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NONISOCLINIC 2-CODIMENSIONAL 4-WEBS OF MAXIMUM 2-RANK

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ABSTRACT. In recent papers, the author has proved that 4-webs $W(4,2,2)$ of codimension 2 and maximum 2-rank on a 4-dimensional differentiable manifold are exceptional in the sense that they are not necessarily algebraizable, while maximum 2-rank 2-codimensional d-webs $W(d,2,2)$, $d > 4$, are algebraizable. Examples of exceptional isoclinic webs $W(4,2,2)$ were given in those papers. In the present paper, the author proves that a polynomial nonisoclinic 3-web $W(3,2,2)$ cannot be extended to a nonisoclinic 4-web $W(4,2,2)$ and constructs an example of a nonisoclinic 4-web $W(4,2,2)$ of maximum 2-rank.

1. Preliminaries and introduction. A 4-*web* $W(4,2,2)$ of codimension 2 is given in an open domain D of a differentiable 4-dimensional manifold X^4 by four 2-codimensional foliations X_a, $a = 1,2,3,4$, in D if the tangent 2-planes to the leaves (web surfaces) of X_a passing through any point of D are in general position.

Note that the first number in the notation $W(4,2,2)$ gives the number of foliations, the third one means the codimension, and the second number is the ratio of the dimension of the ambient manifold and the codimension.

Two webs $W(4,2,2)$ and $\tilde{W}(4,2,2)$ are *equivalent* if there exists a local diffeomorphism $\phi\colon D \to \tilde{D}$ mapping the foliations of W into the foliations of $\tilde{W}$.

The foliations X_a can be given by four completely integrable systems of Pfaffian equations $\underset{a}{\omega}^i = 0$, $a = 1,2,3,4$, $i = 1,2$, where the forms $\underset{1}{\omega}^i$ and $\underset{2}{\omega}^i$ and basis forms of X^4 and

$$(1.1)\qquad \begin{cases} -\underset{3}{\omega}^i = \underset{1}{\omega}^i + \underset{2}{\omega}^i, & -\underset{4}{\omega}^i = \lambda^i_j \underset{1}{\omega}^j + \underset{2}{\omega}^i, \qquad i,j = 1,2, \\ \det(\lambda^i_j) \neq 0, & \det(\delta^i_j - \lambda^i_j) \neq 0. \end{cases}$$

The quantities λ^i_j form a (1,1)-tensor. It is called the *basis affinor* of $W(4,2,2)$ [**7, 8**].

For $x \in D \subset X^4$ we have

$$(1.2)\qquad dx = \underset{1}{\omega}^i e^1_i + \underset{2}{\omega}^i e^2_i.$$

It follows from (1.1) and (1.2) that the vectors $e^2_i, e^1_i, e^3_i = e^1_i - e^2_i$ and $e^4_i = e^1_i - \lambda^j_i e^2_j$ are tangent vectors to the leaves V_1, V_2, V_3, and V_4 at the point x.

Let V be a 2-dimensional surface in D which is determined by the system $\gamma\underset{1}{\omega}^i + \underset{2}{\omega}^i = 0$ where γ is a function of a point $x \in D$. On the surface V we have $dx = \underset{1}{\omega}^i(e^1_i - \gamma e^2_i)$.

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A web $W(4,2,2)$ whose basis affinor λ^i_j is scalar;

$$\lambda^i_j = \delta^i_j \lambda, \tag{1.3}$$

is said to be an *almost Grassmannizable* web. We will denote it by $AGW(4,2,2)$.

The vectors $\xi^a = \xi^i e^a_i$, $a = 1,2,3,4$, are tangent to the leaves V_a at the point x. For $AGW(4,2,2)$ they lie in a 2-plane. The bivector $\xi^1 \wedge \xi^2$ determined by ξ^a is said to be a *transversal bivector* of $AGW(4,2,2)$. Equation (1.2) shows that the tangent plane of V intersects $\xi^1 \wedge \xi^2$ in the direction of the vector $\xi = \xi^i(e^1_i - \gamma e^2_i)$. The anharmonic ratio of ξ and ξ^1, ξ^2, ξ^3 (ξ^4) is equal to γ (γ/λ) and does not depend on ξ^i. The surface V is called an *isoclinic surface* of $AGW(4,2,2)$.

A web $AGW(4,2,2)$ is said to be *isoclinic* if there exists a one-parameter family of isoclinic surfaces through any point $x \in D$.

A web $AGW(4,2,2)$ is said to be *transversally geodesic* if for any $\xi^1 \wedge \xi^2$ there exists a two-dimensional surface U tangent to $\xi^1 \wedge \xi^2$ at x and each $\xi^1 \wedge \xi^2$ is tangent to one and only one U.

A web $AGW(4,2,2)$ which is isoclinic and transversally geodesic is a *Grassmannizable web*, i.e., it is equivalent to a *Grassmann* 4-*web* formed by four foliations of Schubert varieties of codimension 2 on the Grassmannian $G(1,3)$ in a 5-dimensional projective space $\mathbf{P}^5$. Each foliation X_a of Schubert varieties is the image of the bundles of straight lines of a three-dimensional projective space $\mathbf{P}^3$ whose vertices are on a surface V_a in $\mathbf{P}^3$. If the surfaces V_a belong to an algebraic surface V^2_4 of degree 4, the Grassmann web is *algebraic*. A Grassmannizable web which is equivalent to an algebraic web is said to be *algebraizable*.

Suppose that the leaves of the foliations X_a of a web $W(4,2,2)$ are level sets $u^i_a(x) = \text{const}$ of functions $u^i_a(x)$, $x \in D$. The functions $u^i_a(x)$ are defined up to a local diffeomorphism in the space of u^i_a.

An equation of the form

$$\sum_{a=1}^{4} f_a(u^j_a)\, du^1_a \wedge du^2_a = 0 \tag{1.4}$$

is said to be an *abelian* 2-*equation*. The number R_2 of linearly independent abelian 2-equations is called the 2-*rank* of $W(4,2,2)$ (see [**13**]).

Two fundamental problems in web geometry are: finding an upper bound for the rank, and the determination of the maximum rank webs. For d-webs $W(d,n,1)$ of codimension one in X^n the first problem was solved by S. S. Chern [**3**] and the second by S. S. Chern and P. A. Griffiths [**4**, **5**]. For d-webs $W(d,n,r)$ of codimension $r > 1$ in X^{nr} the first problem was solved by S. S. Chern and P. A. Griffiths [**6**]. The author solved both problems for webs $AGW(d,2,2), d > 4$, and $W(4,2,2)$ (see [**9**, **10**]). In particular, it was proved in [**9**, **10**] that:

(i) A web $AGW(d,2,2)$, $d > 4$, is of maximum 2-rank if and only if it is algebraizable.

(ii) A nonisoclinic web $W(4,2,2)$ is of maximum 2-rank if and only if it is an almost Grassmannizable web for which any of the four affine connections indicated in [**9**, **10**] is equiaffine.

This shows that webs $W(4,2,2)$ of maximum 2-rank are exceptional in the sense that they are not necessarily algebraizable while a web $AGW(d,2,2)$, $d > 4$, is of maximum 2-rank if and only if it is algebraizable.

Statement (ii) presents a geometric description of nonisoclinic webs W(4, 2, 2) of maximum 2-rank. However, the existence of such webs was not discussed in [**9**, **10**]. In the recent papers [**11**, **12**], the author proved the existence of isoclinic webs of maximum 2-rank presenting a step-by-step construction of such webs and realizing this construction in three examples.

The purpose of this paper is to prove the existence of nonisoclinic webs W(4, 2, 2) of maximum 2-rank by construction of an example.

2. Nonisoclinic webs W(4, 2, 2) **of maximum 2-rank.** Since a web W(4, 2, 2) of maximum 2-rank is always almost Grassmannizable (see [**9**, **10**]), its basis affinor has the form (1.3) and equations (1.1) can be written in the form

$$-\underset{3}{\omega}^i = \underset{1}{\omega}^i + \underset{2}{\omega}^i, \quad -\underset{4}{\omega}^i = \lambda \underset{1}{\omega}^i + \underset{2}{\omega}^i, \qquad \lambda \neq 0, 1. \tag{2.1}$$

In addition, if a web AGW(4, 2, 2) is nonisoclinic, we have (see [**9**, **10**])

$$d\underset{1}{\omega}^i = \underset{1}{\omega}^j \wedge \omega_j^i + a_j \underset{1}{\omega}^j \wedge \underset{1}{\omega}^i, \quad d\underset{2}{\omega}^i = \underset{2}{\omega}^j \wedge \omega_j^i - a_j \underset{2}{\omega}^j \wedge \underset{2}{\omega}^i, \tag{2.2}$$

$$d\omega_j^i - \omega_j^k \wedge \omega_k^i = b^i_{jkl} \underset{1}{\omega}^k \wedge \underset{2}{\omega}^l, \tag{2.3}$$

$$b^i_{[jkl]} = 0, \tag{2.4}$$

$$d\lambda = \lambda(b_i - a_i)\underset{1}{\omega}^i + (b_i - \lambda a_i)\underset{2}{\omega}^i, \tag{2.5}$$

$$da_i - a_j \omega_i^j = p_{ij} \underset{1}{\omega}^j + q_{ij} \underset{2}{\omega}^j, \tag{2.6}$$

$$db_i - b_j \omega_i^j = [b_i(b_j - a_j) + \lambda(b_{ji} + p_{ij} - q_{ji})]\underset{1}{\omega}^j + b_{ij}\underset{2}{\omega}^j, \tag{2.7}$$

$$b_{[ij]} = p_{[ij]} = \lambda q_{[ij]}, \tag{2.8}$$

$$\nabla p_{ij} = \underset{1}{p}_{ijk} \underset{1}{\omega}^k + \underset{2}{p}_{ijk} \underset{2}{\omega}^k, \quad \nabla q_{ij} = \underset{1}{q}_{ijk} \underset{1}{\omega}^k + \underset{2}{q}_{ijk} \underset{2}{\omega}^k, \tag{2.9}$$

$$\begin{cases} \underset{1}{p}_{i[jk]} + p_{i[j} a_{k]} = 0, & \underset{2}{q}_{i[jk]} - q_{i[j} a_{k]} = 0, \\ \underset{2}{p}_{ijk} - \underset{2}{q}_{ikj} + a_m b^m_{ijk} = 0, & \end{cases} \tag{2.10}$$

where d in (2.2) and (2.3) is the symbol of the exterior differential, $[ij]$ and $[jkl]$ in (2.8), (2.10) and (2.4) mean the alternation with respect to i, j and j, k, l correspondingly, and

$$\nabla p_{ij} = dp_{ij} - p_{kj}\omega_i^k - p_{ik}\omega_j^k, \quad \nabla q_{ij} = dq_{ij} - q_{kj}\omega_i^k - q_{ik}\omega_j^k.$$

The quantities

$$a^i_{jk} = a_{[j}\delta^i_{k]} \tag{2.11}$$

and b^i_{jkl} are the *torsion* and *curvature tensors of* AGW(4, 2, 2).

Denote

$$p_{[12]} = p, \quad q_{[12]} = q. \tag{2.12}$$

The 3-subweb [1, 2, 3] defined by the foliations X_1, X_2, and X_3 is nonisoclinic if and only if (see [**9**] or [**10**])

$$p \neq 0 \quad \text{or} \quad q \neq 0. \tag{2.13}$$

This 3-web can be uniquely extended to a nonisoclinic AGW(4, 2, 2) if and only if two conditions are satisfied. The first condition has the form of inequalities:

$$p \neq 0, \quad q \neq 0, \quad p \neq q. \tag{2.14}$$

This follows from (2.1) since $X_4 \neq X_\alpha$, $\alpha = 1, 2, 3$. This condition allows us to find λ,

$$\lambda = p/q, \tag{2.15}$$

and guarantees that $\lambda \neq 0, \infty, 1$.

To obtain the second condition, we need to find dp and dq. Differentiation of (2.12) by means of (2.9) gives

$$dp = p\omega_i^i + \underset{1}{p_i}\underset{1}{\omega}^i + \underset{2}{p_i}\underset{2}{\omega}^i, \quad dq = q\omega_i^i + \underset{1}{q_i}\underset{1}{\omega}^i + \underset{2}{q_i}\underset{2}{\omega}^i, \tag{2.16}$$

where

$$\underset{k}{p_i} = \underset{k}{p}_{[12]i}, \quad \underset{k}{q_i} = \underset{k}{q}_{[12]i}, \qquad i, k = 1, 2.$$

Differentiation of (2.15) by means of (2.16) leads to

$$\underset{1}{p_i} - \lambda \underset{1}{q_i} = p(b_i - a_i), \quad \underset{2}{p_i} - \lambda \underset{2}{q_i} = qb_i - pa_i. \tag{2.17}$$

Eliminating a_i and b_i from equations (2.17), we get the second condition:

$$q(q\underset{1}{p_i} - p\underset{1}{q_i}) - p(q\underset{2}{p_i} - p\underset{2}{q_i}) = pq(p-q)a_i. \tag{2.18}$$

If a 3-web W(3, 2, 2) satisfying (2.14) and (2.18) is given, one can find λ from (2.15), b_i from (2.5), and b_{ij} from (2.7). Thus, *a nonisoclinic three-web* W(3, 2, 2) *satisfying* (2.14) *and* (2.18) *can be uniquely extended to a nonisoclinic* AGW(4, 2, 2).

Such a nonisoclinic AGW(4, 2, 2) is of maximum 2-rank $\pi(4, 2, 2) = 1$ if and only if (see [**9**, **10**])

$$b_{kij}^k = b_{ij} - q_{ij}. \tag{2.19}$$

The only abelian 2-equation for a web W(4, 2, 2) of maximum 2-rank is (see [**9**, **10**])

$$\begin{aligned}(\lambda - \lambda^2)\sigma\underset{1}{\omega}^1 \wedge \underset{1}{\omega}^2 + (\lambda - 1)\sigma\underset{2}{\omega}^1 \wedge \underset{2}{\omega}^2 - \lambda\sigma(\underset{1}{\omega}^1 + \underset{2}{\omega}^1) \wedge (\underset{1}{\omega}^2 + \underset{2}{\omega}^2) \\ + \sigma(\lambda\underset{1}{\omega}^1 + \underset{2}{\omega}^1) \wedge (\lambda\underset{1}{\omega}^2 + \underset{2}{\omega}^2) = 0,\end{aligned} \tag{2.20}$$

where each term is a closed 2-form and σ is a solution of the completely integrable equation

$$d\ln[\sigma(\lambda - 1)] = \omega_i^i + (a_i - b_i/\lambda)\underset{2}{\omega}^i. \tag{2.21}$$

Note that (2.20) is an identity, and it is an abelian 2-equation for a nonisoclinic web W(4, 2, 2) of maximum 2-rank only under conditions (2.19), (2.21).

3. Examples of nonisoclinic webs W(3, 2, 2) **and nonisoclinic webs** W(4, 2, 2) **of maximum 2-rank.** The main goal of the present paper is to construct examples of nonisoclinic webs W(3, 2, 2) satisfying (2.14) and (2.18) and nonisoclinic webs W(4, 2, 2) of maximum 2-rank.

If we succeed in finding a nonisoclinic 3-web satisfying (2.14) and (2.18), we can extend it to a nonisoclinic W(4, 2, 2) by finding λ, b_i, and b_{ij} from (2.15), (2.5), and (2.7) and eventually by finding equations of the foliation X_4 from the system

$$\lambda \underset{1}{\omega}^i + \underset{2}{\omega}^i = 0. \tag{3.1}$$

We will suppose that three foliations X_1, X_2, and X_3 of the nonisoclinic 3-web are given as level sets $u^i_\alpha = \text{const}$, $\alpha = 1, 2, 3$, of the following functions:

$$X_1\colon u^i_1 = x^i; \quad X_2\colon u^i_2 = y^i; \quad X_3\colon u^i_3 = f^i(x^j, y^k), \qquad i, j, k = 1, 2. \tag{3.2}$$

The forms $\underset{\alpha}{\omega}^i$, $\alpha = 1, 2, 3$, ω^i_j, and the functions a^i_{jk} can be found by means of the following formulas (see [**2**]):

$$\underset{1}{\omega}^i = \bar{f}^i_j\, dx^j, \quad \underset{2}{\omega}^i = \tilde{f}^i_j\, dy^j, \quad \underset{3}{\omega}^i = -df^i, \tag{3.3}$$

where

$$\bar{f}^i_j = \partial f^i/\partial x^j, \quad \tilde{f}^i_j = \partial f^i/\partial y^j, \quad \det(\bar{f}^i_j) \neq 0, \quad \det(\tilde{f}^i_j) \neq 0,$$

and

$$\Gamma^i_{jk} = (-\partial^2 f^i/\partial x^l \partial y^m)\bar{g}^l_j \tilde{g}^m_k \quad (\text{where } \bar{g} = \bar{f}^{-1} \text{ and } \tilde{g} = \tilde{f}^{-1}), \tag{3.4}$$

$$\omega^i_j = \Gamma^i_{kj}\underset{1}{\omega}^k + \Gamma^i_{jk}\underset{2}{\omega}^k, \tag{3.5}$$

$$a^i_{jk} = \Gamma^i_{[jk]}. \tag{3.6}$$

The functions b^i_{jkl}, a_i, p_{ij}, q_{ij}, p, q, $\underset{i}{p}_j$, and $\underset{1}{q}_j$ can be easily calculated from (2.3), (2.11), (2.6), (2.12), and (2.16) after which conditions (2.14) and (2.18) should be checked. If they are satisfied, then λ, b_i, and b_{ij} can be found as indicated above and equations (3.1) should be integrated. It gives a nonisoclinic web W(4, 2, 2). If the latter satisfies (2.19), it is a nonisoclinic web W(4, 2, 2) of maximum 2-rank, and the only abelian 2-equation admitted by it can be found from (2.20) and (2.21).

Let us realize the outlined procedure by considering a few examples. In these examples the foliations X_1, X_2, and X_3 will be defined by (3.2) and we will specify the functions $f^i(x^j, y^k)$. The author has already found examples of isoclinic 3-webs [**10**, **11**, **12**] and nonisoclinic 3-webs W(3, 2, 2) that cannot be extended to a nonisoclinic 4-web W(4, 2, 2) because for them one of the conditions (2.14) or (2.18) fails [**10**].

We will write below the equations of the foliation X_3 of the latter webs and the reason why they cannot be extended to a W(4, 2, 2):

1°. $u^1_3 = x^1y^1 - x^2y^2 = \text{const}$, $u^2_3 = x^1y^2 + x^2y^1 = \text{const}$, $p = 0, q \neq 0$.

2°. $u^1_3 = x^2 \exp(x^1y^1) = \text{const}$, $u^2_3 = x^2 + y^2 = \text{const}$, $p \neq 0, q = 0$.

3°. $u^1_3 = x^1 + y^1 = \text{const}$, $u^2_3 = x^1y^1 + x^2y^2 = \text{const}$, $p = q \neq 0$.

4°. $u^1_3 = (x^1 + y^1)^3/6 + [(x^1)^2 + (y^1)^2 + 2x^2y^2]/2 = \text{const}$, $u^2_3 = x^2 + y^2 = \text{const}$, $p \neq 0$, $q \neq 0$, $p \neq q$, (2.18) fails.

We will give one more example of the third kind that includes a wide class of 3-webs.

EXAMPLE 1. *Polynomial* 3-*webs.* We will call a web W(3, 2, 2) *polynomial* if it is defined by

$$X_3\colon u^i_3 = x^i + y^i + c^i_{jk}x^jy^k = \text{const}, \qquad c^i_{jk} = \text{const}. \tag{3.7}$$

Note that the Taylor expansions of functions $u_3^i = f^i(x^j, y^k)$ can always be reduced to a canonical form [1]. Equation (3.7) is a particular case of this canonical form when it contains only terms of the first and second degree and the coefficients are constants.

For a polynomial web, using (3.3)–(3.6) and (2.11), we obtain

$$\text{(3.8)}\quad \begin{cases} a_1 = [c_{21}^2 - c_{12}^2 + (c_{11}^2 c_{22}^2 - c_{12}^2 c_{21}^2)(y^1 - x^1) + (c_{12}^1 c_{11}^2 - c_{11}^1 c_{12}^2)x^1 \\ \qquad + (c_{22}^1 c_{11}^2 - c_{21}^1 c_{12}^2)x^2 + (c_{11}^1 c_{21}^2 - c_{21}^1 c_{11}^2)y^1 \\ \qquad\qquad + (c_{12}^1 c_{21}^2 - c_{22}^1 c_{11}^2)y^2]/(\Delta_1\Delta_2), \\ a_2 = [c_{12}^1 - c_{21}^1 + (c_{11}^1 c_{22}^1 - c_{12}^1 c_{21}^1)(y^2 - x^2) + (c_{22}^1 c_{11}^2 - c_{21}^1 c_{12}^2)x^1 \\ \qquad + (c_{22}^1 c_{21}^2 - c_{21}^1 c_{22}^2)x^2 + (c_{12}^1 c_{21}^2 - c_{22}^1 c_{11}^2)y^1 \\ \qquad\qquad + (c_{12}^1 c_{22}^2 - c_{22}^1 c_{12}^2)y^2]/(\Delta_1\Delta_2), \end{cases}$$

where

$$\Delta_1 = (1 + c_{1i}^1 y^i)(1 + c_{2j}^2 y^j) - c_{1i}^2 c_{2j}^1 y^i y^j,$$
$$\Delta_2 = (1 + c_{i1}^1 x^i)(1 + c_{j2}^2 x^j) - c_{i1}^2 c_{j2}^1 x^i x^j.$$

Using (2.12) and (3.8), one can get p and q after lengthy calculations and conclude that the condition

$$\text{(3.9)}\qquad (c_{22}^2 + c_{12}^1)(c_{12}^2 - c_{21}^2) + (c_{11}^1 + c_{21}^2)(c_{12}^1 - c_{21}^1) \neq 0$$

is sufficient to satisfy $p \neq 0$ and $q \neq 0$. Therefore, *a polynomial web* (3.7) *satisfying* (3.9) *is nonisoclinic.*

However, for any c_{jk}^i we have $p = q$, the condition (2.14) fails, and *polynomial* 3-*webs* (3.7) *cannot be extended to a nonisoclinic* W(4, 2, 2).

In conclusion we give an example of a nonisoclinic W(3, 2, 2) that can be extended to a nonisoclinic W(4, 2, 2) and the extended W(4, 2, 2) is of maximum 2-rank.

EXAMPLE 2. Suppose that the foliation X_3 of W(3, 2, 2) is defined by

$$\text{(3.10)}\quad X_3\colon u_3^1 = x^1 + y^1 + (x^1)^2 y^2/2 = \text{const}, \quad u_3^2 = x^2 + y^2 - x^1(y^2)^2/2 = \text{const}.$$

Again using (3.3)–(3.6), (2.11)–(2.12), and (2.16), we obtain

$$\text{(3.11)}\qquad a_1 = y^2/[\Delta(2-\Delta)], \qquad a_2 = x^1/[\Delta(2-\Delta)],$$

$$\text{(3.12)}\quad \begin{cases} p_{11} = 2(y^2)^2(\Delta - 1)/[\Delta^3(2-\Delta)^2], & p_{21} = (\Delta^2 - 2\Delta + 2)/[\Delta^3(2-\Delta)^2], \\ q_{22} = 2(x^1)^2(\Delta - 1)/[\Delta^2(2-\Delta)^3], & q_{12} = (\Delta^2 - 2\Delta + 2)/[\Delta^2(2-\Delta)^3], \\ p_{12} = q_{11} = p_{22} = q_{21} = 0, \end{cases}$$

$$\text{(3.13)}\quad p = -(\Delta^2 - 2\Delta + 2)/[2\Delta^3(2-\Delta)^2], \qquad q = (\Delta^2 - 2\Delta + 2)/[2\Delta^2(2-\Delta)^3],$$

$$\text{(3.14)}\quad \begin{cases} \underset{1}{p_2} = \underset{1}{q_2} = \underset{2}{p_1} = \underset{2}{q_1} = 0, \\ \underset{1}{p_1} = y^2(-\Delta^3 + 4\Delta^2 - 8\Delta + 6)/[\Delta^5(2-\Delta)^3], \\ \underset{2}{p_2} = x^1(-\Delta^3 + 3\Delta^2 - 6\Delta + 4)/[\Delta^4(2-\Delta)^4], \\ \underset{1}{q_1} = y^2(\Delta^3 - 4\Delta^2 + 8\Delta - 6)/[\Delta^4(2-\Delta)^4], \\ \underset{2}{q_2} = x^1(\Delta^3 - 2\Delta^2 + 4\Delta - 2)/[\Delta^3(2-\Delta)^5], \end{cases}$$

where $\Delta = 1 + x^1y^2$. It follows from (3.13) and (3.14) that conditions (2.14) and (2.18) are satisfied. Therefore equations (3.10) define *a nonisoclinic* 3-*web* W(3, 2, 2) *that can be extended to a nonisoclinic* 4-*web* W(4, 2, 2).

To find the extension, we determine from (2.15) and (3.13) that

$$\lambda = 1 - 2/\Delta \tag{3.15}$$

and from (2.5) and (2.7) that

$$b_1 = -y^2/\Delta^2, \qquad b_2 = x^1/(2\Delta - \Delta^2), \tag{3.16}$$

$$b_{11} = b_{21} = 0, \quad b_{12} = -p_{21}, \quad b_{22} = q_{22}, \tag{3.17}$$

where p_{ij} and q_{ij} are given by (3.12).

Using (3.15), (3.3), and (3.10), we can write equations (3.1) of the foliation X_4 in the form

$$d[x^1(x^1y^2/2 - 1) + y^1] = 0, \qquad d[-y^2(x^1y^2/2 + 1) + x^2] = 0. \tag{3.18}$$

It follows from (3.18) that the foliation X_4 is defined by

$$X_4 \colon u_4^1 = -x^1 + y^1 + (x^1)^2y^2/2 = \text{const}, \qquad u_4^2 = x^2 - y^2 - x^1(y^2)^2/2 = \text{const}. \tag{3.19}$$

To check whether the web defined by (3.10) and (3.19) is of maximum 2-rank or not, we find $d\omega_k^k$ by means of (3.5) and compare it with (2.3). It gives

$$b_{k11}^k = b_{k21}^k = b_{k22}^k = 0, \qquad b_{k12}^k = 2(-\Delta^2 + 2\Delta - 2)/[\Delta^3(2 - \Delta)^3]. \tag{3.20}$$

Equations (3.20), (3.17), and (3.12) show that condition (2.19) is satisfied. Thus, *the* 4-*web defined by* (3.10) *and* (3.19) *is of maximum* 2-*rank.*

To find the only abelian 2-equation admitted by this web, we integrate (2.21) where λ, ω_i^i, a_i, and b_i are defined by (3.15), (3.5), (3.11), and (3.16). Up to a constant factor, the solution is

$$\sigma = \Delta/[2(\Delta - 2)]. \tag{3.21}$$

Substituting λ from (3.15) and σ from (3.21) into (2.20), we obtain the only abelian 2-equation in the form:

$$(2/\Delta)\underset{1}{\omega}^1 \wedge \underset{1}{\omega}^2 - (2/(\Delta - 2))\underset{2}{\omega}^1 \wedge \underset{2}{\omega}^1 - \underset{3}{\omega}^1 \wedge \underset{3}{\omega}^2 + (\Delta/(\Delta - 2))\underset{4}{\omega}^1 \wedge \underset{4}{\omega}^2 = 0, \tag{3.22}$$

where each term is a closed 2-form [**9**, **10**].

Using (3.3), we can write (3.22) in the form of (1.4):

$$2\,dx^1 \wedge dx^2 + 2dy^1 \wedge dy^2 - du_3^1 \wedge du_3^2 + du_4^1 \wedge du_4^2 = 0. \tag{3.23}$$

Note that *Example* 2 *is the first known example of a nonisoclinic web* W(4, 2, 2) *of maximum* 2-*rank.*

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