I N F O R M A T I K

Universität des Saarlandes
FR 6.2 Informatik

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Exercises for Optimization

1. Assignment Due 29.04.2005

Exercise 1 ($9 \times 1 + 2$ *Points*)
Let $A = (a_1, \ldots, a_n)$ be a non-singular $n \times n$ real matrix with columns a_i and let $\mathrm{adj}(A)$ denote the $n \times n$ matrix with $\mathrm{adj}(A)_{ji} = (-1)^{i+j} \det(m_{ij}(A))$ where $m_{ij}(A)$ is the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from A. The matrix $\mathrm{adj}(A)$ is called adjugate ("Adjungierte") of A.

a) Give the definition of linear independence.

b) State "the" two formulae for computing the determinant of a matrix.

c) Which of the following statements are true for all $A, B \in \mathbb{R}^{n\times n}$? (without proof)

   (a) $\det A = 0 \Leftrightarrow$ columns (rows) of A are linearly dependent
   (b) $\det(A + B) = \det A + \det B$
   (c) $\det(A \cdot B) = \det A \cdot \det B$

d) Show $|\det A| \leq n! \, \|A\|_\infty^n$. ($\| \cdot \|_\infty$ denotes the maximum absolute value in a matrix or vector.)

e) Let $b \in \mathbb{R}^n$. Show that $\mathrm{adj}(A) \cdot b = \begin{pmatrix} \det(m_1(A, b)) \\ \vdots \\ \det(m_n(A, b)) \end{pmatrix}$ where $m_i(A, b)$ is obtained from A by replacing column i by vector b.

f) Show $\mathrm{adj}(A) \cdot A = I \cdot \det A$ where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

g) State and prove Cramer's Rule. (*Hint.* Deduce $\frac{1}{\det A} A \cdot \mathrm{adj}(A) = I$ from f).)

h) Suppose A is an integer matrix. Show A^{-1} is an integer matrix if and only if $|\det A| = 1$.
   (*Remark.* A is called *unimodular* if $A, A^{-1} \in \mathbb{Z}^{n\times n}$.)

i) Suppose A and b have integer entries. Let x be such that $Ax = b$. Show that each x_i is rational. Let $x_i = p_i/q_i$ for integers p_i, q_i with $\gcd(p_i, q_i) = 1$. Give upper bounds for $|p_i|$ and $|q_i|$ in terms of n,$\|A\|_\infty$ and $\|b\|_\infty$.
   (*Remark.* This shows that the *representation size* of x is polynomial in the representation size of A and b.)

j) Show that for any matrix $A \in \mathbb{R}^{n\times m}$ there is a matrix Y such that

$$\{Ax \mid x \in R^m\} = \{b \in \mathbb{R}^n \mid Y^T b = 0\}.$$

   Can you find a geometric interpretation? (*Hint.* This is not related to Cramer's Rule.)

Exercise 2 (4 × 1 *Points*)
For the following exercises only use that the dual of a linear program of the form

$$(P) : \text{maximize } c^T x \text{ subject to } Ax \leq b, x \geq 0$$

is the linear program

$$(D) : \text{minimze } y^T b \text{ subject to } y^T A \geq c^T, y \geq 0.$$

a) Let x and y be feasible solutions to (P) and (D), respectively. Prove algebraically that $c^T x$ is at most $y^T b$. (*Remark.* This fact is called *weak duality.*)

b) Transform the problem (D) to an equivalent problem (D') of the same form than (P).

c) Determine the dual of (D'). Call this problem (P').

d) What is the relation between (P) and (P')?

Exercise 3 (1 + 1 + 2 + 1 *Points*)
Consider the linear problem

$$(P) : \text{maximize } cx_1 + dx_2 \text{ subject to } x_1 \leq 1, x_2 \leq 1, x_1 + x_2 \leq 1$$

a) Graph the feasible region.

b) Determine the dual problem (D) of (P). (*Hint.* $\min\limits_{y^T A = c^T, y \geq 0} y^T b$ is dual to $\max\limits_{Ax \leq b} c^T x$.)

c) For each c, d show whether (P)

   (a) is infeasible,
   (b) is unbounded,
   (c) has exactly one optimum solution or
   (d) has more than one optimum solution.

   In case of *c*) or *d*), compute an optimum solution and prove optimality using the dual.

d) Consider optimum solutions x and y for (P) and (D), respectively. Can you observe a connection between the positive entries of y and the constraints in (P) that are fulfilled with equality by x?

   What happens to (D) if (P) is unbounded?