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A Remark on Algebraic Closure and Orthogonality

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Abstract We show that if T is a stable theory with ndop and ndidip, then $|T|^+$-primary models over free trees are $|T|^+$-minimal over the tree. As a corollary we show, for example, that if T is a stable theory and for all nonempty X, $acl(X) = \cup_{x \in X} acl(\{x\})$, then T is superstable or it has dop or didip.

These notes aroused from the authors attempt to understand the nonorthogonality relation between a type and a set in the strictly stable case. It seems that the behavior of the algebraic closure operation plays a role here. So it is reasonable to ask, do we understand the relation, if the algebraic closure operation is as simple as possible? The answer is yes.

Throughout this paper we assume that T is a stable theory. We write s-saturated, s-primary, and so on, for $F^s_{|T|^+}$-saturated, $F^s_{|T|^+}$-primary, and so on.

Definition 1

(i) Let $P = (P, <)$ be a tree without branches of length $> \omega$ and g be a function from P to the subsets of the monster model. We say that (P, g) is a free tree if the following hold:
  (a) for all $u \in P$, $g(u)$ is an s-saturated model,
  (b) if $u < v$ then $g(u) \subseteq g(v)$,
  (c) if u is an immediate successor of v, then $g(u) \downarrow_{g(v)} \cup \{g(w) | w \in P, w \not\geq u\}$.

(ii) (Shelah [1]) We say that T has didip, if there are a regular infinite cardinal κ, s-saturated models $\mathcal{A}_i$, $i \leq \kappa$, and nonalgebraic $p \in S(\mathcal{A}_\kappa)$ such that
  (a) for all $i < j \leq \kappa$, $\mathcal{A}_i \subseteq \mathcal{A}_j$,
  (b) for limit $i \leq \kappa$, $\mathcal{A}_i$ is s-primary over $\cup_{j<i} \mathcal{A}_j$,
  (c) for all $i < \kappa$, p is orthogonal to $\mathcal{A}_i$.

If T does not have didip, then we say that T has ndidip.

Didip is a nonstructure property for strictly stable theories. Superstability implies ndidip.

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Notice that the following theorem can not be proven as the related result is proven in the superstable case. This is because there may not be enough regular types. However, instead of the direct proof given below, the theorem follows rather easily by using the 'nonorthogonality calculus' (and minimal covers) from the unpublished theses by Hernandez. In [1], (end of §X.2) one instance of the theorem is mentioned.

Theorem 2 *Assume T is stable and has ndop and ndidip. If (P, g) is a free tree, $\mathcal{A}$ is s-primary over $\cup_{u \in P} g(u)$ and $p \in S(\mathcal{A})$, then there is $u \in P$ such that p is not orthogonal to $g(u)$.*

We first prove the following lemma.

Lemma 3 *Assume T is stable.*

(i) Assume T has ndop and $\mathcal{A}_i$, $i < 4$, B and $p \in S(\mathcal{A}_3 \cup B)$ are such that

- *(a) $\mathcal{A}_i$, $i < 4$, are s-saturated, $\mathcal{A}_0 \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$ and $\mathcal{A}_1 \downarrow_{\mathcal{A}_0} \mathcal{A}_2$,*
- *(b) $\mathcal{A}_3$ is s-primary over $\mathcal{A}_1 \cup \mathcal{A}_2$,*
- *(c) p is not orthogonal to $\mathcal{A}_3$*

Then p is not orthogonal to $\mathcal{A}_1$ or to $\mathcal{A}_2$.

(ii) Assume T has ndidip and $\mathcal{A}_i$, $i \leq \alpha$, B and $p \in S(\mathcal{A}_\alpha \cup B)$ are such that

- *(a) α is limit, for all $i \leq \alpha$, $\mathcal{A}_i$ is s-saturated, for all $i < j \leq \alpha$, $\mathcal{A}_i \subseteq \mathcal{A}_j$ and for limit $i \leq \alpha$, $\mathcal{A}_i$ is s-primary over $\cup_{j<i} \mathcal{A}_j$,*
- *(b) p is not orthogonal to $\mathcal{A}_\alpha$.*

Then there is $i < \alpha$ such that p is not orthogonal to $\mathcal{A}_i$.

Proof: We prove (i); (ii) is similar. Assume not. By extending B if necessary, we can find finite sequences a and b such that $b \downarrow_{\mathcal{A}_3} B$, $b \not\downarrow_{\mathcal{A}_3 \cup B} a$ and $t(a, \mathcal{A}_3 \cup B)$ is orthogonal to $\mathcal{A}_1$ and $\mathcal{A}_2$. By induction on $i < \kappa(T)$, we define $\mathcal{A}^i_j$, $j < 4$, so that

1. for all $j < 4$, $\mathcal{A}^0_j = \mathcal{A}_j$ and for all $i < k < \kappa(T)$, $\mathcal{A}^i_j \subseteq \mathcal{A}^k_j$,
2. for all $i < k < \kappa(T)$ and $j, j' \in \{1, 2\}$, $j \neq j'$, $\mathcal{A}^k_j \downarrow_{\mathcal{A}^i_j} \mathcal{A}^i_3 \cup \mathcal{A}^k_{j'}$ and $a \cup B \downarrow_{\mathcal{A}_3} \mathcal{A}^i_3$,
3. for all $i \leq k < \kappa(T)$, $\mathcal{A}^k_3$ is s-primary over $\mathcal{A}^k_1 \cup \mathcal{A}^k_2 \cup \bigcup_{j<i} \mathcal{A}^j_3$ (so, especially, $\mathcal{A}^k_3$ is s-primary over $\mathcal{A}^k_1 \cup \mathcal{A}^k_2$),
4. $b \downarrow_{\mathcal{A}^i_3} B$, $b \not\downarrow_{\mathcal{A}^i_3} \mathcal{A}^{i+1}_3$ and $b \not\downarrow_{\mathcal{A}^i_3 \cup B} a$.

This contradicts the definition of $\kappa(T)$.

$i = 0$: By (1) we must let $\mathcal{A}^0_j = \mathcal{A}_j$, for all $j < 4$. Clearly these satisfy (1)–(4).

$i = k + 1$: By (4), $t(b, \mathcal{A}^k_3)$ is not algebraic, so by ndop $t(b, \mathcal{A}^k_3)$ is not orthogonal to, for example, $\mathcal{A}^3_1$. Choose c so that $c \downarrow_{\mathcal{A}^k_1} \mathcal{A}^k_3$, $c \not\downarrow_{\mathcal{A}^k_3} b$, and $c \downarrow_{\mathcal{A}^k_3 \cup b} B$. Let $\mathcal{A}^i_0 = \mathcal{A}_0$, $\mathcal{A}^i_1$ be s-primary over $\mathcal{A}^k_1 \cup c$, $\mathcal{A}^i_2 = \mathcal{A}^k_2$, and $\mathcal{A}^i_3$ be s-primary over $\mathcal{A}^i_1 \cup \mathcal{A}^k_3$. Clearly (1) and (3) hold. Also $\mathcal{A}^i_1 \downarrow_{\mathcal{A}^k_1} \mathcal{A}^k_3 \cup B$, and since by (2), $t(a, \mathcal{A}^k_3 \cup B)$ is orthogonal to $\mathcal{A}^k_1$, $a \downarrow_{\mathcal{A}^k_3 \cup B} \mathcal{A}^i_3$. So (2) holds. (4) now follows easily.

i is limit: We let $\mathcal{A}^i_0 = \mathcal{A}_0$ and for $j \in \{1, 2\}$, we let $\mathcal{A}^i_j$ be s-primary model over $\cup_{k<i} \mathcal{A}^k_j$. Then by (2) and (3), for all $j < i$, $\mathcal{A}^i_1 \cup \mathcal{A}^i_2 \cup \bigcup_{k<i} \mathcal{A}^k_3$ is s-constructible over

$\mathcal{A}_1^i \cup \mathcal{A}_2^i \cup \bigcup_{k<j} \mathcal{A}_3^k$. So we let $\mathcal{A}_3^i$ be an s-primary model over $\mathcal{A}_1^i \cup \mathcal{A}_2^i \cup \bigcup_{k<i} \mathcal{A}_3^k$. Clearly (1)–(4) hold. □

Proof of Theorem 2: Let $P = \{u_i | \, i < \alpha\}$ be an enumeration of P such that if $u_i < u_i$ then $i < j$. Let $\mathcal{A}_0 = \emptyset$, for all $i < \alpha$, let $\mathcal{A}_{i+1}$ be s-primary over $\mathcal{A}_i \cup g(u_i)$, and if $i \le \alpha$ is limit, then let $\mathcal{A}_i$ be s-primary over $\cup_{j<i} \mathcal{A}_j$. Then $\mathcal{A}_\alpha$ is s-primary over $\cup_{u \in P} g(u)$. By the uniqueness of s-primary models, we can choose these so that $\mathcal{A}_\alpha = \mathcal{A}$. Let $i \le \alpha$ be the least such that p is not orthogonal to $\mathcal{A}_i$. Clearly we may assume that $i > 1$. By Lemma 3(ii), i is not limit. So $i = j+1$ for some $j \ge 1$. Then u_j is not the root. Let v be the immediate predecessor of u_j. Then $\mathcal{A}_i$ is s-primary over $\mathcal{A}_j \cup g(u_j)$. Clearly we may assume that $\mathcal{A}_i \ne g(u_j)$. It is easy to see that $g(u_j) \downarrow_{g(v)} \mathcal{A}_j$. So by Lemma 3(i) and the choice of i, p is not orthogonal to $g(u_j)$. □

Corollary 4 *Assume T is stable and has ndop and ndidip. Then s-primary models over free trees are s-minimal over the tree.*

Proof: By Theorem 2, this can be proved as the related results in [1], (Lemma 3.3). □

It should be pointed out that we do not work in M^{eq}; that is, in the following definition algebraic closure means just the ordinary algebraic closure.

Definition 5 We say that algebraic closure is trivial if for all X, Y, and Z the following holds: if $Y \downarrow_X Z$, then $acl(X \cup Y \cup Z) = acl(X \cup Y) \cup acl(X \cup Z)$.

For example, if for all X, $acl(X) = X$ or for all nonempty X, $acl(X) = \cup_{x \in X} acl(\{x\})$, then algebraic closure is trivial. So if T is the canonical example of a stable unsuperstable theory, then algebraic closure is trivial.

Lemma 6 *Assume T is stable and has ndop, ndidip, and algebraic closure is trivial. Then for all sets $\mathcal{A} \subseteq \mathcal{B}$ and singletons a, if $a \not\downarrow_{\mathcal{A}} \mathcal{B}$ then $t(a, \mathcal{B})$ is orthogonal to $\mathcal{A}$.*

Proof: Assume not. Let $\mathcal{A}$, $\mathcal{B}$, and a exemplify this. Clearly we may assume that $\mathcal{A}$ and $\mathcal{B}$ are s-saturated. By induction on $i < \kappa(T)$, we construct free trees (P_i, g_i) and s-primary models $\mathcal{A}_i$ over the tree such that for all $i < \kappa(T)$, $t(a, \mathcal{A}_i)$ is not algebraic and if $i = j+1$, then $a \not\downarrow_{\mathcal{A}_j} \mathcal{A}_i$. This contradicts the definition of $\kappa(T)$.

$i = 0$: Since $t(a, \mathcal{B})$ is not orthogonal to $\mathcal{A}$, we can find b such that $b \downarrow_{\mathcal{A}} \mathcal{B}$ and $b \not\downarrow_{\mathcal{B}} a$. Let C be s-primary over $\mathcal{A} \cup b$ and $\mathcal{D}$ s-primary over $C \cup \mathcal{B}$. Furthermore, choose these so that $a \downarrow_{\mathcal{B} \cup b} \mathcal{D}$. Clearly $a \notin \mathcal{B}$ and since $b \downarrow_{\mathcal{A}} \mathcal{B}$ but $a \not\downarrow_{\mathcal{A}} \mathcal{B}$, $a \notin acl(\mathcal{A} \cup b)$ and so $a \notin acl(\mathcal{B} \cup b)$. So $t(a, \mathcal{D})$ is not algebraic. Notice also that $a \not\downarrow_C \mathcal{B}$. We let $P_0 = \{x, y, z\}$, x is the root and y and z are immediate successors of x. $g_0(x) = \mathcal{A}$, $g_0(y) = C$, and $g_0(z) = \mathcal{B}$. Finally we let $\mathcal{A}_0 = \mathcal{D}$.

$i = j+1$: By Theorem 2, there is $t \in P_j$ such that $t(a, \mathcal{A}_j)$ is not orthogonal to $g_j(t)$. Choose b so that $b \downarrow_{g_j(t)} \mathcal{A}_j$ and $b \not\downarrow_{\mathcal{A}_j} a$. Since $a \not\downarrow_{g_j(t)} \mathcal{B} \cup C$ (C is defined the case $i = 0$ above), $a \notin acl(g_j(t) \cup b)$ and so $a \notin acl(\mathcal{A}_j \cup b)$. Choose $P_i = P_j \cup \{t'\}$, where t' is a new immediate successor of t. Let g_i be such that $g_i \restriction P_j = g_j$, and $g_i(t')$ is an s-primary model over $g_j(t) \cup b$. Let $\mathcal{A}_i$ be an s-primary model over $\mathcal{A}_j \cup g_i(t')$. It is easy to see that $\mathcal{A}_i$ is s-primary over $\cup_{u \in P_i} g_i(u)$. Furthermore, choose these so that $a \downarrow_{\mathcal{A}_j \cup b} \mathcal{A}_i$. Then $t(a, \mathcal{A}_i)$ is not algebraic.

i is limit: We let $P_i = \cup_{j<i} P_j$ $g_i = \cup_{j<i} g_j$ and $\mathcal{A}_i$ be an s-primary model over $\cup_{j<i}\mathcal{A}_j$. It is not hard to see that $\cup_{j<i}\mathcal{A}_j$ is s-constructible over $\cup_{u\in P_i} g_i(u)$ and so $\mathcal{A}_i$ is s-primary over $\cup_{u\in P_i} g_i(u)$. Furthermore, we choose $\mathcal{A}_i$ so that $a \downarrow_{\cup_{j<i}\mathcal{A}_j} \mathcal{A}_i$. Clearly $t(a, \mathcal{A}_i)$ is not algebraic. □

Corollary 7 *If T is stable and algebraic closure is trivial, then T is superstable or it has dop or didip.*

Proof: Assume T is stable with ndop and ndidip and algebraic closure is trivial. We show that T is superstable. We first prove the following claim.

Claim 8 *There are no singleton a and sets $\mathcal{A}_i$, $i < \omega$, such that for all $i < \omega$, $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$ and $a \not\downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$.*

Proof: Assume not. Clearly we may assume that for all $i < \omega$, $\mathcal{A}_i$ is s-saturated. Let $\mathcal{A}$ be an s-primary model over $\cup_{j<\omega}\mathcal{A}_j$ such that $a \downarrow_{\cup_{j<\omega}\mathcal{A}_j} \mathcal{A}$. Then $t(a, \mathcal{A})$ is not algebraic and by Lemma 6, it is orthogonal to every $\mathcal{A}_i$, $i < \omega$. This contradicts the assumption that T has ndidip. □

As in [1], Claim 8 implies that if $|B| \geq 2^{|T|}$, then $|S_1(B)| = |\mathcal{B}|$. Clearly this implies that T is superstable. □

So if T is stable and algebraic closure is trivial, then either we have a structure theorem for F^a_ω-saturated models or nonstructure theorem for F^s_λ-saturated models for all λ.

Corollary 9 *If T is superstable, has ndop, and algebraic closure is trivial, then every nonalgebraic type in one variable is regular.*

Proof: Immediate by Lemma 6. □

We finish these notes by showing that if T is superstable with ndop and algebraic closure is trivial, then it is trivial to find a decomposition for an s-saturated model. Notice that the same holds also for F^a_w-saturated models.

Corollary 10 *Assume T is a superstable theory with ndop and that algebraic closure is trivial. Let (P, g) be a free tree, $\mathcal{A}$ be s-primary over $\bigcup_{i\in P} g(i)$ and $p \in S_1(\mathcal{A})$ be nonalgebraic. Then there is $i \in P$ such that p does not fork over $g(i)$ and if $j \in P$ is such that $j < i$, then p is orthogonal to $g(j)$ (and so p is orthogonal to $g(j)$ for all $j \not\geq i$).*

Proof: Let $i \in P$ be minimal such that p is not orthogonal to $g(i)$. Then the latter claim holds. By Lemma 6, also the first claim holds. □

REFERENCES

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