‘Calculus’ as Method or ‘Calculus’ as Rules? Boole against Frege on a systematic method for logic
Documents

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Les documents suivants sont destinés à servir de support à la séance du 7 décembre. Leur lecture préalable n’est pas requise, surtout pas celle du troisième ! À toutes fins utiles, la dernière section contient des éléments de commentaire de ces trois textes.

Table des matières

1 Boole, *An Investigation of the Laws of Thought* (1854), p. 8-11

[T]he requirements of a general method in Logic seem to be the following:–

1st. As the conclusion must express a relation among the whole or among a part of the elements involved in the premises, it is requisite that we should possess the means of eliminating those elements which we desire not to appear in the conclusion, and of determining the whole amount of relation implied by the premises among the elements which we wish to retain. Those elements which do not present themselves in the conclusion are, in the language of the common Logic, called middle terms; and the species of elimination exemplified in treatises on Logic consists in deducing from two propositions, containing a common element or middle term, a conclusion connecting the two remaining terms. But the problem of elimination, as contemplated in this work, possesses a much wider scope. It proposes not merely the elimination of one middle term from two propositions, but the elimination generally of middle terms from propositions, without regard to the number of either of them, or to the nature of their connexion. To this object neither the processes of Logic nor those of Algebra, in their actual state, present any strict parallel. In the latter science the problem of elimination is known to be limited in the following manner:–From two equations we can eliminate one symbol of quantity; from three equations two symbols; and, generally, from n equations $n-1$ symbols. But though this condition, necessary in Algebra, seems to prevail in the existing Logic also, it has no essential place in Logic as a science. There, no relation whatever can be proved to prevail between the number of terms to be eliminated and the number of propositions from which the elimination is to be effected. From the equation representing a single proposition, any number of symbols representing terms or elements in Logic may be eliminated; and from any number of equations representing propositions, one or any other number of symbols of this kind may be eliminated in a similar manner. For such elimination there exists one general process applicable to all cases. This is one of the many remarkable consequences of that distinguishing law of the symbols of Logic, to which attention has been already directed.

2ndly. It should be within the province of a general method in Logic to express the final relation among the elements of the conclusion by any admissible *kind* of proposition, or in any selected *order* of terms. Among varieties of kind we may reckon those which logicians have designated by the terms categorical, hypothetical, disjunctive, &c. To a choice or selection in the order of the terms, we may refer whatsoever is dependent upon the appearance of particular elements in the subject or in the predicate, in the antecedent or in the consequent, of that proposition which forms the "conclusion." But waiving the language of the schools, let us consider what really distinct species of problems may present themselves to our notice. We have seen that the elements of the final or inferred relation may either be *things* or *propositions*. Suppose the former case; then it might be required to deduce from the premises a definition or description of some one thing, or class of things, constituting an element of the conclusion in terms of the other things involved in it. Or we might form the conception of some thing or class of things, involving more than one of the elements of the conclusion, and require its expression in terms of the other elements. Again, suppose the elements retained in the conclusion to be propositions, we might desire to ascertain such points as the following, viz., Whether, in virtue of the premises, any of those propositions, taken singly, are true or false?–Whether particular combinations of them are true or false?–Whether, assuming a particular proposition to be true, any consequences will follow, and if so, what consequences, with respect to the other propositions?–Whether any particular condition being assumed with reference to certain of the propositions, any consequences, and what consequences, will follow with respect to the others? and so on. I say that these are general questions, which it should fall within the scope or province of a general method in Logic to solve. Perhaps we might include them all under this one statement of the final problem of practical Logic. Given a set of premises expressing relations among certain elements, whether things or propositions: required explicitly the whole relation consequent among *any* of those elements under any proposed conditions, and in any proposed form. That this problem, under all its aspects, is resolvable, will hereafter appear. But it is not for the sake of noticing this fact, that the above inquiry into the nature and the functions of a general method in Logic has been introduced. It is necessary that the reader should apprehend what are the specific ends of the investigation upon which we are entering, as well as the principles which are to guide us to the attainment of them.

9. Possibly it may here be said that the Logic of Aristotle, in its rules of syllogism and conversion, sets forth the elementary processes of which all reasoning consists, and that beyond these there is neither scope nor occasion for a general method. I have no desire to point out the defects of the common Logic, nor do I wish to refer to it any further than is necessary, in order to place in its true light the nature of the present treatise. With this end alone in view, I would remark:–1st. That syllogism, conversion, &c., are not the ultimate processes of Logic. It will be shown in this treatise that they are founded upon, and are resolvable into, ulterior and more simple processes which constitute the real elements of method in Logic. Nor is it true in fact that all inference is reducible to the particular forms of syllogism and conversion.–*Vide* Chap. xv. 2ndly. If all inference were reducible to these two processes (and it has been maintained that it is reducible to syllogism alone), there would still exist the same necessity for a general method. For it would

still be requisite to determine in what order the processes should succeed each other, as well as their particular nature, in order that the desired relation should be obtained. By the desired relation I mean that full relation which, in virtue of the premises, connects any elements selected out of the premises at will, and which, moreover, expresses that relation in any desired form and order. If we may judge from the mathematical sciences, which are the most perfect examples of method known, this *directive* function of Method constitutes its chief office and distinction. The fundamental processes of arithmetic, for instance, are in themselves but the elements of a possible science. To assign their nature is the first business of its method, but to arrange their succession is its subsequent and higher function. In the more complex examples of logical deduction, and especially in those which form a basis for the solution of difficult questions in the theory of Probabilities, the aid of a directive method, such as a Calculus alone can supply, is indispensable.

2 Boole, idem, p. 146-149 : un exemple de problème et sa solution

Ex. 5. Let the observation of a class of natural productions be supposed to have led to the following general results.

1st, That in whichsoever of these productions the properties A and C are missing, the property E is found, together with one of the properties B and D, but not with both.

2nd, That wherever the properties A and D are found while E is missing, the properties B and C will either both be found, or both be missing.

3rd, That wherever the property A is found in conjunction with either B or E, or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E, or both of them.

Let it then be required to ascertain, first, what in any particular instance may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D; also whether any relations exist independently among the properties B, C, and D. Secondly, what may be concluded in like manner respecting the property B, and the properties A, C, and D.

It will be observed, that in each of the three data, the information conveyed respecting the properties A, B, C, and D, is complicated with another element, E, about which we desire to say nothing in our conclusion. It will hence be requisite to eliminate the symbol representing the property E from the system of equations, by which the given propositions will be expressed.

Let us represent the property A by x, B by y, C by z, D by w, E by v. The data are

$$\bar{x}\bar{z} = qv(y\bar{w} + w\bar{y}); \tag{1}$$

$$\bar{v}xw = q(yz + \bar{y}\bar{z}); \tag{2}$$

$$xy + xv\bar{y} = w\bar{z} + z\bar{w}; \tag{3}$$

$\bar{x}$ standing for $1 - x$, &c., and q being an indefinite class symbol. Eliminating q separately from the first and second equations, and adding the results to the third equation reduced by (5), Chap.VIII., we get

$$\begin{aligned}\bar{x}\bar{z}(1 - vy\bar{w} - vw\bar{y}) + \bar{v}xw(y\bar{z} + z\bar{y}) + (xy + xv\bar{y})(wz + \bar{w}\bar{z})\\ + (w\bar{z} + z\bar{w})(1 - xy - xv\bar{y}) = 0.\end{aligned} \tag{4}$$

From this equation v must be eliminated, and the value of x determined from the result. For effecting this object, it will be convenient to employ the method of Prop. 3 of the present chapter.

Let then the result of elimination be represented by the equation

$$Ex + E'(l - x) = 0.$$

To find E make $x = 1$ in the first member of (4), we find

$$\bar{v}w(y\bar{z} + z\bar{y}) + (y + v\bar{y})(wz + \bar{w}\bar{z}) + (w\bar{z} + z\bar{w})\bar{v}\bar{y}.$$

Eliminating v, we have

$$(wz + \bar{w}\bar{z})\left\{w(y\bar{z} + z\bar{y}) + y(wz + \bar{w}\bar{z}) + \bar{y}(w\bar{z} + z\bar{w})\right\};$$

which, on actual multiplication, in accordance with the conditions $w\bar{w} = 0$, $z\bar{z} = 0$, &c., gives

$$E = wz + y\bar{w}\bar{z}$$

Next, to find E' make $x = 0$ in (4), we have

$$z(1 - vy\bar{w} - v\bar{y}w) + w\bar{z} + z\bar{w}.$$

whence, eliminating v, and reducing the result by Propositions 1 and 2, we find

$$E' = w\bar{z} + z\bar{w} + \bar{y}\bar{w}\bar{z};$$

and, therefore, finally we have

$$(wz + y\bar{w}\bar{z})x + (w\bar{z} + z\bar{w} + \bar{y}\bar{w}\bar{z})\bar{x} = 0; \tag{5}$$

from which

$$x = \frac{w\bar{z} + z\bar{w} + \bar{y}\bar{w}\bar{z}}{w\bar{z} + z\bar{w} + \bar{y}\bar{w}\bar{z} - wz - y\bar{w}\bar{z}}$$

wherefore, by development,

$$\begin{aligned} x = {} & 0yzw + yz\bar{w} + y\bar{z}w + 0y\bar{z}\bar{w} \\ & + 0\bar{y}zw + \bar{y}z\bar{w} + \bar{y}\bar{z}w + \bar{y}\bar{z}\bar{x}; \end{aligned}$$

or, collecting the terms in vertical columns,

$$x = z\bar{w} + \bar{z}w + \bar{y}\bar{z}\bar{w}; \tag{6}$$

the interpretation of which is–

In whatever substances the property A is found, there will also be found either the property C or the property D, but not both, or else the properties B, C, and D, will all be wanting. And conversely, where either the property C or the property D is found singly, or the properties B, C, and D, are together missing, there the property A will be found.

It also appears that there is no independent relation among the properties B, C, and D.

Secondly, we are to find y. Now developing (5) with respect to that symbol,

$$(xwz + x\bar{w}\bar{z} + \bar{x}w\bar{z} + \bar{x}z\bar{w})y + (xwz + \bar{x}w\bar{z} + \bar{x}z\bar{w} + \bar{x}\bar{z}\bar{w})\bar{y} = 0;$$

whence, proceeding as before,

$$y = \bar{x}\bar{w}\bar{z} + \frac{0}{0}(\bar{x}wz + xw\bar{z} + xz\bar{w}), \tag{7}$$

$$\begin{aligned} xzw &= 0; \quad (8) \\ \bar{x}z\bar{z}w &= 0; \quad (9) \\ \bar{x}z\bar{w} &= 0; \quad (10) \end{aligned}$$

From (10) reduced by solution to the form

$$\bar{x}z = \frac{0}{0}w;$$

we have the independent relation,–*If the property A is absent and C present, D is present.* Again, by addition and solution (8) and (9) give

$$xz + \bar{x}\bar{z} = \frac{0}{0}\bar{w}.$$

Whence we have for the general solution and the remaining independent relation :

1st. *If the property B be present in one of the productions, either the properties A, C, and D, are all absent, or some one alone of them is absent.* And conversely, *if they are all absent it may be concluded that the property A is present* (7).

2nd. *If A and C are both present or both absent, D will be absent, quite independently of the presence or absence of B* (8) and (9).

I have not attempted to verify these conclusions.

3 Frege, « Booles rechnende Logik und die Begriffsschrift » (1880/1881, publié à titre posthume) : solution du même problème

3.1 Version originale, in *Nachgelassene Schriften* (1969), p. 44-51

sind, und dass man Gleiches durch Gleiches überall ersetzen darf. Schröder führt sie unter seinen dreizehn Axiomen nicht auf, obgleich kein Grund ist, sie auszulassen, wenn man doch einmal selbstverständliche logische Sätze aufzählt[1]). Er wendet also eigentlich fünfzehn Axiome an. Ich führe in meiner Begriffsschrift neun Grundgesetze auf, dazu kommen noch als in Worte gefasste Regeln, die wesentlich durch die gewählte Bezeichnungsweise bestimmt sind, folgende:

1) Was auf den Inhaltsstrich folgt, muss einen beurteilbaren Inhalt haben. (S. 2)
2) Die Regel des Schliessens.
3) Es müssen die deutschen Buchstaben verschieden gewählt werden, wenn das Gebiet des einen, das des andern einschliesst.*) (S. 21)
4) Regel über die Einführung deutscher Buchstaben an die Stelle lateinischer (S. 21).
5) Regel über Aussonderung einer Bedingung aus dem Gebiete eines deutschen Buchstabens. (S. 21)

Was ich über die Anwendung der kleinen griechischen Buchstaben gesagt habe, kann hier ungezählt bleiben, weil es ausserhalb des Gebietes liegt, in dem eine Vergleichbarkeit mit Booles Formelsprache statt hat. So würde ich mit vierzehn ursprünglichen Sätzen ein etwas weiteres Gebiet beherrschen als Schröder mit fünfzehn. Ich habe aber jetzt erkannt, dass die beiden Grundgesetze der Inhaltsgleichheit ganz entbehrt[2]), und dass die drei Grundgesetze der Verneinung auf zwei zurückgeführt werden können. Nach dieser Vereinfachung brauche ich nur noch elf ursprüngliche Sätze. Hierin sehe ich den Erfolg meines Strebens nach Einfachheit der Urbestandteile und nach Lückenlosigkeit der Beweise. So setzte ich an die Stelle der in den Wortsprachen ins Unbestimmte fortwuchernden logischen Formen wenige, und das scheint mir für die Sicherheit der Gedankenbewegung wesentlich; denn nur das Endliche und Bestimmte kann unmittelbar erfasst werden, und je geringer an Zahl die ursprünglichen Sätze sind, desto vollkommener können sie beherrscht werden.

Da demnach die booleschen Rechnungen mit den Ableitungen nicht vergleichbar sind, die ich in der Begriffsschrift gegeben habe, so mag es nicht unangemessen sein, hier ein Beispiel vorzuführen, für das die Vergleichbarkeit besteht. Es wäre nicht zu verwundern, und ich könnte es ohne Bedauern zugeben, wenn die boolesche Logik für die Lösung solcher Aufgaben, auf die sie besonders angelegt [ist], oder die für sie besonders erdacht sind, geeigneter wäre, als meine Begriffsschrift. Aber dies ist vielleicht nicht einmal der Fall. Da diese Frage indessen für mich von geringer Wichtigkeit ist, will ich mich

* Diese Regel ist in der ersten eigentlich schon enthalten.

[1] Im Manuskript steht: „auszählt".
[2] Cf. oben p. 40.

darauf beschränken, eine von Boole*), dann von Schröder**), und Wundt***) behandelte Aufgabe mit der Begriffsschrift zu lösen und ganz kurz den Unterschied von der booleschen Weise anzudeuten:

Die Aufgabe ist nach der Wiedergabe Schröders[2]) folgende: Es werde angenommen, dass die Beobachtung einer Klasse von Erscheinungen (Natur- oder Kunsterzeugnissen, z.B. Substanzen) zu den folgenden allgemeinen Ergebnissen geführt hat.

α) Dass, in welchem von diesen Erzeugnissen auch die Merkmale oder Eigenschaften A und C gleichzeitig fehlen, das Merkmal E gefunden wird zusammen mit einem der beiden Merkmale B und D, aber nicht mit beiden.

β) Dass, wo immer die Merkmale A und D in Abwesenheit von E gleichzeitig auftreten, die Merkmale B und C entweder beide sich vorfinden oder beide fehlen.

γ) Dass überall, wo das Merkmal A mit dem B oder E [oder] mit beiden zusammen besteht, auch entweder das Merkmal C vorkommt oder das D, aber nicht beide. Und umgekehrt, überall, wo von den Merkmalen C und D das eine ohne das andere wahrgenommen wird, da soll auch das Merkmal A in Verbindung mit B oder mit E oder mit beiden zugleich auftreten.

Es möge nun ermittelt werden:

1) was in jedem Falle aus der Gegenwart des Merkmals A in Bezug auf die Merkmale B, C und D geschlossen werden kann,

2) ob irgendwelche Beziehungen unabhängig von der An- oder Abwesenheit der übrigen Merkmale zwischen denjenigen[3]) der Merkmale B, C, D bestehen und welche etwa,

3) was aus dem Vorhandensein des Merkmals B in Bezug auf A, C, D folgt,

4) was für A, C, D an sich folgt.

Bei der Auflösung bediene ich mich der entsprechenden grossen Buchstaben[4]) in der Weise, dass z.B. *A* den Umstand bedeutet, das Merkmal A finde sich an dem betrachteten Gegenstande vor.

* A.a.O., S. 146f.
** Der Operationskreis des Logikkalkuls, S. 25f.
*** Logik, I.[1]) S. 356.

[1] Frege bezieht sich auf die 1. Auflage von 1880.

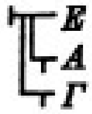
[2] Bei *Schröder*, *Der Operationskreis des Logikkalkuls* (Leipzig 1877), p. 25, Zeile 13 von unten, muß es statt „*C*" heißen: „*E*". In: *Vorlesungen über die Algebra der Logik*, Bd. I (Leipzig 1890), p. 523, Zeile 1 von oben ist der Fehler verbessert worden.

[3] Gemeint ist: „der An- oder Abwesenheit".

[4] Gemeint ist wohl, wie sich aus dem Folgenden ergibt: „der entsprechenden grossen griechischen Buchstaben". Die griechischen Buchstaben sind hier durch Kursivdruck von den lateinischen Buchstaben abgehoben. Das Manuskript schreibt diese Unterscheidung nicht vor.

Ich übersetze zunächst die einzelnen Daten.

α) Die Verneinung von A und Γ soll die Bejahung von E zur Folge haben (1).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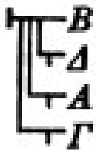
Die Verneinung von A und Γ soll die Bejahung eines der beiden, B oder Δ, zur Folge haben (2);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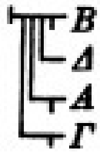
aber es sollen nicht B und Δ gleichzeitig bei der Verneinung von A und Γ statt haben können (3).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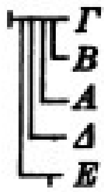
β) Bei der Bejahung von A und Δ und der Verneinung von E, sollen B und Γ entweder beide bejaht oder beide verneint werden; d.h. wenn B bejaht ist, soll auch Γ bejaht sein(4);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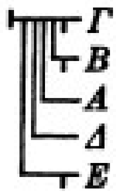
wenn aber B verneint ist, soll auch Γ verneint sein (5).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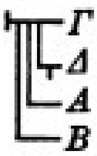
γ) Dies zerlege ich zunächst.

γ_1) Wenn A und B bejaht sind, soll auch Γ oder Δ bejaht sein (6), aber nicht beide (7).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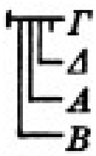
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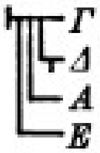
γ_2) Dieselben Folgen sollen stattfinden, wenn A und E bejaht sind[:] (8) und (9).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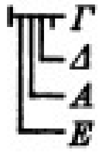
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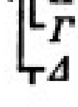
γ_3) Wenn Γ bejaht und Δ verneint ist, soll A bejaht sein (10). Da Γ schon eine Bedingung ist, so ist selbstverständlich auch eins von beiden bejaht; denn Γ ist es jedenfalls.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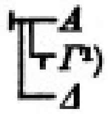
γ_4) Wenn Γ verneint und Δ bejaht ist, soll A bejaht sein (11), und es soll auch eines von beiden, B oder E bejaht sein. Das kann hier nur B sein (12).[2])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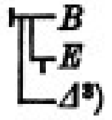
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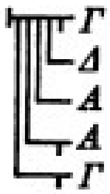
Dies sind die Daten. Die erste Frage wird zum Teil schon durch (6) und (7) beantwortet. Die übrigen Daten sind entweder deshalb keine Antworten auf diese Frage, weil sie wie (2) und (3) das verneinte statt das bejahte A oder wie (10), (11) und (12) A überhaupt nicht als Bedingung enthalten, oder weil sie wie (4), (5), (8), (9) ausser A, B, Γ, Δ noch E enthalten. Es fragt sich, ob E aus einigen der letzteren etwa weggeschafft werden könne. Dies kann geschehen, wenn E in einem Urteile wie in (1) als Folge, in einem andern wie in (9) als Bedingung auftritt. Man schreibt dann (9) bis auf die Bedingung E unverändert hin und ersetzt diese durch die beiden Bedingungen, von denen E in (1) abhängt. Das ergibt (13). Dieses Urteil ist aus zwei Gründen unabhängig von den Bedeutungen von A, Γ und Δ erfüllt, erstens weil als Bedingung von Γ ─┬─ Γ ─┬─ selbst auftritt, zweitens weil unter den Bedingungen zwei sich widersprechende A und ─┬─ A vorkommen. Als Folge von zwei sich widersprechenden Bedingungen kann nämlich jeder beliebige beurteilbare Inhalt ohne Fehler gesetzt werden.*) Daher gibt (13) über den Inhalt von A, Γ und Δ keine Auskunft. In ähnlicher Weise wie mit (1) und (9) kann man mit (1) und (8) verfahren. Der blosse Anblick der Formel überzeugt indessen davon, dass auch hierdurch kein inhaltlicher Aufschluss gewonnen wird, da unter den Bedingungen des Ergebnisses wieder die sich widerprechenden A und ─┬─ A vorkommen würden. Aus zwei Urteilen, in denen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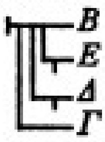
* Begriffsschrift, Formel (36) S. 45.

[1] Im Manuskript steht: „E".

[2] (γ_3) und (γ_4) sind nicht ganz verständlich. Formel (12) ist falsch, in dem Sinne, daß sie nicht aus der Konjunktion von (α) und (β) und (γ) folgt, insbesondere also auch nicht aus (γ) allein. Anstelle der Formel (12) (die später in der Zusammenfassung auf p. 50 von Frege wiederholt wird) sollten eigentlich folgende Formeln stehen: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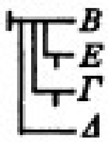
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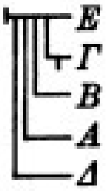
Frege macht in seinen folgenden Überlegungen von der falschen Formel (12) keinen Gebrauch, sondern nur von den übrigen Formeln. Eventuell hat das Versehen in Formel (11) Freges Fehler verursacht.

[3] Im Manuskript steht: „A".

wie in (8) und (9) *E* beide Mal bejaht als Bedingung vorkommt, kann es nicht weggeschafft werden, wohl aber, wenn es in dem einen von zwei Urteilen bejaht, in dem andern verneint als Bedingung wie in (8) und (4) vorkommt. Man kann nämlich ein Urteil mit einer verneinten Bedingung umformen, indem man das, was verneint Bedingung war, bejaht zur Folge und dafür das, was Folge war, verneint zur Bedingung macht.*) So gehen die Formeln (4) und (5) über in (14) und (15). Jedes dieser beiden Urteile kann man nun mit jedem der Urteile (8) und (9) zur Wegschaffung von *E* verbinden. Durch den blossen Anblick der Formeln überzeugt man sich leicht,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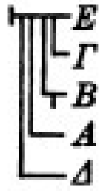
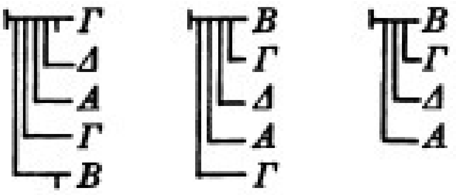

(15

dass die aus (8) und (14), (8) und (15), (9) und (14) zu gewinnenden Ergebnisse wie oben als unabhängig von den Inhalten erfüllt keinen Aufschluss über diese enthalten. Aus (9) und (15) dagegen erhalten wir nebenstehende Formel. Hier können die doppelt vorkommenden Bedingungen *Α* und *Δ* vereinigt werden. Auch die beiden *Γ* können in eins zusammengezogen werden, indem man zunächst wie vorhin *B* zur Folge und *Γ* zur Bedingung macht und nun *Γ* einmal weglässt (16). Dies ist die dritte Antwort auf die erste Frage, und in den Urteilen (6), (7) und (16) ist Alles enthalten, was in Bezug auf die erste Frage aus den Daten gewonnen werden kann. Es ist höchstens noch eine Formänderung dadurch möglich, dass man noch einen Buchstaben, etwa *B*, wegschafft. (6) und (16) liefern kein brauchbares Ergebnis. Aus (7) und (16) erhält man Nebenstehendes, was wie vorhin vereinfacht (17) ergibt. Dies lehrt, dass bei der Gegenwart des Merkmals A die Merkmale C und D einander ausschliessen. (6) zeigt dann, dass eins der beiden Merkmale C und D vorhanden ist, wenn ausser dem Merkmal A auch B eintrifft.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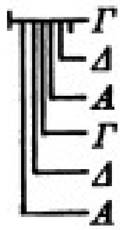
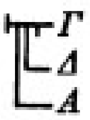
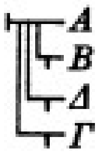
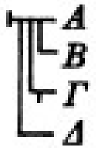



(17

* Begriffsschrift, Formeln (33) und (34), S. 44f.

Ich gehe zu der zweiten Frage über. Um zu entscheiden, ob Beziehungen zwischen *B*, *Γ*, *Δ* unabhängig von *A* und *E* bestehen, muss man Letztere aus den Daten wegschaffen und sehen, ob die erhaltenen Ergebnisse etwas mehr als logisch Selbstverständliches enthalten. Statt der *E* enthaltenden Daten können wir sofort das schon gefundene (16) benutzen. Wir haben demnach aus (2), (3), (6), (7), (10), (11), (16) *A* wegzuschaffen. Wir bringen zuvor (2) und (3) in die Formen (18) und (19). Man kann nun verbinden

 (18

 (19

(6) mit (10) oder (7) mit (10) oder (16) mit (10) oder
(6) „ (11) „ (7) „ (11) „ (16) „ (11) „
(6) „ (17) „ (7) „ (17) „ (16) „ (17) „
(6) „ (18) „ (7) „ (18) „ (16) „ (18),

welche Paare, wie der Anblick der Formeln lehrt, sämtlich unabhängig vom Inhalte erfüllte Ergebnisse liefern. Die zweite Frage ist daher zu verneinen.

Die Antwort auf die dritte Frage ist in den Formeln (6), (7) und (19) enthalten.[1]) Aus (7) und (19) ist zu entnehmen, dass wenn ausser B auch noch das Merkmal D zutrifft, von den Merkmalen A und C eins vorhanden sein muss, aber nicht beide. (6) zeigt, dass bei Abwesenheit von D entweder auch A fehlt, oder A und C gleichzeitig vorhanden sind.

Die Beantwortung der vierten Frage erfordert die Wegschaffung von *B* aus (2), (3), (6), (7) und (16). Statt (3) nehmen wir (19). Von den möglichen Verbindungen

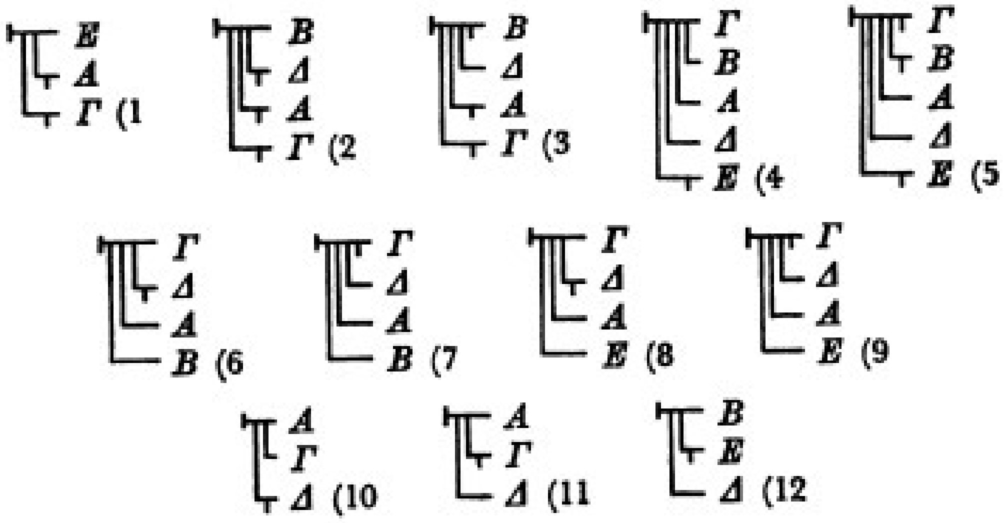
(2) mit (6), (13) mit (6),
(2) „ (7), (13) „ (7),
(2) „ (19), (13) „ (19)

ist nur die vorletzte zu brauchen und schon zur Bildung von (17) benutzt. Die Antwort auf die vierte Frage ist demnach, dass die Merkmale A, C und D nicht zugleich vorhanden sein können, und dass, wie (10) und (11) lehren, die Anwesenheit eines der Merkmale C und D ohne das andere die Anwesenheit von A zur Folge hat.

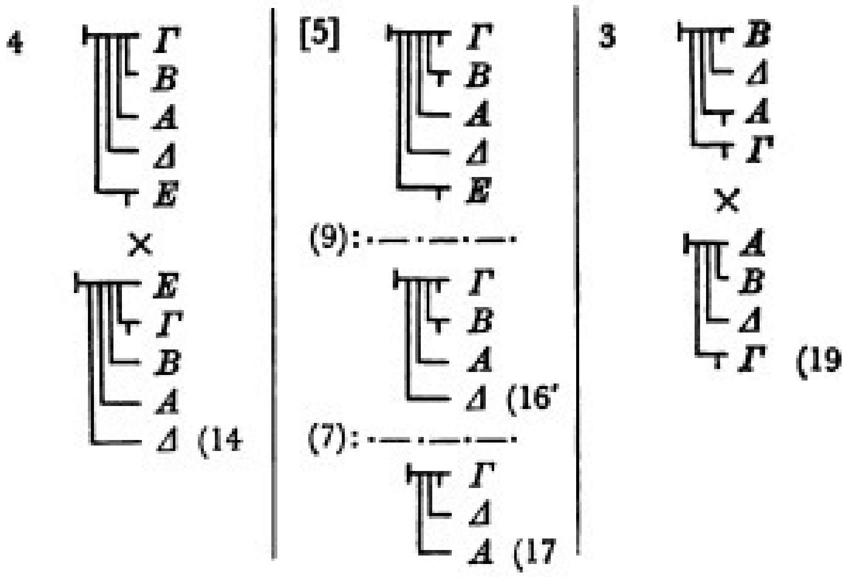
Diese Auflösung erfordert fast gar keine theoretischen Vorbereitungen. Alles, was an Algorithmus erforderlich war, habe ich nebenbei vorgeführt, wodurch vielleicht der Schein einer grössern Länge entstanden ist. Ich will deshalb noch in der Kürze die Daten und die Rechnung übersichtlich zusammenstellen.

[1] Frege interpretiert offenbar die 3. Frage so, daß nur nach den Folgerungen gefragt wird, die das Vorhandensein des Merkmals B wirklich voraussetzen, d.h. mit dem Nichtvorliegen von B unverträglich sind. Daher führt er (10) und (11) nicht als Teil der Antwort auf die 3. Frage auf.

Daten.



Rechnung.



Hierbei deutet das zwischen zwei Formeln gesetzte × den vorhin ausführlich besprochenen Übergang an.[1]) Das Zeichen —·—·—·, welches zwischen (5) und (16′) sowie zwischen (16′) und (17) steht, weist auf eine Regel hin, die den oben eingeschlagenen weitern Weg abkürzt. Sie lautet so:

Wenn zwei Urteile (z.B. (5) und (9)) in der Folge (—┬—Γ) übereinstimmen und je einen von zwei einander widersprechenden Inhalten (E und —┬—E) als Bedingung enthalten, so kann man ein neues Urteil (16′) bilden, indem man der gemeinsamen Folge (—┬—Γ) die Bedingungen der beiden ersten Urteile ((5) und (9)) mit der Ausnahme der beiden einander widersprechenden (E

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[1] Es handelt sich um die p. 48 angeführte Kontraposition.

und $\top\!\!-E$) gibt, dabei aber die beiden Urteilen gemeinsamen Bedingungen (A und Δ) nur einmal hinschreibt.

(16′) ist von (16) nicht wesentlich verschieden.

Die Antwort auf die erste Frage ist in (16) und (17) enthalten;
die " " dritte " " " (6), (7) & (19) "
" " " vierte " " " (10), (11) & (17) "
die zweite Frage ist zu verneinen.

Während bei Boole die Vereinigung verschiedener Urteile zu Gesamtausdrücken vorherrscht, zerlege ich die Daten in einfache Urteile, die dann zum Teil schon Antworten auf die Fragen sind. Darauf wähle ich aus den einfachen Urteilen die aus, welche für die erforderlichen Wegschaffungen geeignet sind, und erhalte so die noch übrigen Antworten, welche dann auch nur das enthalten, wonach gefragt war.

Ich glaube hiermit gezeigt zu haben, dass wenn wirklich einmal in der Wissenschaft ähnliche Aufgaben ihre Lösung heischen sollten, die Begriffsschrift sie ohne Schwierigkeiten werde bewältigen können. Man sieht aber auch, dass ihre eigentliche Kraft, die in der Bezeichnung der Allgemeinheit, dem Functionsbegriffe, der Möglichkeit beruht, verwickeltere Ausdrücke an die Stelle zu setzen, wo hier einfache Buchstaben stehen, dabei in keiner Weise zur Geltung kommt.

Es möge noch eine Bemerkung über das Äussere meiner Begriffsschrift hinzugefügt werden.

Schröder wirft mir vor, dass ich abweichend vom Gewöhnlichen die Schreibung von oben nach unten der von links nach rechts vorziehe. In Wahrheit stehe ich mit Üblichem ganz im Einklange; denn auch in einer arithmetischen Ableitung lässt man die einzelnen Gleichungen von oben nach unten aufeinanderfolgen. Jede Gleichung ist aber ein beurteilbarer Inhalt oder ein Urteil wie auch jede Ungleichung, jede Congruenz u.s.f.. Was ich nun untereinander setze, sind auch beurteilbare Inhalte oder Urteile. Jener Schein des Ungewöhnlichen entsteht dann, wenn man die einfachen beurteilbaren Inhalte nur durch einzelne Buchstaben andeutet. Sobald sie ausführlich hingeschrieben werden, was in den Anwendungen fast immer geschehen wird, dehnt sich ein jeder von links nach rechts in einer Zeile aus, und die einzelnen folgen von oben nach unten auf einander. Hierdurch wird der Vorteil ausgenutzt, den eine Formelsprache in der zweifachen Ausdehnung der Schreibfläche vor der in der einfach ausgedehnten Zeit erscheinenden Wortsprache voraus hat. Boole braucht nicht für jeden einfachen beurteilbaren Inhalt eine Zeile in Anspruch zu nehmen, weil er nicht daran dachte, ihn weitläufiger als durch einen einzigen Buchstaben darzustellen. Die Folge davon ist, dass eine grosse Unübersichtlichkeit entstehen würde, wenn man nachträglich für diese einzelnen Buchstaben ganze Formeln einführen wollte.

Ich glaube, in dieser Abhandlung folgendes nachgewiesen zu haben:

1) Meine Begriffsschrift hat ein weiteres Ziel als die Boolesche Logik, indem sie in Verbindung mit arithmetischen und geometrischen Zeichen die Darstellung eines Inhaltes ermöglichen will.

3.2 Trad. anglaise, in *Posthumous Writings* (1979), p. 39-45

this is possible, the sentences derived by me, with an equally complete chain of inference. This wouldn't be afforded by 'mental multiplying out'. You also need the sentence that you may interchange two sides of an equation, and that equals may always be substituted for equals. Schröder does not include these among his thirteen axioms, although there is no justification for leaving them out, even if you regard them as self-evident truths of logic. And so he really uses fifteen axioms. In my *Begriffsschrift* I laid down nine axioms, to which we must add the rules set out in words, than in essentials are determined by the modes of designation adopted. They are as follows:

(1) What follows the content-stroke must be a content of possible judgement (p. 2).
(2) The rule of inference.
(3) Different gothic letters are to be chosen when one occurs within the scope of another* (p. 21).
(4) A rule for replacing roman letters by gothic (p. 21).
(5) A rule for exporting a condition outside the scope of a gothic letter (p. 21).

We may ignore here what I have to say about the use of Greek small letters, since it lies outside the domain within which we may compare the concept-script and Boole's formula-language. So with 14 primitive sentences I command a somewhat wider domain than does Schröder with 15. But I have since seen that the two basic laws for identity are completely dispensable, and that we may reduce the three basic laws for negation to two. After this simplification I need only 11 basic sentences. I see in this the success of my endeavour to have simple primitive constituents and proofs free from gaps. And so I replace the logical forms which in prose proliferate indefinitely by a few. This seems to me essential if our trains of thought are to be relied on; for only what is finite and determinate can be taken in at once, and the fewer the number of primitive sentences, the more perfect a mastery can we have of them.

Since, then, Boolean computations cannot be compared with the derivations I gave in the *Begriffsschrift*, it may not be out of place to introduce here an example where there can be a comparison. It would not be surprising and I could happily concede the point, if Boolean logic were better suited than my concept-script to solve the kind of problems it was specifically designed for, or for which it was specifically invented. But maybe not even this is the case. Since the question involved is for me one of slight importance, I will confine myself to using the concept-script to solve a problem that has been treated by Boole,** then by Schröder,*** and then Wundt,**** while very briefly indicating how it differs from Boole's method.

* Strictly, this rule is implicit in the first.
** *Op. cit.* pp. 146 f.
*** *Der Operationskreis des Logikkalkuls*, pp. 25 f.
**** *Logik* I, p. 356.

In Schröder's formulation, the problem is as follows. Suppose we observe a class of phenomena (natural kinds or artefacts, e.g. substances) and arrive at the following general results:

(*α*) If the characteristics or properties A and C are simultaneously absent from any of the phenomena, the property E is found together with either the property B or the property D but not both.

(*β*) Wherever A and D are found together in the absence of E, B and C are either both present or both absent.

(*γ*) Wherever A is found together with B or E or both, either C or D is to be found but not both. And conversely wherever one of C, D is found without the other, then A is to be found together with B or E or both.

We now have to find:

(1) What in general can be inferred about B, C and D from the presence of A,

(2) Whether any relations whatever hold between the presence or absence of B, C and D independently of the presence or absence of the remaining properties,

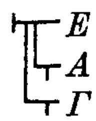
(3) What follows for A, C and D from the presence of B,

(4) What follows for A, C and D considered in themselves.

In the solution I use the corresponding Greek capitals so that e.g. A means the circumstance that the property A is to be found in the object under consideration.

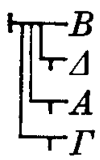
I first translate the individual data.

(*α*) The denial of A and Γ has a consequence the affirmation of E (1).



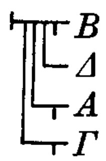
(1)

The denial of A and Γ has as a consequence the affirmation of one of the two B or Δ (2);



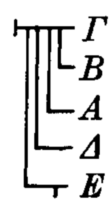
(2)

but it is impossible to have B and Δ together with the denial of A and Γ (3).



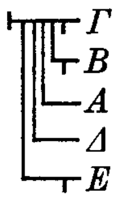
(3)

(*β*) If A and Δ are both to be affirmed and E denied, B and Γ are either both to be affirmed or both denied; that is, if B is affirmed, Γ is also to be affirmed (4);



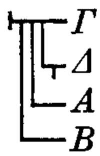
(4)

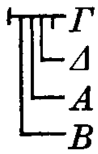
but if B is denied, Γ is also to be denied (5).


(5)

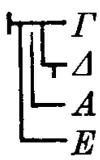
(γ) I begin by breaking this down.

(γ_1) If A and B are affirmed, Γ or Δ is to be affirmed (6) but not both (7).


(6)


(7)

(γ_2) The same result holds if A and E are affirmed: (8) and (9).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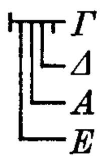
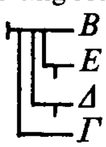
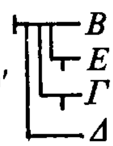

(9)

(γ_3) If Γ is affirmed and Δ denied, A is to be affirmed (10). Since Γ is already a condition one of the two is also to be affirmed, for at least Γ is.[1]


(10)

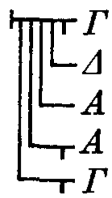
[1] γ_3 and γ_4 are evidently faulty: no single interpretation will make them consistent either with the problem or for that matter internally. They do not even as they stand make clear sense. The most likely hypothesis to explain Frege's mistakes here is that while working at the problem, he sometimes approached it as it stands in the text, and sometimes following Schröder's version which contained a misprint of 'C' for 'E' in the last clause of (γ). E.g. the prose of γ_3 tallies with the Schröder misprinted version, but γ_4 is a hybrid of the two readings. The resulting formulae are wrong in the sense that (10) needs supplementing by (10)′  and (12) replacing by (12)′ . (11) is written here, but not subsequently, by Frege in the form  which we have followed the German editors in emending as above. There is a further slip in that Frege here writes his (12) with A for the Δ, where we have similarly corrected it. Luckily, despite this morass of confusion it has no effect on the solution of the problem: e.g. Frege makes no further use of his defective (12) and as far as we can see the formulae omitted by Frege would contribute nothing to the resolution of this problem. He does indeed get the problem right, although the correctness of his solution is put in jeopardy by the fact that he does not of course show that (10)′ and (12)′ yield nothing further of relevance to the questions asked. It does not seem worth exploring this minor and irritating matter further. (trans.)

(γ_4) If Γ is denied and Δ affirmed, A is to be affirmed (11) and one of the two B or E is also to be affirmed. Here that can only be B (12).


(11)


(12)

These are the data. The first question is already answered in part by (6) and (7). The remaining data yield no answers to this question, either because, as in (2) or (3) they do not contain the affirmation of A but its denial as a condition, or, as in (10), (11), (12), do not contain A as a condition at all, or because, as in (4), (5), (8), (9) they contain E as well as A, B, Γ, Δ. The question arises whether E could perhaps be eliminated from some of these last. This can be done if E is a consequence in one judgement, as it is in (1), and a condition in another as it is in (9). We then write (9) unaltered as far as the condition E and replace this by the two conditions on which E depends in (1). This yields (13). This judgement is satisfied no matter what the meanings of A, Γ and Δ, first because $-\!\!\top\!\!-\Gamma$ appears as a condition of $-\!\!\top\!\!-\Gamma$ itself, and secondly because two of the conditions, A and $-\!\!\top\!\!-A$, are contradictory. For you obtain a truth by putting an arbitrary content of possible judgement as the consequence of two contradictory conditions.* Thus (13) gives no information about the contents A, Γ and Δ. We may proceed with (1) and (8) as with (1) and (9). But we only have to glance at the formula to convince ourselves that we do not get any information by doing this either, since the resulting formula once again contains two contradictory conditions. Whereas in (8) and (9) the affirmation of E occurs as a condition in both judgements, it cannot be eliminated, but where its affirmation is a condition in one and its denial in the other, as in (8) and (4) it may. That is, you can transform a judgement with something denied as a condition by presenting the affirmation of the condition denied as a consequence and making the denial of the original consequence a condition.** So (4) and (5) may be transformed into (14) and (15). Each of these two judgements can now be combined with either of the judgements (8) and (9) in order to eliminate E. We need only glance at the formulae to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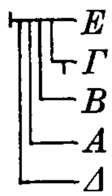
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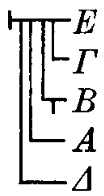
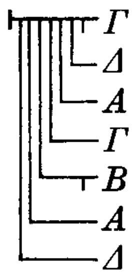
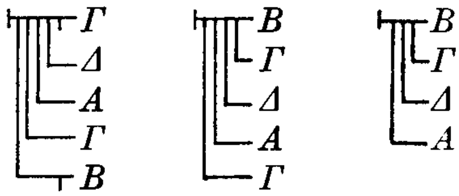

(15)

* *Begriffsschrift* Formula (36), p. 45.

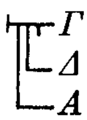
** *Begriffsschrift Formulae* (33) and (34), pp. 44 f.

convince ourselves that, as above, the results to be obtained from (8) and (14), (8) and (15), and (9) and (14) do not tell us anything which doesn't hold independently of the contents. But from (9) and (15) we obtain the formula opposite. Here the antecedents that occur twice over, *A* and *Δ*, can be assimilated. The two *Γ* can also be assimilated by first, as before making *B* the consequence and *Γ* the condition and now dropping one of the *Γ*s (16). This is the third answer to the first question and the judgements (6), (7) and (16) contain everything that is to be obtained from the data in answer to the first question. At most, it might still be possible to give the results in a different form by eliminating a letter—*B*, say. (6) and (16) yield no result of value. (7) and (16) give us the formula alongside; which gives us (17) when simplified as above. This tells us that when the property A is present, the properties C and D exclude one another. (6) then shows that one of the two properties C and D is present when, besides A, B is also present.



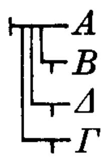


(16)

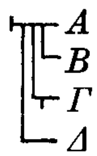


(17)

I move on to the second question. To decide whether any relations hold between *B*, *Γ* and *Δ* independently of *A* and *E*, we must eliminate the latter from the data and see whether the results contain anything other than logical platitudes. Instead of the data containing *E* we may straight off use the formula (16) we have already discovered. We have accordingly to eliminate *A* from (2), (3), (6), (7), (10), (11) and (16). We begin by transforming (2) and (3) into (18) and (19).



(18)



(19)

We may now combine

(6) with (10) or (7) with (10) or (16) with (10) or
(6) with (11) or (7) with (11) or (16) with (11) or

(6) with (17) or (7) with (17) or (16) with (17) or
(6) with (18) or (7) with (18) or (16) with (18).

A glance at the formulae shows that these pairs yield results that hold independently of the contents. Hence the second question is to be answered negatively.

The answer to the third question is contained in (6), (7) and (19). We may infer from (7) and (19), that if, as well as B, the property D is present, one of the properties A and C must be present, but not both. (6) shows that if D is absent, either A is also absent or A and C are both present.

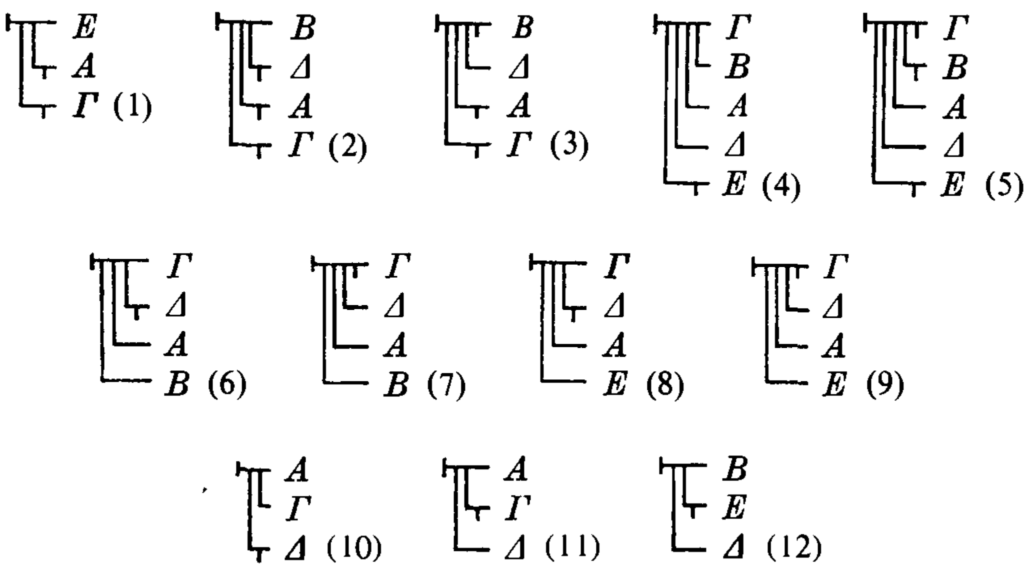
To answer the fourth question we need to eliminate *B* from (2), (3), (6), (7) and (16). We adopt (19) in place of (3). Of the possible combinations

(2) with (6),	(16) with (6),
(2) with (7),	(16) with (7),
(2) with (19),	(16) with (19)

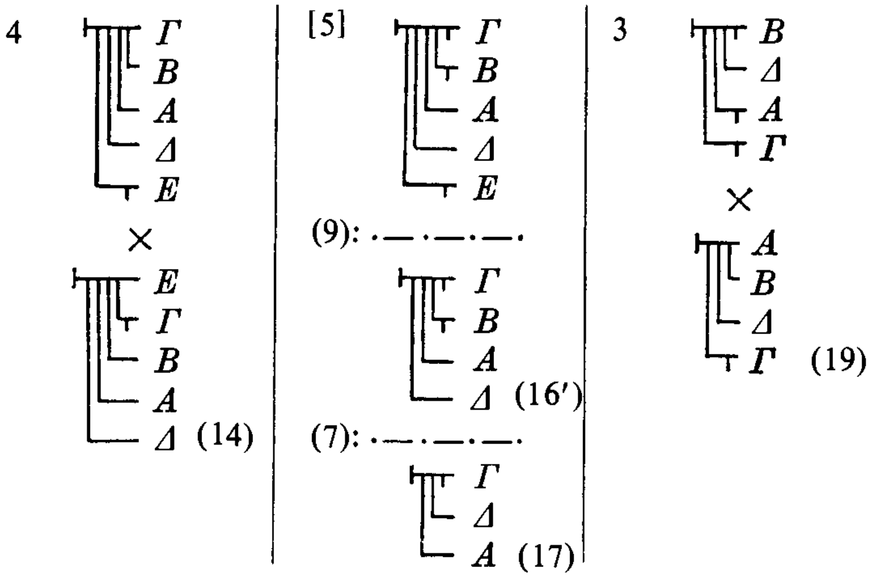
only the last but one is of any use, and has already been used to give us (17). So the answer to the fourth question is that the properties A, C and D cannot all be present at once, and that as (10) and (11) show, the presence of one of the properties C and D without the other implies the presence of A.

This solution requires practically no theoretical preparation at all. I have accompanied the account with every rule required for solving the problem, and this may have created the impression of a greater length than the true one. So I will now collate the data and computation in brief and in a surveyable form:

Data



Computation



Here the × between two formulae indicates the transition spelt out above. The sign ·—·—·—· that stands between (5) and (16′) and between (16′) and (17) indicates a rule that abbreviates the other route followed above. It runs as follows:

If two judgements (e.g. (5) and (9)) have a common consequence (—┬—*Γ*), and one has a condition contradicting a condition of the other (*E* and —┬—*E*), we may form a new judgement (16′), by attaching to the common consequence (—┬—*Γ*), the conditions of the two original judgements ((5) and (9)) minus the contradictory ones (*E* and —┬—*E*), but in which conditions common to both judgements are only written once (*A* and *Δ*).

(16′) isn't essentially different from (16).

The answer to the first question is contained in (6) and (17);

The answer to the third question is contained in (6), (7) and (19);

The answer to the fourth question is contained in (10), (11) and (17);

The answer to the second question is in the negative.

Whereas the dominant procedure in Boole is the unification of different judgements into a single expression, I analyse the data into simple judgements, which are then in part already answers to the questions. I then select from the simple judgements those lending themselves to the eliminations needed, and so arrive at the rest of the answers. These will then only contain what we wanted to find out.

I believe that I have in this way shown that if in fact science were to require the solution of such problems, the concept-script can cope with them without any difficulty. But we see too that, in all this, its real power, which resides in the designation of generality, the concept of a function, in the possibility of putting more complicated expressions in the positions here occupied by simple letters, in no way comes into its own.

4 Aide à la lecture (éléments de commentaire des textes précédents, issus d'un brouillon de DW et Dirk Schlimm)

4.1 Boole's conception of the problem of logic [cf. texte n°1]

To understand the goal-directed nature of Boole's logical method, we first need to explain what the problems are that his method is intended to solve. As it happens, Boole believed that there is a general form common to all logical problems. The easiest way to grasp this form is as a broad generalization of syllogistic, which was Boole's starting point. Accordingly, this section builds up to his conception of logical problems through successive generalizations from a simple syllogistic case.

Take two sentences in Aristotelian subject-predicate form, like 'All horses are mammals' and 'All mammals are animals', that have a term in common (here 'mammals'). The standard problem of syllogistic inference (as Boole construes it) [1] is to find what relation, if any, follows between 'horses' and 'animals', eliminating the term 'mammals' which is already present in both premises. (In this instance, the conclusion sought is 'All horses are animals'.) Importantly, this does not amount to finding *all* possible consequences of the initial sentences, of which there are many others, like 'Some mammals are horses': we are only interested in 'horses' and 'animals'. Nor would just any consequence do as long as it relates 'horses' and 'animals': neither 'Everything that is a horse but not an animal does not exist' nor 'Some horses are animals' would do, the first because it lacks the expected Aristotelian form, the second because it is not as strong as possible. Syllogistic inference, in short, comes with a constrained specification of the expected solution. Boole's problems do likewise, but in a broader setting.

First, Boole's propositions are more general than the Aristotelian subject-predicate forms, and are expressed as symbolic equations rather than in natural language. This aspect of Boole's work is well-known, so we shall be quick. Let us start with an example Boole discusses repeatedly in his *Investigation of the Laws of Thought*, namely the definition of *wealth* offered by the economist William Nassau Senior:

> Wealth consists of things transferable, limited in supply, and either productive of pleasure or preventive of pain. (Boole 1854, p. 59)

To express this definition symbolically, Boole introduces symbols that denote *classes* of things; in our case, he writes w for wealth, t for things transferable, s for things limited in supply, p for things productive of pleasure and r for things preventive of pain. He then uses operations analogous to those of algebra. The juxtaposition of two class symbols, akin to algebraic multiplication, denotes the class of things common to both (in contemporary terms, their *intersection*), so that st denotes things both transferable and limited in supply. The addition of two class symbols expresses the class formed by taking the elements of both together (today called their *union*), except that Boole only allows this operation on classes that are disjoint, that is, have no elements in common. Thus, if nothing is both 'productive of pleasure' and 'preventive of pain', $p + r$ would correspond to things that are either one or the other. The subtraction of two class symbols $x - y$ expresses the class of elements of x that are not elements of y, where it is assumed that y is included in x. Finally, the symbol 1 denotes the universe of discourse, so that for instance $1 - p$ denotes things *not* productive of pleasure. (Boole sometimes abbreviates $1 - p$ as $\bar{p}$.) This allows Boole to express Senior's definition as:

$$w = st\{p + r(1 - p)\}$$

or in words: wealth is things that are at the same time limited in supply, transferable, and either productive of pleasure, or preventive of pain and not productive of pleasure (the complexity

1. We are here glossing on Boole's own account (see in particular chap. XV, pp. 226–242), and not aiming for a historically accurate rendition of the goals of syllogistic.

of this last clause being required by Boole's restriction that addition can only be performed on disjoint classes). One last device that Boole has to introduce is 'indefinite' class symbols (which he often writes v, but sometimes with other letters as well, such as q), that is symbols denoting an unknown class. These allow him to represent inclusions like 'All horses are mammals' in the form of equalities, such as $h = vm$ (where h stands for the class of horses, m for that of mammals and v for an indefinite class); such equations can also be understood as 'conditionals' rather than inclusions, e.g., as 'If something is a horse, then it is a mammal'. [2]

Second, Boole admits not just two premises involving three terms, as in syllogistic inference, but any number of premises involving any number of terms. The problem analogous to syllogistic inference then becomes that of finding a relation – or more precisely, the strongest possible relation – among any number of terms selected among those appearing in the premises. One example Boole discusses is a piece of reasoning from Aristotle's *Nicomachean Ethics.* [3] Aristotle asks whether virtue is a passion, a faculty, or a habit. The six premises, as reconstructed by Boole, express various properties of virtue as well as of passions, faculties, and habits, involving several auxiliary properties, for instance 'things according to which we are praised or blamed' (according to Aristotle, we can be praised for our virtue, but not for our inborn faculties). The goal here, in Boole's terms, is to find the relation between virtue, passions, faculties, and habits, eliminating all other terms – the conclusion, as it turns out, being that virtue is a habit, but not a faculty nor a passion.

Third, the broader range of possible propositional forms in Boole's system allows him to put further constraints on the conclusion that is sought. In the syllogistic case, it is simply expected that the conclusion will be in one of the Aristotelian forms ('All A is B', 'Some A is B' or their negations). [4] Boole is able to be more specific. In the example of virtue, what is expected – and can be provided by Boole's method – is not just any equation linking virtue (denoted by v) with passions (p), faculties (f), and habits (h), but rather an equation of the form $v = \ldots$ (in which the right-hand side only contains p, f and h). This is the simplest and most common case, but other forms can also be requested (and obtained). Returning to Senior's definition of wealth, one could for instance ask about what can be concluded about 'wealth that is preventive of pain' in terms of 'things transferrable' and 'things limited in supply' – in other words, ask for a conclusion of the form $wr = \ldots$, where the right-hand side only contains s and t. [5] In Boole's words, the relation sought is 'that full relation which, in virtue of the premises, connects any elements selected out of the premises at will, and which, moreover, expresses that relation in any desired form and order'. [6]

2. In fact, Boole uses indefinite class symbols ambiguously, a difficulty that we shall point out here but ignore in the sequel, as it does not bear on our main points. In most settings, he takes such symbols to be absolutely indefinite, that is, to denote a class that can be empty, equal to the full universe of discourse or anything in between. But Boole also translates the Aristotelian form 'Some A is B' as '$va = vb$', in which – if the traditional interpretation of such forms is to be preserved – v has to be interpreted as an indefinite *non-empty* class. For a careful discussion written from a Boolean perspective, see Venn (1881, chap. VI–VII).

3. Boole (1854, pp. 134–137). We follow Boole's rendition of Aristotle; the passage in question is *Nicomachean Ethics* II.5 (1105b20–1106a15).

4. In traditional tables of the canonical syllogistic forms, there are other restrictions, which Boole neglects (it is expected that the major term will come first, for instance). The discrepancy arises because Boole does not fully do justice to traditional logic: he takes syllogistic's *classification* of inference forms as a full-fledged theory of reasoning, whereas traditional textbooks would also contain a broader theory exploring how non-canonical pieces of reasoning are to be brought into one of the standard forms. This need not detain us further, as we are only concerned with Boole's own portrayal of syllogistic.

5. There is yet another form that can be requested of the conclusion: one may want the list of those combinations of the selected terms that are *excluded* by the premises, that is, which correspond to classes that the premises force to be empty (for more on this, see Section **??** below). In Boole's system, this amount to seeking an equation of the form $V = 0$, in which V is a sum of combinations of class symbols or their negations; such an equation is equivalent to having each of the members of the sum be separately equal to 0. This form is the most exhaustive, being equivalent to the premises (or to the premises once elimination has been performed, if some terms have been eliminated), whereas equations of the form $x = \ldots$ will usually be weaker.

6. Boole (1854, p. 10).

We have focused so far on what Boole calls 'primary propositions', in which symbols denote classes. His system can also treat 'secondary propositions', in which the symbols already denote propositions; it is this 'secondary' part of his system that is closest to our contemporary propositional calculus. Boole derives his secondary propositions from his primary class-based ones by introducing, for a given proposition X, a class symbol x denoting 'that portion of time for which the proposition X is true'.[7] This *extensional* account of propositions allows treating relations between propositions just like relations between classes. The only differences lie in the interpretation of equations: in this new context, $x = 0$ and $x = 1$ mean that the proposition X is (always) true and (always) false, respectively; equations containing an indefinite class symbol, such as $x = vy$, are interpreted as implications ('If X, then Y'). As far as the general formulation of logical problems is concerned, however, the move to 'secondary' propositions changes very little: premises are still expressed by equations, and the goal is again to obtain an equation of a specified form relating a subset of the symbols appearing in the premises.[8]

4.2 A sample problem [cf. texte n°2]

Let us now turn to the sample problem that will be discussed in more details below. Among the examples discussed by Boole in his *Investigation of the Laws of Thought*, it is of particular interest, not just because it is one of the most intricate, but also because – for this very reason – it was repeatedly addressed by later authors, including Schröder, MacColl, and Frege,[9] to show that their system was able to do as much as Boole's. Its formulation is quite abstract. It is about a class of 'natural productions' (which, in this particular case, will serve as the universe of discourse) whose members can display five properties A, B, C, D and E, with three relations between them which will serve as premises.

Before turning to the premises, a caveat is in order. Symbolically, Boole writes x for the property A, y for B, and so on. Strictly speaking, his method only requires referring back to the meaning of the symbols in the first and last steps (when initially translating the premises into symbols, and when interpreting the final equation), so the discrepancy between the names of the properties and the corresponding symbols is tolerable. Moreover, this discrepancy is justified by Boole's algebraic model, in which it is customary, since Descartes, to write the unknowns using letters from the end of the alphabet and the known (such as coefficients) using letters from the beginning. Nevertheless, as later authors – in particular Schröder and through him Frege, which we shall discuss at length – revert to the more straightforward convention of writing a for property A, etc., keeping Boole's notation would make the discussion below exceedingly confusing. In breach of the spirit of our paper, which strives to be faithful to the algebraic spirit of Boole's method, we therefore altered his choice of symbols. We also chose to effect a minor change in Schröder's notation: in this problem, Boole writes the negation of a symbol a as as $\bar{a}$, while Schröder writes it a_1, and we chose to write $\bar{a}$ throughout. All other notations are unchanged.

The three premises of Boole's problem, then, are the following:[10]

i. 'That in whichsoever of these productions the properties A and C are missing, the property E is found, together with one of the properties B and D, but not with both.' In symbols,

$$\bar{a}\bar{c} = qe(b\bar{d} + d\bar{b})$$

where q is an indefinite class symbol, which can be read equivalently as an inclusion ('the

7. Boole (1854, p. 165).

8. Boole (1854, pp. 178–179).

9. Schröder (1877, pp. 25–28), Lotze (1884, pp. 219–221) = Lotze ([1880] 1912, pp. 265–267), Wundt (1880, vol. 1, pp. 356–357), Frege (1979, pp. 39–45) = Frege (1969, pp. 45–51), Venn (1881, pp. 280–281), and McColl (1878, pp. 23–25). That this single problem has been solved by logicians of various outlooks whose solutions would repay further comparison has already been noted by Gabriel (1989, p. XXIII).

10. See Boole (1854), 146–147.

class of productions without properties A and C is *a certain part of* the class of productions with property E etc.') or as a conditional ('if a production lacks properties A and C, then it has property E etc.').

ii. 'That wherever the properties A and D are found while E is missing, the properties B and C will either both be found, or both be missing.' In symbols,

$$\bar{e}ad = q(bc + \bar{b}\bar{c}).$$

iii. 'That wherever the property A is found in conjunction with either B or E, or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E, or both of them.' This premise is formulated as an equivalence (double implication); Boole thus translates it as an equality without indefinite class symbols:

$$ab + ae\bar{b} = d\bar{c} + c\bar{d}.$$

It may seem more straightforward to write this as Schröder later does:

$$a(b + e) = d\bar{c} + c\bar{d}$$

The reason for the difference is that Boole's '+' only allows for the addition of disjoint classes: writing $ae\bar{b}$ guarantees that it is disjoint from ab. Schröder, who adopts an inclusive interpretation of '+', can accordingly dispense with this extra factor.

Now, Boole asks for the relation between the properties A, B, C, and D, – thus eliminating E [11] – and this in two different forms. First, with interest for 'what may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D', the relation is sought in the form $a = \ldots$; second, looking for 'what may be concluded in like manner respecting the property B, and the properties A, C, and D', the relation is sought in the form $b = \ldots$.

Additionally, Boole asks for 'whether any relations exist independently among the properties B, C, and D' (which are used to express A in the first half of the problem) and likewise among the properties A, C, and D. While these two questions can be seen as instances of the general problem of logic (the first amounts to eliminating both A and E from the premises, the second both B and E, but with no particular form prescribed for the relation sought between the remaining terms), their presence here is somewhat peculiar. They can be read as asking for what information has had to be discarded about the relation between – taking the first case as an example – A, B, C and D in order to express it under the particular form $a = \ldots$. The main reason why they appear here, however, may just be that Boole's method for obtaining the solution $a = \ldots$ gives this further relation for free, as we shall see.

4.3 Boole's solution [cf. texte n°2]

In order to bring to the fore the goal-directedness of Boole's method, we briefly describe how he applies it to the foregoing problem. As Frege used Schröder's modified treatment as well, we briefly discuss it, too. Before delving into technicalities, we start with synopsis of the solution, broken down into four steps – assuming each premise has already been translated into symbols, as done above. This should be enough to drive home our main point, and readers in a hurry may then skip the details.

11. As Boole puts it, 'It will be observed, that in each of the three data, the information conveyed respecting the properties A, B, C, and D, is complicated with another element, E, about which we desire to say nothing in our conclusion. It will hence be requisite to eliminate the symbol representing the property E […]' (Boole 1854, p. 146).

1. Transform and bring together the premises, so as to obtain a single equation of the form $E = 0$.
2. Eliminate the terms that should not appear in the solution (in our case e). This leads to a new equation $E' = 0$, where E' does not contain e.
3. If the desired solution is of the form $a = \ldots$ (say), first factor the preceding equation by a and $\bar{a}$, yielding in our case

$$(dc + b\bar{d}\bar{c})a + (d\bar{c} + c\bar{d} + \bar{b}\bar{d}\bar{c})\bar{a} = 0, \tag{11}$$

then expand $\bar{a}$ as $1 - a$ and proceed as if algebraically solving for a:

$$a = \frac{d\bar{c} + c\bar{d} + \bar{b}\bar{d}\bar{c}}{d\bar{c} + c\bar{d} + \bar{b}\bar{d}\bar{c} - dc - b\bar{d}\bar{c}}.$$

As is the case here, this typically results in division signs of rather unclear meaning on the right-hand side. Schröder, whose solution follows Boole's up to this point (in broad outline at least), avoids this murky division and stops at equation (11) above.
4. At this stage, Boole and Schröder split ways. Boole performs a process he calls 'development' to get rid of the fraction he has just introduced; generically, this gets the right-hand side into the strange-looking form

$$U + \frac{1}{1}V + \frac{1}{0}W,$$

where U, V and W are sums of terms: for our problem,

$$a = c\bar{d} + \bar{c}d + \bar{b}\bar{c}\bar{d} \qquad \text{(i.e., V = W = 0)}, \tag{12}$$

$$b = \bar{a}\bar{d}\bar{c} + \frac{0}{0}(\bar{a}dc + ad\bar{c} + ac\bar{d}) + \frac{1}{0}(acd + \bar{a}\bar{c}d + \bar{a}c\bar{d}). \tag{13}$$

Equation (12) has no unusual symbols, hence straightforwardly answers the question (in words, property A is to be found exactly when one but not both of properties C and D are found, or when none of B, C and D are found). When, as in equation (13), V and W do not vanish, Boole does two things: he interprets $\frac{1}{1}$ as an indeterminate class symbol and splits off the term $\frac{1}{0}W$ into a separate equation $W = 0$. His interpretation of (13), then, is that B has the same extension as the class expressed by U plus *part of* the class expressed by V (remember that indeterminate class symbols are used for inclusions), and that the equation $W = 0$ expresses the *independent relations* between A, C and D (asked for in the statement of the problem, above).
Schröder, for his part, avoids Boole's perplexing symbolic manipulations by way of a general theorem, which allows jumping straight from equations like (11) to solutions equivalent to Boole's. In essence (taking for instance the first question, aiming at $a = \ldots$), from an equation of the form

$$Sa + T\bar{a} = 0,$$

Schröder directly expresses our U, V and W above in terms of S and T:

$$U = T, \qquad V = \bar{S}, \qquad W = ST$$

so that one gets the full solution $a = T + u\bar{S}$ (with u an indeterminate class symbol) and the indendent relation $ST = 0$.

For simplicity, we described Boole's method in the context of a particular problem, but its outline is general, with minor variants to cover special cases. [12] Now, the two features of Boole's

12. For instance, as discussed in the previous section, one might ask for the expression of ab in terms of c and

approach we want to highlight is that it is *systematic* and *goal-directed*. As shown in the previous system, Boole delineates a well-defined class of problems, and the method we just sketched allows for the systematic solution of any of them, guided by the particular problem to be solved. In the sketch above, steps 2 and 3 are where this goal-dependence appears: the elimination of unwanted terms depends on the particular relation sought, as do the algebraic manipulations of step 3, where our mastery of first-degree equations points us to the transformations needed to solve for a particular variable.

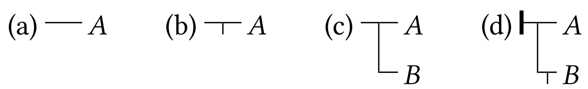
4.4 Frege's solution [cf. texte n°3]

At the end of a manuscript comparing his *Begriffsschrift* with Boole's system, Frege tackles the very problem discussed above in order to show that 'if in fact science were to require the solution of such problems, the concept-script would be able to cope with them without any difficulty'. [13] Yet as we shall see, his solution is rather haphazard, and is much more akin to an exhaustive (though intelligent) search through the space of all possible proofs from the premises than to Boole's algebraic method.

As a preliminary, we need to dispose of two slight complications. First, Frege changes notations a little: he uses the Greek capital letters A, B, Γ, Δ, E to refer to the presence of the properties A, B, C, D, E respectively. To avoid needlessly complicating the comparison, we have decided to revert to Roman capitals. Second, as pointed out by the editors of Frege's manuscript, some mistakes in his premises – partly due to his following Schröder, whose initial phrasing of the problem contains a minor misprint – appear to put his solution in jeopardy, even though he gets to the right conclusions (unsurprisingly, given that he had the correct answer at hand). This, however, is inessential: Frege's solution can be corrected and carried through with only minor changes, and we shall proceed assuming such amendments (spelled out in footnotes below).

Frege starts his solution by decomposing Boole's three premises into as many as thirteen; [14] as he puts it, 'whereas the dominant procedure in Boole is the unification of different judgements into a single expression, I analyse the data into simple judgements' [15]. To understand what Frege's 'simple' judgements are, we need to briefly review his notation, which we shall do through a few examples. [16]

In Frege's *Begriffsschrift* notation, the formulas below stand for, respectively, (a) the proposition A; (b) the negation of A; (c) the material conditional we would write as $B \longrightarrow A$, which is Frege's only device for combining different propositions into more complex ones (conjunction and disjunction being obtained using the conditional together with negation). [17]



Frege explicates the conditional (c) as the proposition that one cannot have that B is asserted and A denied; this parallels the truth-conditional analysis of the conditional as excluding a single one of the four possible combinations of truth-values for A and B (but phrased in terms of 'assertion' and 'denial' rather than of truth and falsity). [18] In fact, in his manuscript on Boole, Frege argues

d, that is, seek a relation of the form $ab = \ldots$ instead of the simpler $a = \ldots$ or $b = \ldots$. Boole would then introduce an auxiliary term t with an additional premise $t = ab$, then eliminate a and b as well as other unwanted terms and proceed as above, seeking a solution of the form $t = \ldots$, where t can ultimately be replaced by ab again. (Boole 1854, pp. 140–142.)

13. Frege (1979, p. 45) = Frege (1969, p. 51).
14. This is assuming his solution is corrected as per footnote 21 below; his own version only has twelve premises.
15. Frege (1979, p. 45) = Frege (1969, p. 51).
16. For a quick introduction to Frege's *Begriffsschrift* notation, see for instance Plato (2017, chap. 4) or Schlimm (2018, pp. 54–65).
17. Frege (1993, pp. 11–12) = Frege (1972, pp. 121–122); see also Schlimm (2018).
18. Frege (1993, p. 5) = Frege (1972, pp. 114–115).

that it is precisely because the conditional only excludes one out of four such combinations that it is 'simpler' than Boole's equality sign (indeed, a Boolean equality $a = b$, which in today's notation corresponds to two conditionals $A \longrightarrow B$ and $B \longrightarrow A$, excludes not just one but two possible combinations).[19] Finally, a thick vertical stroke to the left of a proposition, as in (d), turns it into a *judgement*, that is, means that the proposition is *asserted*. Putting everything together, (d) thus stands for the judgement that 'A and B cannot both be denied'.[20]

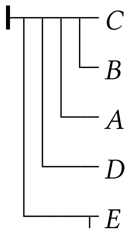
What Frege calls 'simple' judgements in the context of our problem are formed from conditionals like in (c) or (d), but stacked. Take for example Boole's premise (ii), that 'wherever the properties A and D are found while E is missing, the properties B and C will either both be found, or both be missing', which he wrote $\bar{e}ad = q(bc + \bar{b}\bar{c})$. The indefinite class symbol on the right-hand side allows translating this as a conditional, namely, in contemporary notation,

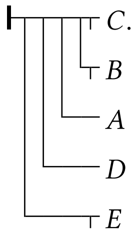
$$(\neg E \wedge A \wedge D) \longrightarrow ((B \wedge C) \vee (\neg B \wedge \neg C)).$$

To understand Frege's translation, notice, first, that that the consequent states that B and C always go together, and so is equivalent to the conjunction of $B \longrightarrow C$ and $\neg B \longrightarrow \neg C$. Splitting this consequent, this leads to the two formulas

$$(\neg E \wedge A \wedge D) \longrightarrow (B \longrightarrow C) \quad \text{and} \quad (\neg E \wedge A \wedge D) \longrightarrow (\neg B \longrightarrow \neg C).$$

Finally, a conjunctive antecedent like $\neg E \wedge A \wedge D$ can be replaced by nested conditionals (the order of nesting being indifferent); hence Frege's translations:

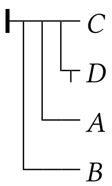
(4) 

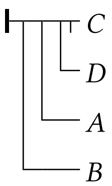
(5) 

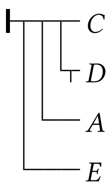
Let us take a more intricate example, namely Boole's premise (iii), which he wrote $ab + ae\bar{b} = d\bar{c} + c\bar{d}$. First, it has the form of an equation without indefinite class symbols, so that in modern terms, it is an equivalence and needs to be split into two implications. The first one can be translated as

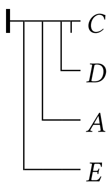
$$((A \wedge B) \vee (A \wedge E \wedge \neg B)) \longrightarrow ((D \wedge \neg C) \vee (C \wedge \neg D)).$$

As above, the consequent may be broken down into $D \longrightarrow \neg C$ and $\neg D \longrightarrow C$. But in this case, the disjunction in the antecedent also requires splitting up, namely into the judgements that the consequent holds given A and B, and that it holds given A and E. All in all, the first implication of Boole's (iii) yields four different judgements:

(6) 

(7) 

(8) 

(9) 

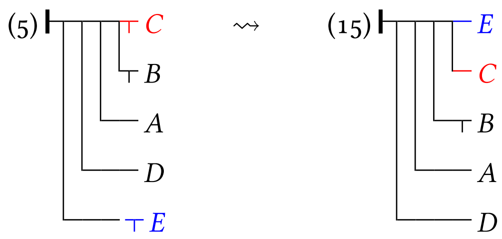
The second implication of (iii) – Frege's mistakes aside – also yields four judgements, which makes eight in total for a single Boolean equation.

In what sense does Frege see his version of the premises as 'simple'? Remember that he dubbed the conditional 'simple' because it excluded a single combination of assertions and denials

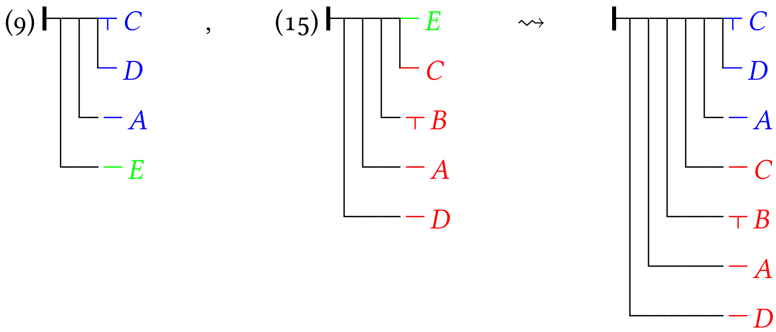
19. Frege (1979, p. 36) = Frege (1969, p. 40).
20. Frege (1993, pp. 10–11) = Frege (1972, p. 121).

of the terms involved. The same can be said here, only with more propositions. For instance, (7) excludes that B, A, D be asserted and the negation of C denied, that is, excludes that B, A, D, and C all be asserted together. In this sense, Frege's premises are similar to the 'atomic denials' into which the 'combinatorial' solutions discussed above broke down the data of the problem. The main difference is that Frege's simple judgements do not always contain all five of the terms involved, as shown by formulas (6)–(9); still, they can easily be used to generate the full list of combinations excluded by the premises. Frege, however, uses the data in a strikingly different way: as will appear presently, his approach could be described as an *inferential* recasting of the combinatorial solutions.

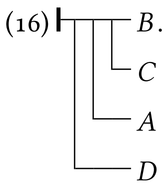
To solve the problem from his list of 'simple judgements', Frege needs two kinds of transformations, which we may call contraposition and *modus ponens*. First, contraposition: in a stacked conditional, the consequent (written on the top line) may be switched with any antecedent (written on any of the other lines) while negating both. For instance, Frege transforms formula (5) above into (15) (the numbering is still his):



(As mentioned already, the order of antecedents does not matter in such formulas: lines other than the top one can be reordered freely.) Second, *modus ponens*: two conditionals can be combined when the consequent of one is among the antecedents of the other, as E (shown in green) is (9) and (15) above:



Here, the antecedents of E in (15), shown in red, have been plugged into (9) at the place of E. The result can then be simplified, using contraposition to switch the consequent $\top C$ with $\top B$ and eliminating redundant antecedents; hence, still using Frege's own numbering,



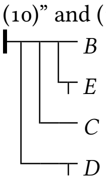
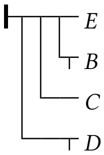
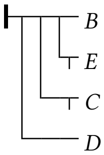
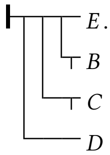
With these tools in hand, we can tackle Boole's first question, namely to find what follows from A regarding B, C and D. Frege's strategy is to search for every possible judgement inferrable from the premises that has A as antecedent and does not contain E. Two such judgements (to wit, (6) and (7) above) are already found among the premises. For the rest, as Frege's system has a single inference rule to combine different judgements – the one we called *modus ponens* –, the problem boils down to surveying every possible application of it. This is exactly what Frege does: he looks for every possible applications of *modus ponens* that would eliminate *E* from some premises. In order to do this efficiently, Frege first uses the rule we called contraposition to rewrite the premises so that *E* never appears negated – he thus transforms (5) into (15), as shown above, and proceeds similarly for (4).

At this stage, the premises can be sorted into three groups, according to the occurence of *E*: those that have *E* as an antecedent, those that have it as consequent, and those in which it does not occur at all. Members of the third group are either already part of the solution, if they contain *A*, like (6) and (7), or are of no use if they do not. As for the rest, *modus ponens* allows combining every premise of the first group with every premise of the second group. Most of these combinations yield judgements which, in Frege's words, hold 'independently of the contents' – that is, tautologies (this can happen either because the consequent is already among the antecedents, or because two of the antecedents contradict each other, like — *B* and ⊤ *B*). As it happens, in our case, ten combinations have to be surveyed, [21] and the only fruitful one is the combination of (9) and (15) shown above, which yields (16). In the end, the full solution is given by (6), (7), and (16). [22] (In passing, note that Frege's solution is, in fact, weaker than Boole's: in contemporary terms, the latter – being an equation with no indefinite class terms – corresponds to a biconditional of which the former is only the first half.)

The nature of Frege's solution should be clear by now: it is, essentially, a systematic search through a space of possible proofs. Despite the superficial similarities, noted above, of his 'simple judgements' with combinations of terms excluded by Boole's premises, the spirit of his solution is inferential rather than combinatorial: his goal is, essentially, to show that Boole's problem can be solved through simple logical inferences, with no tailor-made method and, as he puts it, 'practically no theoretical preparation at all'. [23]

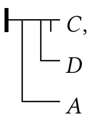
Admittedly, approaching the *Begriffsschrift* through the lens of Boole's problem may seem unfair. After all, before he offers his own solution Frege writes that 'it would not be surprising, and I would have no reluctance to concede the point, if Boolean logic were better suited than my concept-script to solve the kind of problems it was specifically designed for, or that were invented for it' (though he adds that 'perhaps not even this is the case'). [24] Among the points he makes in his comparison with Boole, he also describes the fact that his system can handle such problems just as well as Boole's as 'the point to which I attach least importance' [25] (since in his eyes they are of little use anyway). Yet approaching Frege's system on its own terms only confirms that it contains nothing comparable to Boole's conception of what a logical problem is.

21. This is assuming that Frege's mistakes are corrected. For the record, here is how this should be done. The editors (see Frege 1979, note 1 p. 41) suggest *adding* a premise (10)' and *replacing* Frege's premise (12) by their (12)' (shown below). Additionally, the following amendments are required. First, one should introduce equivalent variants of (10)' and (12)', namely (10)" and (12)":

(10)'  (10)"  (12)'  (12)" 

Second, in Frege's solution of the first question, one should check that combining either of (10)" or (12)" with either of (8) or (9) only produces tautologies. Third, in the list at the top of p. 44, all occurences of '(17)' should be replaced by '(19)'.

22. In fact, Frege notices that a further simplification is possible: *B* can be eliminated by *modus ponens* from (7) and (16), yielding

(17) 

so that (6) and (17) are enough to give the full solution to Boole's first question.

23. Frege (1979, p. 44) = Frege (1969, p. 49).

24. Frege (1979, 39, tr. alt.) = Frege (1969, p. 44).

25. Frege (1979, p. 46) = Frege (1969, p. 52).

In fact, as the rest of the piece on Boole makes abundantly clear, the *Begriffsschrift* is not meant to solve problems at all: it is a visual tool to scrutinize concepts and inferences, as we shall see below.