Definition of limit using supremums and infimums

Suppose (x_n) is a sequence of real numbers, and x is a real number. Here is an alternate way to define the statement "x is a limit of (x_n)", using the notions of supremum and infimum of a set. It's logically equivalent to the definition given in the text.

For each natural number K, define the set S_K of real numbers by

$$S_K = \{|x_n - x| : n \geq K\}.$$

That is, S_K is the set containing all the numbers $|x_K - x|$, $|x_{K+1} - x|$, $|x_{K+2} - x|$, and so on. Notice that S_K is a set, not a sequence. The numbers in S_K are not assumed to be in any particular order, and it's perfectly possible, for example, that S_K has only one element in it. Also notice that

$$S_1 \supseteq S_2 \supseteq S_3 \supseteq \ldots,$$

or in other words, S_2 is a subset of S_1, S_3 is a subset of S_2, S_4 is a subset of S_3, and so on.

Now consider S_1, which is equal to the set $\{|x_n - x| : n \in \mathbf{N}\}$. There are two possibilities: either S_1 is bounded above, or S_1 is not bounded above. If S_1 is not bounded above, we define the statement "x is a limit of (x_n)" to be false. So now it only remains to define what "x is a limit of (x_n)" means when S_1 is bounded above.

Since

$$S_1 \supseteq S_2 \supseteq S_3 \supseteq \ldots,$$

we know that S_K is a subset of S_1 for all $K \in \mathbf{N}$. So when S_1 is bounded above, then S_K is also bounded above for all $K \in \mathbf{N}$. Therefore we know from the Completeness Property of real numbers that each S_K has a supremum, and in this case we define the real number u_K by

$$u_K = \sup S_K.$$

Next, we define the set of real numbers U by

$$U = \{u_K : K \in \mathbf{N}\}.$$

Notice that from the definition of u_K it is clear that $u_K \geq 0$ for all $K \in \mathbf{N}$, so 0 is a lower bound of U. We now define the statement "x is a limit of (x_n)" to be true if and only if 0 is the greatest lower bound of U; that is, if and only if

$$\inf U = 0.$$

To summarize, then, the statement "x is a limit of (x_n)" is true if and only if (1) S_1 is bounded above, and (2) $\inf U = 0$. It is a good exercise in logic to check that this is equivalent to the definition of "x is a limit of (x_n)" given in the text.

The sets S_K and numbers u_K defined above are used often in analysis when dealing with a sequence whose limit (or lack of limits) is under investigation, although we won't have occasion to use them much this semester. The number w defined by

$$\begin{aligned} w &= \inf U \\ &= \inf\{u_K : K \in \mathbf{N}\} \\ &= \inf\{\sup S_K : K \in \mathbf{N}\} \\ &= \inf\{\sup\{|x_n - x| : n \geq K\} : K \in \mathbf{N}\} \end{aligned}$$

is called the "limit superior" of the sequence $(|x_n - x|)$, or "$\limsup |x_n - x|$" for short. One can see easily that w exists whenever (x_n) is bounded. Therefore we can summarize the above discussion in the following statement:

$$\lim(x_n) = x \text{ if and only if } (x_n) \text{ is bounded and } \limsup |x_n - x| = 0,$$

or, more explicitly,

$$\lim(x_n) = x \text{ if and only if } (x_n) \text{ is bounded and } \inf\{\sup\{|x_n - x| : n \geq K\} : K \in \mathbf{N}\} = 0.$$

Thus we have achieved the goal of defining limits of sequences in terms of supremums and infimums of sets.