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Cohomology Rings of Fine Quiver Moduli are Tautologically Presented

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Quiver Settings

- **Quiver**: directed finite graph Q, vertices Q_0, arrows Q_1
- **Dimension vector** for a quiver Q: tuple d of positive integers d_i (all $i \in Q_0$)
- **Stability condition**: a linear map $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$

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Remark

In the following: Q, d, θ fixed with

- Q acyclic
- $\theta(d) = 0$
- d is θ-coprime, i.e. $\theta(d') \neq 0$ for all $0 \leq d' \leq d$ with $0 \neq d' \neq d$

Representations

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Definition

A **representation** M of Q consists of

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- linear maps $M_\alpha : M_i \to M_j$ (all $\alpha : i \to j$).

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Representations of Q form a $\mathbb{C}$-linear abelian category.

Representations

Definition

A **representation** M of (Q, d) consists of

- vector spaces M_i (all $i \in Q_0$) of dimension d_i and
- linear maps $M_\alpha : M_i \to M_j$ (all $\alpha : i \to j$).

Representations

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Definition

A **representation** M of (Q, d) over a variety X consists of

- vector bundles M_i on X (all $i \in Q_0$) of rank d_i and
- bundle maps $M_\alpha : M_i \to M_j$ (all $\alpha : i \to j$).

Moduli space

Definition

A **moduli space** of (Q, d) is a variety Y, whose points parametrize the isomorphism classes of representations of (Q, d) (in a functorial way).

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Problem: This hardly ever exists.

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A representation M of (Q, d) is called θ-**semi-stable** if for every sub-representation M' of M, we have $\theta(d') \geq 0$ (where d' denotes the dimension vector of M').

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Universal representation

Let Y be a moduli space of (Q, d, θ).

Definition

A **universal representation** U of (Q, d, θ) is a representation of (Q, d) over Y such that for every $[M] \in Y$, we have $U_{[M]} \cong M$.

What we know so far

Results about cohomology rings of moduli spaces:

- **Kirwan '84**: Cohomology of Quotients in Symplectic Geometry. Nearly explicit description of cohomology ring of moduli space of n ordered points in $\mathbb{P}^1$ modulo Sl_2.
- **Haussmann, Knudson '98**: Calculate above cohomology ring explicitly using Gröbner bases.
- **Ellingsrud, Stromme '89**: General result for cohomology ring of GIT-quotient.

Properties of moduli spaces

Remember: Q acyclic, d is θ-coprime

Facts

- **King**: There exists a moduli space Y for (Q, d, θ) and a universal representation U.

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  Thus, Y fulfills Poincaré duality, i.e. $H^{2r-i}(Y) \cong H_i(Y)$
  (r = complex dimension of Y).

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- **Ellingsrud-Stromme**: Y is an even-cohomology space.
- **King-Walter**: The cohomology ring $H(Y) := H^{2\bullet}(Y;\mathbb{Q})$ is generated by the Chern classes $c_\nu(U_i)$ ($i \in Q_0$, $1 \leq \nu \leq d_i$) of the universal representation U.

Aim

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Fix Q, d and θ. Let Y be the moduli space and U the universal representation. Remember:

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Goal

Find a ”natural” defining set of relations between the generators $c_\nu(U_i)$ of $H(Y)$.

Result

Let Q, d and θ as above. Let Y be the moduli space and U a universal rep.

Result

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Let Q, d and θ as above. Let Y be the moduli space and U a universal rep.

Construct a flag bundle $\mathrm{Fl}(U) \to Y$ with complete flags $\mathscr{U}_i^{\bullet}$ of $(U_i)_{\mathrm{Fl}(U)}$ (all vertices i).

Result

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Let Q, d and θ as above. Let Y be the moduli space and U a universal rep.
Construct flag bundle $\mathrm{Fl}(U)$ with flags $\mathscr{U}_i^{\bullet}$. Define maps

$$\varphi_\alpha^{d'} : \mathscr{U}_i^{d'_i} \hookrightarrow (U_i)_{\mathrm{Fl}(U)} \to (U_j)_{\mathrm{Fl}(U)} \twoheadrightarrow (U_j)_{\mathrm{Fl}(U)}/\mathscr{U}_j^{d'_j}$$

for every arrow $\alpha : i \to j$ and for every $0 \leq d' \leq d$ with $\theta(d') < 0$.

Result

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for every arrow $\alpha : i \to j$ and for every $0 \leq d' \leq d$ with $\theta(d') < 0$. Call such d' **forbidden**.

Result

Let Q, d and θ as above. Let Y be the moduli space and U a universal rep.
Construct flag bundle $\mathrm{Fl}(U)$ with flags $\mathscr{U}_i^{\bullet}$. Define maps $\varphi_\alpha^{d'}$ for α and forbidden d'.
Show that for every $p \in \mathrm{Fl}(U)$, there ex. arrow α with $(\varphi_\alpha^{d'})_p \not\equiv 0$.

Degeneracy Classes (after Fulton)

Let $\varphi : E \to F$ be a map of vector bundles on X. Let $\mathscr{E}^{\bullet}$ and $\mathscr{F}^{\bullet}$ be complete filtrations of E and F, resp. and let $\xi_i := c_1(\mathscr{E}^i/\mathscr{E}^{i-1})$ and $\eta_j := c_1(\mathscr{F}^j/\mathscr{F}^{j-1})$ (i.e. ξ_i and η_j are the Chern roots of E and F, resp.).

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Definition

Call $\mathbb{Z}(\varphi) := \prod_i \prod_j (\eta_j - \xi_i) \in H(X)$ the **degeneracy class** of φ.

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Remark

Let $Z(\varphi)$ be the closed subset of X of all x with $\varphi_x : E_x \to F_x$ identically zero.
Then $\mathbb{Z}(\varphi)$ has an inverse image under $H(Z(\varphi)) \to H(X)$.

Result (continued)

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Let Q, d and θ as above. Let Y be the moduli space and U a universal rep.
Construct flag bundle $\mathrm{Fl}(U)$ with flags $\mathcal{U}_i^{\bullet}$. Define maps $\varphi_\alpha^{d'}$ for α and "forbidden" d'.
For all $p \in \mathrm{Fl}(U)$ there ex. α with $(\varphi_\alpha^{d'})_p \not\equiv 0$.

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For all $p \in \mathrm{Fl}(U)$ there ex. α with $(\varphi_\alpha^{d'})_p \not\equiv 0$, i.e. $\bigcap_\alpha Z(\varphi_\alpha^{d'}) = \emptyset$.

Result (continued)

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Fact (Grothendieck)

$H(\mathrm{Fl}(U))$ is a free $H(Y)$-module with basis elements $\xi^\lambda := \prod_{i,\nu} \xi_{i,\nu}^{\lambda_{i,\nu}}$ where $0 \le \lambda_{i,\nu} < \nu$.

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For all $p \in \mathrm{Fl}(U)$ there ex. α with $(\varphi_\alpha^{d'})_p \not\equiv 0$, thus

$$0 = \prod_\alpha \mathbb{Z}(\varphi_\alpha^{d'}) = \sum_\lambda \tau^\lambda(d') \cdot \xi^\lambda$$

for some polynomial expressions $\tau^\lambda(d')$ in Chern classes $c_\nu(U_i)$.

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Construct flag bundle $\mathrm{Fl}(U)$ with flags $\mathscr{U}_i^{\bullet}$. Define maps $\varphi_\alpha^{d'}$ for α and "forbidden" d'.
Exist $\tau^\lambda(d')$ with $0 = \prod_\alpha \mathbb{Z}(\varphi_\alpha^{d'}) = \sum_\lambda \tau^\lambda(d')\, \xi^\lambda$. We call these $\tau^\lambda(d')$ **tautological relations**.

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Theorem

$H(Y)$ is the quotient of the polynomial algebra over $\mathbb{Q}$ in $c_{i,\nu}$ (i vertex of Q and $1 \leq \nu \leq d_i$) modulo the relations

$$\left(\tau^\lambda(d')\right)(c_{i,\nu} \mid i, \nu) = 0$$

(all λ, all "forbidden" d') and one linear relation among the $c_{i,1}$'s.

An Example

Let $(Q, d) : \quad \underset{2}{\bullet} \;\overset{\displaystyle\frown}{\underset{\displaystyle\smile}{\longrightarrow}}\; \underset{3}{\bullet}$ and $\theta(m, n) = 2n - 3m$.

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Source q, sink s and arrows $\alpha_1, \alpha_2, \alpha_3$.

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Fix Y and U.

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Fix Y and U. Aside: $\dim Y = 6$ and Betti numbers $1\ 1\ 3\ 3\ 3\ 1\ 1$. A cell decomposition is known.

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Basis: $\xi_2^{\lambda_2}\eta_2^{\mu_2}\eta_3^{\mu_3}$ with $\lambda_2, \eta_2 = 0, 1$ and $\mu_3 = 0, 1, 2$.

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Forbidden: $d' = (1, 1)$ and $d' = (2, 2)$.

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$$0 = \mathbb{Z}(\varphi_{\alpha_1}) \cdot \mathbb{Z}(\varphi_{\alpha_2}) \cdot \mathbb{Z}(\varphi_{\alpha_3})$$

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$$0 = \mathbb{Z}(\varphi_{\alpha_1}) \cdot \mathbb{Z}(\varphi_{\alpha_2}) \cdot \mathbb{Z}(\varphi_{\alpha_3}) = (\eta_3 - \xi_1)^3 \cdot (\eta_3 - \xi_2)^3$$

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1. $d' = (2, 2)$. Then

$$
\begin{aligned}
0 &= \mathbb{Z}(\varphi_{\alpha_1}) \cdot \mathbb{Z}(\varphi_{\alpha_2}) \cdot \mathbb{Z}(\varphi_{\alpha_3}) = (\eta_3 - \xi_1)^3 \cdot (\eta_3 - \xi_2)^3 \\
&= \tau^{0,0,0}(2, 2) + \tau^{0,0,1}(2, 2) \cdot \eta_3 + \tau^{0,0,2}(2, 2) \cdot \eta_3^2
\end{aligned}
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3. Linear relation: $x_1 = y_1$.

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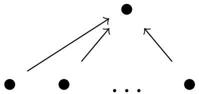
Simplification yields: $H(Y) \cong \mathbb{Q}[x_2, y_1, y_2, y_3]/\mathfrak{a}$, where $\mathfrak{a}$ is generated by

- $3x_2^2 - 3x_2y_2 + y_2^2 - y_1y_3$,
- $(3x_2 - 2y_2)y_3$,
- $x_2^3 - y_1y_2y_3 + y_3^2$,
- $-4x_2y_1 + y_1^3 + 3y_3$,
- $3x_2^2 - x_2y_1^2$,
- $3x_2^2 + x_2y_2 - y_1^2y_2$,
- $x_2y_1y_2 - 3y_2y_3$,
- $3y_1^2 - 5y_2y_3$, and
- $x_2^3 - x_2y_1y_3$.

Another example

Let Q :  with $m = 2r + 1$ sources, $d = (1, \ldots, 1,\ 2)$ and $\theta(a) = ma_s - 2(a_{q_1} + \ldots + a_{q_m})$.

Let Y moduli space and fix a universal rep. U.

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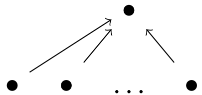
Let Q :  with $m = 2r + 1$ sources, $d = (1, \ldots, 1,\ 2)$ and $\theta(a) = ma_s - 2(a_{q_1} + \ldots + a_{q_m})$.

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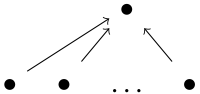
Let Q :  with $m = 2r + 1$ sources, $d = (1, \ldots, 1,\ 2)$ and $\theta(a) = ma_s - 2(a_{q_1} + \ldots + a_{q_m})$.

Let Y moduli space and fix a universal rep. U. Y is isomorphic to moduli space of n ordered points in $\mathbb{P}^1$ modulo Sl_2.

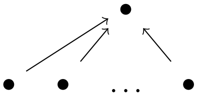
Proposition

$H(Y) \cong \mathbb{Q}[x_1, \ldots, x_{m-1}, y]/\mathfrak{a}$, where $\mathfrak{a}$ generated by

- $x_i(y - x_i)$ (all $1 \le i \le m - 1$),
- $\prod_{i \in I'}(y - x_i)$, and
- $\sum_{j=0}^{l-1} (-1)^j\, y^{l-1-j} \sum_{J \subseteq I : \sharp J = j} \prod_{i \in J} x_i$

all $I', I \subseteq \{1, \ldots, m-1\}$ with $\sharp I' \ge r$ and $l := \sharp I > r$.

Another example

Let Q :  with $m = 2r + 1$ sources, $d = (1, \dots, 1,\ 2)$ and $\theta(a) = ma_s - 2(a_{q_1} + \dots + a_{q_m})$.

Let Y moduli space and fix a universal rep. U. Y is isomorphic to moduli space of n ordered points in $\mathbb{P}^1$ modulo Sl_2.

Corollary (cf. Kirwan, using HN methods)

The Poincaré polynomial of $H(Y)$ is

$$\sum_{n=0}^{2(r-1)} \left(\sum_{\nu=0}^{\min\{n, r-1-n\}} \binom{2r}{\nu} \right) t^n.$$

Thank you!