# Pointed fusion categories over non-algebraically-closed fields

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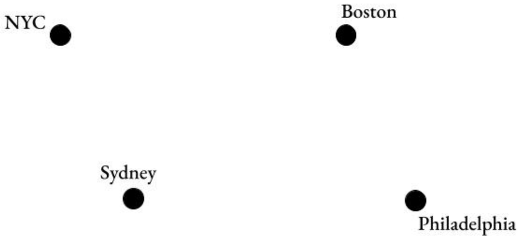
October 15, 2022
MIT PRIMES Conference

# Category of Cities & Trains



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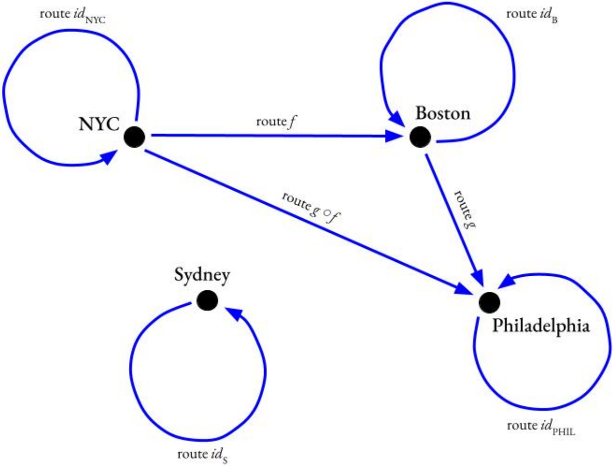


- objects: **NYC, Boston, Phil., Sydney**
- morphisms: **routes**
  - $f$
  - $g$
  - $g \circ f$
  - $id_{NYC}$
  - $id_B$
  - $id_S$
  - $id_{PHIL}$

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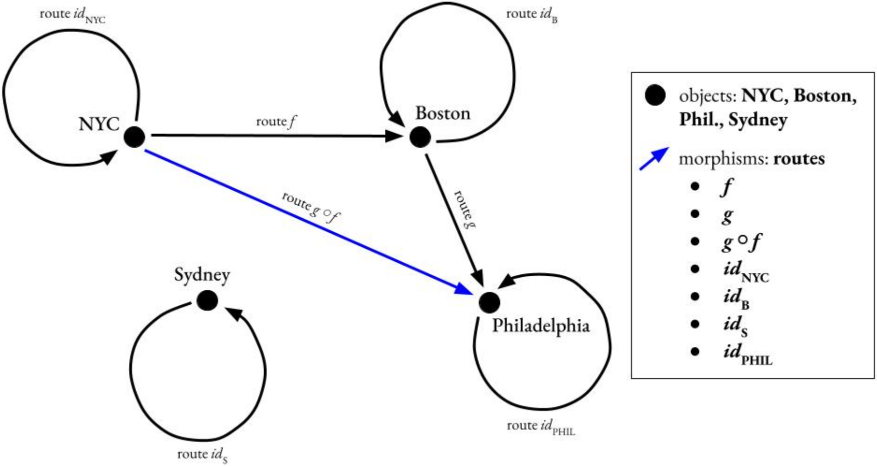
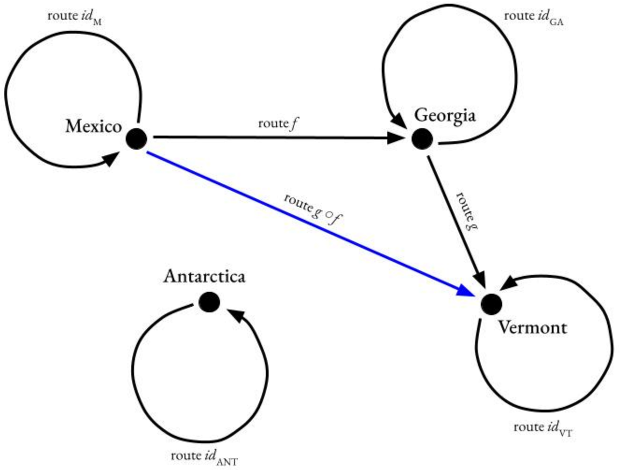


# Category of Butterfly Locations & Migration Patterns




- objects: **Mexico, GA, VT, Antarctica**
- morphisms: **routes**
  - $f$
  - $g$
  - $g \circ f$
  - $id_M$
  - $id_{GA}$
  - $id_{VT}$
  - $id_{ANT}$

# Why study categories?

- If we prove a statement about categories of locations & transportation between them in general, then that statement holds for cities & train routes, butterflies & migration, and more.
- Categories provide a consistent and unifying language for mathematics.

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- Categories provide a consistent and unifying language for mathematics.
- If we prove that a certain type of category always has property A, then we instantly learn about a huge crop of mathematical objects.

*Objects in the category A* always satisfy **Property B** $\longrightarrow$ $\mathbb{Z}, \mathbb{R}$, and *the permutations of a Rubik's cube* (objects in the category of groups) satisfy the **Isomorphism Theorems**

# Formal Definition of a Category

## Definition

A **category** $\mathcal{C}$ consists of

- a set of objects $\mathrm{Ob}(\mathcal{C})$ and
- for any two objects $A$ and $B$, a set of morphisms $\mathcal{C}(A, B)$,

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## Example

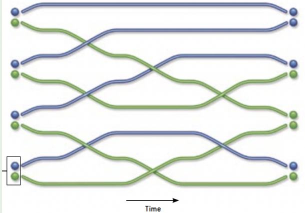
Let $k$ be a field. The category $k$-**Vec** consists of

- finite-dimensional $k$-vector spaces as objects. For instance, $\{0\}$ and $k^2$ are objects.
- $k$-linear maps as morphisms. A map $f$ from an $k$-vector space $V$ to an $k$-vector space $W$ is *$k$-linear* if $c \cdot f(x) = f(cx)$ for any $c \in k$.
  - For instance, the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(a, b) = (2a, 2b)$ is an $\mathbb{R}$-linear map.

# Why study fusion categories?

## Example (Topological Quantum Computing)

- Quantum computers use *qubits*, which can be represented by *anyons*. Operations on these qubits are performed by *braiding* anyons.
- Anyons can be represented as objects in a category.




- Anyons come in pairs; we need each object $X$ to have an "inverse."
- Two anyons can fuse into a new anyon; we need operations $\oplus$ and $\otimes$.

$$X \otimes Y := 2Z \oplus 3W$$

tells us that $\frac{2}{5}$-ths of the time, they fuse into the $Z$ anyon; otherwise, they fuse into the $W$ anyon.

## Question

Anyon systems can be represented by **fusion categories**. Can we classify fusion categories over any field?

An **abelian category** is essentially a category whose objects and morphisms can be "added" with $\oplus$.

**Definition (vague)**

- In a category, a nonzero object $X$ is **simple** if it has no subobjects except the zero object or itself.
- An abelian category is **semisimple** if every object is the direct sum of finitely many simple objects.

Simple objects provide a "basis" for semisimple categories.

The category $k$-$\mathbf{Vec}_G$ is nearly identical to $k$-$\mathbf{Vec}$, except its objects are now $G$-**graded**.

- Let $G$ be a group. The objects are still finite-dimensional $k$-vector spaces $V$; now, they are also equipped with a decomposition

$$V = \oplus_{g \in G} V_g,$$

  where $V_g$ are subspaces of $V$.
- For instance, when $k = \mathbb{R}$ and $G = (\{0, 1\}, +_2)$, an object could be $\mathbb{R} \oplus \mathbb{R}^2$.

| Index $g$ in $G$ | 0 | 1 |
|---|---|---|
| Subspace $V_g$ | **R** | $\mathbf{R}^2$ |

# Example of a Semisimple Abelian Category: $k$-$\mathbf{Vec}_G$

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- Its simple objects are $\delta_g$ for all $g \in G$, defined as

$$(\delta_g)_h = \begin{cases} 0 & \text{if } g \neq h \\ k & \text{if } g = h \end{cases}$$

- It is semisimple because we can write $V = \oplus_{g \in G} \oplus_{i=1}^{\dim V_g} \delta_g$.

## Definition (vague)

A **monoidal category** is essentially a category $\mathcal{C}$ equipped with a multiplication operation and a rule for associativity under such multiplication; that is, it is equipped with

- a *tensor product* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- the *associativity constraint* $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ for any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, and
- an object $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$,

and satisfies the **pentagon axiom**, which is a condition on $\alpha_{X,Y,Z}$.

# Monoidal Categories

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## Example

We define the monoidal category $k$-$\mathbf{Vec}_G$. (We define the following only on simple objects due to its semisimple-ness.)

- The tensor product is defined by $\delta_g \otimes \delta_h = \delta_{gh}$.
- The associativity map is defined by $\alpha_{\delta_g,\delta_h,\delta_i} := \mathrm{id}_{\delta_{ghi}} : (\delta_g \otimes \delta_h) \otimes \delta_i \to \delta_g \otimes (\delta_h \otimes \delta_i)$.
- The unit object $\mathbb{1}$ is $\delta_e$, where $e$ is the identity element in $G$.

# Example of a Monoidal Category: $k$-$\mathbf{Vec}_G^\omega$

We define the monoidal category $k$-$\mathbf{Vec}_G^\omega$, where $\omega \in Z^3(G, k^*)$.

- The set $Z^3(G, k^*)$ consists of all **3-cocycles**, which are the maps $\omega : G \times G \times G \to k^*$ satisfying the following condition for any $g, h, i, j \in G$.

$$\omega(h, i, j)\omega(g, h, ij)\omega(gh, i, j) = \omega(g, hi, j)\omega(g, h, i)$$

The monoidal category $k$-$\mathbf{Vec}_G^\omega$ is identical to $k$-$\mathbf{Vec}_G$, except for its associativity isomorphisms. Instead, for any $g, h, i \in G$, we define

$$\alpha_{\delta_g, \delta_h, \delta_i} := \omega(g, h, i) \cdot \mathrm{id}_{\delta_{ghi}} : (\delta_g \otimes \delta_h) \otimes \delta_i \to \delta_g \otimes (\delta_h \otimes \delta_i).$$

The fact that $\omega$ is a 3-cocycle implies our defined $\alpha$ satisfies the pentagon axiom, as necessary.

**Definition**

A **fusion category over a field** $k$ is a monoidal, abelian, semisimple, $k$-linear, rigid, and finite category whose monoidal unit object $\mathbb{1}$ is simple.

**Definition (vague)**

A category is **pointed** if each of its simple objects $X$ is invertible; in simple terms, there exists an object $Y$ such that $X \otimes Y \cong \mathbb{1}$. Thus, the simple objects in a pointed category $\mathcal{C}$ form a group; call it $G(\mathcal{C})$.

**Example**

An example of a pointed fusion category is $k$-$\mathbf{Vec}_G^\omega$. The "inverse" of $\delta_g$ is $\delta_{g^{-1}}$.

**Theorem**

Let $k$ be an algebraically closed field. Then $\mathcal{C}$ is a pointed fusion category over $k$ if and only if $\mathcal{C}$ is equivalent to $k\text{-}\mathbf{Vec}_G^\omega$ for some group $G$ and $\omega \in Z^3(G, k^*)$.

The equivalence is essentially defined as follows.

$$\mathcal{C} \longrightarrow k\text{-}\mathbf{Vec}_G^\omega$$

$$\text{simple objects: } g \longrightarrow \delta_g$$

$$\text{associativity isomorphisms: } \alpha_{g,h,i} \longrightarrow \alpha_{\delta_g,\delta_h,\delta_i} = \omega(g,h,i) \cdot \mathrm{id}_{\delta_{ghi}}$$

# Our Results

**Research Question**

Can we classify pointed fusion categories when $k$ is not algebraically closed?

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Our result extends the known theorem on the previous slide.

**Theorem**

Let $k$ be a field. Every pointed fusion category $\mathcal{C}$ over $k$ is equivalent to either

1. $k\text{-Vec}^{\alpha}_{G(\mathcal{C})}$, for the constraints $\alpha_{X,Y,Z} = c(X,Y,Z)\mathrm{id}_{(X\otimes Y)\otimes Z}$ for $c \in Z^3(G(\mathcal{C}), k^*)$, or
2. $F\text{-Vec}^{\alpha}_{k,G(\mathcal{C})}$, for some finite field extension $F$ of $k$ and for the constraints $\alpha_{X,Y,Z} = c(X,Y,Z) \triangleright \mathrm{id}_{(X\otimes Y)\otimes Z}$ for $c \in Z^3(G(\mathcal{C}), F^*_{\Diamond})$ and a group action $\Diamond$ of $G(\mathcal{C})$ on $F^*$.

Case 1 arises when $k = \bar{k}$. Case 2 arises when $k \neq \bar{k}$.

# References


- P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik: *Tensor Categories*, Mathematical Surveys and Monographs/American Mathematical Society **205** (2015).
- P. Etingof, D. Nikshych, and V. Ostrik: *On fusion categories*, Annals of Mathematics **162** (2005), 581–642.
- D. Tambara and S. Yamagami: *Tensor categories with fusion rules of self-duality for finite abelian groups*, Journal of Algebra **209** (1998), 692–707.
- E. Rowell and Z. Wang: *Mathematics of topological quantum computing*, Bulletin of the American Mathematical Society **55** (2018), 183–238.

# Acknowledgements

Many thanks go to

- my mentors Prof. Julia Plavnik (IU) and Dr. Sean Sanford (OSU) for their invaluable feedback, guidance, and encouragement.
- Dr. Pavel Etingof, Dr. Slava Gerovitch, Dr. Tanya Khovanova, the MIT Math Department, and the MIT PRIMES program, for providing us with the opportunity to work on this project.
- my mother for her constant support.
- you for listening.