Virtually algebraically fibered congruence subgroups
Geometry & Topology Online
Warwick, April 2020

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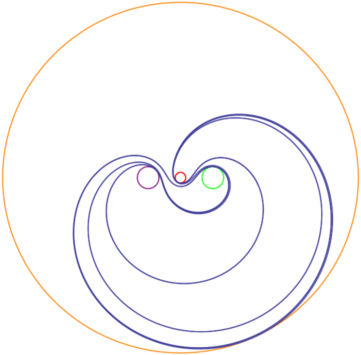


Algebraic fibering

Definition

A group G is called *algebraically fibered* if there is a homomorphism

$$\varphi : G \to \mathbb{Z}$$

with $ker(\varphi)$ finitely generated.

If G is algebraically fibered, then G is also finitely generated.

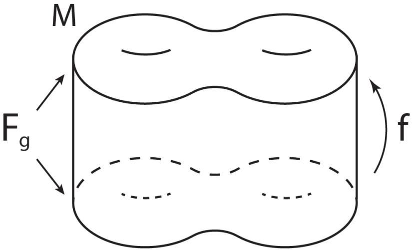
The simplest class of examples of algebraically fibered groups are groups $H \times \mathbb{Z}$, H finitely generated.

Fibering of manifolds

A manifold M *fibers over the circle* S^1 if there is a homeomorphism $f : N \to N$ so that M is homeomorphic to the mapping torus T_f

$$M \cong T_f = \frac{N \times [0,1]}{\{(x,0) \sim (f(x),1)\}}.$$

E.g. if M is 3-dimensional and closed, and $N = F_g$ a surface of genus g.



Fibered implies algebraically fibered

If M is compact, connected, and fibers over S^1, then the fiber N is compact (we may assume connected), and there is a short exact sequence

$$\pi_1(N) \to \pi_1(M) \to \mathbb{Z}.$$

Hence $\pi_1(M)$ is algebraically fibered.

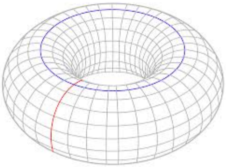
In 1961 John Stallings proved the converse if M is 3-dimensional (compact, irreducible): if $\pi_1(M)$ is algebraically fibered, then M fibers over S^1 with fiber a surface.

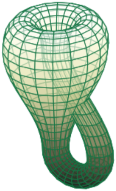
Remark: This does not generalize to higher dimensions without further restrictions on the fundamental group and higher homotopy groups. Thus the notion of algebraic fibering is not a direct analogue of a manifold fibering over S^1.

Fibered 2-dimensional manifolds

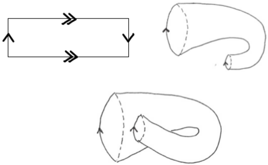
It may be helpful to first consider the 2-dimensional case: which surfaces fiber over the circle?

The only examples of connected surfaces (without boundary) that fiber are the torus and Klein bottle:



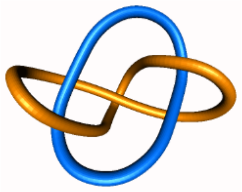


The Klein bottle is the mapping torus of the reflection:



Fibered 3-manifolds

- The Figure 8 knot complement and Whitehead link complements are fibered 3-manifolds (the fiber is a compact surface with boundary obtained by Seifert's algorithm).

- M is **virtually fibered** if there exists a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ fibers
- Thurston asked whether every hyperbolic 3-manifold is virtually fibered?
- This was answered affirmatively in 2012 by myself and Dani Wise.

Rational derived subgroup

Virtual fibering follows from a stronger group-theoretic condition.

Let Γ be a finitely generated group with commutator subgroup $\Gamma^{(1)} = [\Gamma, \Gamma]$ and abelianization

$$\Gamma/\Gamma^{(1)} \cong H_1(\Gamma; \mathbb{Z}).$$

We then define the the *rational commutator subgroup*

$$\Gamma_r^{(1)} = \ker\left(\Gamma \to H_1(\Gamma; \mathbb{Q})\right).$$

Clearly $\Gamma^{(1)} \leq \Gamma_r^{(1)}$ is finite index and $\Gamma/\Gamma_r^{(1)} \cong H_1(\Gamma; \mathbb{Z})/\mathrm{Torsion}$.

RFRS condition

Given a group Γ, let $\{\Gamma_j\}$ be a cofinal tower of finite index subgroups of Γ with $\Gamma_0 = \Gamma$. In other words,

1. $\bigcap \Gamma_j = \{1\}$;
2. Γ_j is a finite index subgroup of Γ;
3. $\Gamma_{j+1} \leq \Gamma_j$ for all j.

We say that $\{\Gamma_j\}$ is a *RFRS tower* if, in addition,

$$(\Gamma_j)_r^{(1)} \leq \Gamma_{j+1} \text{ for all } j \geq 0. \qquad (\star)$$

RFRS implies virtual fibering

Theorem (A. 2007)

If M^3 is closed and aspherical and $\pi_1(M)$ is RFRS, then M virtually fibers.

Theorem (Friedl-Vidussi 2017)

If Γ is a Kähler group, and Γ is RFRS, then either Γ is a surface group, or Γ_j is algebraically fibered for some j.

A Kähler group is the fundamental group of a Kähler manifold, such as a smooth projective algebraic variety (over $\mathbb{C}$).

Theorem (Kielak 2018)

If Γ is RFRS, and $b_1^{(2)}(\Gamma) = 0$, then Γ_j algebraically fibers for some j.

Under these hypotheses, $b_1^{(2)}(\Gamma) = 0$ is essentially equivalent to $\inf_j b_1(\Gamma_j)/[\Gamma : \Gamma_j] = 0$. This condition was known to hold for many groups.

Fibered congruence covers

Consider a subgroup $\Gamma < GL_n(\mathbb{Z}) \subset M_n(\mathbb{Z})$ ($n \times n$ integral matrices), and let p be a prime number. Let $\Gamma(p^j) = \Gamma \cap (I + p^j M_n(\mathbb{Z}))$. This is a principal congruence subgroup of level p^j (if Γ is a congruence arithmetic group).

Then for $j > 0$, $\Gamma(p^j)/\Gamma(p^{j+1}) \cong \mathbb{Z}/p\mathbb{Z}^k$ for some k, an elementary abelian p-group. In particular, $\Gamma(p^j)^{(1)} \leq \Gamma(p^{j+1})$.
Baker and Reid asked when is this a RFRS sequence? I.e. when is $\Gamma(p^j)^{(1)}_r \leq \Gamma(p^{j+1})$?

Short summary of answer:

Theorem (Agol-Stover 2019)

This is (essentially) true if $H_1(\Gamma(p^j); \mathbb{Z})$ has no p-torsion for some j.

3-dimensional examples

The fundamental group of the "Magic manifold" is a principal congruence subgroup.
$\Gamma(\frac{1+\sqrt{-7}}{2})$ inside $\Gamma = \mathrm{PGL}_2(\mathbb{Z}(\frac{1+\sqrt{-7}}{2}))$ This is essentially $O(1,1,1,-7;\mathbb{Z}) < GL(4,\mathbb{Z})$ and the level 2 congruence subgroup.

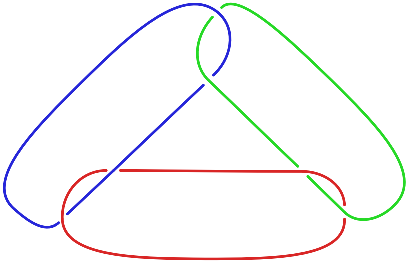


Figure: The magic manifold is the complement of the 3-chain link.

Principal congruence links

More generally, principal congruence link complements were classified by Baker-Goerner-Reid, and occur for discriminants $d = 1, 2, 3, 5, 7, 11, 15, 19, 23, 31, 47, 71$ (principal congruence subgroups of $O(1, 1, 1, -d; \mathbb{Z})$, essentially a Bianchi group).

In the appendix to our paper, Mehmet Sengun computes many more examples where the condition holds, i.e. principal congruence subgroups of Bianchi groups of level p which have no p-torsion.

4-dimensional examples

The group $\Gamma = O(1,1,1,1,-1;\mathbb{Z}) < GL_5(\mathbb{Z})$ is a lattice acting discretely on hyperbolic 4-space.
It is known that Γ is the group generated by reflections in the simplex in hyperbolic 4-space with Coxeter diagram given by

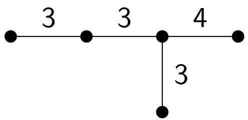


Figure: The Coxeter diagram for $O(4,1;\mathbb{Z})$.

One can check using Magma that $H_1(\Gamma(4);\mathbb{Z}) \cong \mathbb{Z}^{55}$. In particular, it has no 2-torsion. Hence $\Gamma(2^j)$ is algebraically fibered for some j.
Moreover, for $d \not\equiv -1 \pmod 8$, $O(1,1,1,-d;\mathbb{Z})$ is virtually fibered for a congruence subgroup by embedding in Γ.

A 4-dimensional surface

Let $E = \mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive 5^{th} root of unity, and $F = \mathbb{Q}(\sqrt{5})$ be its totally real quadratic subfield. Define $\phi = \frac{1-\sqrt{5}}{2}$ and consider the hermitian form on E^3 with matrix

$$h = \begin{pmatrix} \phi & 1 & 0 \\ 1 & \phi & 1 \\ 0 & 1 & \phi \end{pmatrix}.$$

Let $\Gamma = PU(h, \mathcal{O}_E)$, then this is a lattice defined by Deligne-Mostow acting on complex hyperbolic 4-space. (also moduli space of euclidean cone metrics on S^2 with 5 cone points of angle $8\pi/5$)

Lemma

With notation as above, the congruence subgroup of level $\sqrt{5}\mathcal{O}_E$ in Γ is the commutator subgroup of Γ and its abelianization is isomorphic to $\mathbb{Z}^{60}$.

Hence $\Gamma(\sqrt{5}^j)$ is algebraically fibered for some $j > 0$. This is the first known example of a Kähler RFRS group that is not (virtually) a product of curves.

Conclusion

- Do congruence arithmetic hyperbolic 3-manifolds have a congruence cover which has positive betti number (or stronger fibers over the circle)? This question is of intense investigation by number theorists, and would follow from Langlands functoriality or other number-theoretic conjectures.
- Is there a principal congruence cover of level p which has no p-torsion in H_1? In particular, does this hold for all Bianchi groups?

  Steven Tschantz showed that for $\Gamma = O(1,1,1,1,-1;\mathbb{Z})$, $\Gamma(4) \cong \mathbb{Z}^{256} \times \mathbb{Z}/2\mathbb{Z}$. So our method fails in this case.