QFT Kremniger
1st week Wed 2/6/08

"QFT on Curved Spacetime"
Wald
Hollands - Wald.
Brunetti - Fredenhagen - Verch
Verch.
Weinberg.
Haag.
Araki

States and Observables
Prob. description:

① physical system $\alpha_1, \alpha_2 \cdots$
② measuring devices $Q_1, Q_2, \cdots$
③ observer
④ environment $\Big\}$ ignore

Def $w_\alpha^Q(q) = \lim_{N\to\infty} \frac{n_q}{N}$ n_q is the number of times q occured in N trials.

w_α^Q is a prob. measure on $\mathbb{R}$.
$\sum_q w_\alpha^Q(q) = 1$ $\quad 0 \le w_\alpha^Q(q) \le 1$.

Let Σ be the set of all measured objects.

Def $\alpha_1 \sim \alpha_2$ iff $w_{\alpha_1}^Q(q) = w_{\alpha_2}^Q(q)$ for all Q, q.

an equivalence class will be called a <u>state</u>

Let A be the set of measuring devices we define $Q_1 \sim Q_2$ s.t. ... equiv class = observable.

Given a function $f: \mathbb{R} \to \mathbb{R}$
define $f(Q)$

$$w_\alpha^{f(Q)}(q') = \sum_{(q \mid f(q) = q')} w_\alpha^Q(q).$$

when we have a fin set of observables and Q s.t. $Q_i = f_i(Q)$ $Q_1, \ldots, Q_n$.

then $\{Q_i\}_{i=1,\ldots,n}$ simultaneous observables.

<u>Expectation value</u> $\alpha(Q) = \sum q\, w_\alpha^Q(q)$

if $Q_1, \ldots, Q_n$ simultaneously observable

$$\alpha(\underbrace{c_1 Q_1 + \cdots + c_n Q_n}_{\text{function of several observables}}) = c_1 \alpha(Q_1) + \cdots + c_n \alpha(Q_n)$$

Mixture and pure states

$\alpha_1, \ldots, \alpha_n$ if $w_\alpha^Q(q) = \sum_{k=1} \lambda_k w_{\alpha_k}^Q(q)$ $\lambda_k \geq 0$

$(\alpha(Q) = \lambda_1 \alpha_1(Q) + \cdots + \lambda_n \alpha_n(Q))$ $\sum \lambda_k = 1$

α is said to be a mixture of $\alpha_1, \ldots, \alpha_n$

<u>Assume</u>: for any $\alpha_1, \ldots, \alpha_n$ any mixture is also a state.

We call α a pure state if it is not a nontrivial mixture of other states

An observable Q is called a classical observable if for any pure state

$$\omega_\alpha^Q(q) = \begin{cases} 0 & q \neq q_\alpha \\ 1 & q = q_\alpha \end{cases}$$

if all observables are classical then we get classical physics: the set of all pure states; observables are functions on phase space

general state is given by a measure on phase space $\alpha(Q) = \int Q(w)\, d\alpha(w)$

Hilbert space formulation QM

Axiom: states can be identified with the set of all positive trace class operators ρ on a Hilbert Space H. (separable) $tr(\rho) = 1$

observables are identified with the set of $B_{sa}(H)$ bounded self-adjoint. and $\rho(A) = tr(\rho A)$

$\|A\| = \{\sup \|Av\| \mid \|v\| \le 1\}$

we call A positive if $A = B^* B$ for some bounded B and $A^{1/2}$ exists.

Kreimer QFT
2/8/06

Hilbert space formulation

H sep. Hilb. space

states = set of pos trace class operators ρ s.t. $tr(\rho)=1$.

observables := $B_{sa}(H)$ bounded self-adjoint

expectation $\rho(a) = tr(\rho a)$.

pos operator

$\{v_i\}$ ONB of H.

$tr : B_+(H) \to [0,\infty]$

$tr(A) = \sum_i \langle A v_i, v_i \rangle$

linear

$tr(BB^*) = tr(B^*B)$

$tr(U^*AU) = tr\, A$

$U \in U(H)$

Def A bounded operator A is said to be trace class if

$\|A\|_{tr} = tr\left((A^*A)^{1/2}\right) < \infty$ $\quad T(H)$ is a linear space

$T(H)$ is complete w.r.t. $\|\cdot\|_{tr}$ (Banach space)

if $T \in T(H)$ $\quad A \in B(H)$ $\quad AT, TA \in T(H)$

$\|AT\|_{tr} \leq \|A\| \, \|T\|_{tr}$.

~~T(H) is a~~ $T \in T(H) \Rightarrow T^* \in T(H)$

$T(H)$ is $*$-bi ideal in $B(H)$.

Let $F(H) :=$ (finite) $\{ F \in B(H) \mid \dim \operatorname{Im} F < \infty \}$

$T(H)$ is completion of $F(H)$ w.r.t. $\|\cdot\|_{tr}$

compact $\Rightarrow$ $C(H)$ completion of $F(H)$ w.r.t $\|\cdot\|$.

<u>Claim</u>: $T(H) = C(H)^*$.

$$T(H)^* = B(H)$$

$$\left(B(H)^* = T(H) \oplus C(H)^\perp \right)$$

All but $C(H)$ are von Neumann algebras.

States $\equiv \{ \rho \in T(H) \mid \rho \text{ pos } \operatorname{tr}(\rho) = 1 \}$

"density matrices / statistical ops"

$$\rho = \sum \lambda_i P_{\psi_i} \qquad \lambda_i \in [0,1]$$

$$\sum \lambda_i = 1$$

$\{\psi_i\}$ ONBasis $\qquad P_{\psi_i} = |\psi_i\rangle\langle\psi_i|$

$\rightarrow$ We see that P_v $\|v\| = 1$ this is a pure states. The space of states is the closed convex hull of the pure states.

Observables $(B_{sa}(H))$

$B(H)$ is a *-alg

$\|AB\| \le \|A\|\,\|B\|$

$\|I\| = 1$

Banach alg.

$\|A^*A\| = \|A\|^2$

$\downarrow$

C^*-algebra.

$B_{sa}(H)$ is an $\mathbb{R}$ vs.

$B(H)$ is a complexification of $B_{sa}(H)$,

$AB - BA$

$B^*A^* - A^*B^*$

$BA - AB$

is closed under $[\,,\,]$ $[\,,\,]_+$

$\cancel{AB}(AB - BA)i$ $(AB + BA)$

Def A complex valued function on $B(H)$

φ is called a state if:

① linear

② positive $\varphi(a^*a) \ge 0$

③ normalized $\varphi(1) = 1$.

a state is called normal if for any bounded increasing seq A_n of positive operators,

$\varphi(\sup A_n) = \sup \varphi(A_n)$

Thm $\rho(A) = \mathrm{Tr}(\rho A)$ $\rho \in T(H)$ $tr(\rho) = 1$

is a normal state

Any normal state is of this form

~~Alg. formulation of QM.~~

A physical system is characterized by a C^*-algebra A.

Observables are the self-adjoint elts of A, states are positive linear functional on A $\varphi(1)=1$ (possibly normal).

Gelfand-Naimark-Segal construction of rep

given a state φ we can define a map
$\langle\cdot,\cdot\rangle_\varphi : A \otimes \bar{A} \longrightarrow \mathbb{C}$.

by $\langle A, B\rangle_\varphi = \varphi(B^* A)$

$\langle A, B\rangle_\varphi = \overline{\langle B, A\rangle_\varphi}$ $\quad \forall B,\ \langle B, B\rangle_\varphi \geq 0.$

$|\langle A, B\rangle_\varphi|^2 \leq \langle A, A\rangle_\varphi \langle B, B\rangle_\varphi.$

Def: a representation of a C^*-alg A on a hilbert space H is a map $A \to B(H)$ of C^*-alg.

To any state φ on a C^*-alg. A corresponds a map π_φ on a Hilb H_φ. and a vector $v \in H_\varphi$ st $\varphi(a) = \langle \pi_\varphi(a) v, v\rangle$

v is cyclic ($\pi_\varphi(A)\cdot v$ is dense in H_φ)
This rep is unique up to equiva

2/10/08
Kronniger QFT

GNS - given a state on A
we can define
$\langle a, b \rangle_\varphi = \varphi(b^* a)$

(it can happen that
$\langle a, a \rangle = 0$ $a \neq 0$.

Def a rep of a C^*-alg A
on H is a map
$A \to B(H)$ of C^*-algs.

Thm To every state φ on A corresponds a rep.
π_φ on a Hilb. space H_φ
$v \in H_\varphi$ st.
① $\varphi(a) = \langle \pi_\varphi(a) v, v \rangle$

② v is cyclic for this rep.
$\{\pi_\varphi(a) v\}_{a \in A}$ is dense in H_φ

This rep is unique up to unitary equiv.

Proof of existence define $K_\varphi = \{a \in A \mid \varphi(a^* a) = 0\}$

from Cauchy ineq we can see that
$K_\varphi = \{a \in A \mid \varphi(b^* a) = 0 \quad \forall b \in A\}$

hence K_φ is a left ideal, closed in A.

A/K_φ is a left A-module with a pre-Hilbert space structure.
Define H_φ to be completion of A/K_φ

vac is the image of $1 \in A$ inside $A/K_\varphi \hookrightarrow H_\varphi$.

A rep is called irred if
① $0, H$ are only A-stable subspace.

commutant

$$\Updownarrow \qquad \pi(A)'' = B(H)$$

$$\Updownarrow$$

$$\pi(A)' = \mathbb{C} \cdot Id.$$

Claim: π_φ is irred $\Longleftrightarrow$ φ is a pure state.

States on a comm C^*-alg.

Suppose A is comm C^*-alg.
Claim! the following are equivalent. $\varphi \in A^*$
① φ is a pure state
② φ is mult $\varphi(ab) = \varphi(a)\varphi(b)$.

Proof Suppose φ is a state

$$\begin{array}{ccc} \varphi \text{ is pure} & \Longleftrightarrow & \varphi \text{ is mult.} \\ \Updownarrow & & \Updownarrow \\ H_\varphi \text{ is irred} & \Longleftrightarrow & H_\varphi \text{ is 1-dim} \end{array}$$

∎

<u>Def</u> Let A be a comm C^*-alg.
A character is a mult (nonzero) lin map $\varphi : A \to \mathbb{C}$.
$Spec(A) = \{$set of all characters$\}$

$= \{$set of all pure states$\}$

<u>Thm</u>: Let A be a comm C^*-alg
$X = Spec(A)$ with the weak*-top.

$$N_{\varphi, A_1, \dots, A_n} = \{ \varphi' \mid |\varphi(A_i) - \varphi'(A_i)| < \varepsilon \}$$

X is a loc compact Hausdorff space
~~Haus~~ $A \simeq C_0(X)$ $\times$ closure of compactly supp func.
X is compact $\iff$ A has a unit.

$$a \in A \mapsto \hat{a}(\varphi) = \varphi(a)$$

<u>Th</u> $\left\{ \begin{matrix} \text{loc compact} \\ \text{Hausdorff spaces} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{comm } C^* \\ \text{algebras} \end{matrix} \right\}^{op}$

<u>Cor</u> mixed states on $A \iff$ prob measures on X.

<u>Classical ~~mechanics~~ Quantization</u> X phase space
symp. mfld.

$C_0(X)$ is a poisson algebra.

$A \otimes A \xrightarrow{\{,\}} A$ Lie bracket and biderivation.

antisymm bivector field B on X

$$\{f,g\} = B(df, dg).$$

Suppose A_0 a comm C^*-alg. with a Poisson structure.
By quantization we mean: I interval containing 0 for $\hbar \in I$ we have $A_\hbar$ a C^*-alg.

$$Q_\hbar : A_0 \to A_\hbar \qquad Q_0 = Id$$

① $\forall f$ $\hbar \mapsto \| Q_\hbar(f) \|$ is cont.

② $\lim_{\hbar \to 0} \| Q_\hbar(f) Q_\hbar(g) - Q_\hbar(fg) \| = 0.$

③ $\lim_{\hbar \to 0} \| [Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f,g\}) \| = 0.$

④ for each $\hbar \in I$ $\{ Q_\hbar(f) \mid f \in A_0 \}$ is dense in $A_\hbar$.

we call it strict if $Q_\hbar(A_0)$ is closed under multiplication; nondegen if $\operatorname{Ker} Q_\hbar = 0.$

Kremnizer 02/13/08
QFT

$(\mathbb{R}^{2n}, \omega)$ $C^\infty(M)$ Poisson algebra

$a \in C^\infty(\mathbb{R}^{2n}) \mapsto \hat{a} \in \mathcal{B}(H)$

$[\hat{a}, \hat{b}] = i\hbar \widehat{\{a,b\}}$

Schrödinger $H = L^2(\mathbb{R}^{2n})$ $\quad x \to x$
$\quad p \to i\hbar \frac{\partial}{\partial x}$

<u>No go theorem:</u> This is impossible

<u>Deformation quantization $*$-product</u>

A_0 poisson algebra

$* : A_0[[\hbar]] \hat{\otimes} A_0[[\hbar]] \to A_0[[\hbar]]$ linear over $\mathbb{C}[[\hbar]]$

$a, b \in A_0$

$$a * b = ab + \sum m_i(a,b)\hbar^i$$

$$\frac{a*b - b*a}{\hbar} \bmod \hbar = \{a,b\}_{A_0}$$

<u>Categories</u>: Poisson Category

ob Poisson = Poisson Manifolds.

Mor Poisson = $X \xleftarrow{} S \xrightarrow{} Y$

C^*-alg ob C^*-alg = C^* algebras.

Mor $C^*(A,B) = \{$ Hilbert A-B bimod $\}$

E is a Hilbert bimodule
left A-module, right B module
$\langle\ ,\ \rangle : E \times E \to B$
$\langle a, a\rangle \geq 0$

E A-B bimod, F B-C bimod $\to E \otimes_B F$ A-C bimodule

C^*-alg $\to KK$-cat. Conj: such a functor exists
Quant: Poisson $\to KK$

$A_0 = C_0(X)$ a family of C^*-alg.

$A_\hbar$ $\hbar \in I$ $Q_\hbar : A_0 \to A_\hbar$

1. cont: $\hbar \mapsto \|Q_\hbar(f)\|$ is cont
2. $\lim_{\hbar \to 0} \|Q_\hbar(f)Q_\hbar(g) - Q_\hbar(fg)\|$
3. $\lim_{\hbar \to 0} \|[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\})\|$
4. $\{Q_\hbar(f)\}$ is dense in $A_\hbar$.

Strict: $Q_\hbar(A_0)$ is closed under multiplication nondeg, $\operatorname{Ker} Q_\hbar = 0$.

How to construct quantizations

Def A cont family of C^*-algebras over a locally compact Hausdorff space

(Think of a map $C_0(X) \hookrightarrow Z(A)$)

$$\left\{ A, \{A_x, \varphi_x\}_{x \in X} \right\}$$ field/family

A is a C^*-alg, A_x are C^*-alg
surj mor $\varphi_x : A \to A_x$ s.t.

① $x \mapsto \|\varphi_x(a)\|$ is in $C_0(X)$
for all $a \in A$

② $\|a\| = \sup_{x \in X} \|\varphi_x(a)\|$

③ A is a left $C_0(X)$-module and $\varphi_x(fa) = f(x)\varphi_x(a)$.

(related to gerbes over X).

<u>Thm</u>: Suppose we have a strict quantization of a Poisson alg A_0
Assume $\hbar \mapsto \|Q_\hbar(f)\|$ is in $C_0(\mathbb{X})$
assume, $A_\hbar$ are identified with each other
$\hbar \mapsto Q_\hbar(f)$ are continuous
$\exists!$ cont field of C^*-alg A
whose sections contains all of $\{Q_\hbar(f)\}_\hbar$.

Coherent states and Berezin Quantization
<u>Def</u> Given a cont field of C^*-alg $\{A, \{A_x, \varphi_x\}\}$

QFT 02/15/08

Kronnnizr

H hilbert space (X,μ) $x \in X \mapsto v_x \in H$ unit vector

$|x\rangle$ ← "coherent states"

$$Id_H = \int_X |x\rangle\langle x|$$

$A \in B(H)$ get $\hat{A}(x) = \langle x | Ax \rangle$

$f \to A_f = \int_X f\,|x\rangle\langle x|$ $\quad H \rightleftarrows L^2(X)$

$K(x,y) = \langle x | y \rangle$

Proj. Hilbert space $\mathbb{P}H = SH/U(1)$
This will have quotient topology.

$\psi \in \mathbb{P}H$ $\quad \psi \in SH$

$$N_\psi = \{\varphi \in \mathbb{P}H \mid \langle \varphi, \psi \rangle \neq 0\}$$

$$F_\psi : N_\psi \to \psi^\perp$$

$$F_\psi(\varphi) = \frac{\varphi}{\langle \psi, \varphi \rangle} - \psi = \frac{[\psi]^\perp \varphi}{\langle \psi, \varphi \rangle}$$

$F_\psi(\psi) = 0$ and gives a homeo to the image
So $\mathbb{P}H$ is a hilbert manifold.

$TH = H \times H$ $\quad \varphi \in H \mapsto V(\varphi)_\psi f = \frac{d}{dt} f(\psi + t\varphi)_{t=0}$

H is symp. $\omega(V(\varphi), v(\psi)) = 2\hbar\, \mathrm{Im}\langle\varphi,\psi\rangle$

<u>Remark</u> $A \in B_{s.a.}(H)$ $\quad \hat{A} \in C^\infty(H, \mathbb{R})$

$$\tilde{A}(\psi) := \langle \psi, A\psi \rangle$$

<u>Poisson:</u> $\{\tilde{A}, \tilde{B}\} = \frac{i}{\hbar}\widetilde{[A,B]}$

The vector field corr to $\tilde{A}$ is

$$-V\left(\frac{i}{\hbar}A\psi\right)$$

Schrödinger eqn for $A \longleftrightarrow$ flow along Hamiltonian vector field.

If $V(\varphi)$ is tang to $S(H)$
we get a vector in $T\mathbb{P}H$ $\quad v(\varphi)$
$\psi \in \mathbb{P}H$ $\quad T_\psi \mathbb{P}H = \{ v(iA\psi) \mid A \in B_{sa}(H) \}$

$\mathbb{P}H$ is also symplectic $\quad \omega_\psi(v(iA\psi), v(iB\psi))$
$= -i\hbar\,\widetilde{[A,B]}(\psi)$

(Schrödinger = hamiltonian flow)

<u>Notation</u> if $\rho, \sigma \in \mathbb{P}H$ $\quad p(\rho,\sigma) = |\langle \tilde{\rho}, \tilde{\sigma}\rangle|^2$

<u>Def:</u> A pure state quant. of a symp. mfld S is:
for each $\hbar \in I$ a sep. Hilb space $H_\hbar$
a smooth inj $q_\hbar : S \to \mathbb{P}H_\hbar$ and a measure
$\mu_\hbar$ on S.

① for all $\hbar \in I$ and all $\psi \in \mathbb{P}\mathcal{H}_\hbar$

$$\int_S d\mu_\hbar(s)\, p(q_\hbar(s), \psi) = 1$$

② for all $f \in C_0(S)$

$$\hbar \mapsto \int_S d\mu_\hbar(\sigma)\, p(q_\hbar(\rho), q_\hbar(\sigma))\, f(\sigma)$$

is continuous

$$\lim_{\hbar \to 0} * = f(\rho)$$

③ $\lim_{\hbar \to 0} q_\hbar^* \omega_{\mathbb{P}\mathcal{H}} = \omega_S$

Berezin quantization $\{\mathcal{H}_\hbar, q_\hbar, \mu_\hbar\}$

pure state quant of S.

$f \in L^\infty(S)$

is the following family $\{Q_\hbar^B(f)\}_{\hbar \in I}$

$Q_\hbar^B(f) \in \mathcal{B}(\mathcal{H}_\hbar)$

projection

$$Q_\hbar^B(f) = \int_S d\mu_\hbar(\sigma)\, f(\sigma)\, [q_\hbar(\sigma)]$$

Coherent states: Def: A pure state quant is said to be coherent if $q_\hbar(\sigma) \in \mathbb{P}\mathcal{H}_\hbar$ can be lifted to $\sigma \mapsto \psi_\hbar^\sigma$ a unit vector in $\mathcal{H}_\hbar$.

Fock Space: for any hilbert space H we can define the symmetric Fock space to be $\exp(H) = \bigoplus_{l=0}^{\infty} S^l(H)$

$S^l(H)$ is the invariant subspace of $H^{\otimes l}$ with respect to $\mathfrak{S}(l)$