Élie Cartan's torsion in geometry and in field theory, an essay

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ABSTRACT. We review the application of torsion in field theory. First we show how the notion of torsion emerges in differential geometry. In the context of a Cartan circuit, *torsion* is related to *translations* similar as curvature to rotations. Cartan's investigations started by analyzing Einsteins general relativity theory and by taking recourse to the theory of Cosserat continua. In these continua, the points of which carry independent translational and rotational degrees of freedom, there occur, besides ordinary (force) stresses, additionally *spin moment stresses.* In a 3-dimensional "continuized" (Kröner) crystal with dislocation lines, a linear connection can be introduced that takes the crystal lattice structure as a basis for parallelism. Such a continuum has similar properties as a Cosserat continuum, and the dislocation density is equal to the torsion of this connection. Subsequently, these ideas are applied to 4-dimensional spacetime. A translational gauge theory of gravity is displayed (in a Weitzenböck or teleparallel spacetime) as well as the viable Einstein-Cartan theory (in a Riemann-Cartan spacetime). In both theories, the notion of torsion is contained in an essential way. Cartan's spiral staircase is described as a 3-dimensional Euclidean model for a space with torsion, and eventually some controversial points are discussed regarding the meaning of torsion.

P.A.C.S.: 04.20.Cv; 11.10.-z; 61.72.Lk; 62.20.-x; 02.40.Hw

1 A connection induces torsion and curvature

"...the essential achievement of general relativity, namely to overcome 'rigid' space (ie the inertial frame), is only indirectly *connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the 'displacement field' (Γ^l_{ik}), which expresses the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated*

vectors fixed by the inertial frame (ie the equality of corresponding components) by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of 'rigid' space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular Γ field can be deduced from a Riemannian metric..."

A. Einstein (4 April 1955)[1]

On a differential manifold, we can introduce a linear connection, the components of which are denoted by ${\Gamma_{ij}}^k$. The connection allows a parallel displacement of tensors and, in particular, of vectors, on the manifold. We denote (holonomic) coordinate indices with Latin letters $i, j, k, \dots = 0, 1, 2, \dots, n-1$, where n is the dimension of the manifold. A vector $u = u^k \partial_k$, if parallelly displaced along dx^i, changes according to

$$\delta^{||} u^k = -{\Gamma_{ij}}^k u^j dx^i \,. \tag{1}$$

Based on this formula, it is straightforward to show that a non-vanishing Cartan torsion,[2]

$$ {T_{ij}}^k = {\Gamma_{ij}}^k - {\Gamma_{ji}}^k \equiv 2{\Gamma_{[ij]}}^k \neq 0 \,, \tag{2}$$

breaks infinitesimal parallelograms on the manifold, see Fig.1. Here for antisymmetrization we use the abbreviation $[ij] := \frac{1}{2}(ij - ji)$ and for symmetrization $(ij) := \frac{1}{2}(ij + ji)$, see [88]. There emerges a *closure failure*, i.e., a parallelogram is only closed up to a small translation.

In GR, the connection is identified with the Christoffel symbol ${\Gamma_{ij}}^k = \{{}_i{}^k{}_j\}$ and is as such symmetric $\{{}_i{}^k{}_j\} = \{{}_j{}^k{}_i\}$. In other words, the torsion vanishes in GR.

The torsion surfaces more naturally in a frame formalism. At each point we have a basis of n linearly independent vectors $e_\alpha = {e^i}_\alpha \partial_i$ and the dual basis of covectors $\vartheta^\beta = {e_j}^\beta dx^j$, the so-called coframe, with $e_\alpha \rfloor \vartheta^\beta = \delta^\beta_\alpha$ (the interior product is denoted by $\rfloor$). We denote (anholonomic) frame indices with Greek

[1] Preface in 'Cinquant'anni di Relatività 1905–1955.' M. Pantaleo, ed.. Edizioni Giuntine and Sansoni Editore, Firenze 1955 (translation from the German original by F. Gronwald, D. Hartley, and F.W. Hehl). For the role that generalized connections play in physics, see Mangiarotti and Sardanashvily [58].

[2] According to Kiehn [47], one can distinguish at least five different notions of torsion. In our article, we treat Cartan's torsion of 1922, as it is established in the meantime in differential geometry, see Frankel [21], p.245. We find it disturbing to use the same name for different geometrical objects.

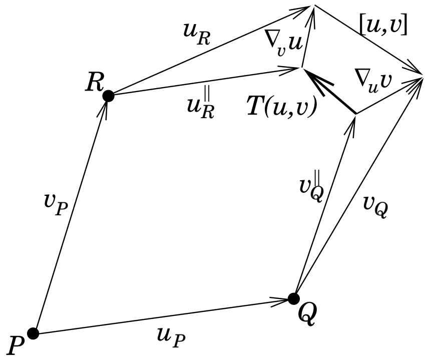

Figure 1: *On the geometrical interpretation of torsion,* see [39]: Two vector fields u and v are given. At a point P, we transport parallelly u and v along v or u, respectively. They become $u_{\mathrm{R}}^{||}$ and $v_{\mathrm{Q}}^{||}$. If a torsion is present, they don't close, that is, a *closure failure* $T(u,v)$ emerges. This is a schematic view. Note that the points R and Q are infinitesimally near to P. A proof can be found in Schouten [88], p.127.

letters $\alpha, \beta, \gamma, \dots = 0, 1, 2, \dots, n-1$. The connection is then introduced as 1-form[3] $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta dx^i$, and, for a form w^A, we can define a covariant exterior derivative according to $Dw^A := dw^A + \rho_B{}^{A\,\alpha}{}_\beta\,\Gamma_\alpha{}^\beta \wedge w^B$. Here the coefficients $\rho_B{}^{A\,\alpha}{}_\beta$ describe the behavior of w^A under linear transformations, for details see [98] and [39], p.199, and $\wedge$ denotes the exterior product. Then the torsion 2-form is defined as

$$T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta\,. \tag{3}$$

If the frames are chosen as coordinate frames, then $d\vartheta^\alpha = 0$ and the definition (3) degenerates to (2). From (3) we can read off that T^α is a kind of a field strength belonging to the 'potential' ϑ^α.

Since we introduced a connection $\Gamma_\alpha{}^\beta$, we can define in the conventional way the RC-curvature,

$$R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta + \Gamma_\gamma{}^\beta \wedge \Gamma_\alpha{}^\gamma\,. \tag{4}$$

[3] The relation between $\Gamma_{i\alpha}{}^\beta$ and the holonomic $\Gamma_{ij}{}^k$ in (1) is $\Gamma_{i\alpha}{}^\beta = e^j{}_\alpha e_k{}^\beta \Gamma_{ij}{}^k + e^j{}_\alpha \partial_i e_j{}^\beta$.

If we differentiate (3) and (4), we find straightforwardly the first and the second Bianchi identities, respectively,[4]

$$DT^{\alpha} = R_{\beta}{}^{\alpha} \wedge \vartheta^{\beta}\,, \qquad DR_{\alpha}{}^{\beta} = 0\,. \tag{5}$$

We can recognize already here, how closely torsion and curvature are interrelated. Moreover, it is clear, that torsion as well as curvature are notions linked to the process of parallel displacement on a manifold and are as such something very particular.

2 Cartan circuit: Translational and rotational misfits

Since the metric plays an essential role in the applications we have in mind, we will now introduce — even though it is not necessary at this stage — besides the connection $\Gamma_{\alpha}{}^{\beta}$, a (symmetric) metric $g_{ij} = g_{ji}$ that determines distances and angles. The line element is given by

$$ds^2 = g_{ij}dx^i \otimes dx^j = g_{\alpha\beta}\vartheta^{\alpha} \otimes \vartheta^{\beta}\,. \tag{6}$$

We assume that the connection is compatible with the metric, i.e., the nonmetricity $Q_{\alpha\beta}$ vanishes:

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} = 0\,. \tag{7}$$

A space fulfilling this condition is called a *Riemann-Cartan* (RC) *space*. We can solve (7) with respect to the symmetric part of the (anholonomic) connection:

$$\Gamma_{(\alpha\beta)} = \frac{1}{2}dg_{\alpha\beta}\,. \tag{8}$$

Furthermore, we will choose an *orthonormal coframe*. We will apply the formalism to the 4-dimensional (4D) spacetime with Lorentzian metric $g_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ or to the 3D space with Euclidean metric $g_{\alpha\beta} = \mathrm{diag}(1,1,1)$. Then, due to (8), we find a vanishing symmetric part of the anholonomic connection. Accordingly, we have in a RC-space as geometrical field variables the orthonormal coframe $\vartheta^{\alpha} = e_i{}^{\alpha}dx^i$ and the metric-compatible connection $\Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta}dx^i = -\Gamma^{\beta\alpha}$.

Now we are prepared to characterize a RC-space in the way Cartan did. Locally a RC-space looks Euclidean, since for any single point P, there exist

[4] In 3 dimensions we have $1 \times (3+3) = 6$ and in 4 dimensions $4 \times (4+6) = 40$ independent components of the Bianchi identities.

coordinates x^i and an orthonormal coframe ϑ^α in a neighborhood of P such that

$$\left\{\begin{matrix} \vartheta^\alpha = \delta_i^\alpha dx^i \\ \Gamma_\alpha{}^\beta = 0 \end{matrix}\right\} \quad \text{at} \quad P\,, \tag{9}$$

where $\Gamma_\alpha{}^\beta$ are the connection 1-forms referred to the coframe ϑ^α, see Hartley [29] for details. Eq.(9) represents, in a RC-space, the anholonomic analogue of the (holonomic) Riemannian normal coordinates of a Riemannian space.

Often it is argued incorrectly that in RC-space normal frames cannot exist, since torsion, as a tensor, cannot be transformed to zero. In this context Riemannian normal coordinates are tacitly assumed and the torsion is 'superimposed'. However, since only a *natural*, i.e., a holonomic or coordinate frame is attached to Riemannian normal coordinates, one is too restrictive in the discussion right from the beginning. And, of course, the curvature is also of tensorial nature – and still Riemannian normal coordinates do exist.

How can a local observer at a point P with coordinates x^i tell whether his or her space carries torsion and/or curvature? The local observer defines a small loop (or a circuit) originating from P and leading back to P. Then he/she *rolls* the local reference space *without sliding* — this is called Cartan displacement — along the loop and adds up successively the small relative translations and rotations, see Cartan [13, 14], Schouten [88], Sharpe [95] or, for a modern application, Wise [106]. As a computation shows, the added up *translation* is a measure for the *torsion* and the *rotation* for the *curvature*. Since the loop encircles a small 2-dimensional area element, Cartan's prescription attaches to an area element a small translation and a small rotation. Thus, torsion T^α and curvature $R^{\alpha\beta} = -R^{\beta\alpha}$ are both 2-forms in any dimensions $n > 1$, the torsion is vector-valued, because of the translation vector, the curvature bivector-valued, because of the rotations.

In this way Cartan visualized a RC-space as consisting of a collection of small Euclidean granules that are translated and rotated with respect to each other. Intuitively it is clear that this procedure of Cartan is similar to what one does in gauge field theory: A rigid (or global) symmetry, here the corresponding Euclidean motions of translation and rotations, is extended to a local symmetry. In four-dimensional spacetime it is the Poincaré (or inhomogeneous Lorentz) group of Minkowski space that is gauged and that yields a RC-spacetime, see [68, 6, 25].

There are two degenerate cases: A RC-space with vanishing torsion is the conventional Riemannian space, a RC-space with vanishing RC-curvature is called a *Weitzenböck space* [105], or a space with teleparallelism. We will come back to this notion later.

We can now list the number of the components of the different geometrical quantities in a RC-space of 3 or 4 dimensions. These numbers are reflecting the $3+3$ generators of the 3D Euclidean group and of the $4+6$ generators of the 4D Poincaré group:

	orthon. cofr. ϑ^α	RC-connection $\Gamma^{\alpha\beta}$	Cartan's torsion T^α	RC-curvature $R^{\alpha\beta}$
$n=3$	$9=3\times 3$	$9=3\times 3$	$9=3\times 3$	$9=3\times 3$
$n=4$	$16=4\times 4$	$24=6\times 4$	$24=6\times 4$	$36=6\times 6$

The results of Secs.1 and 2 can all be proven rigorously. They are all consequences of the introduction of a connection $\Gamma_{ij}{}^k$ and a metric g_{ij}. Let us now turn to a new ideas that influenced Cartan's thinking in the context of RC-geometry.

3 The Cosserat continuum

Cartan, according to his acknowledgment in [12], was inspired by the brothers Cosserat [16] and their theory of a new type of continuum. The classical continuum of elasticity and fluid dynamics consists of unstructured points, and the displacement vector u_i is the only quantity necessary for specifying the deformation. The Cosserats conceived a specific *medium with microstructure,* see [26, 10, 24] and for a historical review [3], consisting of structured points such that, in addition to the displacement field u_i, it is possible to measure the rotation of such a structured point by the bivector field $\omega_{ij}=-\omega_{ji}$, see Fig.2 for a schematic view.

The deformation measures *distortion* β and *contortion* κ of a linear Cosserat continuum are (∇_i is the covariant derivative operator of the Euclidean 3D space)

$$\beta_{ij}=\nabla_i u_j-\omega_{ij}\,, \qquad \omega_{ij}=-\omega_{ji}\,, \tag{10}$$

$$\kappa_{ijk}=\nabla_i\omega_{jk}=-\kappa_{ikj}\,, \tag{11}$$

see Günter [26] and Schaefer [85]. A rigorous derivation of these deformation measures is given in the Appendix. In classical elasticity, the only deformation measure is the strain $\varepsilon_{ij}:=\frac{1}{2}(\beta_{ij}+\beta_{ji})\equiv\beta_{(ij)}=\nabla_{(i}u_{j)}$. Let us visualize these deformations. If the displacement field $u_1\sim x$ and the rotation field $\omega_{ij}=0$, we find $\beta_{11}=\varepsilon_{11}=const$ and $\kappa_{ijk}=0$, see Fig.3. This

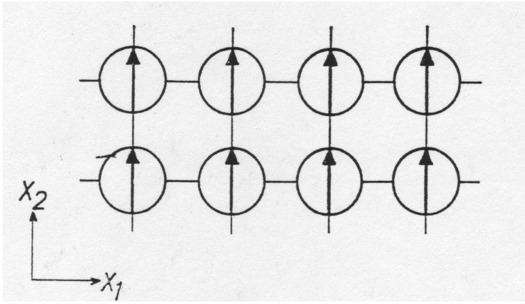

Figure 2: Schematic view on a two-dimensional Cosserat continuum: Undeformed initial state.

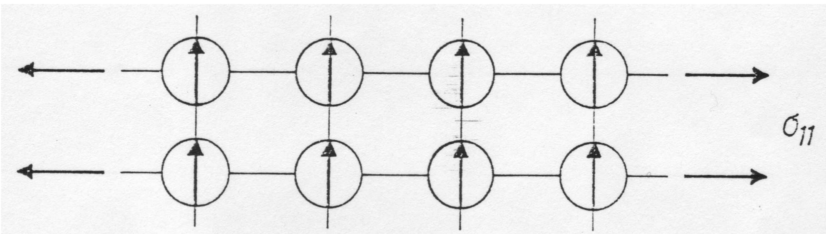

Figure 3: Conventional homogeneous strain ε_{11} of a Cosserat continuum: Distance changes of the "particles" caused by force stress σ_{11}.

homogeneous strain is created by ordinary force stresses. In contrast, if we put $u_i = 0$ and $\omega_{12} \sim x$, then $\beta_{12} = \omega_{12} \sim x$ and $\kappa_{112} \sim const$, see Fig.4. This homogeneous contortion is induced by applied spin moment stresses. Fig.5 depicts the pure constant antisymmetric stress with $\omega_{12} = const$ and Fig.6 the conventional rotation of the particles according to ordinary elasticity. This has to be distinguished carefully from the situation in Fig.4.

Apparently, in addition to the force stress $\overline{\Sigma}_{ij} \sim \delta\mathcal{H}/\delta\beta_{ij}$ (here $\mathcal{H}$ is an elastic potential), which is asymmetric in a Cosserat continuum, i.e., $\overline{\Sigma}_{ij} \neq \overline{\Sigma}_{ji}$, we have as new response the *spin moment* stress $\overline{\tau}_{ijk} \sim \delta\mathcal{H}/\delta\kappa_{kji}$. Hence (force) stress $\overline{\Sigma}_{ij}$ and *spin moment* stress $\overline{\tau}_{ijk}$ characterize a Cosserat continuum from the static side. We used bars for denoting stress and spin moment stress specifically in 3D.

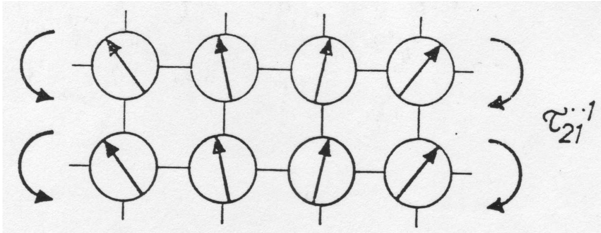

Figure 4: Homogeneous contortion κ_{112} of a Cosserat continuum: Orientation changes of the "particles" caused by spin moment stress ${\tau_{21}}^{1}$.

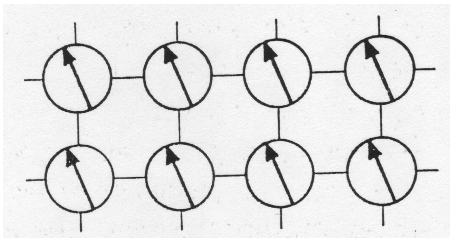

Figure 5: Homogeneous Cosserat rotation ω_{12} of the "particles" of a Cosserat continuum caused by the antisymmetric piece of the stress $\Sigma_{[12]}$.

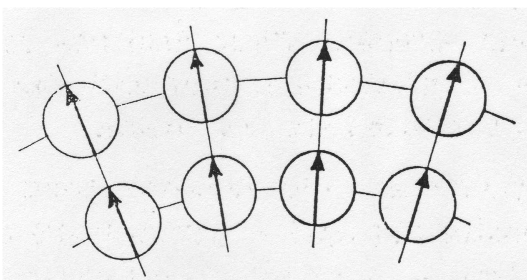

Figure 6: Conventional rotation $\partial_{[1}u_{2]}$ of the "particles" of a Cosserat continuum caused by an inhomogeneous strain.

Only in 3D, a rotation can be described by a vector according to $\omega^i = \frac{1}{2}\epsilon^{ijk}\omega_{jk}$, where $\epsilon_{ijk} = 0, +1, -1$ is the totally antisymmetric 3D permutation symbol. We chose here the bivector description such that the discussion becomes independent of the dimension of the continuum considered. Even though there exist 1D Cosserat continua (wires and beams) and 2D ones (plates and shells), we will concentrate here, exactly as Cartan did, on 3D Cosserat continua.

The equilibrium conditions for forces and moments read[5]

$$\nabla_j \overline{\Sigma}_i{}^j + f_i = 0, \qquad \nabla_k \overline{\tau}_{ij}{}^k - \overline{\Sigma}_{[ij]} + m_{ij} = 0, \tag{12}$$

where f_i are the volume forces and $m_{ij} = -m_{ji}$ volume moments. They correspond to translational and rotational Noether identities. In classical elasticity and in fluid dynamics, $\overline{\tau}_{ij}{}^k = 0$ and $m_{ij} = 0$; thus, the stress is symmetric, $\overline{\Sigma}_{[ij]} = 0$, and then denoted by $\overline{\sigma}_{ij}$; for early investigations of asymmetric stress and energy-momentum tensors, see Costa de Beauregard [17].

Nowadays the Cosserat continuum finds many applications. As one example we may mention the work of Zeghadi et al. [108] who take the grains of a metallic polycrystal as (structured) Cosserat particles and develop a linear Cosserat theory with the constitutive laws $\overline{\Sigma}_{ij} \sim \beta_{ij}$ and $\overline{\tau}_{ijk} \sim \kappa_{kji}$.

The Riemannian space is the analogue of the body of classical continuum theory: points and their relative distances is all what is needed to describe it geometrically; the analogue of the strain ε_{ij} of classical elasticity is the difference between the metric tensor g_{ij} of the Riemannian space and a flat background metric. In GR, a symmetric "stress" $\sigma_{ij} = \sigma_{ji}$ is the response of the matter Lagrangian to a variation of the metric g_{ij}.

A RC-space can be realized by a generalized Cosserat continuum. The "deformation measures" $\vartheta^\alpha = e_i{}^\alpha dx^i$ and $\Gamma^{\alpha\beta} = \Gamma_i{}^{\alpha\beta} dx^i = -\Gamma^{\beta\alpha}$ of a RC-space correspond to those of a Cosserat continuum according to[6]

$$\delta e_i{}^\alpha \to \beta_{ij}\,, \qquad \delta\Gamma_i{}^{\alpha\beta} \to \kappa_{ijk}\,. \tag{15}$$

[5] In exterior calculus we have $D\Sigma_\alpha + f_\alpha = 0$ and $D\tau_{\alpha\beta} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} + m_{\alpha\beta} = 0$. These relations are valid in all dimensions $n \geq 1$, see [24]. In 3 dimensions we have $3+3$ and in 4 dimensions $4+6$ independent components of the "equilibrium" conditions.

[6] This can be seen from the response of the coframe $e_i{}^\alpha$ and the Lorentz connection $\Gamma_i{}^{\alpha\beta}$ in a RC-space to a local Poincaré gauge transformation consisting of small translations ϵ^α and small Lorentz transformations $\omega^{\alpha\beta}$,

$$\delta e_i{}^\alpha = -D_i\epsilon^a + e_i{}^\gamma \omega_\gamma{}^\alpha - \epsilon^\gamma T_{\gamma i}{}^\alpha\,, \tag{13}$$

$$\delta\Gamma_i{}^{\alpha\beta} = -D_i\omega^{\alpha\beta} - \epsilon^\gamma R_{\gamma i}{}^{\alpha\beta}\,, \tag{14}$$

see [35], Eqs.(4.33),(4.32); here $D_i := \partial_i \rfloor D$ are the components of the exterior covariant derivative. The second term on the right-hand-side of (13) is due to the semi-direct product structure

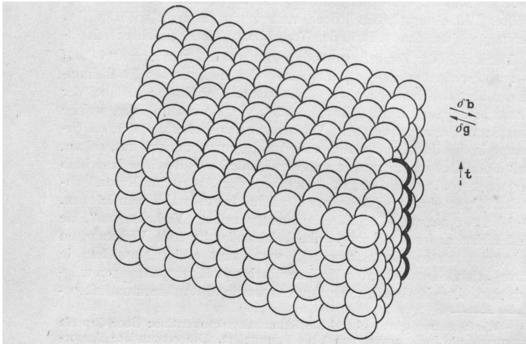

Figure 7: *Edge dislocation* after Kröner [50]: The dislocation line is parallel to the vector $\mathbf{t}$. The Burgers vector $\delta\mathbf{b}$, characterizing the missing half-plane, is perpendicular to $\mathbf{t}$. The vector $\delta\mathbf{g}$ characterizes the gliding of the dislocation as it enters the ideal crystal.

However, the coframe ϑ^α and the connection $\Gamma^{\alpha\beta}$ cannot be derived from a displacement field u_i and a rotation field ω_{ij}, as in (10),(11). Such a generalized Cosserat continuum is called incompatible, since the deformation measures β_{ij} and κ_{ijk} don't fulfill the so-called compatibility conditions

$$\nabla_{[i}\beta_{j]k} + \kappa_{[ij]k} = 0\,, \qquad \nabla_{[i}\kappa_{j]kl} = 0\,, \tag{16}$$

see Günther [26] and Schaefer [85, 86]. They guarantee that the "potentials" u_i and ω_{ij} can be introduced in the way as it is done in (10),(11). Still, also in the RC-space, as *incompatible* Cosserat continuum, we have, besides the force stress $\overline{\Sigma}_\alpha{}^i \sim \delta\mathcal{H}/\delta e_i{}^\alpha$, the spin moment stress $\overline{\tau}_{\alpha\beta}{}^i \sim \delta\mathcal{H}/\delta\Gamma_i{}^{\alpha\beta}$. And in the geometro-physical interpretation of the structures of the RC-space, Cartan apparently made use of these results of the brothers Cosserat.

In 4D, the stress $\overline{\Sigma}_\alpha{}^i$ corresponds to energy-momentum[7] $\Sigma_\alpha{}^i$ and the spin moment stress $\overline{\tau}_{\alpha\beta}{}^i$ to spin angular momentum $\tau_{\alpha\beta}{}^i$. Accordingly, Cartan enriched the Riemannian space of GR geometrically by the *torsion* $T_{ij}{}^\alpha$ and statically (or dynamically) by the *spin angular momentum* $\tau_{\alpha\beta}{}^i$ of matter.

of the Poincaré group. If we put torsion and curvature to zero, these formulas are analogous to (10),(11).

[7]This is well-known from classical electrodynamics: The 3D Maxwell stress generalizes, in 4D, to the energy-momentum tensor of the electromagnetic field, see [39].

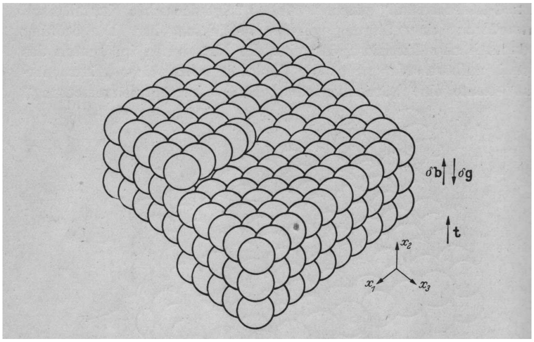

Figure 8: *Screw dislocation* after Kröner [50]: Here the Burgers vector is parallel to $\mathbf{t}$.

4 A rule in three dimensions: Dislocation density equals torsion

In the 1930s, the concept of a crystal dislocation was introduced in order to understand the plastic deformation of crystalline solids, as, for instance, of iron. Dislocations are one-dimensional lattice defects. Basically, there exist two types of dislocations, edge and screw dislocation, see Weertman & Weertman [102]. In Fig.7, we depicted a three-dimensional view on such an edge dislocation in a cubic primitive crystal. We recognize that one atomic half-plane has been moved to the right-hand-side of the crystal. The missing half-plane is characterized by the Burgers vector that is perpendicular to the dislocation line. The screw dislocation of Fig.8 has again a Burgers vector, but in this case it is parallel to the dislocation line. In the framework of classical elasticity, at the beginning of the last century, theories of the elastic field of singular defect lines had been developed by Volterra, Somigliana, and others, see Nabarro [66] and Puntigam & Soleng [81]. These theories could be used to compute the far-field of a crystal dislocation successfully. For more recent developments in this field, one may quote Malyshev [57], who went beyond the linear approximation.

If sufficiently many dislocations populate a crystal, then a continuum or field theory of dislocations is appropriate, see Kröner's theory of a continuized crystal [52]. In order to give an idea of such an approach, let us look at a cubic crystal in which several dislocations are present, see Fig.11. By averaging over, we can define a dislocation density tensor ${\alpha_{ij}}^k = -{\alpha_{ji}}^k$. The indices ij denote the area element, here the 12-plane, and k the direction of the Burgers

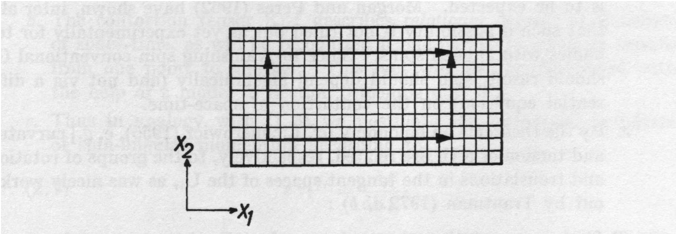

Figure 9: *The ideal cubic crystal in the undeformed state,* see [34]: A "small" parallelogram has been drawn.

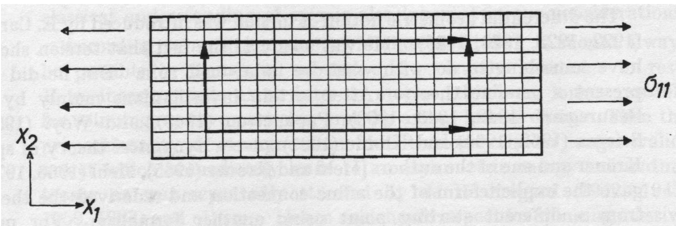

Figure 10: Homogeneously strained crystal caused by force stress σ_{11}: The average distances of the lattice points change. The parallelogram remains closed.

vector, here only the component δb^1. Thus, in Fig.11, only the $\alpha_{12}{}^1 = -\alpha_{21}{}^1$ components are nonvanishing.

Already in 1953, Nye [70] was able to derive a relation between the dislocation density α_{ijk} and the contortion tensor K_{ijk}, which describes the relative rotations between neighboring lattice planes:

$$K_{ijk} = -\alpha_{ijk} + \alpha_{jki} - \alpha_{kij} = -K_{ikj}\,. \tag{17}$$

On purpose we took here the letter K similar to the contortional measure κ of a Cosserat continuum, see (11). In Fig.11, according to Eq.(17), only $K_{121} = -K_{211} \neq 0$: We have rotations in the 12-plane if we go along the x_1-direction.

At the same time it becomes clear that, from a macroscopic, i.e., continuum theoretical view, the response of the crystal to its contortion induced by the dislocations are spin moment stresses $\tau_{ij}{}^k$, as indicated in Fig.11, see [36]. This is the new type of spin moment stress that already surfaced in the Cosserat con-

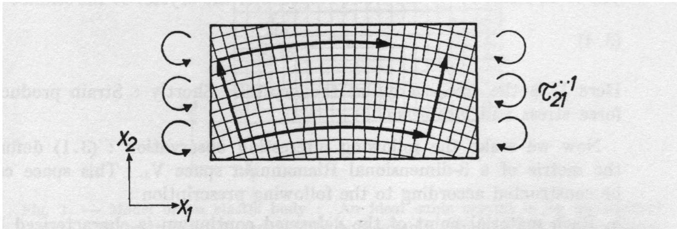

Figure 11: Deformation of a cubic crystal by edge dislocations of type ${\alpha_{12}}^1$: The relative orientations of the lattice planes in 2-direction change. A vector in x_2-direction will rotate, if parallelly displaced along the x_1-direction. As a consequence a contortion κ_{112} emerges and the closure failure of the "infinitesimal" parallelogram occurs.

tinuum in Fig.4. It is obvious, if one enriches the geometry by adding *torsion* to curvature, then on the dynamical side one should allow, besides stress (in 4D energy-momentum), *spin moment stress* (in 4D spin angular momentum).

The ideal reference crystal, in the sense of Cartan, is the undeformed crystal of Fig.9. One can imagine to roll it along the dislocated crystal in Fig.11. Then the closure failure of Fig.11 is determined, provided we define the connection with respect to the lattice vectors. In dislocation theory, this is known as the Frank-Burgers circuit, the closure failure as the Burgers vector. The cracking of a small parallelogram, defined in the undeformed crystal in Fig.9 and left untouched by the strain in Fig.10, can be recognized in Fig.11. Clearly, this procedure is isomorphic to the Cartan circuit, as has been proven by Kondo (1952) [49], Bilby et al. [5], and Kröner [50, 51]. Thus, it is an established fact that *dislocation density* and *torsion* in three dimensions can be used synonymously.

We recognize that at each point in a crystal with dislocations a lattice direction is well-defined, see Fig.11. In other words, a global teleparallelism is provided thereby reducing the RC-space to a Weitzenböck space with vanishing RC-curvature, see Fig.12. It can be shown [89] that the connection of a Weitzenböck space can always be represented in terms of the components of the frame $e_\alpha = {e^k}_\alpha \partial_k$ and the coframe $\vartheta^\alpha = {e_j}^\alpha dx^j$ as

$$\Gamma_{ij}{}^k = e^k{}_\alpha \, \partial_i e_j{}^\alpha \, . \tag{18}$$

Accordingly, on the one hand a dislocated crystal carries a torsion (that is, a

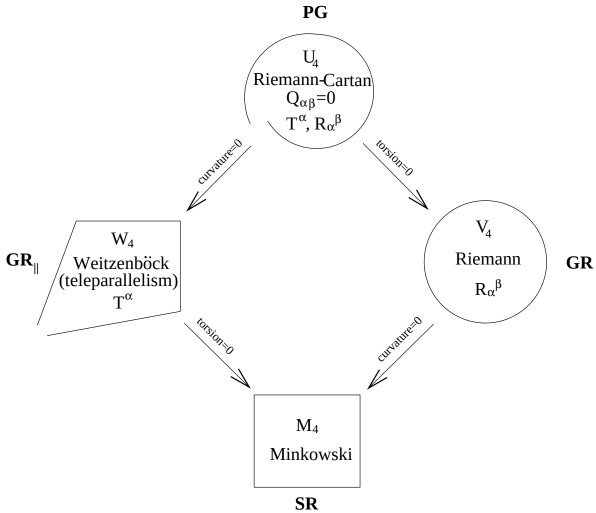

Figure 12: A Riemann-Cartan space and its special cases of a Weitzenböck and a Riemannian space.

dislocation density), on the other hand it provides a teleparallelism or defines a Weitzenböck space (that is, a space with vanishing RC-curvature), see, e.g., the discussion of Kröner [53].

5 Translation gauge theory of continuously distributed dislocations

What are then the deformational measures in the field theory of dislocations, see Kröner [50, 51, 52, 53]? Clearly, torsion α or contortion K must be one measure, but what about the distortion? We turn to the fundamental work of Lazar [54, 55], Katanaev [45], and Malyshev [57] on the 3D translational gauge approach to dislocation theory. The underlying geometrical structure of the theory is the affine tangent bundle $A(M)$ over the 3-dimensional base space M. It arises when one replaces at every point of M the usual tangent space by an affine tangent space. In the affine space, one can perform translations of the points and vectors, and in this way the translation group T_3 is realized as an *internal symmetry*.

The full description of the corresponding scheme requires the formalism of fiber bundles and connections on fiber bundles, see, among many others, the early work on this subject by Cho [15], also the recent important work of Tresguerres [100], and the references given therein. Here we only briefly formulate the general ideas and basic results of the translational gauge approach.

In accordance with the general gauge-theoretic scheme, to the three generators P_α of the translation group there corresponds a Lie algebra-valued 1-form $\Gamma^{(T)} = \Gamma_i^{(T)\alpha} P_\alpha\, dx^i$ as the translational gauge field potential. Under translations $y^\alpha \to y^\alpha + \epsilon^\alpha$ in the affine tangent space, it transforms like a connection

$$\delta\Gamma_i^{(T)\alpha} = -\partial_i \epsilon^\alpha. \tag{19}$$

Since T_3 is Abelian, i.e., translations commute with each other, there is no homogeneous term in this transformation law. Thus, it resembles the phase transformation of an electromagnetic potential. For the same reason, the gauge field strength $F^{(T)\alpha} = d\Gamma^{(T)\alpha} = \frac{1}{2} F_{ij}^{(T)\alpha}\, dx^i \wedge dx^j$ is formally reminiscent of a generalized electromagnetic field strength. This analogy was extensively used by Itin [41, 42].

In addition to the translational gauge field, another important structure is a field ξ^α defined as a local section of the affine tangent bundle. Geometrically, this field determines the "origin" of the affine spaces; it is known as Cartan's "radius vector". Under the gauge transformation (translation) it changes as $\xi^\alpha \to \xi^\alpha + \epsilon^\alpha$. However, the combination $e_i{}^\alpha = \partial_i \xi^\alpha + \Gamma_i^{(T)\alpha}$ is obviously *gauge invariant*, see [38], Eq.(3.3.1). In a rigorous gauge-theoretic framework, the 1-form $\vartheta^\alpha = e_i{}^\alpha dx^i = d\xi^\alpha + \Gamma^{(T)\alpha}$ arises as the nonlinear translational gauge field with ξ^α interpreted as the Goldstone field describing the spontaneous breaking of the translational symmetry.

We can consistently treat $\vartheta^\alpha = e_i{}^\alpha dx^i$ as the coframe of our 3D manifold. Then the translational gauge field strength is actually the anholonomity 2-form of this coframe: $F^{(T)\alpha} = d\Gamma^{(T)\alpha} = d\vartheta^\alpha$. Collecting our results, we have the deformation measures

$$e_i{}^\alpha = \partial_i \xi^\alpha + \Gamma_i^{(T)\alpha}\,, \tag{20}$$

$$F^{(T)\alpha} = d\Gamma^{(T)\alpha} = d\vartheta^\alpha\,. \tag{21}$$

If, in linear approximation, we compare these measures with the Cosserat deformation measure (10),(11), then we find, in generalization of the Cosserat structure,

$$e_i{}^\alpha \to \beta_{ij} \quad \text{(distortion)}\,, \tag{22}$$

$$\xi^\alpha \to u_i \quad \text{(displacement)}\,, \tag{23}$$

$$\Gamma_i^{(T)\alpha} \to \omega_{ij} \quad (!)\,, \tag{24}$$

$$F_{ij}^{(T)\alpha} \to \kappa_{kji} \quad \text{(contortion)}\,. \tag{25}$$

Here $F_{ij}^{(T)\alpha} \sim \alpha_{ijk}$ represents the dislocation density (torsion). Hence (25) represents Nye's relation (17), and the second deformational measure of dislocation theory with its 9 independent components corresponds to the contortion of the Cosserat theory. However, as we can recognize from (24), the dislocated continuum requires a more general description. The 3 component Cosserat rotation $\omega_{ij} = -\omega_{ji}$ is substituted by the asymmetric 9 component (translational gauge) potential $\Gamma_i^{(T)\alpha}$. Still, the distortion β_{ij} carries also 9 independent components and the corresponding static response is represented by the asymmetric force stress $\overline{\Sigma}_{ij} \sim \delta\mathcal{H}/\delta\beta_{ij}$.

If the second deformation measure in dislocation theory were, similar to the Cosserat theory, the gradient of $\Gamma_i^{(T)\alpha}$, i.e., $\partial_j\ \Gamma_i^{(T)\alpha}$, it would have 27 independent components and the static responses would be represented hyperstresses with and without moments, see [24]. However, as it turns out, see (25) — and this is very decisive — it is the dislocation density (torsion), i.e., the *curl* of $\Gamma_i^{(T)\alpha}$, with only 9 independent components that plays a role. For this reason, the static response in dislocation theory are again, as in a Cosserat continuum, just spin moment stresses $\overline{\tau}_{ijk} \sim \delta\mathcal{H}/\delta K_{kji}$, see [36]. Note that $\overline{\tau}_{ijk}$ is equivalent to[8] $\overline{\mu}_{ijk} \sim \delta\mathcal{H}/\delta\alpha_{kji}$. Thus, in dislocation theory as well as in the Cosserat continuum, we have the same type of stresses $\overline{\Sigma}_{\alpha}$ and $\overline{\tau}_{\alpha\beta}$ in spite of the newly emerging 9 component field $\Gamma_i^{(T)\alpha}$.

Continuum theories of *moving dislocations* are still a developing subject, see, e.g., Lazar [54] and Lazar & Anastassiadis [55] (and the literature quoted therein). Probably it is fair to say that they didn't find too many real applications so far. Nevertheless, the identification of the dislocation density with the torsion is invariably a cornerstone of all these theories.

6 Translational gauge theory of gravity

The construction of a translation gauge theory does not depend on the dimension of the underlying space. Hence we can take in 4D spacetime the same fundamental formulas (20),(21). Incidentally, the construction of the gauge theory for the group of translations is quite nontrivial because the local spacetime translations look very similar to the diffeomorphisms of spacetime. They are, however, different [101, 100]. The underlying geometrical structure of the theory is, as explained in the previous section, the affine tangent bundle. The corresponding translational connection is the 1-form $\Gamma_i^{(T)\alpha}dx^i$ with the transformation law (19). Now, however, the Latin and Greek indices run from 0 to 3.

[8] In 4D it is called the spin energy potential, see [38], Eqs.(5.1.24) and (5.1.22).

With the help of the Goldstone type field ξ^α, the translational gauge field gives rise to the coframe $\vartheta^\alpha = e_i{}^\alpha dx^i$ as described in (20). The anholonomity 2-form $F^{(T)\alpha}$ is the corresponding translational gauge field strength (21). The gravitational theories based on the coframe as the fundamental field have long history. The early coframe (or so-called vierbein, or tetrad, or teleparallel) gravity models were developed by Møller [64], Pellegrini and Plebański [74], Kaempfer [44], Hayashi and Shirafuji [30], to mention but a few. The first fiber bundle formulation was provided by Cho [15]. The dynamical contents of the model was later studied by Schweizer et al. [92], Nitsch and Hehl [69], Meyer [61], and more recent advances can be found in Aldrovandi and Pereira [1], Andrade and Pereira [2], Gronwald [23], Itin [42, 43], Maluf and da Rocha-Neto [56], Muench [65], Obukhov and Pereira [72], and Schucking and Surowitz [90, 91].

The Yang-Mills type Lagrangian 4-form for the translational gauge field ϑ^α is constructed as the sum of the quadratic invariants of the field strength:

$$\tilde{V}(\vartheta, d\vartheta) = -\frac{1}{2\kappa}\, F^{(T)\alpha} \wedge {}^\star \left(\sum_{I=1}^{3} a_I\, {}^{(I)}F^{(T)}_\alpha\right). \tag{26}$$

Here $\kappa = 8\pi G/c^3$, and * denotes the Hodge dual of the Minkowski flat metric $g_{\alpha\beta} = o_{\alpha\beta} := \mathrm{diag}(-1,1,1,1)$, that is used also to raise and lower the Greek (local frame) indices. As it is well known, we can decompose the field strength $F^{(T)\alpha}$ into the three irreducible pieces of the field strength:

$$^{(1)}F^{(T)\alpha} := F^{(T)\alpha} - {}^{(2)}F^{(T)\alpha} - {}^{(3)}F^{(T)\alpha}, \tag{27}$$

$$^{(2)}F^{(T)\alpha} := \frac{1}{3}\, \vartheta^\alpha \wedge \left(e_\beta \rfloor F^{(T)\beta}\right), \tag{28}$$

$$^{(3)}F^{(T)\alpha} := \frac{1}{3}\, e^\alpha \rfloor \left(\vartheta^\beta \wedge F^{(T)}_\beta\right), \tag{29}$$

i.e., the tensor part, the trace, and the axial trace, respectively.

There are three coupling constants in this theory, in general: a_1, a_2, a_3. In accordance with the general Lagrange-Noether scheme [23, 38] one derives from (26) the translational excitation 2-form and the canonical energy-momentum 3-form:

$$\tilde{H}_\alpha = -\frac{\partial \tilde{V}}{\partial F^{(T)\alpha}} = \frac{1}{\kappa}\, {}^\star \left(\sum_{I=1}^{3} a_I\, {}^{(I)}F^{(T)}_\alpha\right), \tag{30}$$

$$\tilde{E}_\alpha = \frac{\partial \tilde{V}}{\partial \vartheta^\alpha} = e_\alpha \rfloor \tilde{V} + (e_\alpha \rfloor F^{(T)\beta}) \wedge \tilde{H}_\beta. \tag{31}$$

Accordingly, the variation of the total Lagrangian $L = \tilde{V} + L_{\rm mat}$ with respect to the tetrad results in the gravitational field equations

$$d\tilde{H}_\alpha - \tilde{E}_\alpha = \Sigma_\alpha, \tag{32}$$

with the canonical energy-momentum current 3-form of matter

$$\Sigma_\alpha := \frac{\delta L_{\rm mat}}{\delta \vartheta^\alpha} \tag{33}$$

as the source.

The coframe models do not possess any other symmetry except the diffeomorphism invariance and the invariance under the *rigid* Lorentz rotations of the tetrads. However, for a special choice of the coupling constants,

$$a_1 = 1, \qquad a_2 = -\,2, \qquad a_3 = -\,\frac{1}{2}, \tag{34}$$

the field equations turn out to be invariant under the *local Lorentz* transformations $\vartheta^\alpha \longrightarrow L^\alpha{}_\beta(x)\vartheta^\beta$ with the matrices $L^\alpha{}_\beta(x)$ arbitrary functions of the spacetime coordinates. At the same time, one can demonstrate that the tetrad field equations (32) are then recast *identically* into the form of Einstein's equation (here $\eta_{\alpha\beta\gamma} = {}^\star\,(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)$ and * denotes the Hodge star operator):

$$\frac{1}{2\kappa}\,\eta_{\alpha\beta\gamma} \wedge \tilde{R}^{\beta\gamma} = \Sigma_\alpha. \tag{35}$$

Here $\tilde{R}_\alpha{}^\beta = d\tilde{\Gamma}_\alpha{}^\beta + \tilde{\Gamma}_\gamma{}^\beta \wedge \tilde{\Gamma}_\alpha{}^\gamma$ is the Riemannian curvature of the Christoffel connection

$$\tilde{\Gamma}_{\alpha\beta} := \frac{1}{2}\left[e_\alpha \rfloor F^{(T)}_\beta - e_\beta \rfloor F^{(T)}_\alpha - (e_\alpha \rfloor e_\beta \rfloor F^{(T)}_\gamma) \wedge \vartheta^\gamma\right]. \tag{36}$$

For that reason, the coframe gravity model with the choice (34) is usually called a *teleparallel equivalent* of general relativity theory.

7 Einstein-Cartan theory of gravity

Einstein-Cartan (EC) theory is an extension of Einstein's general relativity, in which the local Lorentz symmetry, which appears to be accidental in the teleparallel equivalent model above, is taken seriously as a fundamental feature of the gravitational theory.

One can naturally arrive at the EC-theory by using the heuristic arguments based on the mapping of the Noether to Bianchi identities, as shown in McCrea et al. [37, 60]. Similar are the thoughts of Ruggiero and Tartaglia [82], who consider the EC-theory as a defect type theory; see also Hammond [27], Ryder and Shapiro [83], and Trautman [97, 99].

However, the most rigorous derivation is based on the gauge approach for the Poincaré group (see [23, 38], for example), in which the gauge potentials are the coframe ϑ^α and the Lorentz connection $\Gamma_\alpha{}^\beta$. They correspond to the translational and the Lorentz subgroups of the Poincaré group, respectively.

The dynamics of the gravitational field is described in this model by the Hilbert-Einstein Lagrangian plus, in general, a cosmological term:

$$V = -\,\frac{1}{2\kappa}\left(\eta_{\alpha\beta}\wedge R^{\alpha\beta} - 2\lambda\eta\right). \tag{37}$$

Here η is the volume 4-form and $\eta_{\alpha\beta} = {}^\star\left(\vartheta_\alpha\wedge\vartheta_\beta\right)$. The field equations arise from the variations of the total Lagrangian $V_{\rm tot} = V + L_{\rm mat}$ with respect to the coframe and connection, see Sciama [93] and Kibble [46]:

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\wedge R^{\beta\gamma} - \lambda\eta_\alpha = \kappa\Sigma_\alpha, \tag{38}$$

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\wedge T^{\gamma} = \kappa\tau_{\alpha\beta}. \tag{39}$$

Here in addition to the canonical energy-momentum current (33), the canonical spin current 3-form of matter

$$\tau^\alpha{}_\beta := \frac{\delta L_{\rm mat}}{\delta\Gamma_\alpha{}^\beta} \tag{40}$$

arises as the source of the gravitational field. Two sources Σ_α and $\tau^\alpha{}_\beta$ satisfy the identities ("covariant conservation laws") that follows from the Noether theorem for the invariance of the theory under diffeomorphisms and the local Lorentz group:

$$D\Sigma_\alpha = (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta + (e_\alpha\rfloor R_{\beta\gamma})\wedge\tau^{\beta\gamma}\,, \tag{41}$$

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha}\wedge\Sigma_{\beta]} = 0. \tag{42}$$

When the matter has no spin, $\tau^\alpha{}_\beta = 0$, the second (Cartan's) field equation (39) yields the zero spacetime torsion, $T^\alpha = 0$. As a result, the Riemann-Cartan curvature $R^{\beta\gamma}$ reduces to the Riemannian curvature $\tilde{R}^{\beta\gamma}$, and the first field equation (38) reduces to Einstein's equation (35) of general relativity theory. Physical effects of classical and quantum matter in the EC-theory are overviewed in [35, 94]. There emerges, as compared to general relativity, an additional spin-spin contact interaction of gravitational origin that only plays a role at extremely high matter densities.

Blagojević et al. [7] found in 3D gravity with torsion an interesting quantum effect: The black hole entropy depends on the torsional degrees of freedom.

8 Poincaré gauge theory and metric-affine gravity

Einstein-Cartan theory, outlined in Sec. 7, represents a degenerate Poincaré gauge model in which spin couples algebraically to the Lorentz connection. As a result, torsion is a nonpropagating field and vanishes identically outside the material sources.

Things are however different in the Yang-Mills type models of the Poincaré gravity based on the quadratic Lagrangians in torsion and curvature. These models are discussed by Hehl [31], Ponomariov and Obukhov [76], Gronwald and Hehl [25], see also a recent review by Obukhov [71].

The general Lagrangian which is at most *quadratic* (q) in the Poincaré gauge field strengths – in the torsion and the curvature – reads

$$V_{\rm q}=-\,\frac{1}{2\kappa}\,\left[a_0\,R^{\alpha\beta}\wedge\eta_{\alpha\beta}-2\lambda\,\eta+T^\alpha\wedge{}^*\left(\sum_{I=1}^{3}a_I\,{}^{(I)}T_\alpha\right)\right]$$
$$-\,\frac{1}{2}\,R^{\alpha\beta}\wedge{}^*\left(\sum_{J=1}^{6}b_J\,{}^{(J)}R_{\alpha\beta}\right). \qquad (43)$$

We use the unit system in which the dimension of the gravitational constant is $[\kappa]=\ell^2$ with the unit length ℓ. The coupling constants a_0,a_1,a_2,a_3 and $b_1,...,b_6$ are *dimensionless,* whereas $[\lambda]=\ell^{-2}$. These coupling constants determine the particle contents of the quadratic Poincaré gauge models. The three irreducible parts of the torsion ${}^{(I)}T_\alpha$ are defined along the pattern (27)-(29), whereas the irreducible decomposition of the curvature into the six pieces ${}^{(J)}R_{\alpha\beta}$ is given in [38]. The Lagrangian (43) has the general structure similar to that of the Yang–Mills Lagrangian for the gauge theory of internal symmetry group.

The Poincaré gauge field equations are derived from the total Lagrangian $V_{\rm q}+L_{\rm mat}$ from the variations with respect to the coframe and connection. They read explicitly

$$DH_\alpha-E_\alpha=\Sigma_\alpha\,, \qquad (44)$$
$$DH^\alpha{}_\beta-E^\alpha{}_\beta=\tau^\alpha{}_\beta\,. \qquad (45)$$

The right-hand sides describe the material sources of the Poincaré gauge gravity: the canonical energy–momentum (33) and the spin (40) three–forms. The left-hand sides are constructed from the *gauge field momenta* 2-forms

$$H_\alpha := -\frac{\partial V_{\rm q}}{\partial T^\alpha}\,, \qquad H^\alpha{}_\beta := -\frac{\partial V_{\rm q}}{\partial R_\alpha{}^\beta}\,, \tag{46}$$

and the *canonical* 3–forms of the energy-momentum and spin of the gauge gravitational field

$$E_\alpha := \frac{\partial V_{\rm q}}{\partial \vartheta^\alpha} = e_\alpha \rfloor V_{\rm q} + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge H^\beta{}_\gamma, \tag{47}$$

$$E^{\alpha\beta} := \frac{\partial V_{\rm q}}{\partial \Gamma_{\alpha\beta}} = -\vartheta^{[\alpha} \wedge H^{\beta]}. \tag{48}$$

The class of gravitational models (43) has a rich geometrical and physical structure. Depending on the choice of the coupling constants a_0, a_1, a_2, a_3 and $b_1, ..., b_6$, the field equations (44) and (45) admit black hole, cosmological, and wave solutions that generalize the general-relativistic solutions of Einstein's theory at small distances. On large time and space scales, the physical predictions of Poincaré gravity agree (in the generic case of the coupling constants a_I and b_J) with the results of general relativity, see [31, 25, 38, 71].

The Cosserat medium in elasticity theory and the physical sources in the Poincaré gauge gravity deal with the material continua and bodies, the elements of which have rigid microstructure. A further generalization is possible when the matter elements possess *deformable microstructure*. In elasticity theory this is the case, for example, in Mindlin's 3-dimensional continuum with microstructure [62]. In 4 dimensions, the corresponding counterpart arises as *metric-affine gravity* (MAG) theory. The proper framework is then the gauge theory based on the general affine symmetry group [38]. The geometry of such an elastic medium and of the spacetime in MAG is characterized, in addition to the curvature and torsion, by a nontrivial *nonmetricity*.

9 Cartan's spiral staircase: A 3D Euclidean model for a space with torsion

Apparently in order to visualize torsion in a simple 3D model, see Fig.13, Cartan proposed a certain construction that, in his own (translated) words of 1922 [12], reads as follows:

> "…imagine a space F which corresponds point by point with a Euclidean space E, the correspondence preserving distances. The

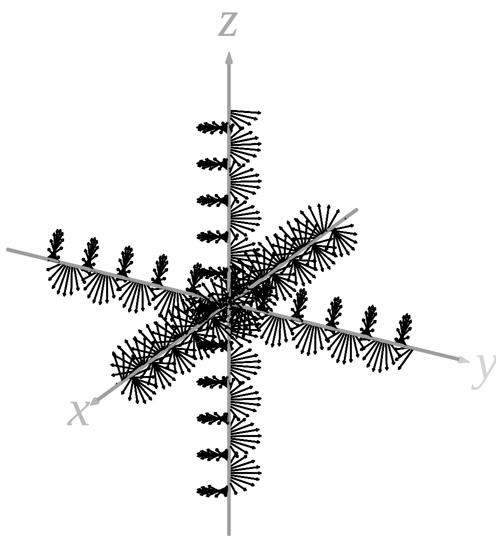

Figure 13: *Cartan's spiral staircase*, see García et al. [22]. Cartan's rules [12] for the introduction of a non-Euclidean connection in a 3D Euclidean space are as follows: (i) A vector which is parallelly transported along itself does not change (cf. a vector directed and transported in x-direction). (ii) A vector that is orthogonal to the direction of transport rotates with a prescribed constant "velocity" (cf. a vector in y–direction transported in x–direction). The winding sense around the three coordinate axes is always positive.

> difference between the two spaces is the following: two orthogonal triads issuing from two points A and A' infinitesimally nearby in F will be parallel when the corresponding triads in E may be deduced one from the other by a given helicoidal displacement (of right–handed sense, for example), having as its axis the line joining the origins. The straight lines in F thus correspond to the straight lines in E: They are geodesics. The space F thus defined admits a six parameter group of transformations; it would be our ordinary space as viewed by observers whose perceptions have been twisted. Mechanically, it corresponds to a medium having constant pressure and constant internal torque."

One can show [22] that Cartan's prescription yields a trivial coframe and a constant connection,

$$\vartheta^\alpha = \delta_i^\alpha \, dx^i \,, \qquad \Gamma^{\alpha\beta} = \frac{\mathcal{T}}{\ell}\, \eta^{\alpha\beta} \,, \tag{49}$$

with the 1-form $\eta^{\alpha\beta} = {}^\star \left(\vartheta^\alpha \wedge \vartheta^\beta\right)$ and * as the Hodge star operator; moreover, $\mathcal{T}$ and ℓ are constants. The components of the connection are totally antisymmetric, $\Gamma_{\gamma\alpha\beta} = e_\gamma \rfloor \Gamma_{\alpha\beta} = (\mathcal{T}/\ell)\, \eta_{\gamma\alpha\beta}$. Thus, autoparallels and geodesics coincide. Accordingly, in the spiral staircase, extremals are *Euclidean* straight lines. This is apparent in Cartan's construction. By simple algebra we find for the torsion, the Riemannian curvature, and the Riemann-Cartan curvature, respectively,

$$T^\alpha = 2\, \frac{\mathcal{T}}{\ell}\, {}^\star\vartheta^\alpha \,, \qquad \widetilde{R}^{\alpha\beta} = 0 \,, \qquad R^{\alpha\beta} = -\frac{\mathcal{T}^2}{\ell^2}\, \vartheta^\alpha \wedge \vartheta^\beta \,. \tag{50}$$

In components, the torsion tensor reads $T_{[\alpha\beta\gamma]} = 2\frac{\mathcal{T}}{\ell}\epsilon_{\alpha\beta\gamma}$, with the totally antisymmetric Levi-Civita symbol $\epsilon_{\alpha\beta\gamma}$; the pitch of the helices is proportional to the constant $\mathcal{T}$.

For a *solid state physicist* it is immediately clear that the geometry in Fig.13 represents a set of three perpendicular constant 'forests' of *screw* dislocations of equal strength. Hence Cartan thought in terms of screw dislocations without knowing them! Of course, the totally antisymmetric part of the dislocation density $\alpha_{[ijk]}$ is an irreducible piece of the torsion which has one independent component. Wouldn't it be interesting to find this spiral staircase as an exact solution in dislocation gauge theory? Since only one irreducible piece of the dislocation density (torsion) is involved, this should be possible.

Cartan apparently had in mind a 3D space with Euclidean signature. For an alternative interpretation of Cartan's spiral staircase we consider the 3D Einstein–Cartan field equations without cosmological constant:

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\,R^{\beta\gamma} = \ell\,\overline{\Sigma}_\alpha\,, \tag{51}$$

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\,T^{\gamma} = \ell\,\overline{\tau}_{\alpha\beta}\,. \tag{52}$$

The coframe and the connection of (49), Euclidean signature assumed, form a solution of the Einstein–Cartan field equations *with matter* provided the energy–momentum current (for Euclidean signature the force stress tensor $\mathfrak{t}_\alpha{}^\beta$) and the spin current (here the torque or spin moment stress tensor $\mathfrak{s}_{\alpha\beta}{}^\gamma$) are constant,

$$\overline{\Sigma}_\alpha =: \mathfrak{t}_\alpha{}^\beta\,\eta_\beta = -\frac{\mathcal{T}^2}{\ell^3}\,\eta_\alpha \quad \text{and} \quad \overline{\tau}_{\alpha\beta} =: \mathfrak{s}_{\alpha\beta}{}^\gamma\,\eta_\gamma = -\frac{\mathcal{T}}{\ell^2}\,\vartheta_{\alpha\beta}\,. \tag{53}$$

Inversion yields

$$\mathfrak{t}_\alpha{}^\beta = -\frac{\mathcal{T}^2}{\ell^3}\,\delta_\alpha^\beta\,, \qquad \mathfrak{s}_{\alpha\beta\gamma} = -\frac{\mathcal{T}}{\ell^2}\,\eta_{\alpha\beta\gamma}\,. \tag{54}$$

We find a constant hydrostatic *pressure* $-\mathcal{T}^2/\ell^3$ and a constant *torque* $-\mathcal{T}/\ell^2$, exactly as foreseen by Cartan.

By studying the spiral staircase and reading also more in the Cartan book [13], it becomes clear that Cartan's intuition worked in 3D (and not in 4D). This led Cartan to a decisive mistake in this context. Take the energy-momentum law in a 4D RC-space, if the mater field equation is fulfilled, see (41),

$$D\Sigma_\alpha = (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta + (e_\alpha\rfloor R_{\beta\gamma})\wedge\tau^{\beta\gamma}\,, \qquad \mathbf{4D}\,. \tag{55}$$

Note the Lorentz type forces on the right-hand-side, in particular the last term representing a Mathisson-Papapetrou type of force with curvature $\times$ spin. However, straightforward algebra yields, for 3D,

$$D\overline{\Sigma}_\alpha = 0\,, \qquad \mathbf{3D}\,. \tag{56}$$

Cartan assumed incorrectly that (56) is also valid in four dimensions. For that reason he ran into difficulties with his 4D gravitational theory that includes (56) and came, after his 1923/1924 papers, never back to his (truncated Einstein-Cartan-)theory. Hence intuition (without algebra) can even lead the greatest mathematical minds astray.

10 Some controversial points

In more physically oriented papers, the authors treat the question of the possible existence of a torsion as a dynamical one. Hanson and Regge [28], e.g., open their paper with the statement: *"We suggest that the absence of torsion in conventional gravity could in fact be dynamical. A gravitational Meissner effect might produce instanton-like vortices of nonzero torsion concentrated at four-dimensional points..."* Accordingly they study certain dynamical models in order to find a possible answer for this question. We don't follow this train of thought. However, such a model building is a desirable feature.

In contrast, in the literature there are numerous statements about a possible torsion of the spacetime manifold that don't stand a closer examination. Let us quote some examples:

1. *Ohanian and Ruffini* [73] claim that the Einstein-Cartan theory is defective, see ref.[73], pp. 311 and 312. Since this is a widely read and, otherwise, excellent textbook, we would just like to comment on their arguments, see also [32]:

   "If $\Gamma^{\beta}{}_{\nu\mu}$ were not symmetric, the parallelogram would fail to close. This would mean that the geometry of the curved spacetime differs from a flat geometry even on a small scale – the curved spacetime would not be approximated locally by a flat spacetime."

   Equation (9), see also the paper of Hartley [29], disproves the Ohanian and Ruffini statement right away. In (9) it is clearly displayed that the Riemann-Cartan geometry is Euclidean 'in the infinitesimal'. And this was, as we discussed in Sec.2, one of the guiding principles of Cartan.

   "...we do not know the 'genuine' spin content of elementary particles..."

   According to present day wisdom, matter is built up from quarks and leptons. No substructures have been found so far. According to the mass-spin classification of the Poincaré group and the experimental information of lepton and hadron collisions etc., leptons and quarks turn out to be fermions with spin 1/2 (obeying the Pauli principle). As long as we accept the (local) Poincaré group as a decisive structure for describing elementary particles, there can be no doubt what spin really is. And abandoning the Poincaré group would result in an overhaul of (locally valid) special relativity theory.

   The nucleon is a composite particle and things related to the build-up of its spin are not clear so far. But we do know that we can treat it as a fermion with spin $1/2$. As long as this can be taken for granted, at least in an effective sense, we know its spin and therefore its torsion content.

2. *Carroll* [11] argues in his book on p.190 as follows:

   "...Thus, we do not really lose any generality by considering theories of torsion-free connections (which lead to GR) plus any number of tensor fields, which we can name what we like. Similar considerations..."

   (i) This opinion is often expressed by particle physicists who don't think too profoundly about geometry. As we saw in Secs.1 and 2, the torsion tensor is not *any* tensor, but it is a particular tensor related to the *translation group.* A torsion tensor cracks infinitesimal parallelograms, see Fig.1. A parallelogram is deeply related to the geometry of a manifold with a linear connection. The closure failure of a parallelogram can only be created by a distinctive geometrical quantity, namely the torsion tensor — and not by any other tensor. This fact alone makes Carroll's argument defective.

   (ii) Another way of saying this is that torsion affects the Bianchi identities (5). This cannot be done by any other tensor, apart from the curvature tensor. Moreover, as we saw in Sec.5, the torsion is the field strength belonging to the translation group.

   (iii) In particular, as Sciama [93] has shown, an independent Lorentz connection couples to the spin of a matter field in a similar way as the coframe couples to the energy-momentum of matter. This shows too that a splitting off of the Levi-Civita connection is of no use in such a context. The Einstein-Cartan theory of gravity is a viable gravitational theory. If one studies its variational principle etc., then one will recognize that the splitting technique advised by Carroll messes up the whole structure.

   (iv) If one minimally couples to a connection, it is decisive which connection one really has. Of course, one can couple minimally to the Levi-Civita connection and add later nonminimal $\sim$ (torsion)^2 pieces thereby transforming a minimal to a nonminimal coupling; also here one messes up the structure. Minimal coupling would lose its heuristic power.

3. *Kleinert and Shabanov* [48] postulate that a scalar particle moves in a Riemann-Cartan space along an autoparallel. However, the equations of motion cannot be postulated freely, they have rather to be determined from the energy-momentum and the angular momentum laws of the underlying theory. Then it turns out that a scalar particle can only 'feel' the Riemannian metric of spacetime, it is totally insensitive to a possibly

existent torsion (and nonmetricity) of spacetime. This has been proven, e.g., by Yasskin and Stoeger [107], Ne'eman and Hehl [67], and by Puetzfeld and Obukhov [80].

4. *Weinberg* [103] wrote an article about "Einstein's mistakes". In a response, Becker [4] argued that for "generalizing general relativity" one should allow torsion and teleparallelism. Weinberg's response [104] was as follows:

   "I may be missing the point of Robert Becker's remarks, but I have never understood what is so important physically about the possibility of torsion in differential geometry. The difference between an affine connection with torsion and the usual torsion-free Christoffel symbol is just a tensor, and of course general relativity in itself does not constrain the tensors that might be added to any dynamical theory. What difference does it make whether one says that a theory has torsion, or that the affine connection is the Christoffel symbol but happens to be accompanied in the equations of the theory by a certain tensor? The first alternative may offer the opportunity of a different geometrical interpretation of the theory, but it is still the same theory."

   This statement of Weinberg was answered by one of us, see [104]. We argued, as in this essay, that torsion is related to the translation group and that it is, in fact, the translation gauge field strength. Moreover, we pointed out the existence of a new spin-spin contact interaction in the EC-theory and that torsion could be measured by the precession of nuclear spins.

   Weinberg's answer was:

   "Sorry, I still don't get it. Is there any physical principle, such as a principle of invariance, that would require the Christoffel symbol to be accompanied by some specific additional tensor? Or that would forbid it? And if there is such a principle, does it have any other testable consequences?"

   The physical principle Weinberg is looking for is *translational gauge invariance*, see Sec.6. And the testable consequences are related to the new spin-spin contact interaction and to the precession of elementary particle spins in torsion fields.

5. *Mao, Tegmark, Guth, and Cabi* [59] claim that torsion can be measured by means of the Gravity Probe B experiment. This is totally incorrect since the sensitive pieces of this gyroscope experiment, the rotating

quartz balls, don't carry uncompensated elementary particle spin. If the balls were made of polarized elementary particle spins, that is, if one had a *nuclear gyroscope*, see Simpson [96], as they were constructed for inertial platforms, then the gyroscope would be sensitive to torsion. As mentioned in the last point regarding Kleinert et al., an equation of motion in a general relativistic type of field theory has to be derived from the energy-momentum and angular momentum laws, see Yasskin and Stoeger [107] and Puetzfeld and Obukhov [79, 80]. Then it turns out that measuring torsion requires elementary spin — there is no other way.

6. *Torsion in string theory?* Quite some time ago it was noticed by Scherk and Schwarz [87] that the low-energy effective string theory can be elegantly reformulated in geometrical terms by using a non-Riemannian connection. The graviton field, the dilaton field, and the antisymmetric tensor field (2-form B), which represent the massless modes of the closed string, then give rise to a spacetime with torsion and nonmetricity. In particular, the 3-form $H = dB$ is interpreted in this picture as one of the irreducible parts (namely, the axial trace part, cf. (29)) of the spacetime torsion. Later this idea was extended to interpret the dilaton field as the potential for the (Weyl) nonmetricity, see [18, 84, 77], for example.

   Another formal observation reveals certain mathematical advantages in discussing compactification schemes with torsion for the higher-dimensional string models, see [8, 75].

   It is however unclear whether some fundamental principle or model underlies these formal observations. The geometrical interpretation of this kind is certainly interesting, but one should take it with a grain of salt. The qualitative difference (from the elastic models with defects and the gauge gravity models) lies in the fact that the field H, although viewed as torsion, is not an independent variable in this approach but arises from the potential 2-form B. Consistent with this view is Polchinski's definition of string torsion in his glossary, see [75], p.514: "**torsion** *a term applied in various 3-form field strengths, so called because they appear in covariant derivatives in combination with the Christoffel connection.*"

   Thus, the notion of torsion in string theory is used in an unorthodox way and should not be mixed up with Cartan's torsion of 1922.

7. In the past, there have been several attempts to relate the torsion of spacetime to electromagnetism. A recent approach is the one of Evans [19, 20], who tried to construct a unified field theory. As we have seen

in Secs.1 and 2, torsion is irresolvably tied to the notion of a translation. Thus, torsion has nothing to do with internal (unitary) symmetry groups. We have shown in two separate papers [33, 40] that Evans' theory is untenable.

11 Outlook

In three-dimensional dislocated crystals, the equality of the dislocation density and torsion is an established fact. In four dimensions, with respect to the experimental predictions, the Einstein-Cartan theory is a viable gravity model that is presently indistinguishable from Einstein's general relativity. The contact character of the spin-connection interaction and the smallness of Newton's gravitational coupling constant underlies this fact for macroscopic distances and large times.

Sciama was the first in 1961 to derive the field equations (38),(39) in tensor notation [93]; in 1979 he passed the following judgment (private communication): "The idea that spin gives rise to torsion should not be regarded as an ad hoc modification of general relativity. On the contrary, it has a deep group-theoretical and geometric basis. If the history had been reversed and the spin of the electron discovered before 1915, I have little doubts that Einstein would have wanted to include torsion in his original formulation of general relativity. On the other hand, the numerical differences which arise are normally very small, so that the advantages of including torsion are entirely theoretical."

However, the quadratic Poincaré gauge models and their generalizations in the framework of MAG predict propagating torsion (and nonmetricity) modes which can potentially be detected on extremely small scales (high energies). The appropriate physical conditions may occur during the early stages of the cosmological evolution of the universe, see, e.g., Minkevich [63], Puetzfeld [78], and Brechet, Hobson, and Lasenby [9].

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Appendix: Derivation of the deformation measures of a Cosserat continuum

Let us consider a 3D Euclidean space. Its geometrical structure is determined by the 1-form fields of the coframe $\overset{\circ}{\vartheta}{}^{\alpha}$ and the connection $\overset{\circ}{\Gamma}{}_{\alpha}{}^{\beta}$. They satisfy the trivial Cartan relations:

$$d\overset{\circ}{\vartheta}{}^{\alpha} + \overset{\circ}{\Gamma}{}_{\beta}{}^{\alpha} \wedge \overset{\circ}{\vartheta}{}^{\beta} = \overset{\circ}{T}{}^{\alpha} = 0, \tag{57}$$

$$d\overset{\circ}{\Gamma}{}_{\alpha}{}^{\beta} + \overset{\circ}{\Gamma}{}_{\gamma}{}^{\beta} \wedge \overset{\circ}{\Gamma}{}_{\alpha}{}^{\gamma} = \overset{\circ}{R}{}_{\alpha}{}^{\beta} = 0. \tag{58}$$

The right-hand sides, given by the torsion and the curvature 2-forms, respectively, vanish for the Euclidean space.

We now consider an infinitesimal deformation of this manifold produced by the "generalized gauge transformation" which is defined as a combination of the diffeomorphism and of the local rotation. The diffeomorphism is generated by some vector field, whereas the rotation is given by the 3×3 matrix which acts on the anholonomic (Greek indices) components. We assume that a deformation is small which means that we only need to consider the infinitesimal diffeomorphism and rotational transformations. By definition, the deformation is the sum of the two infinitesimal gauge transformations:

$$\beta^{\alpha} := \Delta\overset{\circ}{\vartheta}{}^{\alpha} = \delta_{\text{diff}}\overset{\circ}{\vartheta}{}^{\alpha} + \delta_{\text{rot}}\overset{\circ}{\vartheta}{}^{\alpha}, \tag{59}$$

$$\kappa_{\alpha}{}^{\beta} := \Delta\overset{\circ}{\Gamma}{}_{\alpha}{}^{\beta} = \delta_{\text{diff}}\overset{\circ}{\Gamma}{}_{\alpha}{}^{\beta} + \delta_{\text{rot}}\overset{\circ}{\Gamma}{}_{\alpha}{}^{\beta}. \tag{60}$$

Let u be an arbitrary vector field, and we recall that a diffeomorphism, generated by it, is described by the Lie derivative along this vector field, i.e., $\delta_{\text{diff}} = \ell_u = du\rfloor + u\rfloor d$. As for the local rotations, they are given by the standard transformation formulas,

$$\delta_{\text{rot}}\overset{\circ}{\vartheta}{}^{\alpha} = \varepsilon^{\alpha}{}_{\beta}\,\overset{\circ}{\vartheta}{}^{\beta}, \quad \delta_{\text{rot}}\overset{\circ}{\Gamma}{}_{\alpha}{}^{\beta} = -\overset{\circ}{D}\varepsilon^{\beta}{}_{\alpha}. \tag{61}$$

Here $\overset{\circ}{D}$ is the covariant derivative defined by the connection $\overset{\circ}{\Gamma}$. For the Lie derivative of the coframe we find (with $u^{\alpha} = u\rfloor\overset{\circ}{\vartheta}{}^{\alpha}$)

$$\begin{aligned}
\ell_u\overset{\circ}{\vartheta}{}^{\alpha} &= d\,u^{\alpha} + u\rfloor d\,\overset{\circ}{\vartheta}{}^{\alpha} \\
&= d\,u^{\alpha} - u\rfloor(\overset{\circ}{\Gamma}{}_{\beta}{}^{\alpha} \wedge \overset{\circ}{\vartheta}{}^{\beta}) + u\rfloor\overset{\circ}{T}{}^{\alpha} \\
&= d\,u^{\alpha} + \overset{\circ}{\Gamma}{}_{\beta}{}^{\alpha}\,u^{\beta} - (u\rfloor\overset{\circ}{\Gamma}{}_{\beta}{}^{\alpha})\,\overset{\circ}{\vartheta}{}^{\beta}.
\end{aligned} \tag{62}$$

We used here (57) because the space is Euclidean. Substituting (62) together with (61) into (59), we find for the translational deformation

$$\beta^\alpha = \overset{\circ}{D} u^\alpha - \omega^\alpha{}_\beta \, \overset{\circ}{\vartheta}{}^\beta . \tag{63}$$

Here we introduced $\omega^\alpha{}_\beta := u \rfloor \overset{\circ}{\Gamma}_\beta{}^\alpha - \varepsilon^\alpha{}_\beta$.

Analogously we have for the Lie derivative of the connection

$$\begin{aligned} \ell_u \overset{\circ}{\Gamma}_\beta{}^\alpha &= d\,(u \rfloor \overset{\circ}{\Gamma}_\beta{}^\alpha) + u \rfloor d\, \overset{\circ}{\Gamma}_\beta{}^\alpha \\ &= d\,(u \rfloor \overset{\circ}{\Gamma}_\beta{}^\alpha) - u \rfloor (\overset{\circ}{\Gamma}_\gamma{}^\alpha \wedge \overset{\circ}{\Gamma}_\beta{}^\gamma) + u \rfloor \overset{\circ}{R}_\beta{}^\alpha \\ &= d\,(u \rfloor \overset{\circ}{\Gamma}_\beta{}^\alpha) + \overset{\circ}{\Gamma}_\gamma{}^\alpha (u \rfloor \overset{\circ}{\Gamma}_\beta{}^\gamma) - \overset{\circ}{\Gamma}_\beta{}^\gamma (u \rfloor \overset{\circ}{\Gamma}_\gamma{}^\alpha). \end{aligned} \tag{64}$$

We again used here (58) for the Euclidean space. Now, substituting (64) together with (61) into (60), we find for the rotational deformation

$$\kappa_\alpha{}^\beta = \overset{\circ}{D} \omega^\beta{}_\alpha . \tag{65}$$

We thus recovered the deformation measures (10),(11) of the linear Cosserat continuum. Using local coordinates, we expand $\overset{\circ}{\vartheta}{}^\alpha = \overset{\circ}{e}_i{}^\alpha dx^i$, and then (63) and (65) reduce in tensor components to

$$\beta_i{}^j = \overset{\circ}{\nabla}_i u^j - \omega^j{}_i , \tag{66}$$

$$\kappa_{ij}{}^k = \overset{\circ}{\nabla}_i \omega^k{}_j . \tag{67}$$

Thus, the deformation measures of the Cosserat continuum are literally given by the deformations of coframe and connection (59),(60).

The compatibility conditions (16) can be derived from (63) and (65) by applying the covariant derivative. The result reads

$$\overset{\circ}{D} \beta^\alpha + \kappa_\beta{}^\alpha \wedge \overset{\circ}{\vartheta}{}^\beta = 0, \qquad \overset{\circ}{D} \kappa_\beta{}^\alpha = 0. \tag{68}$$

The crucial point is that the geometry of the space is Euclidean and flat.

When, however, the space has a nontrivial Riemann-Cartan geometry with the coframe ϑ^α and connection $\Gamma_\alpha{}^\beta$ satisfying Cartan's structure equations with the nontrivial torsion T^α and curvature $R_\alpha{}^\beta$, the deformation measures are given by

$$\beta^\alpha = D u^\alpha - \omega^\alpha{}_\beta \, \vartheta^\beta + u \rfloor T^\alpha , \tag{69}$$

$$\kappa_\alpha{}^\beta = D \omega^\beta{}_\alpha + u \rfloor R_\alpha{}^\beta , \tag{70}$$

and they no longer satisfy the compatibility conditions (68). In 4D, after suitably adjusting the signs, Eqs.(69) and (70) coincide with the Poincaré gauge transformations (13), (14).

References

[1] R. Aldrovandi and J.G. Pereira, *An introduction to teleparallel gravity*, (2001-2007) Lecture notes, unpublished (112 pages), available from http://www.ift.unesp.br/users/jpereira/classnotes.html

[2] V.C. de Andrade and J.G. Pereira, *Gravitational Lorentz force and the description of the gravitational interaction*, Phys. Rev. **D56** (1997) 4689–4695.

[3] J. Badur and H. Stumpf, *On the influence of E. and F. Cosserat on modern continuum mechanics and field theory,* University of Bochum, Institute for Mechanics, Communication number **72** (December 1989) 39 pages.

[4] R.E. Becker, *Letter,* Physics Today **59** (April 2006) 14–15.

[5] B.A. Bilby, R. Bullough and E. Smith, *Continuous distributions of dislocations: a new application of the methods of Non-Riemannian geometry,* Proc. Roy. Soc. (London) **A 231** (1955) 263–273.

[6] M. Blagojević, *Gravitation and Gauge Symmetries,* IoP, Bristol, UK (2002).

[7] M. Blagojević and B. Cvetković, *Black hole entropy from the boundary conformal structure in 3D gravity with torsion,* JHEP **0610** (2006) 005 (12 pages) [arXiv:gr-qc/0606086].

[8] E. Braaten, T.L. Curtright, and C.K. Zachos, *Torsion and geometrostasis in non-linear sigma models*, Nucl. Phys. **B260** (1985) 630–688.

[9] S.D. Brechet, M.P. Hobson, A.N. Lasenby, *Weyssenhoff fluid dynamics in general relativity using a 1+3 covariant approach*, Class. Quant. Gravity, to be published (Dec. 2007).

[10] G. Capriz, *Continua with Microstructure*, Springer Tracts Nat. Phil. Vol. **35** (1989)

[11] S.M. Carroll, *Spacetime and Geometry,* An Introduction to General Relativity, Addison Wesley, San Francisco (2004).

[12] É. Cartan, *Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion*, C.R. Acad. Sci. (Paris) **174** (1922) 593–595; English translation by G.D. Kerlick: *On a generalization of the notion of Riemann curvature and spaces with torsion*, in: *"Proc. of the 6th Course of Internat. School on Cosmology and Gravitation: Spin, Torsion, Rotation, and Supergravity" (Erice, 1979) Eds. P.G.Bergmann and V.De Sabbata* (Plenum: New York, 1980) 489–491; with subsequent comments of A. Trautman, *Comments on the paper by Élie Cartan: Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion*, pp. 493–496.

[13] É. Cartan, *On Manifolds with an Affine Connection and the Theory of General Relativity*, English transl. of the French original by A. Magnon and A. Ashtekar, Bibliopolis, Napoli (1986).

[14] É. Cartan, *Riemannian Geometry in an Orthogonal Frame*, from lectures delivered in 1926/7, transl. from the Russian by V.V. Goldberg, World Scientific, New Jersey (2001).

[15] Y.M. Cho, *Einstein Lagrangian as the translational Yang-Mills Lagrangian*, Phys. Rev. **D14** (1976) 2521–2525.

[16] E. et F. Cosserat, *Théorie des corps déformables*, Hermann, Paris (1909), translated into English by D. Delphenich (2007).

[17] O. Costa de Beauregard, *Translational inertial spin effect*, Phys. Rev. **129** (1963) 466–471.

[18] T. Dereli and R.W. Tucker, *An Einstein-Hilbert action for axi-dilaton gravity in four dimensions*, Class. Quantum Grav. **12** (1995) L31–L36.

[19] M.W. Evans, *A generally covariant field equation for gravitation and electromagnetism*, Foundations of Physics Letters **16** (2003) 369–377.

[20] M.W. Evans, *The spinning and curving of spacetime: The electromagnetic and gravitational fields in the Evans field theory,* Foundations of Physics Letters **18** (2005) 431–454.

[21] T. Frankel, *The Geometry of Physics*, 2nd ed., Cambridge, UK (2004).

[22] A.A. García, F.W. Hehl, C. Heinicke and A. Macías, *Exact vacuum solution of a (1+2)-dimensional Poincaré gauge theory: BTZ solution with torsion,* Phys. Rev. D **67** (2003) 124016 (7 Pages) [arXiv:gr-qc/0302097].

[23] F. Gronwald, *Metric-affine gauge theory of gravity: I. fundamental structure and field equations*, Int. J. Mod. Phys. **D6** (1997) 263–303.

[24] F. Gronwald and F.W. Hehl: *Stress and hyperstress as fundamental concepts in continuum mechanics and in relativistic field theory.* In: 'Advances in Modern Continuum Dynamics', International Conference in Memory of Antonio Signorini, Isola d'Elba, June 1991. G. Ferrarese, ed. (Pitagora Editrice, Bologna, 1993) pp. 1–32; arXiv.org/abs/gr-qc/9701054.

[25] F. Gronwald and F.W. Hehl, *On the gauge aspects of gravity,* in *Proc. Int. School of Cosm. & Gravit.* 14$^{\text{th}}$ Course: Quantum Gravity. Held in Erice, Italy. Proceedings, P.G. Bergmann et al. (eds.). World Scientific, Singapore (1996) pp. 148–198; http://arxiv.org/abs/gr-qc/9602013.

[26] W. Günther, *Zur Statik und Kinematik des Cosseratschen Kontinuums,* Abh. Braunschw. Wiss. Ges. **10** (1958) 195–213.

[27] R.T. Hammond: *New fields in general relativity.* Contemporary Physics **36** (1995) 103–114.

[28] A.J. Hanson and T. Regge, *Torsion and quantum gravity*, in Lecture Notes in Physics (Springer) **94** (1979) 354–360.

[29] D. Hartley, *Normal frames for non-Riemannian connections*, Class. Quantum Grav. **12** (1995) L103–L105.

[30] K. Hayashi and T. Shirafuji, *New general relativity*, Phys. Rev. **D19** (1979) 3524–3553.

[31] F.W. Hehl, *Fermions and Gravity.* Colloque du Centenaire de la Naissance d'Albert Einstein au Collège de France, June 1979, pp.119–148. Edition du Centre National Recherche Scientifique, Paris (1980).

[32] F.W. Hehl, *Alternative gravitational theories in four dimensions,* in: Proc. 8th M. Grossmann Meeting, T. Piran, ed., World Scientific, Singapore (1998); arXiv:gr-qc/9712096 (10 pages).

[33] F.W. Hehl, *An assessment of Evans' unified field theory I,* Foundations of Physics, to be published (2007/8); arXiv:physics/0703116 (36 pages).

[34] F.W. Hehl and P. von der Heyde, *Spin and the structure of spacetime,* Ann. Inst. H. Poincaré **A19** (1973) 179–196.

[35] F.W. Hehl, P. von der Heyde, G.D. Kerlick, and J.M. Nester, *General relativity with spin and torsion: Foundations and prospects*, Rev. Mod. Phys. **48** (1976) 393–416.

[36] F.W. Hehl and E. Kröner, *On the constitutive law of an elastic medium with moment stresses (in German),* Z. f. Naturf. **20a** (1965) 336–350.

[37] F.W. Hehl and J.D. McCrea, *Bianchi identities and the automatic conservation of energy–momentum and angular momentum in general-relativistic field theories,* Foundations of Physics **16** (1986) 267–293.

[38] F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne'eman: *Metric-affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance*. Phys. Rep. **258** (1995) 1–171.

[39] F.W. Hehl and Yu.N. Obukhov, *Foundations of Classical Electrodynamics: Charge, Flux, and Metric* (Birkhäuser: Boston, MA, 2003).

[40] F.W. Hehl and Yu.N. Obukhov, *An assessment of Evans' unified field theory II,* Foundations of Physics, in press (2007/8); arXiv:physics/0703117 (11 pages).

[41] Y. Itin, *Energy-momentum current for coframe gravity,* Class. Quant. Grav. **19** (2002) 173–190 [arXiv:gr-qc/0111036].

[42] Y. Itin, *Noether currents and charges for Maxwell-like Lagrangians,* J. Phys. A: Math. Gen. **36** (2003) 8867–8883.

[43] Y. Itin, *Weak field reduction in teleparallel coframe gravity: Vacuum case,* J. Math. Phys. **46** (2005) 012501 (14 pages) [arXiv:gr-qc/0409021].

[44] F.A. Kaempfer, *Vierbein field theory of gravity,* Phys. Rev. **165** (1968) 1420-1423.

[45] M. O. Katanaev, *Geometric Theory of Defects,* Phys. Usp. **48** (2005) 675–701 [Usp. Fiz. Nauk **175** (2005) 705–733] [arXiv:cond-mat/0407469].

[46] T.W.B. Kibble, *Lorentz invariance and the gravitational field,* J. Math. Phys. **2** (1961) 212–221.

[47] R.M. Kiehn, *The many faces of torsion,* Preprint (22 pages) September 2007.

[48] H. Kleinert and S.V. Shabanov, *Spaces with torsion from embedding, and the special role of autoparallel trajectories,* Phys. Lett. **B428** (1998) 315–321. See also H. Kleinert: *Gauge Fields in Condensed Matter.* Vol.I: Superflow and

Vortex Lines. Vol.II: Stresses and Defects, World Scientific, Singapore (1989) and H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 4th ed., World Scientific, Singapore (2004); http://www.physik.fu-berlin.de/˜kleinert/re.html#B8 .

[49] K. Kondo, *On the geometrical and physical foundations of the theory of yielding,* in: *Proc. 2nd Japan Nat. Congr. Applied Mechanics, Tokyo,* pp.41–47 (1952).

[50] E. Kröner, *Kontinuumstheorie der Versetzungen und Eigenspannungen,* Erg. Ang. Math. (Springer) Vol. **5** (1958).

[51] E. Kröner, *Continuum theory of defects*, in: *Physics of Defects*, Les Houches, Session XXXV, 1980, R.Balian et al., eds., North-Holland, Amsterdam (1981) p. 215.

[52] E. Kröner, *The continuized crystal — a bridge between micro– and macromechanics*, Z. angew. Math. Mech. (ZAMM) **66** (1986) T284.

[53] E. Kröner, *A variational principle in nonlinear dislocation theory,* in: Proc. 2nd Int. Conf. Nonlin. Mechanics, Chien Wei-zang, ed., Peking University Press, Beijing (1993) pp.59–64.

[54] M. Lazar, *Dislocation theory as a 3-dimensional translation gauge theory*, Ann. Phys. (Leipzig) **9** (2000) 461–473; arxiv.org/abs/cond-mat/0006280.

[55] M. Lazar and C. Anastassiadis, *The gauge theory of dislocations: static solutions of screw and edge dislocations,* preprint, TU Darmstadt, October 2007. (38 pages).

[56] J.W. Maluf and J.F. da Rocha-Neto, *Hamiltonian formulation of general relativity in the teleparallel geometry*, Phys. Rev. **D64** (2001) 084014 (8 pages).

[57] C. Malyshev, *The Einsteinian T(3)-gauge approach and the stress tensor of the screw dislocation in the second order: avoiding the cut-off at the core*, J. Phys. **A 40** (2007) 10657–10684.

[58] L. Mangiarotti and G. Sardanashvily, *Connections in Classical and Quantum Field Theory,* World Scientific, Singapore (2000).

[59] Y. Mao, M. Tegmark, A. Guth and S. Cabi, *Constraining torsion with Gravity Probe B,* Phys. Rev. **D** (2007/8) to be published, arXiv:gr-qc/0608121.

[60] J.D. McCrea, F.W. Hehl, and E.W. Mielke, *Mapping Noether identities into Bianchi identities in general relativistic field theories of gravity and in the field theory of static lattice defects,* Int. J. Theor. Phys. **29** (1990) 1185–1206.

[61] H. Meyer, *Møller's tetrad theory of gravitation as a special case of Poincaré theory – a coincidence?,* Gen. Relat. Grav. **14** (1982) 531–548.

[62] R.D. Mindlin, *Micro-structure in linear elasticity,* Arch. Rat. Mech. Anal. **16** (1964) 51–78.

[63] A.V. Minkevich, *Gravitation, cosmology and space-time torsion,* arxiv.org/abs/0709.4337 (10 pages).

[64] C. Møller, *Conservation laws and absolute parallelism in general relativity*, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, no. 10 (1961) 1–50.

[65] U. Muench, *Über teleparallele Gravitationstheorien*, Diploma Thesis, University of Cologne (1997).

[66] F.R.N. Nabarro, *Theory on Crystal Dislocations*, Dover, N.Y. (1987).

[67] Y. Ne'eman and F.W. Hehl, *Test matter in a spacetime with nonmetricity,* Class. Quant. Grav. **14** (1997) A251–A260 [arXiv:gr-qc/9604047].

[68] J. M. Nester, *Gravity, torsion and gauge theory,* in: *Introduction to Kaluza–Klein theories*, H.C. Lee, ed. (World Scientific, Singapore (1984) pp.83–115.

[69] J. Nitsch and F. W. Hehl, *Translational gauge theory of gravity: post-Newtonian approximation and spin precession,* Phys. Lett. **B90** (1980) 98–102.

[70] J.F. Nye, *Some geometrical relations in dislocated crystals,* Acta metallurgica **1** (1953) 153–162.

[71] Yu.N. Obukhov, *Poincaré gauge gravity: Selected topics,* Int. J. Geom. Meth. Mod. Phys. **3** (2006) 95–138; [arXiv:gr-qc/0601090].

[72] Yu.N. Obukhov and J.G. Pereira, *Metric-affine approach to teleparallel gravity,* Phys. Rev. **D67** (2003) 044016 (17 pages).

[73] H.C. Ohanian and R. Ruffini, *Gravitation and Spacetime,* 2nd ed., Norton, New York (1994).

[74] C. Pellegrini and J. Plebanski, *Tetrad fields and gravitational fields*, Mat. Fys. Skr. Dan. Vid. Selsk. **2**, no. 4 (1963) 1–39.

[75] J. Polchinski, *String Theory Volume II, Superstring Theory and Beyond,* Cambridge University Press, Cambridge, UK (1998).

[76] V.N. Ponomariov and Yu. Obukhov: *The generalized Einstein-Maxwell theory of gravitation.* Gen. Relat. Grav. **14** (1982) 309–330.

[77] D.S. Popović and B. Sazdović, *The geometrical form for the string space-time action*, Eur. Phys. J. **C50** (2007) 683–689.

[78] D. Puetzfeld, *Status of non-Riemannian cosmology, New Astronomy Reviews* **49** (2005) 59-64.

[79] D. Puetzfeld and Yu.N. Obukhov, *Probing non-Riemannian spacetime geometry,* arXiv:0708.1926 [gr-qc] (4 pages).

[80] D. Puetzfeld and Yu.N. Obukhov, *Propagation equations for deformable test bodies with microstructure in extended theories of gravity*, Phys. Rev. **D76** (2007) 084025 (20 pages).

[81] R.A. Puntigam and H.H. Soleng, *Volterra distortions, spinning strings, and cosmic defects,* Class. Quant. Grav. **14** (1997) 1129–1149 [arXiv:gr-qc/9604057].

[82] M.L. Ruggiero and A. Tartaglia, *Einstein-Cartan theory as a theory of defects in space-time,* Am. J. Phys. **71** (2003) 1303.

[83] L.H. Ryder and I.L. Shapiro, *On the interaction of massive spinor particles with external electromagnetic and torsion fields,* Phys. Lett. **A247** (1998) 21–26. [arXiv:hep-th/9805138].

[84] A. Saa, *A geometrical action for dilaton gravity*, Class. Quantum Grav. **12** (1995) L85–L88.

[85] H. Schaefer, *Das Cosserat-Kontinuum*, ZAMM **47** (1967) 485–498.

[86] H. Schaefer, *Die Motorfelder des dreidimensionalen Cosserat-Kontinuums im Kalkül der Differentialformen,* Int. Centre for Mechanical Sciences (CISM), Udine, Italy, Courses and Lectures (L. Sobrero, ed.) No. **19** (60 pages) (1970).

[87] J. Scherk and J.H. Schwarz, *Dual models and the geometry of space-time,* Phys. Lett. **B52** (1974) 347–350.

[88] J.A. Schouten, *Ricci-Calculus*, 2nd ed., Springer, Berlin (1954).

[89] E. Schrödinger: *Space–Time Structure,* reprinted with corrections, Cambridge University Press, Cambridge (1960).

[90] E.L. Schucking, *The homogeneous gravitational field,* Foundations of Physics **15** (1985) 571–577.

[91] E.L. Schucking and E.J. Surowitz, *Einstein's apple: his first principle of equivalence*, arXiv:gr-qc/0703149 (30 pages).

[92] M. Schweizer, N. Straumann and A. Wipf, *Post-Newtonian generation of gravitational waves in a theory of gravity with torsion,* Gen. Rel. Grav. **12** (1980) 951–961.

[93] D.W. Sciama, *On the analogy between charge and spin in general relativity*, in: *Recent Developments of General Relativity*, Pergamon, London (1962) p.415.

[94] I. L. Shapiro, *Physical aspects of the space-time torsion,* Phys. Rept. **357** (2002) 113–213 [arXiv:hep-th/0103093].

[95] R.W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program,* Springer, New York (1997).

[96] J.H. Simpson, *Nuclear gyroscopes,* Astronautics & Aeronautics, October 1964, pp.42–48.

[97] A. Trautman, *Fiber bundles, gauge fields and gravitation*, in: *"General Relativity and Gravitation: One Hundred Years after the Birth of Albert Einstein" A.Held, ed.* (Plenum: New York, 1980) vol. **1**, 287–308.

[98] A. Trautman, *Differential Geometry for Physicists, Stony Brook Lectures* Bibliopolis, Napoli (1984).

[99] A. Trautman, *Einstein-Cartan theory,* in *Encyclopedia of Math. Physics,* J.-P. Francoise et al., eds., Elsevier, Oxford (2006) pp. 189–195; arXiv.org/gr-qc/0606062.

[100] R. Tresguerres, *Translations and dynamics*, arXiv:0707.0296 [gr-qc] (21 pages).

[101] R. Tresguerres and E.W. Mielke, *Gravitational Goldstone fields from affine gauge theory,* Phys. Rev. D **62** (2000) 044004 (7 pages); [arXiv:gr-qc/0007072].

[102] J. and J.A. Weertman, *Elementary Dislocation Theory*, MacMillan, London (1969).

[103] S. Weinberg, *Einstein's mistakes,* **58** (November 2005) 31–35.

[104] S. Weinberg, *Weinberg replies,* Physics Today **59** (April 2006) 15–16 (last paragraph of the reply); F.W. Hehl and S. Weinberg, *Note on the torsion tensor,* Physics Today **60** (March 2007) 16.

[105] R. Weitzenböck, *Invarianten-Theorie*, Noordhoff, Groningen (1923) 416 pp.

[106] D.K. Wise, *MacDowell-Mansouri gravity and Cartan geometry,* arXiv.org/gr-qc/0611154.

[107] P.B. Yasskin and W.R. Stoeger, *Propagation equations for test bodies with spin and rotation in theories of gravity with torsion*, Phys. Rev. **D21** (1980) 2081–2094.

[108] A. Zeghadi, S. Forest, A.-F. Gourgues, O. Bouaziz, *Cosserat continuum modelling of grain size effects in metal polycrystals,* PAMM – Proc. Appl. Math. Mech. **5** (2005) 79–82.

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