Neron-Severi group for nonalgebraic elliptic surfaces II: non-kählerian case

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0. Introduction

In this paper we shall study the Neron-Severi group for non-kählerian elliptic surfaces. In a previous paper, see [2], we obtained a description of the Neron-Severi group for elliptic bundles.

In section one we prove for convenience the (generally) known result that every non-kählerian elliptic surface is a quasi-bundle (with the general fibre an elliptic curve E). Then we construct a (finite) cyclic cover of a (non-kählerian) quasi-bundle, which is an elliptic bundle (for the definitions see the first section).

In section two we describe the Neron-Severi group (modulo torsion) for a non-kählerian elliptic surface by using the corresponding result from [2] for elliptic bundles. We state the main result (Theorem 5):
"For a non-kählerian elliptic surface $X \to B$, we have that the group $NS(X) \otimes \mathbb{Q}$ is isomorphic to the group $Hom(J_B, E) \otimes \mathbb{Q}$, where J_B is the Jacobian variety of the curve B and $Hom(J_B, E)$ is the group of the morphisms of abelian varieties"

We mention that in [3], by using the result from [2], we gave an explicit description of the Picard group for a primary Kodaira surface.

For the case of algebraic quasi-bundles, see [12], which focuses on the divisibility properties of a general fibre of a quasi-bundle.

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1. Quasi-bundles

All varieties will be defined over the field $\mathbb{C}$ of complex numbers.

An *elliptic surface* $\varphi : X \to B$ is a proper, connected, holomorphic map from a (compact, connected, smooth) surface X to a (compact, connected, smooth) curve B, such that the general fibre X_b ($b \in B$) is non-singular elliptic (the holomorphic structure may depend on b). We shall always assume that φ is relatively minimal, i.e. all fibres are free of (-1)-curves.

Let $F = \sum n_i D_i$ be a singular fibre of φ, where D_i' s are the irreducible reduced components and the n_i' s are their multiplicities. Let m denotes the greatest common divisor of the n_i' s. If $m \geq 2$, then the fibre F is called *multiple fibre of multiplicity* m and we will write $F = mD$, where $D = \sum (n_i/m) D_i$.

An elliptic surface $\varphi : X \to B$ is called a *quasi-bundle* if all smooth fibres are pairwise isomorphic, and the only singular fibres are multiples of smooth (elliptic) curves. If moreover φ has no singular fibres then $\varphi : X \to B$ is said to be a *fibre bundle*.

Let E be an elliptic curve and let us consider its universal covering sequence

$$(1) \qquad 0 \to \Gamma \to \mathbb{C} \to E \to 0, \quad \Gamma \cong \mathbb{Z}^2 .$$

An *elliptic bundle* $\varphi : X \to B$ is a principal fibre bundle whose typical fibre and structure group are the elliptic curve E. These holomorphic fibre bundles are classified by the cohomology set $H^1(\mathcal{E}_B)$, where $\mathcal{E}_B$ is the sheaf of germs of local holomorphic maps from B to E. To describe $H^1(\mathcal{E}_B)$ one use the exact cohomology sequence

$$(2) \qquad H^1(B, \Gamma) \to H^1(B, \mathcal{O}_B) \to H^1(\mathcal{E}_B) \xrightarrow{c} H^2(B, \Gamma) \to o$$

induced by (1); see, for example, [1] , Chapter V.5.

Let $\varphi : X \to B$ be a non-kählerian elliptic surface. We need the following (generally) known result (for algebraic elliptic surfaces see, for example, [12]):

Lemma 1 *A non-kählerian elliptic surface $\varphi : X \to B$ is a quasi-bundle.*

Proof: In view of a result of Miyaoka (see [8]) the first Betti number is odd. By the Theorem 2.6 of Chapter IV in [1] one has $b_1(X) = 2q(X) - 1$, where $q(X) = h^{0,1} = dim\ H^1(X, \mathcal{O}_X)$ is the irregularity of X.

Denote by $m_1D_1, ..., m_tD_t$ all multiple fibres of φ and let E be any smooth fibre. Define

$$G(\varphi) := Coker(\mathbf{Z} \to \oplus \mathbf{Z}_{m_i}), \;\; 1 \to (1, ..., 1).$$

Then there exists an exact sequence

$$(3) \qquad H_1(E, \mathbf{Z}) \to H_1(X, \mathbf{Z}) \to H_1(B, \mathbf{Z}) \times G(\varphi) \to o \;\; ,$$

induced by φ and the inclusion of E in X (see [11], Theorem 1.3). Since $b_1(X)$ is odd and $rank H_1(E, \mathbf{Z}) = 2$, it follows that the rank of the image of $H_1(E, \mathbb{Z})$ is one. We get $b_1(X) = 2g(B) + 1$, hence $q(X) - g(B) = 1$ ($g(B)$ is the genus of the curve B).

The first terms of the Leray spectral sequence

$$E_2^{p,q} = H^p(B, R^q\varphi_*\mathcal{O}_X) \longrightarrow H^{p+q}(X, \mathcal{O}_X)$$

yield the exact sequence

$$(4) \qquad 0 \to H^1(B, \varphi_*\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(B, R^1\varphi_*\mathcal{O}_X) \to 0$$

Since $\varphi_*\mathcal{O}_X \cong \mathcal{O}_B$ we get

$$(5) \qquad h^0(B, R^1\varphi_*\mathcal{O}_X) = q(X) - g(B) = 1$$

and

$$\chi(\mathcal{O}_X) = deg(R^1\varphi_*\mathcal{O}_X)^\vee$$

(see [1] , Chapter V, Proposition 12.2).

By relative duality (see [1]), one has:

$$\varphi_*(\omega_{X/B}) \cong (R^1\varphi_*\mathcal{O}_X)^\vee,$$

where " $\vee$ " denotes dual as $\mathcal{O}_B$-module. The Theorem 18.2, Chapter III, in [1] shows that

$$deg(R^1\varphi_*\mathcal{O}_X)^\vee = deg(\varphi_*(\omega_{X/B})) \geq 0$$

and this degree vanishes if and only if all the smooth fibres of φ are isomorphic and the singular fibres are of type mI_0, i.e. multiples of smooth (elliptic) curves.

Suppose $deg(\varphi_*(\omega_{X/B})) > 0$. Since $deg(R^1\varphi_*\mathcal{O}_X) < 0$, then $h^0(B, R^1\varphi_*\mathcal{O}_X) = 0$ and we get a contradiction with (5). It follows that $\chi(\mathcal{O}_X) = 0$ and $\varphi : X \to B$ is a quasi-bundle. ∎

Lemma 2 *Let $\varphi : X \to B$ be a quasi-bundle with the first Betti number odd. Let $m_1 D_1, ..., m_t D_t$ be all multiple fibres of φ and let m denote the least common multiple of $m_1, ..., m_t$. Then there exist an elliptic bundle $\psi : Y \to C$, with the first Betti number odd, and two cyclic coverings $\varepsilon : C \to B, \pi : Y \to X$, both with group $\mathbb{Z}_m$, such that $\varphi \circ \pi = \varepsilon \circ \psi$.*

Proof: Choose an integer $e \geq 0$ such that m divides $t + e$. Let $\varphi(D_i) = P_i \in B$, $i = 1, ..., t$, and take distinct points $P_{t+1}, ..., P_{t+e} \in B$, which are different from $P_i, i = 1, ..., t$. Then there is at least one line bundle L on B with

$$L^{\otimes m} \cong \mathcal{O}_B(P_1 + ... + P_{t+e}).$$

Such an L defines a cyclic covering $\varepsilon : C \to B$ of degree m, totally ramified at $P_1, ..., P_{t+e}$ (see [1],Chapter I, Lemma 17.1).

Let Y be the normalization of $X \times_B C$. Then Y is smooth by [1], Chapter III, Proposition 9.1, and there exists a cyclic covering map $\pi : Y \to X$ with group $G \cong \mathbb{Z}_m$, i.e. $X = Y/G$.
We get the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ \psi \downarrow & & \downarrow \varphi \\ C & \xrightarrow{\varepsilon} & B \end{array}$$

For this construction see [12] or [1], Chapter III, Theorem 10.3. Again, by [1] , Chapter III, Proposition 9.1, one sees that $\psi : Y \to C$ is a fibre bundle. Denote by E a smooth fibre of φ and by $\tilde{E}$ a connected component of $\pi^{-1}(E)$. Then $\tilde{E}$ is a fibre of ψ and the restriction $\tilde{E} \to E$ of π is an isomorphism.

Now, we can apply to this situation the results of Kodaira in [6], [7]. The elliptic surface Y has no singular fibres, hence the local monodromy is trivial. Then the functional invariant J of Y is constant. Since $X = Y/G$ is not a deformation of an algebraic surface, by Theorems 14.5 and 14.6 in [7], we deduce that the homological invariant of Y is trivial too (the global monodromy is trivial). It follows that $\psi : Y \to C$ is an elliptic bundle defined by an element $\xi \in H^1(\mathcal{E}_B)$. From the proof of the Theorem 14.7 in [7] we get $c(\xi) \neq 0$ hence, by the Theorem 11.9 in [7] (or, by the Proposition 5.3, Chapter V, in [1]), we obtain that $b_1(Y)$ is odd. ∎

2. The Neron-Severi group

Let $\varphi : X \to B$ be a non-kählerian elliptic surface. From the previous section we get the diagram:

$$\begin{array}{ccc} Pic(X) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \\ \pi^* \downarrow & & \downarrow \pi^* \\ Pic(Y) & \xrightarrow{c_1} & H^2(Y,\mathbb{Z}) \end{array}$$

Let $G \cong \mathbb{Z}_m$ be the finite group of the cyclic covering $\pi : Y \to X$.

Lemma 3 *With the above notations we have the isomorphism*

$$NS(X) \otimes \mathbb{Q} \cong NS(Y)^G \otimes \mathbb{Q},$$

where $NX(Y)^G$ is the subgroup of invariants of the Neron-Severi group $NS(Y)$.

Proof: Obviously, $\pi^*(NS(X)) \subset NS(Y)$. For any $g \in G$ we denote also by g the covering transformation $g : Y \to Y$ and we have $\pi \circ g = \pi$. Then, for any $\gamma \in NS(X)$, it follows $g^*(\pi^*(\gamma)) = \pi^*(\gamma)$, i.e.

$$\pi^*(NS(X)) \subset NS(Y)^G. \tag{6}$$

Now, let β be an element of $NS(Y)^G$ and write $\beta = c_1(\mathcal{M})$ with $\mathcal{M}$ a line bundle on Y. Take the line bundle on Y defined by

$$\otimes_{g\in G}(g^*\mathcal{M}) := \mathcal{N}.$$

Since $c_1(g^*\mathcal{M}) = g^*(c_1(\mathcal{M})) = g^*(\beta) = \beta$, we get $c_1(\mathcal{N}) = m\beta$. Because $\mathcal{N}$ is a G-sheaf we have an action of G on $\pi_*(\mathcal{N})$. But G is an abelian group, so the sheaf of invariants $\pi_*(\mathcal{N})^G$ of $\pi_*(\mathcal{N})$ is a line bundle (the invariant summand in the splitting of $\pi_*(\mathcal{N})$ as a direct sum according to the characters of G); see [9], [4] or, for a systematic study of abelian covers [10].

Taking the zero divisor D of the natural morphism of line bundles

$$\pi^*(\pi_*(\mathcal{N})^G) \to \mathcal{N},$$

we get the isomorphism

$$\mathcal{N} \cong \pi^*(\mathcal{L}) \otimes \mathcal{O}_Y(D),$$

where $\mathcal{L} := \pi_*(\mathcal{N})^G$. But D is a sum of fibres of the non-kählerian elliptic bundle $Y \to C$ and, by [2] the Chern class $c_1(\mathcal{O}_Y(D))$ is a torsion element in $NS(Y)$. Let k be an integer such that $kc_1(\mathcal{O}_Y(D)) = 0$ in $NS(Y)$. Then we have

$$km\beta = kc_1(\pi^*(\mathcal{L})) = \pi^*(c_1(\mathcal{L}^{\otimes k})),$$

i.e.

$$kmNS(Y)^G \subset \pi^*(NS(X)) \subset NS(Y). \tag{7}$$

By [5], we know that there exists an homomorphism

$$\mu : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z}),$$

such that $\mu \circ \pi^*(\gamma) = m\gamma$ for $\gamma \in H^2(X, \mathbb{Z})$. If $\gamma \in Ker(\pi^*)$, then $m\gamma = 0$, i.e. γ is a torsion element of $H^2(X, \mathbb{Z})$. From this fact and from (7) we get

$$NS(X) \otimes \mathbb{Q} \cong NS(Y)^G \otimes \mathbb{Q},$$

the desired isomorphism. ∎

Lemma 4 *Let $\varepsilon : C \to B$ be a cyclic covering (with group $G \cong \mathbb{Z}_m$) of curves and let E be an elliptic curve. Then there exists an exact sequence of groups*

$$0 \to Hom(J_B, E) \to Hom(J_C, E)^G \to Hom(G, E),$$

where J_B, resp. J_C, is the Jacobian variety of the curve B, resp. C.

Proof: By a suitable choice we can suppose that we have the diagram:

$$\begin{array}{ccc} C & \hookrightarrow & J_C \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ B & \hookrightarrow & J_B \end{array}$$

Let h be the genus of the curve C and let $\alpha = x_1 + \ldots + x_h \in J_C$, where $x_1, \ldots, x_h \in C$. If $f \in Hom(J_C, E)^G$, then for all $g \in G$

$$f(gx_1 + \ldots + gx_h) = f(x_1 + \ldots + x_h).$$

Take $x_1 = x \in C$ an arbitrary element and $x_2 = \ldots = x_h = 0 \in C \subset J_C$. Then we get

$$f(gx + (h-1)g0) = f(x) = \tilde{f}(x),$$

where $\tilde{f}$ is the restriction of f to C. It follows

$$\tilde{f}(gx) - \tilde{f}(x) = a_g \in E,$$

for all $x \in C$. Thus we obtain a map $(g \to a_g) : G \to E$. But

$$\tilde{f}(gg'x) - \tilde{f}(x) = \tilde{f}(gg'x) - \tilde{f}(g'x) + \tilde{f}(g'x) - \tilde{f}(x),$$

i.e.

$$a_{gg'} = a_g + a_{g'},$$

and so $(g \to a_g)$ is a homomorphism $a^f : G \to E$. Clearly, $a^f = 0$ iff $\tilde{f}(gx) = \tilde{f}(x)$ for all $g \in G$ and all $x \in C$, i.e. iff there exists $\tilde{u} : B \to E$ such that $\tilde{f} = \tilde{u} \circ \varepsilon$. But the elements of a curve generates (as a group) the corresponding Jacobian, so we have $f = u \circ \varepsilon$,where $u : J_B \to E$ is a morphism.∎

Theorem 5 *Let $X \to B$ be a non-kählerian elliptic surface. Then we have the isomorphism*

$$NS(X) \otimes \mathbb{Q} \cong Hom(J_B, E) \otimes \mathbb{Q},$$

where J_B is the Jacobian variety of the curve B and $Hom(J_B, E)$ is the group of the morphisms of abelian varieties.

Proof: By the previous results and the Theorem 3.1 in [2].∎

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