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Semantics for Fragments of Propositional Intuitionistic Logic

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Abstract

In 1932, Gödel proved that there is no finite semantics for propositional intuitionistic logic. We consider all fragments of propositional intuitionistic logic and check in each case whether a finite semantics exists. This note may fulfill a didactic goal, as little logic and algebra are required.

Keywords: intuitionistic logic, many-valued logics

1 Introduction

As is well known, propositional classical logic has a finite semantics. In the beginning of the 1920s, mathematicians like Kolmogorov, Glivenko, and Heyting began to study intuitionistic logic, at that time sometimes called "the logic of M. Brouwer" (see [12], [4], and [10], respectively). The natural question arised whether also propositional intuitionistic logic had a finite semantics. In 1932, Gödel proved that *not* to be the case. In his words:

"*Es gibt keine Realisierung mit* endlich *vielen Elementen (Wahrheitswerten), für welche die und nur die in H beweisbaren Formeln erfüllt sind*..." (see [6]),

where H refers to the axiomatic system for intuitionistic logic set up by Heyting in 1930 (see [10]). In fact, Gödel's argument also holds for positive logic, that is, the conjunction-disjunction-conditional fragment of intuitionistic logic. So, there is no finite semantics for the (usual axiomatic system of the) mentioned fragment. Shortly afterwards, in 1933, Gödel himself proved that the conjunction-negation fragments of intuitionistic and classical logic coincide when only considering derivable formulas (see [7]). This implies that the conjunction-negation fragment of intuitionistic logic *does* have a finite semantics for the case of derivable formulas. However, Gödel's result does not hold when premisses are considered (just note that the Double Negation Law holds in classical logic, but not in intuitionistic logic). In particular, in this note we will consider the conjunction-negation fragment also when having premisses in order to see whether it has a finite semantics. In general, it is natural to try to answer the same question regarding every fragment of intuitionistic logic, including the trivial case of the fragment given by the language with no connectives, which will be denoted $\varnothing$. This we do in the present note. All languages considered appear pictorially in Figure 1.

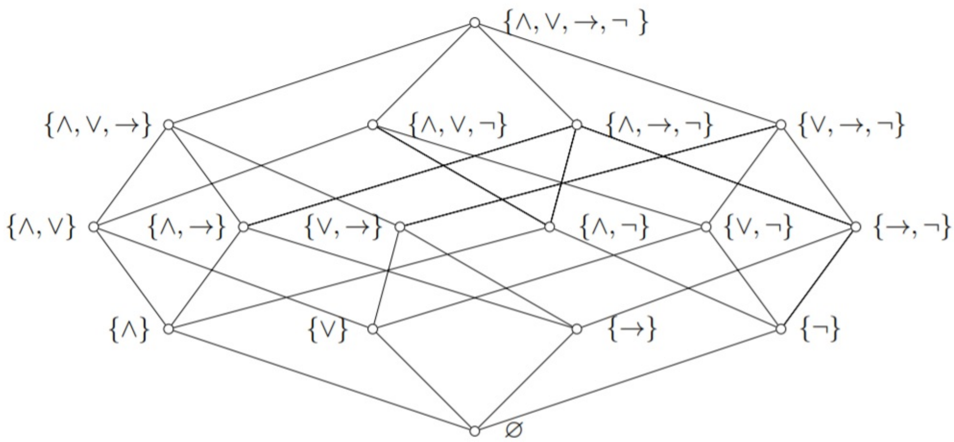

Figure 1: The sixteen languages to be considered

In Section 2, we state our prerequisites. In Section 3, we use Gödel's argument in order to prove that any fragment having the conditional, in particular positive and intuitionistic logics, do not have a finite semantics. There are only eight fragments left to consider. In Section 4, we see that the conjunction-disjunction fragments and the fragments contained in it, *do* have a finite semantics. In Section 5, we consider the disjunction-negation and the conjunction-disjunction-negation fragments. Using an argument similar to the one in Section 3, we see that there is no finite semantics for those fragments. Finally, in Section 6, we consider the negation and conjunction-negation fragments, finding in those cases a finite semantics.

As little knowledge of logic and algebra are required, we think this note may fulfill the following didactic goal. Sometimes Gentzen's Natural Deduction is used to introduce logic. In that context, in order to decide if certain formula follows from a given set of formulas in a given fragment, it is natural to inquire if the fragment involved has a finite semantics.

The main purpose of this paper is to state and prove Theorems 3.7, 5.7, and 6.5.

2 Prerequisites

In this section we consider the basic syntactic and semantic notions. Also, we fix the notation to be used.

2.1 Formulas and derivations

Any set included in the set of connectives $\{\wedge, \vee, \rightarrow, \neg\}$ is called a *language*. Given a language L, we use the notation $\mathfrak{F}_L$ for the set of *formulas* obtained in the usual way from the set of (propositional) letters Π applying the connectives in L. Recall that in intuitionistic logic the given connectives cannot be defined from each other. Also, whenever we mention intuitionistic (classical) logic, we mean *propositional* intuitionistic (classical) logic.

Given any language L, we talk of the *L-fragment* (of intuitionistic logic), which is defined by the consequence relation resulting from the rules given for the conectives in L. In the case of fragments with the conditional, it is possible to use a Frege-style axiomatization, for example, the axiomatization of intuitionistic logic given by Heyting in [10] that Gödel used in [6]. For one contemporary version of that axiomatization the reader may see [2, Section 11.1]. In the case of any fragment, we use Gentzen's Natural Deduction rules (see [3, p. 186]):

$$(\wedge\mathrm{I})\ \frac{\varphi \qquad \psi}{\varphi \wedge \psi}\,, \quad (\wedge\mathrm{E})\ \frac{\varphi \wedge \psi}{\varphi}\,, \quad (\wedge\mathrm{E})\ \frac{\varphi \wedge \psi}{\psi}\,,$$

$$(\vee\mathrm{I})\ \frac{\varphi}{\varphi \vee \psi}\,, \quad (\vee\mathrm{I})\ \frac{\psi}{\varphi \vee \psi}\,, \quad (\vee\mathrm{E})\ \frac{\varphi \vee \psi \qquad \begin{matrix}[\varphi]\\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ \chi\end{matrix}}{\chi}\,,$$

$$(\rightarrow\mathrm{I})\ \frac{\begin{matrix}[\varphi]\\ \psi\end{matrix}}{\varphi \rightarrow \psi} \quad \text{and} \quad (\rightarrow\mathrm{E})\ \frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}\,,$$

where φ, ψ, and χ are formulas and the letters I and E stand for introduction and elimination, respectively. Moreover, as we do not use $\perp$, we need something like the following two rules in the case of negation, for introduction and elimination, respectively:

$$(\neg\mathrm{I})\ \frac{\begin{matrix}[\varphi]\\ \psi\end{matrix} \qquad \begin{matrix}[\varphi]\\ \neg\psi\end{matrix}}{\neg\varphi} \quad \text{and} \quad (\neg\mathrm{E})\ \frac{\varphi \qquad \neg\varphi}{\psi}\,,$$

where φ and ψ are any formulas.

Note that one is allowed to use assumptions. That means, for example, that rule ($\vee$E) should be understood as follows:

$$(\vee\mathrm{E})\ \frac{\varphi\vee\psi \qquad \begin{matrix}\Gamma,[\varphi]\\ \chi\end{matrix} \qquad \begin{matrix}\Gamma,[\psi]\\ \chi\end{matrix}}{\chi}\,,$$

where φ, ψ, and χ are formulas and Γ is any set of formulas.

Given a language L and $\Gamma\cup\{\varphi\}\subseteq\mathfrak{F}_L$, we write $\Gamma\vdash_L\varphi$ and say that φ is *a syntactic consequence of* Γ in the L-fragment, meaning that there exists a derivation of φ from Γ, where a derivation, roughly speaking, is a finite tree constructed using only the given rules for the connectives in L (for a detailed definition, see [13, p.24]). Also, we sometimes write $\vdash_i$ or $\vdash_c$ meaning that there exists a derivation using intuitionistic logic (that is, all the given rules) or classical logic, respectively. By *classical* logic we mean that we may use all the given rules plus, for example, the rule

$$(\mathrm{TND})\ \frac{}{\varphi\vee\neg\varphi}\,,$$

where φ is any formula and (TND) stands for *tertium non datur*.

Note, for instance, that facts like $\varphi\vdash_L\varphi$ and if $\Gamma\vdash_L\varphi$, then $\Gamma,\psi\vdash_L\varphi$ hold for any language L, in particular when $L=\varnothing$, no rule being needed. For those facts we use the notation (SC).

Finally, note that, unlike Gödel, who only considered derivable formulas, in the definition of syntactic consequence we need to consider the given set Γ of formulas. This is due to the fact that a fragment may not have the conditional as one of its connectives, so that it will not enjoy the Deduction Theorem.

2.2 Semantics

Let us now consider the concepts of poset, algebra, matrix, and semantic consequence for a given matrix. In particular, we define Heyting and Boolean algebras.

Given a set S and a binary relation R on S, we say that R is an *order* on S iff R is reflexive, antisymmetric, and transitive.

Definition 2.1 *A* poset *is a pair* $\langle P;\leq\rangle$ *such that*

(i) P *is a non-empty set* and
(ii) $\leq$ *is an order on* P.

We frequently use the following particular case of a poset.

Definition 2.2 *A poset* $(P; \leq)$ *is a* chain *iff it holds that* $x \leq y$ *or* $y \leq x$ *for any elements* $x, y \in P$.

Definition 2.3 *Let* L *be a language.* An algebra (for L) *is a pair* $\mathbf{A}_L = \langle A; O_L \rangle$ *such that*

(i) A *is a non-empty set* and
(ii) O_L *is a set of operations on* A *for every connective in* L.

In what follows we use the same symbols for both the connectives and their corresponding operations. This ambiguity should not cause any problem.

Example 2.4 *A* lattice *is an algebra* $\langle L; \wedge, \vee \rangle$ *such that the following facts hold for any* $x, y, z \in L$*:*

$$
\begin{array}{ll}
x \wedge x = x, & x \vee x = x, \\
x \wedge y = y \wedge x, & x \vee y = y \vee x, \\
x \wedge (y \wedge z) = (x \wedge y) \wedge z, & x \vee (y \vee z) = (x \vee y) \vee z, \\
x \wedge (x \vee y) = x, & x \vee (x \wedge y) = x.
\end{array}
$$

Given a lattice $\langle L; \wedge, \vee \rangle$, note that it can be proved that the binary relation $\leq$ on L defined by $x \leq y$ iff $x \wedge y = x$ is an order (exercise). Consequently, $\langle L; \leq \rangle$ is a poset.

Definition 2.5 *A lattice* $\langle L; \wedge, \vee \rangle$ *is* distributive *iff it holds that* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, *for any* $x, y, z \in L$.

Definition 2.6 *A* Heyting algebra *is an algebra* $\langle L; \wedge, \vee, \to, \neg \rangle$ *such that the algebra* $\langle L; \wedge, \vee \rangle$ *is a distributive lattice and the following two conditions hold, for all* $x, y, z \in L$*:*

$z \leq x \to y$ *iff* $z \wedge x \leq y$ and
$z \leq \neg x$ *iff* $z \wedge x \leq y$, *for all* $y \in L$.

Definition 2.7 *A* Boolean algebra *is a Heyting algebra* $\langle H; \wedge, \vee, \to, \neg \rangle$ *such that it holds that* $y \leq x \vee \neg x$, *for any* $x, y \in H$.

Remark 2.8 *A Boolean algebra is usually defined without using* $\to$*. We are using* $\to$ *in order to make clear the connection between Boolean algebras and our definition of classical logic, which we gave in the previous subsection.*

Definition 2.9 *Let* L *be a language.* A matrix for L *is a triple* $\mathbf{M}_L = \langle M, D, O_L \rangle$ *such that*

(i) $\langle M; O_L \rangle$ *is an algebra for* L and
(ii) $D \subseteq M$ *is a set of* designated *values.*

Definition 2.10 *Take a language L, $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_L$, and let $\mathbf{M}_L = \langle M, D, O_L \rangle$ be a matrix for L. Then, we say that φ is* a semantic consequence of Γ *for $\mathbf{M}_L$ (and use the notation $\Gamma \vDash_{\mathbf{M}_L} \varphi$) iff for every assignment $v : \Pi \to M$, the unique homomorphism $\bar{v} : \mathfrak{F}_L \to M$ satisfies that if $\bar{v}\psi \in D$, for all $\psi \in \Gamma$, then $\bar{v}\varphi \in D$.*

Under "having a finite semantics" we understand the same as Gödel, that is, we use the following definition.

Definition 2.11 *Let L be a language. We say that the L-fragment has* a finite semantics *iff there exists a matrix $\mathbf{M}_L$ with a finite set of values such that for every $\Gamma \cup \{\varphi\} \in \mathfrak{F}_L$ it holds that $\Gamma \vdash_L \varphi$ iff $\Gamma \vDash_{\mathbf{M}_L} \varphi$.*

It should be clear that, for each fragment, we will be looking for *one* matrix (not for a class of matrices) and, moreover, for a *finite* one.

In this paper we assume to be known that intuitionistic and classical logic are sound and complete relatively to the class of Heyting algebras and to the two-element Boolean algebra, respectively. That is, we assume that the following two theorems hold. In the statement of the first theorem, given the language $L = \{\wedge, \vee, \to, \neg\}$ and $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_L$, notation $\Gamma \vDash_{\mathscr{H}} \varphi$ means that *for every* Heyting algebra with universe H and for every assignment $v : \Pi \to H$, the unique homomorphism $\bar{v} : \mathfrak{F}_L \to H$ satisfies that if $\bar{v}\psi = 1$, for all $\psi \in \Gamma$, then $\bar{v}\varphi = 1$.

Theorem 2.12 *Let L be the language $\{\wedge, \vee, \to, \neg\}$ and $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_L$. Then, it holds that $\Gamma \vdash_i \varphi$ iff $\Gamma \vDash_{\mathscr{H}} \varphi$.*

In the statement of the second theorem, $\mathbf{2}_{\{\wedge,\vee,\to,\neg\}}$ denotes the matrix with universe $\{0, 1\}$ taking 1 as only designated element and operations $\wedge$, $\vee$, $\to$, and $\neg$ defined as in the case of a Boolean algebra. Formally, $\mathbf{2}_{\{\wedge,\vee,\to,\neg\}} = \langle \{0, 1\}, \{1\}, \{\wedge, \vee, \to, \neg\} \rangle$, such that $\langle \{0, 1\}, \{\wedge, \vee, \to, \neg\} \rangle$ is a Boolean algebra.

Theorem 2.13 *Let L be the language $\{\wedge, \vee, \to \neg\}$ and $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_L$. Then, $\Gamma \vdash_c \varphi$ iff $\Gamma \vDash_{\mathbf{2}_{\{\wedge,\vee,\to,\neg\}}} \varphi$.*

Remark 2.14 *Note that Theorem 2.13 requires just one matrix, whereas Theorem 2.12, as emphasized, makes use of the hole class of Heyting algebras.*

3 On fragments with the conditional

In his proof that there is no finite semantics for the syntactic consequence of intuitionistic logic, Gödel constructed a formula using, among others, the disjunction connective. Now, in this section our goal is to deal with fragments that have the conditional connective, but this may be the only one. Let us see in detail that, instead of disjunctions, we may use formulas of the form $(\varphi \to \psi) \to \psi$.

Lemma 3.1 *Let L be a language with $\to$. Then, for any $\varphi, \psi \in \mathfrak{F}_L$, we have* (i) $\varphi \vdash_L \varphi$, (ii) $\vdash_L \varphi \to \varphi$, and (iii) $\varphi, \varphi \to \psi \vdash_L \psi$.

Proof. For (i) use (SC). Then, (ii) follows from (i) using ($\to$I). In order to prove (iii), just use ($\to$E). ■

As already said, in what follows we use Gödel's argument with formulas of a different form. In order to do that, we use the following abbreviation:

$$\varphi \dot{\vee} \psi := (\varphi \to \psi) \to \psi.$$

For example, the formula $[(p_3 \to p_2)\dot{\vee}(p_3 \to p_1)]\dot{\vee}(p_2 \to p_1)$ denotes the formula

$$([([(p_3 \to p_2) \to (p_3 \to p_1)] \to (p_3 \to p_1)) \to (p_2 \to p_1)] \to (p_2 \to p_1)).$$

Note that $\dot{\vee}$ is neither commutative nor associative. We omit parentheses assuming *association to the left*. So, instead of the given formula, we may as well write

$$(p_3 \to p_2)\dot{\vee}(p_3 \to p_1)\dot{\vee}(p_2 \to p_1).$$

We use the following lemma.

Lemma 3.2 *Let L be a language with $\to$ and $\varphi \in \mathfrak{F}_L$ such that $\vdash_L \varphi$. Then* (i) $\vdash_L \varphi\dot{\vee}\psi$, *for any formula $\psi \in \mathfrak{F}_L$,* (ii) $\vdash_L \psi\dot{\vee}\varphi$, *for any formula $\psi \in \mathfrak{F}_L$,* and (iii) *If $\psi = \cdots \dot{\vee}\ \varphi\ \dot{\vee} \cdots$, where the given dots may be empty at the beginning or the end, then $\vdash_L \psi$.*

Proof. (i) By ($\to$E) we have $\varphi, \varphi \to \psi \vdash_L \psi$. Then, using ($\to$I) it follows that $\varphi \vdash_L (\varphi \to \psi) \to \psi$. As we have $\vdash_L \varphi$, using (SC) it follows that $\vdash_L (\varphi \to \psi) \to \psi$. (ii) As we have $\vdash_L \varphi$, by (SC) it follows that $\psi \to \varphi \vdash_L \varphi$. Then, by ($\to$I) it follows that $\vdash_L (\psi \to \varphi) \to \varphi$. Part (iii) follows using parts (i) and (ii). ■

Now, let us turn to algebraic considerations, where we use the $\dot{\vee}$ notation in a way analogous to the corresponding connective.

Lemma 3.3 *Let us have a Heyting algebra with universe A, order $\leq$, and $a, b \in A$. Then,* (i) *If $(A; \leq)$ is a chain and $a < b$, then $b \to a = a$* and (ii) *if $a \leq b$, then $a\dot{\vee}b = b$.*

Proof. (i) It is clear that (1) $b \wedge a \leq a$. Now, let us suppose that $b \wedge c \leq a$, for any $c \in A$. Then, as A is a chain, then either $b \wedge c = b$ or $b \wedge c = c$. Now, as $a < b$, it cannot be the case that $b \wedge c = b$. So, $b \wedge c = c$. Then $c \leq a$. So, we have that, (2) for any $c \in A$, if $b \wedge c \leq a$, then $c \leq a$. From (1) and (2) it follows that $b \to a = a$.

(ii) Assume $a \leq b$. So, $a \to b$ is top. So, $(a \to b) \to b \leq b$. It is also the case that $b \leq (a \to b) \to b$. ■

Similarly to Gödel in [6], in order to prove our first theorem, we divide our task into two propositions. The first one states that, if we had a finite semantics, then certain formulas would be derivable. The second proposition states that those formulas are not derivable.

The strategy of the proof of the first proposition consists, roughly speaking, in finding a formula with more propositional letters than values in the given semantics.

Proposition 3.4 *Let L be a language with $\to$. Suppose that the L-fragment has a finite semantics, say with $n \geq 1$ values. Then, for all $n \geq 1$, it follows that $\vdash_L \alpha_n$, where*

$$\alpha_n = \dot{\bigvee_{1 \leq i < j \leq n+1}} p_j \to p_i.$$

Proof. Let L be a language with $\to$. Suppose that the L-fragment has a semantics with n values, that is, that there exists a matrix $\mathbf{M}_L$ with a set of values M and a natural number n such that $|M| = n$ and such that for every $\Gamma \cup \{\varphi\} \in \mathfrak{F}_L$ it holds that

$$\text{(C)} \quad \Gamma \vdash_L \varphi \text{ iff } \Gamma \vDash_{\mathbf{M}_L} \varphi.$$

Let us take an assignment $v : \Pi \to M$ and let us consider $\bar{v}\alpha_n$. As there are $n+1$ propositional letters in α_n, but only n values in M, there must be letters p_i, p_j such that $vp_i = vp_j$. Now, let us consider the formula $\beta_n = \alpha_n[p_i/p_j]$, where $\alpha_n[p_i/p_j]$ indicates the substitution of all appearances of p_i in α_n for appearances of p_j. It should be clear that $\bar{v}\beta_n = \bar{v}\alpha_n$. Also, $\beta_n = \cdots \dot{\vee} (p_j \to p_j) \dot{\vee} \cdots$, where the given dots may be empty at the beginning or the end. Now, using Lemma 3.1 (ii), it holds that $\vdash_L p_j \to p_j$ and then, using Lemma 3.2 (iii), it follows that $\vdash_L \beta_n$. Then, taking $\Gamma = \varnothing$ in (C), $\vDash_{\mathbf{M}_L} \beta_n$. So, $\vDash_{\mathbf{M}_L} \alpha_n$. So, using (C) in the other direction, $\vdash_L \alpha_n$. ■

Now, let us see that the formulas given in Proposition 3.4 are not derivable.

Proposition 3.5 *Let $1 \leq n$ be a natural number and α_n a formula as in Proposition 3.4. Then, $\nvdash_i \alpha_n$.*

Proof. Let us consider the Heyting algebra with universe $\{1, 2, \ldots, n+1\}$ and operations $\wedge$, $\vee$, $\rightarrow$, and $\neg$ defined by

$x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$,

if $x \leq y$, then $x \rightarrow y = n+1$ else $x \rightarrow y = y$, and

if $x = 1$, then $\neg x = n+1$, else $\neg x = 1$, respectively.

Let us consider any assignment w such that $wp_i = i$, for $1 \leq i \leq n+1$. Then,

$$
\begin{aligned}
\bar{w}\alpha_n &= \dot{\bigvee}_{1 \leq i < j \leq n+1} wp_j \rightarrow wp_i, \\
&= \dot{\bigvee}_{1 \leq i < j \leq n+1} wp_i, \text{(as } wp_i < wp_j, \text{ using Lemma 3.3(i))}, \\
&= wp_1 \dot{\vee} wp_1 \dot{\vee} \cdots \dot{\vee} wp_2 \dot{\vee} wp_2 \cdots \dot{\vee} wp_n, \\
&= wp_1 \dot{\vee} wp_2 \dot{\vee} \cdots \dot{\vee} wp_n, \text{ (by Lemma 3.3(ii))}, \\
&= wp_n, \text{ (by Lemma 3.3(ii))}, \\
&= n, \\
&\neq n+1.
\end{aligned}
$$

Using soundness of intuitionistic logic relatively to Heyting algebras, to finite chains in particular, it follows that $\nvdash_i \alpha_n$. ■

Note that Proposition 3.5 was stated for i, that is, for the fragment with all connectives. So, we immediately get the following fact.

Corollary 3.6 *Let α_n be a formula as in Proposition 3.4. Then, for any language L, we have that $\nvdash_L \alpha_n$, for any natural number $n \geq 1$.*

We finally get our goal in this section.

Theorem 3.7 *Fragments with $\rightarrow$ do not have a finite semantics.*

Proof. Applying Proposition 3.4, the formulas α_n would be derivable, which cannot be the case due to Corollary 3.6. ■

4 On the $\varnothing$, $\{\wedge\}$, $\{\vee\}$, and $\{\wedge, \vee\}$-fragments

It remains to consider the fragments with languages appearing in Figure 2. In this section, we will only consider the fragments with languages $\varnothing$, $\{\wedge\}$, $\{\vee\}$, and $\{\wedge, \vee\}$. The notation $\mathbf{2}_L$ will stand for any matrix of the form

$\langle\{0,1\},\{1\},O_L\rangle$, where $L \subseteq \{\wedge,\vee\}$ and the operations for $\wedge$ and $\vee$ will make them behave as the usual meet and join in a Boolean algebra. Let us first consider the case of the $\{\wedge\}$-fragment.

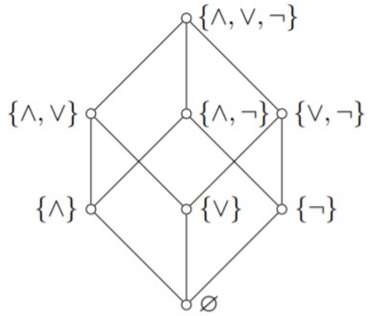

Figure 2: Languages without the conditional

Proposition 4.1 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_{\{\wedge\}}$. *Then,* $\Gamma \vdash_{\{\wedge\}} \varphi$ *iff* $\Gamma \vDash_{\mathbf{2}_{\{\wedge\}}} \varphi$.

Proof. Suppose $\Gamma \vdash_{\{\wedge\}} \varphi$. Then, the reader can easily check that it follows that $\Gamma \vDash_{\mathbf{2}_{\{\wedge\}}} \varphi$. On the other hand, suppose $\Gamma \nvdash_{\{\wedge\}} \varphi$, that is, $\Gamma \nvdash_{\{\wedge\}} p_1 \wedge p_2 \wedge \cdots \wedge p_n$. Then, by ($\wedge$I), there is a letter p_i such that $\Gamma \nvdash_{\{\wedge\}} p_i$. So, by ($\wedge$E), p_i is not a subformula of any formula in Γ. Then, there exists the assignment w such that $wp_i = 0$ and $wp = 1$ for letters p other that p_i. So, $\Gamma \nvDash_{\mathbf{2}_{\{\wedge\}}} \varphi$. ■

The $\varnothing$-fragment is easily dealt with.

Proposition 4.2 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_{\varnothing}$. *Then,* $\Gamma \vdash_{\varnothing} \varphi$ *iff* $\Gamma \vDash_{\mathbf{2}_{\varnothing}} \varphi$.

Proof. Similar to the previous one. ■

Let us now consider the $\{\wedge,\vee\}$-fragment.

Proposition 4.3 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_{\{\wedge,\vee\}}$. *Then,* $\Gamma \vdash_{\{\wedge,\vee\}} \varphi$ *iff* $\Gamma \vDash_{\mathbf{2}_{\{\wedge,\vee\}}} \varphi$.

Proof. If $\Gamma \vdash_{\{\wedge,\vee\}} \varphi$, then, checking that the rules for $\wedge$ and $\vee$ preserve 1 in the matrix $\mathbf{2}_{\{\wedge,\vee\}}$, it follows that $\Gamma \vDash_{\mathbf{2}_{\{\wedge,\vee\}}} \varphi$. On the other hand, suppose $\Gamma \nvdash_{\{\wedge,\vee\}} \varphi$. Then, using the conjunctive normal form theorem (recall that we have distributivity), it follows that φ and every formula in Γ is equivalent to a conjunction of disjunction of letters. Then, by ($\wedge$I), $\Gamma \nvdash_{\{\wedge,\vee\}} \chi$, where $\chi = q_1 \vee \cdots \vee q_n$. Also, as every formula in Γ is a conjunction (of disjunctions), and to have formulas α and β as different premisses is equivalent to having $\alpha \wedge \beta$ as only premiss, then we might as well consider Γ to be a set of disjunctions and call it Δ. Now, by ($\vee$I), it follows that every disjunction in Δ has a letter that does not belong to the set $\{q_1, \ldots, q_n\}$. Consequently, there exists the assignment w such that $wq_i = 0$, for all $1 \leq i \leq n$ and $wp = 1$ for letters p other than the q_i. So, every formula in Δ will have value 1. So, $\Gamma \nvDash_{\mathbf{2}_{\{\wedge,\vee\}}} \varphi$. ■

Finally, having considered the $\{\wedge, \vee\}$-fragment, the $\{\vee\}$-fragment is easily dealt with.

Proposition 4.4 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_{\{\vee\}}$. *Then,* $\Gamma \vdash_{\{\vee\}} \varphi$ *iff* $\Gamma \vDash_{\mathbf{2}_{\{\vee\}}} \varphi$.

Proof. Similar to the end of the proof for the $\{\wedge, \vee\}$-fragment, which was reduced to only having disjunctions. Note that distributivity is not required. ■

5 On the $\{\vee, \neg\}$ and $\{\wedge, \vee, \neg\}$-fragments

It remains to consider the fragments for the languages appearing in Figure 3. In this section we will only consider the $\{\vee, \neg\}$ and $\{\wedge, \vee, \neg\}$-fragments. We will use the following two lemmas.

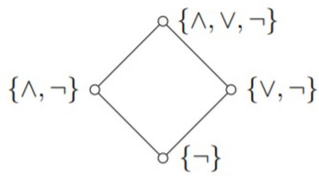

Figure 3: The four remaining languages to be considered

Lemma 5.1 *Let* L *be a language with* $\vee$. *Then,* $\varphi \vdash_L \varphi \vee \psi$ *and* $\psi \vdash_L \varphi \vee \psi$.

Proof. Immediate using ($\vee$I). ■

Lemma 5.2 *Let* L *be a language with* $\vee$ *and* $\neg$. *Then, for any* $\varphi, \psi \in \mathfrak{F}_L$, *we have* (i) *if* $\varphi \vdash_L \psi$, *then* $\neg\psi \vdash_L \neg\varphi$ and (ii) $\vdash_L \neg\neg(\varphi \vee \neg\varphi)$.

Proof. (i) Suppose $\varphi \vdash_L \psi$. Then, by (SC) we have that $\varphi, \neg\psi \vdash_L \psi$. By (SC) we also have that $\varphi, \neg\psi \vdash_L \neg\psi$. Then, using ($\neg$I) it follows that $\neg\psi \vdash_L \neg\varphi$. (ii) Using Lemma 5.1 we have that $\varphi \vdash_L \varphi \vee \neg\varphi$. Then, by part (i), $\neg(\varphi \vee \neg\varphi) \vdash_L \neg\varphi$. Using Lemma 5.1 again, we have that $\neg\varphi \vdash_L \varphi \vee \neg\varphi$. So, using (SC), $\neg(\varphi \vee \neg\varphi) \vdash_L \varphi \vee \neg\varphi$. Now, by (SC) we also have that $\neg(\varphi \vee \neg\varphi) \vdash_L \neg(\varphi \vee \neg\varphi)$. Finally, using $(\neg\mathrm{I})$, we get that $\vdash_L \neg\neg(\varphi \vee \neg\varphi)$. ■

Remark 5.3 *Recall that* tertium non datur *does not hold in intuitionistic logic. However, as seen in part (ii) of Lemma 5.2, its double negation holds in any fragment with* $\vee$ *and* $\neg$.

On the other hand, intuitionistic logic enjoys the Disjunction Property, which does not hold for classical logic.

Lemma 5.4 *Let* $\varphi, \psi \in \mathfrak{F}$. *If* $\vdash_i \varphi \vee \psi$, *then* $\vdash_i \varphi$ *or* $\vdash_i \psi$.

Proof. An algebraic proof runs as follows. If neither $\vdash_i \varphi$ nor $\vdash_i \psi$ hold, then, by completeness, there are Heyting algebras $\mathbf{H_1}$, $\mathbf{H_2}$ and assignments v_1, v_2 such that $\bar{v}_1\varphi \neq 1_{\mathbf{H_1}}$ and $\bar{v}_2\psi \neq 1_{\mathbf{H_2}}$. Now, take the direct product $\mathbf{H_1} \times \mathbf{H_2}$ and add an element which is greater than any element of the universe of the given product. Then, the resulting algebra with the natural assignment will prove that it is not the case that $\vdash_i \varphi \vee \psi$. For details, the reader may see [14]. For other proofs, see [15, Exercise 2.6.7 or sections 5.6 to 5.10]. ■

In order to prove our next theorem, we use the same strategy as in the case of Theorem 3.7.

Proposition 5.5 *Let* L *be a language with* $\vee$ *and* $\neg$. *Suppose that* L *has a finite semantics, say with* $n \geq 1$ *values. Then, the formulas of the following form are derivable in* L:

$$\alpha_n = \bigvee_{1 \leq i < j \leq n+1} \neg\neg(\neg p_i \vee p_j).$$

Proof. Suppose that L is a language with $\vee$ and $\neg$ that has a semantics with n values, that is, that there exists a matrix $\mathbf{M}_L$ with set of values M and a natural number n such that $|M| = n$ and such that for every $\Gamma \cup \{\varphi\} \in \mathfrak{F}_L$ it holds that

$$\text{(C)} \quad \Gamma \vdash_L \varphi \text{ iff } \Gamma \vDash_{\mathbf{M}_L} \varphi.$$

Take an assignment $v : \Pi \to M$ and let us consider $\bar{v}\alpha_n$. As there are $n+1$ propositional letters in α_n, but only n values in M, there must be letters p_i, p_j such that $vp_i = vp_j$. Now, consider the formula $\beta_n = \alpha_n[p_i/p_j]$. It should be clear that $\bar{v}\beta_n = \bar{v}\alpha_n$. Now, it holds that $\vdash_L \neg\neg(\neg p_j \vee p_j)$. Consequently,

$\vdash_L \beta_n$. Then, by (C), $\models_{\mathbf{M}_L} \beta_n$. So, $\models_{\mathbf{M}_L} \alpha_n$. So, using (C) in the other direction, $\vdash_L \alpha_n$. ■

The following fact is easy to prove using the already stated Disjunction Property.

Proposition 5.6 *The formulas of the form given in Proposition 5.5 are not intuitionistically derivable.*

Proof. For every i, j, $i \neq j$, $\neg\neg(\neg p_i \vee p_j)$ is not even classically derivable. So, by Lemma 5.4, it follows that $\nvdash_i \alpha_n$. ■

Theorem 5.7 *Fragments with* $\vee$ *and* $\neg$ *do not have a finite semantics.*

Proof. Applying Proposition 5.5, the formulas of the given form would be derivable, which cannot be the case, because they are not intuitionistically derivable, as stated in Proposition 5.6. ■

6 On the $\{\neg\}$ and $\{\wedge, \neg\}$-fragments

There are only two fragments left to consider, that is, the $\{\neg\}$ and the $\{\wedge, \neg\}$-fragment. In the Introduction we stated that Gödel proved that the set of derivable formulas of the conjunction-negation fragment of intuitionistic logic coincides with the set of classically derivable formulas. This is also stated and proved in detail in [11] (see Corollary to (a2) in p. 493). This implies that the conjunction-negation fragment has a finite semantics with respect to derivable formulas, that is, two-valued classical semantics. The natural question arises whether we also have a finite semantics when having premisses as well. This we solve in this section.

Regarding syntactical matters, in this section we use the following version of the celebrated Glivenko Theorem and also the following Corollary. Glivenko Theorem was originally proved for intuitionistic logic in [5]. Before stating those facts, we define a set of formulas $\Gamma \in \mathfrak{F}_{\{\wedge,\neg\}}$ to be classically (respectively $\{\wedge, \neg\}$-) *consistent* iff from Γ it is not possible to derive a contradiction in classical logic (respectively in the $\{\wedge, \neg\}$-fragment of intuitionistic logic), where by a contradiction we mean a pair $\{\varphi, \neg\varphi\}$ of formulas.

Theorem 6.1 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_{\{\wedge,\neg\}}$. *Then, if* $\Gamma \vdash_c \neg\varphi$, *then* $\Gamma \vdash_{\{\wedge,\neg\}} \neg\varphi$.

By contraposition we get the following result.

Corollary 6.2 *Let* $\Gamma \subseteq \mathfrak{F}_{\{\wedge,\neg\}}$. *If* Γ *is* $\{\wedge,\neg\}$*-consistent, then* Γ *is classically consistent.*

Regarding semantics, we use the concepts of subalgabra and congruence, whose definitions we now state (for details or examples the reader may see [1]).

Definition 6.3 *Given two algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *of the same type, we say that* $\mathbf{B}$ *is* a subalgebra of $\mathbf{A}$ *iff the universe of* $\mathbf{B}$ *is included in the universe of* $\mathbf{A}$ *and every fundamental operation of* $\mathbf{B}$ *is the restriction to the universe of* $\mathbf{B}$ *of the corresponding operation of* $\mathbf{A}$.

Definition 6.4 *Given an algebra* $\mathbf{A} = \langle A; F\rangle$, *we say that* a congruence on $\mathbf{A}$ *is an equivalence relation* E *on* A *such that for every* n*-ary operation* f *in* L *and elements* a_i, b_i *in* A,

if $a_i E b_i$, *for all* $i, 1 \leq i \leq n$, *then* $f(a_1, \ldots, a_n) E f(b_1, \ldots, b_n)$.

The diagonal relation and the all relation are the only trivial congruences.

In this section, $\mathbf{3}_L$ stands for the matrix $\langle\{0, {}^1\!/_2, 1\}; \{1\}; O_L\rangle$, where L is either $\{\wedge,\neg\}$ or $\{\neg\}$ and the operations for $\wedge$ and $\neg$ make them behave as the usual meet and meet complement in a Heyting algebra (we might as well say that $a \wedge b = \min\{a, b\}$, for any a, b in the given set of values; $\neg 0 = 1$, and $\neg {}^1\!/_2 = \neg 1 = 0$). The only non-trivial congruence is given by the ellipses in Figure 4. Note, also, that the algebra $\langle\{0, 1\}; O_L\rangle$, where L is either $\{\wedge,\neg\}$ or $\{\neg\}$ and the operations for $\wedge$ and $\neg$ make them behave as the usual meet and meet complement in a Boolean algebra, is a subalgebra of the algebra of $\mathbf{3}_L$.

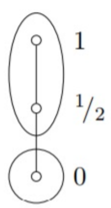

Figure 4: The only non-trivial congruence in the algebra of $\mathbf{3}_L$

Let us now consider the $\{\wedge,\neg\}$-fragment.

Theorem 6.5 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_{\{\wedge,\neg\}}$. *Then,* $\Gamma \vdash_{\{\wedge,\neg\}} \varphi$ *iff* $\Gamma \vDash_{\mathbf{3}_{\{\wedge,\neg\}}} \varphi$.

Proof. $\Rightarrow$) Suppose $\Gamma \vdash_{\{\wedge,\neg\}} \varphi$. It is easily seen that the rules for conjunction and the ($\neg$E)-rule preserve 1. Regarding the ($\neg$I)-rule, it can also be seen that if $\varphi \vDash_{\mathbf{3}_{\{\wedge,\neg\}}} \psi$ and $\varphi \vDash_{\mathbf{3}_{\{\wedge,\neg\}}} \neg\psi$, then $\vDash_{\mathbf{3}} \neg\varphi$. Indeed, suppose (H)

$\varphi \vDash_{\mathbf{3}_{\{\wedge,\neg\}}} \psi$ and $\varphi \vDash_{\mathbf{3}_{\{\wedge,\neg\}}} \neg\psi$. Given an assignment v, we will aim to discard both $\bar{v}\neg\varphi =^1/_2$ and $\bar{v}\neg\varphi = 0$. First, suppose $\bar{v}\neg\varphi =^1/_2$. Then, $\neg\bar{v}\varphi =^1/_2$, which cannot be the case as a negation can only have values 0 or 1. Second, suppose $\bar{v}\neg\varphi = 0$, that is, $\neg\bar{v}\varphi = 0$. Then, either $\bar{v}\varphi = 1$ or $\bar{v}\varphi =^1/_2$. In case $\bar{v}\varphi = 1$, by (H) it follows a contradiction. In case $\bar{v}\varphi =^1/_2$, we define the assignment v' for propositional letters as follows: let $v'p = 0$ if $vp = 0$ and let $v'p = 1$ if vp is either $^1/_2$ or 1. Then, it is easily seen by induction on the formation of φ that if a subformula ψ of φ has value $\bar{v}\psi =^1/_2$, then $\bar{v}'\psi = 1$. In particular, if $\bar{v}\varphi =^1/_2$, then $\bar{v}'\varphi = 1$. So, $\bar{v}'\varphi = 1$. Then, by (H), we get a contradiction.

$\Leftarrow$) Suppose $\Gamma \nvdash_{\{\wedge,\neg\}} \varphi$. There are three cases: (i) $\varphi = \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$, where each φ_i is a either a negation or a letter; (ii) $\varphi = \neg\psi$, for some formula ψ; or (iii) $\varphi = p$, for some letter p.

In case (i), by ($\wedge$I) and ($\wedge$E) we have that $\Gamma \nvdash_{\{\wedge,\neg\}} \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$ iff there is at least an i such that $\Gamma \nvdash_{\{\wedge,\neg\}} \varphi_i$ and such that φ_i is either a negation or a letter. So, this case reduces to either case (ii) or case (iii).

In case (ii), using Theorem 6.1, we have that $\Gamma \nvdash_c \neg\psi$. Then, by completeness of classical logic, $\Gamma \nvDash_{\mathbf{2}_{\{\wedge,\neg\}}} \neg\psi$. So, there is an assignment $v : \Pi \to \{0, 1\}$ such that $\bar{v}\chi = 1$ for all $\chi \in \Gamma$ and $\bar{v}\neg\psi = 0$. Then, as the algebra of the matrix $\mathbf{2}_{\{\wedge,\neg\}}$ is a subalgebra of the algebra of the matrix $\mathbf{3}_{\{\wedge,\neg\}}$, it follows, using the same assignment v, that $\Gamma \nvDash_{\mathbf{3}_{\{\wedge,\neg\}}} \neg\psi$, that is, $\Gamma \nvDash_{\mathbf{3}_{\{\wedge,\neg\}}} \varphi$.

Finally, case (iii) means that $\Gamma \nvdash_{\{\wedge,\neg\}} p$. Then, Γ is $\{\wedge,\neg\}$-consistent. Then, using Corollary 6.2, it follows that Γ is classically consistent. Then, there exists an assignment $v : \Pi \to \{0, 1\}$ such that $\bar{v}\psi = 1$ for all $\psi \in \Gamma$. Now, let us define an assignment $w : \Pi \to \{0,^1/_2, 1\}$ such that $wp_i = vp_i$, for all $p_i \in \Pi$. Then, as the algebra of the matrix $\mathbf{2}_{\{\wedge,\neg\}}$ is a subalgebra of the algebra of the matrix $\mathbf{3}_{\{\wedge,\neg\}}$, it follows that $\bar{w}\psi = 1$ for all $\psi \in \Gamma$. If $wp = 0$, then we are done. In case $wp = 1$, let us define w' like w except for $w'p = {}^1/_2$. It remains to be seen that for every $\psi \in \Gamma$, $\bar{w}'\psi = 1$. Now, as $\Gamma \nvdash_{\{\wedge,\neg\}} p$, then, due to (SC) and ($\wedge$E), every occurrence of p in a formula ψ of Γ must appear in a subformula of ψ of the form $\neg\chi$, that is, it must appear in the scope of a negation. Now, taking θ to be congruence relation given in Figure 4, we have that as $^1/_2\theta 1$, then $\bar{w}'\neg\chi\theta\bar{w}\neg\chi$. As a negation can only have Boolean values (that is, either 0 or 1), then we will have $\bar{w}'\neg\chi = \bar{w}\neg\chi$. Finally, $\bar{w}'\psi = \bar{w}\psi$. So, for every $\psi \in \Gamma$, $\bar{w}'\psi = 1$. ■

Now we can easily deal with the $\{\neg\}$-fragment.

Proposition 6.6 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}_{\{\neg\}}$. *Then,* $\Gamma \vdash_{\{\neg\}} \varphi$ *iff* $\Gamma \vDash_{\mathbf{3}_{\{\neg\}}} \varphi$.

Proof. The left to right direction is the same as in the previous proof. For the other direction, just consider cases (ii) and (iii) in the previous proof. ■

It is clear then, due to propositions 6.5 and 6.6, that both the $\{\wedge, \neg\}$- and the $\{\neg\}$-fragment have a finite semantics.

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