HANDOUT

LOGIC AND SET

MUSTHOFA
MATHEMATHIC EDUCATION
YOGYAKARTA STATE UNIVERSITY

PART 1

- Introduction
- Proposition
- Logical operation and Truth Table
- Tautology
- Contradiction
- Equivalence

INTRODUCTIONS

Many proof in mathematics use logical expressions such as " if …..Then ……" or "If ….. And …….Then …….or …….."

It is therefore necessary to know the cases in which these expressions are either true or false

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PROPOSITIONS

A *proposition* (or *statement*) is a declarative sentence which is true or false, but not both. Consider, for example, the following eight sentences:

(i) Paris is in France.
(ii) $1 + 1 = 2$.
(iii) $2 + 2 = 3$.
(iv) London is in Denmark.
(v) $9 < 6$.
(vi) $x = 2$ is a solution of $x^2 = 4$.
(vii) Where are you going?
(viii) Do your homework.

All of them are propositions except (vii) and (viii). Moreover, (i), (ii), and (vi) are true, whereas, (iii), (iv), and (v) are false.

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LOGICAL OPERATIONS

Conjunction $p \wedge q$

Any two propositions can be combined by the word "and" to form a compound proposition called the *conjunction* of the original propositions. Symbolically,

$$p \wedge q$$

read "p and q", denotes the conjunction of p and q. Since $p \wedge q$ is a proposition it has a truth value, and this truth value depends only on the truth values of p and q.

Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

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Disjunction, $p \vee q$

Any two propositions can be combined by the word "or" to form a compound proposition called the *disjunction* of the original propositions. Symbolically,

$$p \vee q$$

read "p or q", denotes the disjunction of p and q. The truth value of $p \vee q$ depends only on the truth values of p and q as follows.

Truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

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Negation, ¬ *p*

Given any proposition *p*, another proposition, called the *negation* of *p*, can be formed by writing "It is not the case that . . ." or "It is false that . . ." before *p* or, if possible, by inserting in *p* the word "not". Symbolically,

$$\neg p$$

read "not *p*", denotes the negation of *p*. The truth value of ¬ *p* depends on the truth value of *p* as follows.

Truth table

p	$\neg p$
T	F
F	T

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IMPLICATIONS

Many statements in mathematic are of the form if p then q .such statements are called *implications*, and denoted by $p \Rightarrow q$, or $p \rightarrow q$.

Truth table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

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BIIMPLICATIONS

Another common statements in mathematics are of the form " p if and only if q". Such statements are called *biimplications* and are denoted by $p \Leftrightarrow q$, or $p \leftrightarrow q$.

Truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

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TAUTOLOGY

Many propositions are *composite*, that is, composed of *subpropositions* and various connectives discussed subsequently. Such composite propositions are called *compound propositions*. A proposition is said to be *primitive* if it cannot be broken down into simpler propositions, that is, if it is not composite.

Example :

(*a*) "Roses are red and violets are blue" is a compound proposition with subpropositions "Roses are red" and "Violets are blue".

(*b*) "John is intelligent or studies every night" is a compound proposition with subpropositions "John is intelligent" and "John studies every night".

(*c*) The above propositions (i) through (vi) are all primitive propositions; they cannot be broken down into simpler propositions.

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Consider the truth table :

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T	T	T	F	F	T
F	T	F	T	T	T	F
F	F	T	F	F	F	T
F	F	T	F	F	T	F
Step		4	1	3	2	1

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Proposition –(p & -q) can be true or false, but proposition p ∨ - p always true in any conditions. Propositions that that are always true in any conditions are called *tautology.*

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Contradiction

Propositions that are always false in any conditions are called contradiction.
For example, consider the truth table :

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

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Equivalence

Two propositions P(p,q,…) and Q(p,q,….) are said to be equivalent , denoted by P ≡Q if they have identically truth table
Example :

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

So, - (p & q) ≡ -p ∨ -q

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PART 2

- Algebra of propositions
- Inference
- Consistency of Premises
- Indirect proofs(RAA)
- Constant and Variabel
- Quantifiers

Algebra of Propositions

Idempotent laws	
(1*a*) $p \vee p \equiv p$	(1*b*) $p \wedge p \equiv p$
Associative laws	
(2*a*) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2*b*) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws	
(3*a*) $p \vee q \equiv q \vee p$	(3*b*) $p \wedge q \equiv q \wedge p$
Distributive laws	
(4*a*) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4*b*) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws	
(5*a*) $p \vee \mathrm{T} \equiv p$	(5*b*) $p \wedge \mathrm{F} \equiv p$
(6*a*) $p \vee \mathrm{T} \equiv \mathrm{T}$	(6*b*) $p \wedge \mathrm{F} \equiv \mathrm{F}$
Complement laws	
(7*a*) $p \vee \neg p \equiv \mathrm{T}$	(8*a*) $\neg \mathrm{T} \equiv \mathrm{F}$
(7*b*) $p \wedge \neg p \equiv \mathrm{F}$	(8*b*) $\neg \mathrm{F} \equiv \mathrm{T}$
Involution law	
(9) $\neg\neg p \equiv p$	
DeMorgan's laws	
(10*a*) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10*b*) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

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INFERENCE

- Main rules:
  1. Law of Dethacment (modus ponendo ponens)

$$\begin{array}{c} p \\ p \Rightarrow q \\ \hline \therefore q \end{array}$$

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2. Modus tollendo tollens

$$\begin{array}{l} p \Rightarrow q \\ -q \\ \hline \therefore -p \end{array}$$

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3. Syllogism

$$\begin{array}{c} p \Rightarrow q \\ q \Rightarrow r \\ \hline \therefore p \Rightarrow r \end{array}$$

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4. Law of Simplification

$$\frac{p \,\&\, q}{\therefore p}$$

$$\frac{p \,\&\, q}{\therefore q}$$

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5. Law of Addition

$$\frac{p}{\therefore p \vee q}$$

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6.Law of Absurdity

$$\frac{p \Rightarrow q \& -q}{\therefore -p}$$

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Example :

EXAMPLE 1. *If there are no government subsidies of agriculture, then there are government controls of agriculture. If there are government controls of agriculture, there is not an agricultural depression. There is either an agricultural depression or overproduction. As a matter of fact, there is no overproduction. Therefore, there are government subsidies of agriculture.*

We want to derive the conclusion 'There are government subsidies of agriculture' from the four premises given. For clarity, here and subsequently we symbolize the argument; the meaning of the various numerals on the left is explained below.

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1	(1) $-S \rightarrow C$	Premise
2	(2) $C \rightarrow -D$	Premise
3	(3) $D \vee O$	Premise
4	(4) $-O$	Premise
3, 4	(5) D	(3) & (4) tautologically imply (5)
2, 3, 4	(6) $-C$	(2) & (5) tautologically imply (6)
1, 2, 3, 4	(7) S	(1) & (6) tautologically imply (7)

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EXAMPLE 2 (THE NATIONAL LEAGUE RACE). *If the Cards are third, then if the Dodgers are second the Braves will be fourth. Either the Giants will not be first or the Cards will be third. In fact, the Dodgers will be second. Therefore, if the Giants are first, then the Braves will be fourth.*

We use letters 'C', 'D', etc., in the obvious way; thus, C is the sentence 'The Cards are third'.

{1}	(1) $C \rightarrow (D \rightarrow B)$	P
{2}	(2) $-G \vee C$	P
{3}	(3) D	P
{4}	(4) G	P
{2, 4}	(5) C	2, 4 T
{1, 2, 4}	(6) $D \rightarrow B$	1, 5 T
{1, 2, 3, 4}	(7) B	3, 6 T
{1, 2, 3}	(8) $G \rightarrow B$	4, 7 C.P.

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Example 3 (A Horse Race). *If* A *wins, then either* B *or* C *will place. If* B *places, then* A *will not win. If* D *places, then* C *will not. Therefore, if* A *wins,* D *will not place.*

{1}	(1) A → B ∨ C	P
{2}	(2) B → −A	P
{3}	(3) D → −C	P
{4}	(4) A	P
{1, 4}	(5) B ∨ C	1, 4 T (Law of Detachment)
{2, 4}	(6) −B	2, 4 T (Laws of Double Negation and *modus tollendo tollens*)
{1, 2, 4}	(7) C	5, 6 T (*modus tollendo ponens*)
{1, 2, 3, 4}	(8) −D	3, 7 T (Laws of Double Negation and *modus tollendo tollens*)
{1, 2, 3}	(9) A → −D	4, 8 C.P.

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Consistency of Premises.

A set of premises is inconsistent if the premises cannot be true together.

Example 4. *If the contract is valid, then Horatio is liable. If Horatio is liable he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid and the bank will loan him money.*

{1}	(1) V → L	P
{2}	(2) L → B	P
{3}	(3) M → −B	P
{4}	(4) V & M	P
{4}	(5) V	4 T (Law of Simplification)
{1, 2}	(6) V → B	1, 2 T (Law of Hypothetical Syllogism)
{1, 2, 4}	(7) B	5, 6 T
{4}	(8) M	4 T
{3, 4}	(9) −B	3, 8 T
{1, 2, 3, 4}	(10) B & −B	7, 9 T (Law of Adjunction)

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REDUCTIO AD ABSURDUM (RAA)

Derived Rule for Indirect Proofs: R.A.A. *If a contradiction derivable from a set of premises and the negation of a formula* S, *then is derivable from the set of premises alone.*

Example 5. *If twenty-five divisions are enough, then the general will win the battle. Either three wings of tactical air support will be provided, or the general will not win the battle. Also, it is not the case that twenty-five divisions are enough and that three wings of tactical air support will be provided. Therefore, twenty-five divisions are not enough.*

{1}	(1) D → W	P
{2}	(2) A ∨ −W	P
{3}	(3) −(D & A)	P
{4}	(4) D	P
{1, 4}	(5) W	1, 4 T
{1, 2, 4}	(6) A	2, 5 T
{3}	(7) −D ∨ −A	3 T (De Morgan's Laws)
{1, 2, 3, 4}	(8) −D	6, 7 T
{1, 2, 3, 4}	(9) D & −D	4, 8 T
{1, 2, 3}	(10) −D	4, 9 R.A.A.

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Constant and Variabel

Constant :- well- determined meaning
- unchanged throughout the course of the consideration

For example : "number" such as "zero(0)", "one(1)" in arithmatic.

variabel :- do not possess any meaning by themselves
- can be changed

For example : x +5 = 7. Here, x is variabel.

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Quantifiers

Let A be a given set. A *propositional function* (or an *open sentence* or *condition*) defined on A is an expression

$$p(x)$$

which has the property that $p(a)$ is true or false for each $a \in A$. That is, $p(x)$ becomes a statement (with a truth value) whenever any element $a \in A$ is substituted for the variable x. The set A is called the *domain* of $p(x)$, and the set T_p of all elements of A for which $p(a)$ is true is called the *truth set* of $p(x)$. In other words,

$$T_p = \{x : x \in A,\ p(x) \text{ is true}\} \quad \text{or} \quad T_p = \{x : p(x)\}$$

Frequently, when A is some set of numbers, the condition $p(x)$ has the form of an equation or inequality involving the variable x.

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Example : Find the truth set T_p of each propositional function p(x) defined on the set P = {1, 2, 3, …}

(*a*) Let $p(x)$ be "$x + 2 > 7$". Then

$$T_p = \{x : x \in \mathbf{P},\ x + 2 > 7\} = \{6, 7, 8, \ldots\}$$

consisting of all integers greater than 5.

(*b*) Let $p(x)$ be "$x + 5 < 3$". Then

$$T_p = \{x : x \in \mathbf{P},\ x + 5 < 3\} = \varnothing$$

the empty set. In other words, $p(x)$ is not true for any positive integer in **P**.

(*c*) Let $p(x)$ be "$x + 5 > 1$". Then

$$T_p = \{x : x \in \mathbf{P},\ x + 5 > 1\} = \mathbf{P}$$

Thus $p(x)$ is true for every element in **P**.

Remark: The above example shows that if $p(x)$ is a propositional function defined on a set A then $p(x)$ could be true for all $x \in A$, for some $x \in A$, or for no $x \in A$. The next two subsections discusses quantifiers related to such propositional functions.

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Universal Quantifier

Simbol : $\forall$

Consider the expression: $(\forall x \in A)p(x)$

which reads "For every x in A, $p(x)$ is a true statement"

The expression p(x) by its self is open sentence, therefore has no truth value.

Example :

(*a*) The proposition $(\forall n \in \mathbf{P})\ (n + 4 > 3)$ is true since

$$\{n : n + 4 > 3\} = \{1, 2, 3, \ldots\} = \mathbf{P}$$

(*b*) The proposition $(\forall n \in \mathbf{P})\ (n + 2 > 8)$ is false since

$$\{n : n + 2 > 8\} = \{7, 8, \ldots\} \neq \mathbf{P}$$

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Existential Quantifier

Simbol : $\exists$

Consider the expression : $(\exists x \in A)p(x)$

which reads "there exists" or "for some" or "for at least one"

p(x) preceded by the existential quantifier doesn't have a truth value.

Example :

(*a*) The proposition $(\exists n \in \mathbf{P})\ (n + 4 < 7)$ is true since

$$\{n : n + 4 < 7\} = \{1, 2\} \neq \varnothing$$

(*b*) The proposition $(\exists n \in \mathbf{P})\ (n + 6 < 4)$ is false since

$$\{n : n + 6 < 4\} = \varnothing$$

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Write Using Quantifier

Let A = { 2,3,5} and p(x) be the sentence " x is prime "
The proposition "two is prime and three is prime and five is prime "can be denoted :
p(2) & p(3) & p(5) or ($\forall$ a $\in$ A , p(a)

Similarly , the proposition: two is prime or three is prime or five is prime, can be denoted :
P(2) $\vee$ p(3) $\vee$ p(5)
Which equvalent to:
" at least one number in A is prime" or
($\exists$a $\in$A) p(a).

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Negation

Consider the statement: "All math majors are male". Its negation is either of the following equi lent statements:

"It is not the case that all math majors are male"

"There exists at least one math major who is a female (not male)"

Symbolically, using M to denoted the set of math majors, the above can be written as

$$\neg(\forall x \in M)\ (x \text{ is male}) \equiv (\exists x \in M)\ (x \text{ is not male})$$

or, when $p(x)$ denotes "x is male",

$$\neg(\forall x \in M)p(x) \equiv (\exists x \in M)\neg p(x) \quad \text{or} \quad \neg\forall x, p(x) \equiv \exists x \neg p(x)$$

The above is true for any proposition $p(x)$. That is:

$$\neg(\forall x \in A)p(x) \equiv (\exists x \in A)\neg p(x)$$

And

$$\neg(\exists x \in A)p(x) \equiv (\forall x \in A)\neg p(x)$$

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Example :

(a) The following statements are negatives of each other:

"For all positive integers n we have $n + 2 > 8$"

"There exists a positive integer n such that $n + 2 \not> 8$"

(b) The following statements are also negatives of each other:

"There exists a college student who is 60 years old"

"Every college student is not 60 years old"

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PART 3

- Set : Definition and Notations
- Set Operations
- Algebra of Set

definition and notation

The concept of a *set* appears in all branches of mathematics. This concept formalizes the idea of grouping objects together and viewing them as a single entity. This chapter introduces this notion of a set and its members. We also investigate three basic operations on sets, that is, the operations union, intersection, and complement.

A *set* may be viewed as any well-defined collection of objects; the objects are called the *elements* or *members* of the set.

Although we shall study sets as abstract entities, we now list ten examples of sets:

(1) The numbers 1, 3, 7, and 10.
(2) The solutions of the equation $x^2 - 3x - 2 = 0$.
(3) The vowels of the English alphabet: a, e, i, o, u.
(4) The people living on the earth.
(5) The students Tom, Dick, and Harry.
(6) The students absent from school.
(7) The countries England, France, and Denmark.
(8) The capital cities of Europe.
(9) The even integers: 2, 4, 6, ...
(10) The rivers in the United States.

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A set will usually be denoted by a capital letter, such as,

$$A, B, X, Y, \ldots,$$

whereas lower-case letters, $a, b, c, x, y, z, \ldots$ will usually be used to denote elements of sets.

There are essentially two ways to specify a particular set, as indicated above. One way, if possible to list its elements. For example,

$$A = \{a, e, i, o, u\}$$

means that A is the set whose elements are the letters a, e, i, o, u. Note that the elements are separated commas and enclosed in braces { }. This is sometimes called the *tabular form* of a set.

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The second way is to state those properties which characterize the elements in the set, that is, properties held by the members of the set but not by nonmembers. Consider, for example, the expression

$$B = \{x : x \text{ is an even integer}, x > 0\}$$

which reads:

"B is the set of x such that x is an even integer and $x > 0$"

It denotes the set B whose elements are the positive even integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and". This is sometimes called the *set-builder form* or *property method* of specifying a set.

Two sets A and B are *equal*, written $A = B$, if they both have the same elements, that is, if every element which belongs to A also belongs to B, and vice versa. The negation of $A = B$ is written $A \neq B$.

The statement "p is an element of A" or, equivalently, the statement "p belongs to A" is written

$$p \in A$$

We also write

$$a, b \in A$$

to state that both a and b belong to A. The statement that p is not an element of A, that is, the negation of $p \in A$, is written

$$p \notin A$$

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Example :

(*a*) The set A above can also be written as

$$A = \{x : x \text{ is a letter in the English alphabet, } x \text{ is a vowel}\}$$

Observe that $b \notin A$, $e \in A$, and $p \notin A$.

(*b*) We cannot list all the elements of the above set B, although we frequently specify the set by writing

$$B = \{2, 4, 6, \ldots\}$$

where we assume everyone knows what we mean. Observe that $8 \in B$, but $9 \notin B$.

(*c*) Let $E = \{x : x^2 - 3x + 2 = 0\}$. In other words, E consists of those numbers which are solutions of the equation $x^2 - 3x + 2 = 0$, sometimes called the *solution set* of the given equation. Since the solutions are 1 and 2, we could also write $E = \{1, 2\}$.

(*d*) Let $E = \{x : x^2 - 3x + 2 = 0\}$, $F = \{2, 1\}$, and $G = \{1, 2, 2, 1, 6/3\}$. Then $E = F = G$ since each consists precisely of the elements 1 and 2. Observe that a set does not depend on the way in which its elements are displayed. A set remains the same even if its elements are repeated or rearranged.

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Relation between two set

1. Subset

   Set A is called subset of B , if every element of A is contain in B. this relationship is written

   $A \subseteq B$ or $B \supseteq A$.
2. Equality

   Two set A and B are equal if $A \subseteq B$ & $B \subseteq A$.
3. Disjoint Set

   two set A and B are disjoint if they have no elements in common.

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Note :

Some sets of numbers will occur very often in the text, and so we use special symbols for them. Unless otherwise specified, we will let:

N = the set of nonnegative integers: $0, 1, 2, \ldots$
P = the set of positive integers: $1, 2, 3, \ldots$
Z = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$
Q = the set of rational numbers
R = the set of real numbers
C = the set of complex numbers

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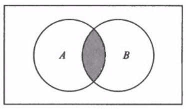
Set operations

1. Intersection

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of all elements which belong both A and B; that is,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



$A \cap B$ is shaded

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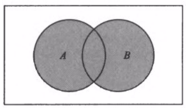
2. Union

The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or B; that is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$



$A \cup B$ is shaded

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Example :

Let $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7\}$, $C = \{2,3,8,9\}$. Then

$$A \cup B = \{1,2,3,4,5,6,7\}, \qquad A \cap B = \{3,4\}$$
$$A \cup C = \{1,2,3,4,8,9\}, \qquad A \cap C = \{2,3\}$$
$$B \cup C = \{2,3,4,5,6,7,8,9\}, \qquad B \cap C = \{3\}$$

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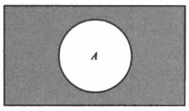
3. Complement

Recall that all sets under consideration at a particular time are subsets of a fixed universal set **U**. The *absolute complement*, or, simply, *complement* of a set A, denoted by A^c, is the set of elements which belong to **U** but which do not belong to A; that is,

$$A^c = \{x : x \in \mathbf{U}, x \notin A\}$$



A^c is shaded

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4. Difference and Symmetric Difference

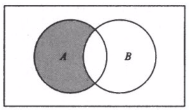
Let A and B be sets. The *relative complement* of B with respect to A or, simply, the *difference* of A and B, denoted by $A \backslash B$, is the set of elements which belong to A but which do not belong to B; that is,

$$A \backslash B = \{x : x \in A, x \notin B\}$$

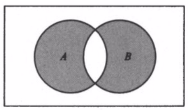
The set $A \backslash B$ is read "A minus B". Many texts denote $A \backslash B$ by $A - B$ or $A \sim B$.

The *symmetric difference* of the sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both A and B. That is,

$$A \oplus B = (A \cup B) \backslash (A \cap B) \quad \text{or} \quad A \oplus B = (A \backslash B) \cup (B \backslash A)$$



$A \backslash B$ is shaded



$A \oplus B$ is shaded

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Algebra of set

Idempotent laws	
(1a) $A \cup A = A$	(1b) $A \cap A = A$
Associative laws	
(2a) $(A \cup B) \cup C = A \cup (B \cup C)$	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	
(3a) $A \cup B = B \cup A$	(3b) $A \cap B = B \cap A$
Distributive laws	
(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	
(5a) $A \cup \varnothing = A$	(5b) $A \cap \mathbf{U} = A$
(6a) $A \cup \mathbf{U} = \mathbf{U}$	(6b) $A \cap \varnothing = \varnothing$
Involution law	
(7) $(A^c)^c = A$	
Complement laws	
(8a) $A \cup A^c = \mathbf{U}$	(8b) $A \cap A^c = \varnothing$
(9a) $\mathbf{U}^c = \varnothing$	(9b) $\varnothing^c = \mathbf{U}$
DeMorgan's laws	
(10a) $(A \cup B)^c = A^c \cap B^c$	(10b) $(A \cap B)^c = A^c \cup B^c$

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PART 4

- Relations
- Type of Relations
- Equivalence Relations
- Functions
- Compositions of Functions
- Type of Functions

Relation on Set

Introduction

The reader is familiar with many relations which are used in mathematics and computer science, e.g., "less than", "is parallel to", "is a subset of", and so on. In a certain sense, these relations consider the existence or nonexistence of certain connections between pairs of objects taken in a definite order. Formally, we define a relation in terms of these "ordered pairs".

Relations, as noted above, will be defined in terms of ordered pairs (a, b) of elements, where a is designated as the first element and b as the second element. Specifically:

$$(a, b) = (c, d) \text{ if and only if } a = c \text{ and } b = d$$

In particular, $(a, b) \neq (b, a)$ unless $a = b$. This contrasts with sets studied in Chapter 1 where the order of elements is irrelevant, for example, $\{3, 5\} = \{5, 3\}$.

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Definition: Let A and B be sets. A *binary relation* or, simply, a *relation* from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a, b) \in R$; we then say "a is *R-related* to b", written $a R b$.
(ii) $(a, b) \notin R$; we then say "a is not *R-related* to b", written $a \not R b$.

The *domain* of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R, and so it is a subset of A; and the *range* of R is the set of all second elements, and so it is a subset of B.

Sometimes R is a relation from a set A to itself, that is, R is a subset of $A^2 = A \times A$. In such a case, we say that R is a relation *on* A.

Although n-ary relations, which involve ordered n-tuples, are introduced in Section 3.11, the term relation shall mean binary relation unless otherwise stated or implied.

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Example :

(a) Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation,

$$1Ry,\ 1Rz,\ 3Ry, \qquad \text{but} \qquad 1\not{R}x,\ 2\not{R}x,\ 2\not{R}y,\ 2\not{R}z,\ 3\not{R}x,\ 3\not{R}z$$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

(b) Suppose we say that two countries are *adjacent* if they have some part of their boundaries in common. Then "is adjacent to" is a relation R on the countries of the earth. Thus:

$$(\text{Italy, Switzerland}) \in R \qquad \text{but} \qquad (\text{Canada, Mexico}) \notin R$$

(c) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of sets A and B, either $A \subseteq B$ or $A \nsubseteq B$.

(d) A familiar relation on the set **Z** of integers is "m divides n". A common notation for this relation is to write $m|n$ when m divides n. Thus $6|30$ but $7 \nmid 25$.

(e) Consider the set L of lines in the plane. Perpendicularity, written $\perp$, is a relation on L. That is, given any pair of lines a and b, either $a \perp b$ or $a \not\perp b$. Similarly, "is parallel to", written $\|$, is a relation on L since either $a \parallel b$ or $a \nparallel b$.

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Type of Relations

Consider a given set A. This section discusses a number of important types of relations which are defined on A.

(1) **Reflexive Relations**: A relation R on a set A is *reflexive* if $a\,R\,a$ for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. Thus R is not reflexive if there exists an $a \in A$ such that $(a, a) \notin R$.

(2) **Symmetric Relations**: A relation R on a set A is *symmetric* if whenever $a\,R\,b$ then $b\,R\,a$, that is, if whenever $(a, b) \in R$, then $(b, a) \in R$. Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

(3) **Antisymmetric Relations**: A relation R on a set A is *antisymmetric* if whenever $a\,R\,b$ and $b\,R\,a$ then $a = b$, that is, if whenever (a, b) and (b, a) belong to R then $a = b$. Thus R is not antisymmetric if there exist $a, b \in A$ such that (a, b) and (b, a) belong to R, but $a \neq b$.

(4) **Transitive Relations**: A relation R on a set A is *transitive* if whenever $a\,R\,b$ and $b\,R\,c$ then $a\,R\,c$, that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$. Thus R is not transitive if there exist $a, b, c \in A$ such that $(a, b), (b, c) \in R$, but $(a, c) \notin R$.

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Examples :

Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$$\begin{aligned}
R_1 &= \{(1,1),\ (1,2),\ (2,3),\ (1,3),\ (4,4)\} \\
R_2 &= \{(1,1),\ (1,2),\ (2,1)\,(2,2),\ (3,3),\ (4,4)\} \\
R_3 &= \{(1,3),\ (2,1)\} \\
R_4 &= \varnothing, \text{ the empty relation} \\
R_5 &= A \times A, \text{ the universal relation}
\end{aligned}$$

Determine which of the relations are: (a) reflexive, (b) symmetric, (c) antisymmetric, (d) transitive.

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Solutions :

(*a*) Since A contains the four elements 1, 2, 3, 4, a relation R on A is reflexive if it contains the four pairs $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$. Thus only R_2 and the universal relation $R_5 = A \times A$ are reflexive. Note that R_1, R_3, and R_4 are not reflexive since, for example, $(2,2)$ does not belong to any of them.

(*b*) R_1 is not symmetric since $(1,2) \in R_1$ but $(2,1) \notin R_1$. R_3 is not symmetric since $(1,3) \in R_3$ but $(3,1) \notin R_3$. The other relations are symmetric.

(*c*) R_2 is not antisymmetric since $(1,2)$ and $(2,1)$ belong to R_2, but $1 \neq 2$. Similarly, the universal relation R_5 is not antisymmetric. All the other relations are antisymmetric.

(*d*) The relation R_3 is not transitive since $(2,1), (1,3) \in R_3$ but $(2,3) \notin R_3$. All the other relations are transitive.

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Equivalence Relations

Consider a nonempty set S. A relation R on S is an *equivalence relation* if R is reflexive, symmetric, and transitive. That is, R is an equivalence relation on S if it has the following three properties:

(1) For every $a \in S$, $a R a$.
(2) If $a R b$, then $b R a$.
(3) If $a R b$ and $b R c$, then $a R c$.

The general idea behind an equivalence relation is that it is a classification of objects which are in some way "alike". In fact, the relation $=$ of equality on any set S is an equivalence relation; that is,

(1) $a = a$ for every $a \in S$.
(2) If $a = b$, then $b = a$.
(3) If $a = b$ and $b = c$, then $a = c$.

For this reason, one frequently uses $\sim$ or $\equiv$ to denote an equivalence relation.

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Examples :

(*a*) Consider the set L of lines and the set T of triangles in the Euclidean plane. The relation "is parallel to or identical to" is an equivalence relation on L, and congruence and similarity are equivalence relations on T.

(*b*) The classification of animals by species, that is, the relation "is of the same species as," is an equivalence relation on the set of animals.

(*c*) The relation $\subseteq$ of set inclusion is not an equivalence relation. It is reflexive and transitive, but it is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$.

(*d*) Let m be a fixed positive integer. Two integers a and b are said to be *congruent modulo m*, written

$$a \equiv b \pmod{m}$$

if m divides $a - b$. For example, for $m = 4$ we have $11 \equiv 3 \pmod 4$ since 4 divides $11 - 3$, and $22 \equiv 6 \pmod 4$ since 4 divides $22 - 6$. This relation of congruence modulo m is an equivalence relation.

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Partitions

Suppose R is an equivalence relation on a set S. For each a in S, let $[a]$ denote the set of elements of S to which a is related under R; that is,

$$[a] = \{x : (a, x) \in R\}$$

We call $[a]$ the *equivalence class* of a in S under R. The collection of all such equivalence classes is denoted by S/R, that is,

$$S/R = \{[a] : a \in S\}$$

It is called the *quotient* set of S by R.

Theorem ☐ Let R be an equivalence relation on a set S. Then the quotient set S/R is a partition of S. Specifically:

(i) For each a in S, we have $a \in [a]$.
(ii) $[a] = [b]$ if and only if $(a, b) \in R$.
(iii) If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

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Examples :

(*a*) Consider the following relation R on $S = \{1, 2, 3, 4\}$:

$$R = \{(1,1),\ (2,2),\ (1,3),\ (3,1),\ (3,3),\ (4,4)\}$$

One can show that R is reflexive, symmetric and transitive, that is, that R is an equivalence relation. Under the relation R,

$$[1] = \{1,3\}, \quad [2] = \{2\}, \quad [3] = \{1,3\}, \quad [4] = \{4\}$$

Observe that $[1] = [3]$ and that $S/R = \{[1], [2], [4]\}$ is a partition of S. One can choose either $\{1, 2, 4\}$ or $\{2, 3, 4\}$ as a system of representatives of the equivalence classes.

(*b*) Let R_5 be the relation on the set $\mathbf{Z}$ of integers defined by

$$x \equiv y \pmod 5$$

which reads "x is congruent to y modulo 5" and which means that the difference $x - y$ is divisible by 5. Then R_5 is an equivalence relation on $\mathbf{Z}$. There are exactly five equivalence classes in the quotient set $\mathbf{Z}/R_5$ as follows:

$$A_0 = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$$
$$A_1 = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$$
$$A_2 = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$$
$$A_3 = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$$
$$A_4 = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$$

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Observe that any integer x, which can be uniquely expressed in the form $x = 5q + r$ where $0 \le r < 5$, is a member of the equivalence class A_r where r is the remainder. As expected, the equivalence classes are disjoint and

$$\mathbf{Z} = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$$

This quotient set $\mathbf{Z}/R_5$ is usually denoted by

$$\mathbf{Z}/5\mathbf{Z} \text{ or simply } \mathbf{Z}_5$$

Usually one chooses $\{0, 1, 2, 3, 4\}$ or $\{-2, -1, 0, 1, 2\}$ as a system of representatives of the equivalence classes.

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FUNCTION

One of the most important concepts in mathematics is that of a function. The terms "map", "mapping", "transformation", and many others mean the same thing; the choice of which word to use in a given situation is usually determined by tradition and the mathematical background of the person using the term.

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a *function* from A into B. The set A is called the *domain* of the function, and the set B is called the *target set*.

Functions are ordinarily denoted by symbols. For example, let f denote a function from A into B. Then we write

$$f: A \to B$$

which is read: "f is a function from A into B", or "f takes A into B", or "f maps A into B".

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Suppose $f: A \to B$ and $a \in A$. Then $f(a)$ [read: "f of a"] will denote the unique element of B which f assigns to a. This element $f(a)$ in B is called the *image* of a under f or the *value* of f at a. We also say that f *sends* or *maps* a into $f(a)$. The set of all such image values is called the *range* or *image* of f, and it is denoted by Ran(f), Im(f) or $f(A)$. That is,

$$\mathrm{Im}(f) = \{b \in B : \text{there exists } a \in A \text{ for which } f(a) = b\}$$

We emphasize that Im(f) is a subset of the target set B.

Frequently, a function can be expressed by means of a mathematical formula. For example, consider the function which sends each real number into its square. We may describe this function by writing

$$f(x) = x^2 \qquad \text{or} \qquad x \mapsto x^2 \qquad \text{or} \qquad y = x^2$$

In the first notation, x is called a *variable* and the letter f denotes the function. In the second notation, the barred arrow $\mapsto$ is read "goes into". In the last notation, x is called the *independent variable* and y is called the *dependent variable* since the value of y will depend on the value of x.

Furthermore, suppose a function is given by a formula in terms of a variable x. Then we assume, unless otherwise stated, that the domain of the function is **R** or the largest subset of **R** for which the formula has meaning, and that the target set is **R**.

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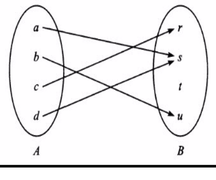
Examples :

1. Consider the function $f(x) = x^3$, i.e., f assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2) = 8$. Similarly, $f(-3) = -27$, and $f(0) = 0$.
2. Let g assign to each country in the world its capital city. Here the domain of g is the set of all the countries in the world, and the target set is the list of cities in the world. The image of France under g is Paris; that is g(France) = Paris. Similarly, g(Denmark) = Copenhagen and g(England) = London.
3. defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$ in the obvious way; that is,

$$f(a) = s, \qquad f(b) = u, \qquad f(c) = r, \qquad f(d) = s$$

The image of f is the set $\{r, s, u\}$. Note that t does not belong to the image of f because t is not the image of any element of A under f.



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COMPOSITION OF FUNCTION

Consider functions $f: A \to B$ and $g: B \to C$, that is, where the target set B of f is the domain of g. This relationship can be pictured by the following diagram:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Let $a \in A$; then its image $f(a)$ under f is in B which is the domain of g. Accordingly, we can find the image of $f(a)$ under the function g, that is, we can find $g(f(a))$. Thus we have a rule which assigns to each element a in A an element $g(f(a))$ in C or, in other words, f and g give rise to a well defined function from A to C. This new function is called the *composition* of f and g, and it is denoted by

$$g \circ f$$

More briefly, if $f: A \to B$ and $g: B \to C$, then we define a new function $g \circ f: A \to C$ by

$$(g \circ f)(a) \equiv g(f(a))$$

Here $\equiv$ is used to mean equal by definition.

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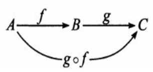
Note that we can now add the function $g \circ f$ to the above diagram of f and g as follows:

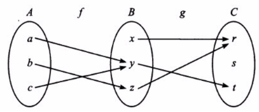


We emphasize that the composition of f and g is written $g \circ f$, and not $f \circ g$; that is, the composition of functions is read from right to left, and not from left to right.

(*a*) Let $f: A \to B$ and $g: B \to C$ be the functions defined by Fig. 4-3. We compute $g \circ f: A \to C$ by its definitio

$$(g \circ f)(a) \equiv g(f(a)) = g(y) = t, \quad (g \circ f)(b) \equiv g(f(b)) = g(z) = r, \quad (g \circ f)(c) \equiv g(f(c)) = g(y) = t$$

Observe that the composition $g \circ f$ is equivalent to "following the arrows" from A to C in the diagrams of th functions f and g.



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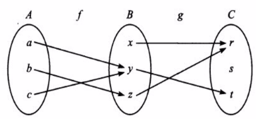
Examples :

(*a*) Let $f: A \to B$ and $g: B \to C$ be the functions defined by Fig. 4-3. We compute $g \circ f: A \to C$ by its definition:

$$(g \circ f)(a) \equiv g(f(a)) = g(y) = t, \quad (g \circ f)(b) \equiv g(f(b)) = g(z) = r, \quad (g \circ f)(c) \equiv g(f(c)) = g(y) = t$$

Observe that the composition $g \circ f$ is equivalent to "following the arrows" from A to C in the diagrams of the functions f and g.



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(*b*) Let $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$ and $g(x) = x + 3$. Then

$$(g \circ f)(2) \equiv g(f(2)) = g(4) = 7; \quad (f \circ g)(2) \equiv f(g(2)) = f(5) = 25$$

Thus the composition functions $g \circ f$ and $f \circ g$ are not the same function. We compute a general formula for these functions:

$$(g \circ f)(x) \equiv g(f(x)) = g(x^2) = x^2 + 3$$
$$(f \circ g)(x) \equiv f(g(x)) = f(x + 3) = (x + 3)^2 = x^2 + 6x + 9$$

(*c*) Consider any function $f : A \to B$. Then one can easily show that

$$f \circ 1_A = f \quad \text{and} \quad 1_B \circ f = f$$

where 1_A and 1_B are the identity functions on A and B, respectively. In other words, the composition of any function with the appropriate identity function is the function itself.

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Type of Function

1. one-to-one function / injective

A function $f: A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing follows:

f is one-to-one if $f(a) = f(a')$ implies $a = a'$

1. Onto function / surjective

A function $f: A \to B$ is said to be an *onto* function if every element of B is the image of some element in A or, in other words, if the image of f is the entire target set B. In such a case we say that f is a function of A onto B or that f maps A onto B. That is:

f maps A onto B if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$

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3. one-to-one correspondence/bijective

If $f: A \to B$ is both one-to-one and onto, then f is called a *one-to-one correspondence* between A and B. This terminology comes from the fact that each element of A will correspond to a unique element of B and vice versa.

Some texts use the term *injective* for a one-to-one function, *surjective* for an onto function, and *bijective* for a one-to-one correspondence.

4. Invertible

A function $f: A \to B$ is said to be *invertible* if its inverse relation f^{-1} is a function from B to A. Equivalently, $f: A \to B$ is *invertible* if there exists a function $f^{-1}: B \to A$, called the *inverse* of f, such that

$$f^{-1} \circ f = 1_A \quad \text{and} \quad f \circ f^{-1} = 1_B$$

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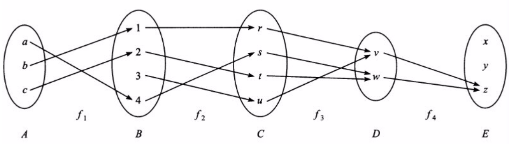
Example :

Consider function f1: A →B, f2 : B → C , f3: C → D , and f4 : D → E defined by figure below :



Note :

- f1 is one-to one but not onto
- f2 is onto and one-to one, hence invertibel
- f3 is onto but not one-to-one
- f4 is neither one-to one nor onto

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PART 5

- Introduction
- Equipotent Set
- Infinite Set
- Denumerable and Countable Set
- Real Number

Introduction

It is natural to ask whether or not two sets have the same number of elements. For finite sets the answer can be found by simply counting the number of elements. For example, each of the sets

$$\{a, b, c, d\}, \qquad \{2, 3, 5, 7\}, \qquad \{x, y, z, t\}$$

has four elements. Thus these sets have the same number of elements. However, it is not always necessary to know the number of elements in two finite sets before we know that they have the same number of elements. For example, if each chair in a room is occupied by exactly one person and there is no one standing, then clearly there are "just as many" people as there are chairs in the room.

The above simple notion, that two sets have "the same number of elements" if their elements can be "paired-off", can also apply to infinite sets. In fact, it has the following startling results:

(*a*) Infinite sets need not have the "same number of elements"; some are "more infinite" than others.
(*b*) There are "just as many" even integers as there are integers, and "just as many" rational numbers **Q** as positive integers **P**.
(*c*) There are "more" points on the real line **R** than there are positive integers **P**; and there are "more" curves in the plane $\mathbf{R}^2$ than there are points in the plane.

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Equipotent Set

Sets A and B are said to have the *same cardinality* or the *same number of elements*, or to be *equipotent*, written

$$A \approx B$$

if there is a function $f: A \to B$ which is bijective, that is, both one-to-one and onto.

The relation $\approx$ of being equipotent is an equivalence relation in any collection of sets. That is:

(i) $A \approx A$ for any set A.
(ii) If $A \approx B$, then $B \approx A$.
(iii) If $A \approx B$ and $B \approx C$, then $A \approx C$.

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Examples :

(*a*) Let A and B be sets with exactly three elements, say,

$$A = \{2, 3, 5\}, \qquad \text{and } B = \{\text{Marc, Erik, Audrey}\}$$

Then clearly we can find a one-to-one correspondence between A and B. For example, we can label the elements of A as the first element, the second element, and the third element, and label B similarly. Then the rule which pairs the first elements of A and B, pairs the second elements of A and B, and pairs the third elements of A and B, that is, the function $f: A \to B$ defined by

$$f(2) = \text{Marc}, \qquad f(3) = \text{Erik}, \qquad f(5) = \text{Audrey}$$

is one-to-one and onto. Thus A and B are equipotent.

The same idea may be used to show that any two finite sets with the same number of elements are equipotent.

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(*b*) Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. Then A and B are not equipotent. For suppose there were a rule for pairing the elements of A and B. If there were four or more pairs, then an element of B would be used twice, and if there were three or fewer pairs then some element of A would not be used. In other words, since A has more elements than B, any function $f: A \to B$ must assign at least two elements of A to the same element of B, and hence f would not be one-to-one.

In a similar way, we can see that any two finite sets with different numbers of elements are not equipotent.

(*c*) Let $\mathbf{I} = [0, 1]$, the closed unit interval, and let S be any other closed interval, say $S = [a, b]$ where $a < b$. The function $f: \mathbf{I} \to S$ defined by

$$f(x) = (b - a)x + a$$

is one-to-one and onto. Thus $\mathbf{I}$ and S have the same cardinality.

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Infinite Set

A set S is infinite if it has same cardinality as a proper subset of itself. Otherwise S is finite.

Example :

Consider any two sets A and B. Let $A' = A \times \{1\}$ and $B' = B \times \{2\}$. Then

$$A \approx A' \quad \text{and} \quad B \approx B'$$

For example, the functions

$$f(a) = (a, 1),\ a \in A \quad \text{and} \quad g(b) = (b, 2),\ b \in B$$

are each bijective. Although A and B need not be disjoint, the sets A' and B' are disjoint, i.e.,

$$A' \cap B' = \varnothing$$

Specifically, each ordered pair in A' has 1 as a second component, whereas each ordered pair in B component.

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Denumerable and Countable Set

- A set D is said to be denumerable or countably infinite if it has same cardinality as N (natural number)
- A set is countable if it is finite or denumerable
- A set is nondenumerable if it is not countable

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Examples :

(a) Any infinite sequence

$$a_1, a_2, a_3, \ldots$$

of distinct elements is countably infinite, for a sequence is essentially a function $f(n) = a_n$ whose domain is **P**. So if the a_n are distinct, the function is one-to-one and onto. Thus each of the following sets is countably infinite:

$$\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$$
$$\{1, -2, 3, - \ldots 1)^{n-1}n, \ldots\}$$
$$\{(1,1), (4,8), (9,27), \ldots, (n^2, n^3), \ldots\}$$

(b) Consider the product set $\mathbf{P} \times \mathbf{P}$ as exhibited in Fig. 6-1. The set $\mathbf{P} \times \mathbf{P}$ can be written as an infinite sequence as follows:

$$\{(1,1),\ (2,1),\ (1,2),\ (1,3),\ (2,2), \ldots\}$$

This sequence is determined by "following the arrows" in Fig. 6-1. Thus $\mathbf{P} \times \mathbf{P}$ is countably infinite for the reasons stated in (a).

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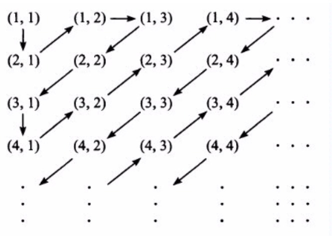

Fig. 6-1

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(*c*) Recall that $\mathbf{N} = \{0, 1, 2, \ldots\} = \mathbf{P} \cup \{0\}$ is the set of natural numbers or nonnegative integers. Now each positive integer $a \in \mathbf{P}$ can be written uniquely in the form

$$a = 2^r(2s + 1)$$

where $r, s \in \mathbf{N}$. Consider the function $f\colon \mathbf{P} \to \mathbf{N} \times \mathbf{N}$ defined by

$$f(a) = (r, s)$$

where r and s are as above. Then f is one-to-one and onto. Thus $\mathbf{N} \times \mathbf{N}$ is denumerable (countably infinite) or, in other words, $\mathbf{N} \times \mathbf{N}$ has the same cardinality as $\mathbf{P}$. Note that $\mathbf{P} \times \mathbf{P}$ is a subset of $\mathbf{N} \times \mathbf{N}$.

Note :

Every infinite set contains a subset which is denumerable.

A subset of a denumerable set is finite or denumerable.

A subset of a countable set is countable.

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Real Number

- Not every infinite set is countable
- Example :

  the unit interval I = (0,1) is nondenumerable.

Proof:

Assume I is denumerable. Then

$$\mathbf{I} = \{x_1, x_2, x_3, \ldots\}$$

that is, the elements of **I** can be written in a sequence.

Now each element in **I** can be written in the form of an infinite decimal as follows:

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$$x_1 = 0.a_{11}a_{12}a_{13}\cdots a_{1n}\cdots$$
$$x_2 = 0.a_{21}a_{22}a_{23}\cdots a_{2n}\cdots$$
....
$$x_n = 0.a_{n1}a_{n2}a_{n3}\cdots a_{nn}\cdots$$
....

where $a_{ij} \in \{0, 1, \ldots, 9\}$ and where each decimal contains an infinite number of nonzero elements. Thus we write 1 as 0.999... and, for those numbers which can be written in the form of a decimal in two ways, for example,

$$1/2 = 0.5000\ldots = 0.4999\ldots$$

(in one of them there is an infinite number of nines and in the other all except a finite set of digits are zeros), we write the infinite decimal in which an infinite number of nines appear.

Now construct the real number

$$y = 0.b_1b_2b_3\cdots b_n\cdots$$

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which will belong to **I**, in the following way:

Choose b_1 so $b_1 \neq a_{11}$ and $b_1 \neq 0$. Choose b_2 so $b_2 \neq a_{22}$ and $b_2 \neq 0$. And so on.

Note $y \neq x_1$ since $b_1 \neq a_{11}$ (and $b_1 \neq 0$); $y \neq x_2$ since $b_2 \neq a_{22}$ (and $b_2 \neq 0$), and so on. That is, $y \neq x_n$ for all $n \in \mathbf{P}$. Thus $y \notin \mathbf{I}$, which contradicts the fact that $y \in \mathbf{I}$. Thus the assumption that **I** is denumerable has led to a contradiction. Consequently, **I** is nondenumerable.

Definition :

A set A is said to have the *power of the continuum* if A has the same cardinality as the unit interval $\mathbf{I} = [0, 1]$.

Besides the unit interval **I**, all the other intervals also have the power of the continuum. There are several such kinds of intervals. Specifically, if a and b are real numbers with $a < b$, then we define:

closed interval:	$[a, b] = \{x \in \mathbf{R} : a \leq x \leq b\}$
open interval:	$(a, b) = \{x \in \mathbf{R} : a < x < b\}$
half-open intervals:	$[a, b) = \{x \in \mathbf{R} : a \leq x < b\}$
	$(a, b] = \{x \in \mathbf{R} : a < x \leq b\}$

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Cardinal Number

Frequently, we want to know the "size" of a given set without necessarily comparing it to another set. For finite sets, there is no difficulty. For example, the set $A = \{a, b, c\}$ has 3 elements. Any other set with 3 elements is equipotent to A. On the other hand, for infinite sets it is not sufficient to just say that the set has infinitely many elements since not all infinite sets are equipotent. To solve this problem, we introduce the concept of a cardinal number.

Each set A is assigned a symbol in such a way that two sets A and B are assigned the same symbol if and only if they are equipotent. This symbol is called the *cardinality* or *cardinal number* of A, and it is denoted by

$$|A|, \qquad n(A), \qquad \text{or} \qquad \text{card}(A)$$

$$|A| = |B| \quad \text{if and only if} \quad A \approx B$$

$$|A| = n \quad \text{if and only if} \quad A \approx \{1, 2, \ldots, n\}$$

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Definition

Cardinal numbers of infinite sets are called *infinite* or *transfinite cardinal numbers*. The cardinal number of the infinite set **P** of positive integers is $\aleph_0$ which is read aleph-nought. This notation was introduced by Cantor.

$$|A| = \aleph_0 \quad \text{if and only if} \quad A \approx \mathbf{P}$$

In particular, we have $|\mathbf{Z}| = \aleph_0$ and $|\mathbf{Q}| = \aleph_0$.

The cardinal number of the unit interval $\mathbf{I} = [0, 1]$ is denoted by: $\mathbf{c}$ and it is called the *power of the continuum*.

$$|A| = \mathbf{c} \quad \text{if and only if} \quad A \approx \mathbf{I}$$

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In particular, we have $|\mathbf{R}| = \mathbf{c}$, and the cardinal number of any interval is $\mathbf{c}$.

The following statements follow directly from the above definitions:

(*a*) A is denumerable or countably infinite means $|A| = \aleph_0$.
(*b*) A is countable means $|A|$ is finite or $|A| = \aleph_0$.
(*c*) A has the power of the continuum means $|A| = \mathbf{c}$.

Definition :

Let A and B be sets. We say that

$$|A| \le |B|$$

if A has the same cardinality as a subset of B or, equivalently, if there exists a one-to-one (injective) function $f: A \to B$.

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Examples :

(*a*) Let A be a proper subset of a finite set B. Clearly, $|A| \le |B|$. Since A is a proper subset of B, where A and B are finite, we know that $|A| \ne |B|$. Thus $|A| < |B|$. In other words, for finite cardinals m and n, we have $m < n$ as cardinal numbers if and only if $m < n$ as nonnegative integers. Accordingly, the inequality relation $\le$ for cardinal numbers is an extension of the inequality relation $\le$ for nonnegative integers.

(*b*) Let n be a finite cardinal. Then $n < \aleph_0$ since any finite set A is equipotent to a subset of **P** and $|A| \ne |\mathbf{P}|$. Thus we may write

$$0 < 1 < 2 < \cdots < \aleph_0$$

(*c*) Consider the set **P** of positive integers and the unit interval **I**, that is, consider the sets

$$\mathbf{P} = \{1, 2, 3, \ldots\} \qquad \text{and} \qquad \mathbf{I} = \{x \in \mathbf{R} : 0 \le x \le 1\}$$

The function $f: \mathbf{P} \to \mathbf{I}$ defined by $f(n) = 1/n$ is one-to-one. Therefore, $|\mathbf{P}| \le |\mathbf{I}|$. On the other hand, by Theorem 6.7, $|\mathbf{P}| \ne |\mathbf{I}|$. Therefore, $\aleph_0 = |\mathbf{P}| < |\mathbf{I}| = \mathbf{c}$. Accordingly, we may now write

$$0 < 1 < 2 < \cdots < \aleph_0 < \mathbf{c}$$

(*d*) Let A be any infinite set. By Theorem 6.2, A contains a subset which is denumerable. Accordingly, for any infinite set A, we always have $\aleph_0 \le |A|$.

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