} = a(1 + \cos t)$

$$\frac{dy}{dt} = a \sin t$$

$$dy/dx = \frac{a \sin t}{a(1 + \cos t)} = \frac{a \cdot 2 \sin t/2 \, \cos t/2}{a \cdot 2 \cos^2 t/2}$$

$$\frac{dy}{dx} = \sin \frac{t}{2}$$

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The table given below is the corresponding values of x, y & dy/dx for t:

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	$-a\pi$	$-a(\frac{\pi}{2}+1)$	0	$a(\frac{\pi}{2}+1)$	$a\pi$
y	$2a$	a	0	a	$2a$
dy/dx	$-\infty$	-1	0	1	∞

Since for $t=-\pi$; we have $x=-a\pi$, $y=2a$ & $\frac{dy}{dx}=-\infty$
We see that the point $B(-a\pi, 2a)$ lie on the curve and the tangent there is y-axis.
When $t=-\pi/2$; we have $x=-a(\frac{\pi}{2}+1)$, $y=a$ & $\frac{dy}{dx}=-1$, so the point $Q\{-a(\frac{\pi}{2}+1), a\}$ lies on the curve and the tangent makes there angle of $3\pi/4$ with x-axis. Also for $t=0$, $x=0$, $y=0$ and $\frac{dy}{dx}=0$; so the point $O(0,0)$ lie on the curve and tangent there is x-axis. Hence, as t varies from $-\pi$ to 0 we get the branch BQO of the curve.
Similarly by considering the values of t from 0 to π we get the branch OPA of the curve. Hence the curve is as shown follows: