A PRIORI LIMITATIONS FOR SOLUTIONS OF MONGE-AMPÈRE EQUATIONS*

BY
HANS LEWY

Introduction. As has been well known since the fundamental papers of S. Bernstein† on the generalization of Dirichlet's principle, one of the main points in proofs of existence for elliptic equations, and the most difficult one, is to obtain suitable estimates for solutions and their derivatives *under the assumption* of their existence, the so-called a priori limitations. Since the purpose of this paper is merely to establish such limitations in the case of analytic elliptic Monge-Ampère equations, we do not intend to repeat here once more the classical reasoning of S. Bernstein.

The method used in this paper will be found different from those generally adopted for similar purposes in the case of quasi-linear equations. It seems to the author that the adaptation of a Monge-Ampère equation to a scheme permitting the application of the sharp a priori limitations for linear equations‡ is efficient as long as there exist estimates for the third derivatives, but is no longer available, if, as in the present paper, we try to *establish* estimates for the third derivatives assuming the knowledge of bounds for the absolute values of the solution and its derivatives up to the second order.

Our method consists in an analytic continuation of the solution into a complex domain similar to that which we introduced in a former paper§ in order to prove the uniqueness of Cauchy's problem. In the case of a Monge-Ampère equation we can, however, use a simpler system of characteristic equations than in the most general non-linear case of analytic elliptic equations. After this has been done a study of the behavior of the analytic continuation is undertaken in order to prove that it has a certain property of "Schlichtheit." This is the principal feature which makes the application of Cauchy's integral formula possible. This Schlichtheit is proved by some simple topological considerations.

---

* Presented to the Society, September 5, 1934; received by the editors October 1, 1934.

† S. Bernstein, *Sur la généralisation du problème de Dirichlet*, Mathematische Annalen, vol. 62 (1906), p. 253, and vol. 69 (1910), p. 82.

‡ See for example Julius Schauder, *Über lineare elliptische Differentialgleichungen zweiter Ordnung*, Mathematische Zeitschrift, vol. 38 (1934), p. 257.

§ Hans Lewy, *Eindeutigkeit der Lösung des Anfangsproblems einer elliptischen Differentialgleichung*, Mathematische Annalen, vol. 104 (1931), p. 325.

In a subsequent paper we intend to give an application of our present results to an important existence problem of differential geometry in the large.

1. We start with a discussion of the analytic continuation of solutions mentioned in the Introduction. Let A, B, C, D, E be analytic functions of the five complex arguments u, v, x, p, q in a complex neighborhood N of a real point $T_0(u_0, v_0, x_0, p_0, q_0)$, $p_0^2+q_0^2\leqq 1$, which assume real values for real values of the arguments. Suppose moreover that at the point T_0 the expression

$$\Delta \equiv 4(AC+DE)-B^2=2\alpha^2, \quad \alpha>0,$$

so that

$$4(AC+DE)\geqq 2\alpha^2.$$

Now let $M>0$ be a given number. Then there exists a sufficiently small subneighborhood N_ϵ of T_0 of the form

$$|u-u_0|\leqq\epsilon, \quad |v-v_0|\leqq\epsilon, \quad |x-x_0-p_0(u-u_0)-q_0(v-v_0)|\leqq 4\epsilon^2M,$$
$$|p-p_0|\leqq 2\epsilon M, \quad |q-q_0|\leqq 2\epsilon M, \quad \epsilon>0,$$

and a constant $K>0$, such that at every point T of N_ϵ,

$$|\Delta|>\alpha^2, \qquad |4(AC+DE)|>\alpha^2, \tag{1}$$

and that all the functions A, B, C, D, E are analytic in N_ϵ and remain, together with their first partial derivatives, less than K in absolute value. We omit the obvious proof and state that the numbers α, ϵ, K are functionals of A, B, C, D, E and functions of the number M. In N_ϵ the two roots of the quadratic equation in λ

$$\lambda^2-\lambda B+(AC+DE)=0 \tag{2}$$

are distinct and given by

$$2\lambda_1=B+i\Delta^{1/2} \text{ and } 2\lambda_2=B-i\Delta^{1/2}.$$

We now take for x an analytic and real-valued function of two real variables u, v, defined in the square

$$S_\epsilon: \qquad |u-u_0|\leqq\epsilon, \qquad |v-v_0|\leqq\epsilon,$$

and denote, as usual, by p, q, r, s, t its first and second partial derivatives. We suppose

$$x(u_0,v_0)=x_0, \qquad p(u_0,v_0)=p_0, \qquad q(u_0,v_0)=q_0,$$

and assume that in the square S_ϵ

$$|r|, \ |s|, \ |t|\leqq M \tag{1.1}$$

and that throughout S_ϵ the Monge-Ampère equation holds:

$$(3) \qquad Ar + Bs + Ct + D(rt - s^2) = E.$$

Condition (1.1) implies that, for every point of S_ϵ, (u, v, x, p, q) lies in N_ϵ. We take two points of S_ϵ, P and Q, and map the line segment PQ affinely onto the line segment from the point $(-1, 1)$ to the point $(1, -1)$ of the line $\gamma+\delta=0$ of the γ, δ-plane. Thus to every point π of the latter segment there belongs a pair of functions $u(\pi)$, $v(\pi)$, which serve to define three more functions of π,

$$x(\pi) = x(u(\pi), v(\pi)), \qquad p(\pi) = p(u(\pi), v(\pi)), \qquad q(\pi) = q(u(\pi), v(\pi)).$$

With that segment as initial curve and $u(\pi)$, $v(\pi)$, $x(\pi)$, $p(\pi)$, $q(\pi)$ as initial values we set up the following hyperbolic system (of characteristic equations of (3)):

$$(4.1) \qquad x_\gamma - pu_\gamma - qv_\gamma = 0,$$

$$(4.2) \qquad \lambda_1 u_\gamma - Av_\gamma - Dq_\gamma = 0,$$

$$(4.3) \qquad \lambda_2 u_\delta - Av_\delta - Dq_\delta = 0,$$

$$(4.4) \qquad - Ev_\gamma + Cq_\gamma + \lambda_1 p_\gamma = 0,$$

$$(4.5) \qquad - Ev_\delta + Cq_\delta + \lambda_2 p_\delta = 0.$$

Concerning this problem we prove the following

THEOREM. *There exists a positive number $\epsilon_1 < \epsilon$, which depends only on ϵ, α, K, M, such that*
(i) *a solution $u(\gamma, \delta)$, $v(\gamma, \delta)$, $\cdots$, $q(\gamma, \delta)$ of* (4.1–4.5) *exists throughout the square $|\gamma| \leqq 1$, $|\delta| \leqq 1$;*
(ii) *for every point of this square $(u, \cdots, q)$ belongs to N_ϵ;*
(iii) *the solution has continuous derivatives with respect to γ and δ and depends analytically on the coordinates of the points P and Q, used in the determination of the initial values;*
(iv) *in the square $|\gamma| \leqq 1$, $|\delta| \leqq 1$ the first derivatives of the functions u, v, x, p, q with respect to γ and δ remain bounded by $2\tau z$, $z = \overline{PQ}$, and moreover*

$$(5) \qquad |u_\gamma(\gamma, \delta) - u_\gamma(\gamma, -\gamma)| \leqq \beta z,$$

provided P and Q lie both in the square

$$S_{\epsilon_1}: \qquad |u - u_0| \leqq \epsilon_1, \qquad |v - v_0| \leqq \epsilon_1;$$

the numbers τ, β are functions of α, K, M, ϵ, to be specified later.

We first derive certain bounds for our functions $x(u, v)$, $p(u, v)$, $q(u, v)$, $r(u, v)$, $s(u, v)$, $t(u, v)$ for real u, v in S_ϵ. In view of (3),

$$\Delta = 4(A + Dt)(C + Dr) - (B - 2Ds)^2.$$

Hence, by (1),

$$4(A + Dt)(C + Dr) > \alpha^2.$$

The factor $A+Dt$, being continuous, never vanishes and may, without loss of generality, be assumed positive. We then have

$$(6) \qquad A + Dt > \frac{\alpha^2}{4K(1 + M)}, \qquad C + Dr > \frac{\alpha^2}{4K(1 + M)},$$

$$(6') \qquad \left(\frac{\frac{B}{2} - Ds}{A + Dt}\right)^2 = \frac{-\Delta}{4(A + Dt)^2} + \frac{C + Dr}{A + Dt} < \frac{C + Dr}{A + Dt} \leqq \frac{4K^2(1 + M)^2}{\alpha^2},$$

$$(7) \qquad |\Delta| \leqq 9K^2, \qquad |\Delta|^{1/2} \leqq 3K,$$

the last inequality being valid in the whole N_ϵ. We now define a non-negative angle $\phi_1 < \pi/2$ by the equation

$$(8) \qquad \tan \phi_1 = \frac{4K(1 + M)}{\alpha}.$$

If $|\phi| \leqq \phi_1$, we have $\cos \phi \geqq \cos \phi_1 > 0$. If, on the other hand, $\pi - \phi_1 \geqq \phi \geqq \phi_1$, we deduce the following inequalities:

$$\tan \phi_1 \leqq \tan \phi, \quad \left|\cos \phi \cdot \left(Ds - \frac{B}{2}\right)\right| \leqq \frac{1}{2} \sin \phi \cdot (A + Dt),$$

$$(9) \qquad \cos \phi \cdot \left(Ds - \frac{B}{2}\right) + \sin \phi \cdot (A + Dt) \geqq \frac{1}{2} \sin \phi \cdot (A + Dt)$$

$$\geqq \frac{1}{2} \sin \phi_1 \cdot (A + Dt) \geqq \frac{\alpha^2 \sin \phi_1}{8K(1 + M)}.$$

In order to solve the above stated hyperbolic problem, we reduce it to a system of five equations of second order. Thus we shall need to know the initial values of the first derivatives of the solution whose existence we are going to establish, along the initial line $\gamma + \delta = 0$. The equation of the line in the u, v plane joining P and Q has the form

$$v \cos \phi - u \sin \phi = \text{const.},$$

where ϕ is the angle of the direction from P to Q with the positive u axis. On the other hand let us denote by (′) the operation of differentiation with respect to the arc length of the line $\gamma + \delta = 0$. We evidently have for any function $f(\gamma, \delta)$, having continuous first derivatives,

$$f' = \frac{1}{2^{1/2}}\left(\frac{\partial}{\partial\gamma} - \frac{\partial}{\partial\delta}\right)f.$$

Thus we find

$$2^{1/2}u' = u_\gamma - u_\delta = \frac{z}{2}\cos\phi, \qquad 2^{1/2}v' = v_\gamma - v_\delta = \frac{z}{2}\sin\phi,$$
$$2^{1/2}p' = \frac{z}{2}(r\cos\phi + s\sin\phi), \quad 2^{1/2}q' = \frac{z}{2}(s\cos\phi + t\sin\phi),$$
$$2^{1/2}x' = p(u_\gamma - u_\delta) + q(v_\gamma - v_\delta).$$

This together with the equations (4.1–4.5) yields the following values for the initial values of u_γ, u_δ, v_γ, v_δ, account being taken of the non-vanishing of the determinant

$$\text{(10)} \quad \begin{vmatrix} 1 & -p & -q & 0 & 0 \\ 0 & \lambda_1 & -A & -D & 0 \\ 0 & \lambda_2 & -A & -D & 0 \\ 0 & 0 & -E & C & \lambda_1 \\ 0 & 0 & -E & C & \lambda_2 \end{vmatrix} = (\lambda_1 - \lambda_2)^2(AC + DE)$$
$$= -\Delta(AC + DE);$$

(11)
$$u_\gamma = \frac{z}{2(\lambda_1 - \lambda_2)}[-(\lambda_2 - Ds)\cos\phi + (A + Dt)\sin\phi],$$
$$u_\delta = \frac{z}{2(\lambda_1 - \lambda_2)}[-(\lambda_1 - Ds)\cos\phi + (A + Dt)\sin\phi],$$
$$v_\gamma = \frac{z}{2(\lambda_1 - \lambda_2)}[-(Dr + C)\cos\phi + (\lambda_1 - Ds)\sin\phi],$$
$$v_\delta = \frac{z}{2(\lambda_1 - \lambda_2)}[-(Dr + C)\cos\phi + (\lambda_2 - Ds)\sin\phi],$$

$$x_\gamma = pu_\gamma + qv_\gamma, \qquad x_\delta = pu_\delta + qv_\delta,$$
$$p_\gamma = ru_\gamma + sv_\gamma, \qquad p_\delta = ru_\delta + sv_\delta,$$
$$q_\gamma = su_\gamma + tv_\gamma, \qquad q_\delta = su_\gamma + tv_\delta.$$

This leads to an important set of inequalities for the real and imaginary parts of u_γ on the initial line $\gamma + \delta = 0$. If $|\phi| \leqq \phi_1$, we have

$$(12_1) \qquad \Re(u_\gamma) = \frac{z}{4}\cos\phi \geqq \frac{z}{4}\cos\phi_1.$$

Similarly for $\pi-\phi_1 \leqq \phi \leqq \pi+\phi_1$,

$$\Re(u_\gamma) = \frac{z}{4}\cos\phi \leqq -\frac{z}{4}\cos\phi_1. \tag{12$_2$}$$

If, on the other hand, $\phi_1 \leqq \phi \leqq \pi-\phi_1$, we find, on account of (9) and (7),

$$-\Im(u_\gamma) \geqq \frac{\alpha^2 \sin\phi_1}{96K^2(1+M)}\cdot z, \tag{12$_3$}$$

and, for $\pi+\phi_1 \leqq \phi \leqq 2\pi-\phi_1$,

$$-\Im(u_\gamma) \leqq \frac{-\alpha^2 \sin\phi_1}{96K^2(1+M)}\cdot z. \tag{12$_4$}$$

This may be expressed in the following form: the entire range of ϕ splits up into four sectors such that within the first sector $\Re(u_\gamma)$ is bounded below by $2\beta z$, within the third sector $\Re(u_\gamma)$ is bounded above by $-2\beta z$, within the second sector $-\Im(u_\gamma)$ is bounded below by $2\beta z$ and within the fourth sector $-\Im(u_\gamma)$ is bounded above by $-2\beta z$. It is sufficient to choose for $2\beta>0$ the smaller one of the two numbers $\frac{1}{4}\cos\phi_1$ and $\alpha^2\sin\phi_1/(96K^2(1+M))$ which both depend only on α, K, M. Hence β depends only on α, K and M.

We notice furthermore that our formulas (11) show the existence of a positive number τ, depending only on α, K, M, ϵ, such that, on $\gamma+\delta=0$,

$$|u_\gamma| < \tau z, \quad |u_\delta| < \tau z, \quad \cdots, \quad |q_\delta| < \tau z. \tag{13}$$

We now indicate briefly a method† of solving the system (4.1–4.5). Differentiate (4.1), (4.2), (4.4) with respect to δ and (4.3) and (4.5) with respect to γ. We obtain five equations containing second derivatives of the unknown functions only of the type $\partial^2/\partial\gamma\partial\delta$. We solve with respect to these, which is possible since $\Delta\neq 0$. The equations so obtained have the form

$$\Delta u_{\gamma\delta} = \text{quadratic form in } u_\gamma, u_\delta, \cdots, q_\gamma, q_\delta; \cdots, \tag{14}$$

with coefficients which are polynomials in A, B, C, D, E and their partial derivatives with respect to u, v, x, p, q, in other words with coefficients limited in absolute value by a suitable polynomial $g(K)>0$ as long as u, v, x, p, q remain within N_ϵ. Under the same condition $|\Delta|>\alpha^2$. We now apply successive approximations to system (14) with initial values as determined above by two points P and Q of S_{ϵ_1} where ϵ_1 satisfies the following inequalities:

$$\begin{gathered} \epsilon_1 < \epsilon, \qquad 800g(K)\tau\cdot 2^{3/2}\epsilon_1 < \alpha^2, \qquad \epsilon_1(1+8\cdot 2^{1/2}\tau) < \epsilon, \\ \epsilon_1(2M+8\cdot 2^{1/2}\tau) < 2M\epsilon, \qquad \epsilon_1(4M\epsilon+24\cdot 2^{1/2}\tau) < 4M\epsilon^2, \\ 800g(K)\tau^2\cdot 2^{3/2}\epsilon_1 < \alpha^2\beta. \end{gathered} \tag{15}$$

† Cf. Hadamard, *Leçons sur le Problème de Cauchy*, Paris, Hermann, 1932, pp. 488–501.

Postponing the definition of the first approximation, let us suppose that any one of the successive approximations exists in $|\gamma| \leqq 1$, $|\delta| \leqq 1$, and that (u, v, x, p, q) is in N_ϵ, while the partial derivatives with respect to γ or δ are in absolute value $\leqq 2\tau z$. Then for the derivatives of the next approximation $u^*, \cdots, q^*$ which certainly exists in $|\gamma| \leqq 1$, $|\delta| \leqq 1$, we find from (14)

$$\left| \overset{*}{u}_{\gamma\delta} \right| \leqq \frac{g(K)}{\alpha^2} \cdot 400\tau^2 z^2.$$

On integrating this and taking into account (13) and the relation $\overline{PQ} = z \leqq 2^{3/2}\epsilon_1$, we have

$$(16) \quad \begin{aligned} &\left| u_\gamma^* \right| \leqq \tau z + \frac{800 g(K)}{\alpha^2} \tau^2 z^2 \leqq \tau z \left(1 + \frac{800 g(K)}{\alpha^2} \tau z \right) \leqq 2\tau z, \\ &\left| u_\delta^* \right| \leqq 2\tau z, \cdots, \left| q_\delta^* \right| \leqq 2\tau z \qquad \text{for } |\gamma| \leqq 1, \ |\delta| \leqq 1. \end{aligned}$$

On the other hand by (1.1) we have for the initial values of the u, v, x, p, q

$$\begin{aligned} &|u - u_0| \leqq \epsilon_1, \ |v - v_0| \leqq \epsilon_1, \ |p - p_0| \leqq 2M\epsilon_1, \ |q - q_0| \leqq 2M\epsilon_1, \\ &|x - x_0 - p_0(u - u_0) - q_0(v - v_0)| \leqq 4M\epsilon_1^2. \end{aligned}$$

Hence on integrating (16) and in view of (15) and $|p_0| \leqq 1$, $|q_0| \leqq 1$, we have for $|\gamma| \leqq 1$, $|\delta| \leqq 1$,

$$(17) \quad \begin{aligned} &|p^* - p_0| \leqq 2M\epsilon_1 + 8\tau \cdot 2^{1/2}\epsilon_1 \leqq 2M\epsilon, \\ &|q^* - q_0| \leqq 2M\epsilon, \\ &|u^* - u_0| \leqq \epsilon_1 + 8\tau \cdot 2^{1/2}\epsilon_1 \leqq \epsilon, \qquad |v^* - v_0| \leqq \epsilon, \\ &|x^* - x_0 - p_0(u^* - u_0) - q_0(v^* - v_0)| \leqq 4M\epsilon_1^2 + 24 \cdot 2^{1/2}\tau\epsilon_1 \leqq 4M\epsilon^2. \end{aligned}$$

This shows that all the functions u^*, v^*, x^*, p^*, q^* remain within N_ϵ, provided we define a suitable first approximation. We do this by setting as a first approximation a solution of the system

$$u_{\gamma\delta} = 0, \ v_{\gamma\delta} = 0, \ \cdots, \ q_{\gamma\delta} = 0,$$

with the given initial data.

Since for this solution the values of $u_\gamma, \cdots, q_\delta$ at any point γ, δ coincide with some initial value, the above inequalities (16), (17) hold for the first approximation and thus for all of them.

It may now be shown that the successive approximations converge uniformly together with their first derivatives to limit functions and we get a solution of the system (4) assuming the given initial values. This solution is the only one having continuous first derivatives and can be differentiated with respect to γ and δ as well as with respect to any parameter as many times as the initial data themselves.†

Obviously the solution satisfies the same inequalities (16) and (17) that are satisfied by all the approximations.

Moreover, by (14),

$$|u_\gamma(\gamma,\delta)-u_\gamma(\gamma,-\gamma)| = \left|\int_\delta^{-\gamma}\frac{\partial^2 u}{\partial\gamma\partial\delta}\,d\delta\right| \leqq 800\,\frac{g(K)}{\alpha^2}\,\tau^2 z^2 \leqq \beta z.$$

This is the desired inequality (5). Thus all the statements of our theorem on page 419 have been proved except the analytical dependence on the coordinates of P and Q.

This last property results from the fact that our initial data are analytic functions of these four coordinates. Let us denote the coordinates of P by a_1 and b_1, those of Q by a_2 and b_2. If we give a_1, b_1, a_2, b_2 complex values, but *very near* to those real values for which we made the above construction, we get new initial data which differ as little as we please from the former ones; we may assume that the inequalities (13) remain valid for the new initial data. If now we construct the solution of the modified initial problem, we get again a solution for the whole square $|\gamma|\leqq 1$, $|\delta|\leqq 1$ and this solution stays as near as we please to the former solution on account of the continuous dependence on the initial data. We thus get a solution depending on four complex parameters and the two real variables γ, δ. Denote by $\eta=\eta_1+i\eta_2$ any one of the complex parameters. The initial values, being analytic functions of η, satisfy the Cauchy-Riemann equations which evidently can be written in the following form using the abbreviation $\nabla=\partial/\partial\eta_1+i\partial/\partial\eta_2$:

$$\nabla u = 0,\quad \nabla v = 0,\ \cdots,\ \nabla q = 0.$$

On the other hand, on operating with ∇ on the system (4.1–4.5) we get

$$\frac{\partial}{\partial\gamma}(\nabla x) - p\frac{\partial}{\partial\gamma}(\nabla u) - q\frac{\partial}{\partial\gamma}(\nabla v) = u_\gamma\nabla p + v_\gamma\nabla q,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-E\frac{\partial}{\partial\delta}(\nabla v) + C\frac{\partial}{\partial\delta}(\nabla q) + \lambda_2\frac{\partial}{\partial\delta}(\nabla p) = v_\gamma\frac{\partial E}{\partial u}\nabla u + \cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

i.e., a hyperbolic system, linear and homogeneous, with a non-vanishing "determinant" (10) for the unknowns $\nabla x, \cdots, \nabla q$. Since a solution of this system is $\nabla x = \cdots = \nabla q = 0$ and the solution is uniquely determined by initial values, we conclude

† For the proof of all these statements, see Hadamard, loc. cit.

$$\nabla x = \nabla u = \cdots = \nabla q = 0.$$

This implies the desired analytic dependence on a_1, b_1, a_2, b_2 in the square $|\gamma| \leqq 1$, $|\delta| \leqq 1$, expressed by means of the Cauchy-Riemann equations.

2. We return to real values of a_1, b_1, a_2, b_2. The manifold of solutions seems to depend on these four variables and the two variables γ, δ. It is possible, however, to reduce the dependence to that of a_1, b_1, a_2, b_2 only. If we substitute for γ a variable $\gamma' = k\gamma + l$ and for δ a variable $\delta' = k'\delta + l'$ with $kk' \neq 0$, a solution of the system (4) becomes a solution of the same system in γ', δ' because of the homogeneity of (4) with respect to the derivatives $\partial/\partial\gamma$, $\partial/\partial\delta$. Now let P', Q' be two points of, and in the same order as, PQ. The range of the initial values of the problem (4) determined by P' and Q' may be transformed by the above substitution into a part of the range of the initial values of the PQ problem. The same must then be true for the range of the corresponding solutions of the $P'Q'$ problem and of the PQ problem. If therefore we agree to consider only the point $\gamma = 1$, $\delta = 1$ of the PQ problem and to write the solution at this point as a function of a_1, b_1, a_2, b_2 only, the whole square will consist of points at which the solution is the same function with different values of the argument a_1, b_1, a_2, b_2. In particular, the line $\delta = 1$ corresponds to fixed (a_1, b_1) while (a_2, b_2) varies along a line through (a_1, b_1). This shows that we have on $\delta = 1$ the following rule: The differentiation $\partial/\partial\gamma$ of our solution, considered as function of γ, δ and of the four parameters a_1, b_1, a_2, b_2, reduces to the operation

$$\frac{z}{2}\left(\cos\phi \frac{\partial}{\partial a_2} + \sin\phi \frac{\partial}{\partial b_2}\right)$$

if we write our solutions as functions of a_1, b_1, a_2, b_2 only. Notice that the solution is defined for $|a_1 - u_0| \leqq \epsilon_1$, $|b_1 - v_0| \leqq \epsilon_1$, $|a_2 - u_0| \leqq \epsilon_1$, $|b_2 - v_0| \leqq \epsilon_1$ and that the originally given functions in the real u, v-plane now appear as functions of the four arguments for coincident P and Q, i.e., for $a_1 = a_2$, $b_1 = b_2$.

3. We now set up another hyperbolic system for two functions $u(\gamma, \delta)$ and $v(\gamma, \delta)$ to be determined for $|\gamma| \leqq 1$, $|\delta| \leqq 1$. The initial values are given on $\gamma + \delta = 0$ in the same way as on page 419, but the equations are

$$(18.1) \qquad (\lambda_1 - Ds)u_\gamma - (A + Dt)v_\gamma = 0,$$

$$(18.2) \qquad (\lambda_2 - Ds)u_\delta - (A + Dt)v_\delta = 0.$$

Here the quantities x, p, q, r, s, t are to be considered as analytic functions of u and v, determined by the originally presented solution of the Monge-Ampère equation. On writing

$$\rho = \frac{\lambda_1 - Ds}{A + Dt}, \qquad \bar{\rho} = \frac{\lambda_2 - Ds}{A + Dt}, \tag{19}$$

this becomes

$$v_\gamma - \rho u_\gamma = 0, \tag{18.3}$$

$$v_\delta - \bar{\rho} u_\delta = 0. \tag{18.4}$$

We conclude

$$\lambda_1 u_\gamma - A v_\gamma - D q_\gamma = 0, \ \lambda_2 u_\delta - A v_\delta - D q_\delta = 0, \ -E v_\gamma + C q_\gamma + \lambda_1 p_\gamma = 0,$$

since, in view of (2) and (3),

$$(\lambda_1 r + Cs)(A + Dt) + (\lambda_1 s + Ct - E)(\lambda_1 - Ds) = 0.$$

Similarly we obtain the relation

$$-E v_\delta + C q_\delta + \lambda_2 p_\delta = 0.$$

We also mention the identity

$$x_\gamma - p u_\gamma - q v_\gamma = 0.$$

Thus the introduction of the new initial problem leads to a set of functions $u(\gamma, \delta), \cdots, q(\gamma, \delta)$ which turns out to be a solution of (4.1–4.5) with the same initial conditions. In view of the uniqueness theorem the two solutions must coincide wherever they exist simultaneously, which is true for P sufficiently near Q.

Equations (18.3) and (18.4) admit of a simple interpretation. For instance, if γ varies alone, the differentials of u and v are connected by the *ordinary* differential equation

$$dv - \rho du = 0.$$

Returning to the variables a_1, b_1, a_2, b_2, we may say that if (a_1, b_1) is fixed while (a_2, b_2) varies along any line through (a_1, b_1), then (u, v) moves on a certain two-dimensional surface M_1 of the four-dimensional complex u, v-space, called characteristic "megaline."* For (a_2, b_2) sufficiently near (a_1, b_1) the equation of M_1 has the form

$$v = \text{analytic function of } u.$$

Thus, for Q sufficiently near P, we may write the corresponding Cauchy-Riemann equations in the form

$$v_{a_2} u_{b_2} - v_{b_2} u_{a_2} = 0. \tag{20}$$

* Hadamard, loc. cit., p. 512.

But in view of the above stated analytical dependence on (a_1, b_1, a_2, b_2) this relation holds on the whole range of the values of (a_1, b_1, a_2, b_2). Similarly we find

$$v_{a_1}u_{b_1} - v_{b_1}u_{a_1} = 0. \tag{21}$$

From (20), (21), (18.3), (18.4) we derive a fact of fundamental importance for our discussion, namely that the *quotient* $u_{a_2}:u_{b_2}$ *is independent of* a_1, b_1.

Indeed, since on M_1 we have $0=dv-\rho du$, the relation

$$v_{a_2} - \rho u_{a_2} = 0 \tag{22.1}$$

is satisfied on M_1 and holds for any M_1, or, in other words, for any a_1, b_1. Similarly

$$v_{b_2} - \rho u_{b_2} = 0, \tag{22.2}$$

$$v_{a_1} - \bar{\rho} u_{a_1} = 0. \tag{23.1}$$

Now differentiate (23.1) with respect to a_2 and b_2, (22.1) with respect to a_1 and (22.2) with respect to a_1. This gives

$$v_{a_1a_2} - \bar{\rho} u_{a_1a_2} - \bar{\rho}_u u_{a_1} u_{a_2} - \bar{\rho}_v u_{a_1} v_{a_2} = 0,$$
$$v_{a_1b_2} - \bar{\rho} u_{a_1b_2} - \bar{\rho}_u u_{a_1} u_{b_2} - \bar{\rho}_v u_{a_1} v_{b_2} = 0,$$

$$v_{a_1a_2} - \rho u_{a_1a_2} - \rho_u u_{a_2} u_{a_1} - \rho_v u_{a_2} v_{a_1} = 0,$$
$$v_{a_1b_2} - \rho u_{a_1b_2} - \rho_u u_{b_2} u_{a_1} - \rho_v u_{b_2} v_{a_1} = 0.$$

By (20)

$$v_{a_1a_2}u_{b_2} - v_{a_1b_2}u_{a_2} - \bar{\rho}(u_{a_1a_2}u_{b_2} - u_{a_1b_2}u_{a_2}) = 0,$$
$$v_{a_1a_2}u_{b_2} - v_{a_1b_2}u_{a_2} - \rho(u_{a_1a_2}u_{b_2} - u_{a_1b_2}u_{a_2}) = 0.$$

$\rho \neq \bar{\rho}$ gives

$$u_{a_1a_2}u_{b_2} - u_{a_1b_2}u_{a_2} = 0,$$

which shows that $u_{a_2}:u_{b_2}$ does not depend on a_1.

In an analogous manner we find that $u_{a_2}:u_{b_2}$ does not depend on b_1. Hence, in order to compute the value of $u_{a_2}:u_{b_2}$ for any (a_1, b_1, a_2, b_2) we may compute it for $a_1=a_2$, $b_1=b_2$, i.e., for (a_2, b_2, a_2, b_2). In view of the known values (11) for the initial values of u_γ and our remark on page 425 we get

$$\begin{aligned} \frac{2}{z} u_\gamma &= \cos\phi \cdot u_{a_2} + \sin\phi \cdot u_{b_2} \\ &= \frac{1}{\lambda_1 - \lambda_2}[-(\lambda_2 - Ds)\cos\phi + (A + Dt)\sin\phi]. \end{aligned} \tag{24}$$

Taking $\phi=0$ and $\phi=\pi/2$ gives finally

$$u_{a_2}:u_{b_2} = -\frac{\lambda_2 - Ds}{A + Dt}. \tag{25}$$

We notice that this expression has a positive imaginary part. Since x, p, q are analytic functions of u and v and thus of u alone on M_1, we have on M_1 the Cauchy-Riemann equations

$$\begin{aligned} x_{a_2}u_{b_2} - x_{b_2}u_{a_2} &= 0, \\ p_{a_2}u_{b_2} - p_{b_2}u_{a_2} &= 0, \\ q_{a_2}u_{b_2} - q_{b_2}u_{a_2} &= 0, \end{aligned} \tag{23}$$

and these relations again hold for all a_1, b_1, a_2, b_2 in question.

4. We now proceed to establish a fact which forms the salient point of the present investigation.

THEOREM. *The function* $u(u_0, v_0, a_2, b_2)$ *is "schlicht" for* $|u-u_0|<2\beta\epsilon_1$, *or, in other words, the equation*

$$u(u_0, v_0, a_2, b_2) = U$$

has one and only one solution (a_2, b_2) *in* $|a_2-u_0| \leqq \epsilon_1$, $|b_2-v_0| \leqq \epsilon_1$ *provided* $|U-u_0| <2\beta\epsilon_1$.

Let us write (5) in the form

$$u_\gamma(\gamma, \delta) = u_\gamma(\gamma, -\gamma) + h, \quad |h| \leqq \beta z.$$

The value $u_\gamma(\gamma, -\gamma)$ satisfies the inequalities (12_1–12_4). Hence, for $|\gamma| \leqq 1$, $|\delta| \leqq 1$,

$$\begin{aligned} \Re(u_\gamma) &\geqq \beta z \quad \text{for} \quad |\phi| \leqq \phi_1, \\ \Re(u_\gamma) &\leqq -\beta z \quad \text{for} \quad \pi-\phi_1 \leqq \phi \leqq \pi+\phi_1, \\ -\Im(u_\gamma) &\geqq \beta z \quad \text{for} \quad \phi_1 \leqq \phi \leqq \pi-\phi_1, \\ -\Im(u_\gamma) &\leqq -\beta z \quad \text{for} \quad \pi+\phi_1 \leqq \phi \leqq 2\pi-\phi_1. \end{aligned} \tag{26}$$

An integration from $\gamma=-1$ to $\gamma=+1$ along $\delta=1$ yields

$$\begin{aligned} \Re[u(1, 1) - u_0] &\geqq 2\beta z \quad \text{for} \quad |\phi| \leqq \phi_1, \\ \Re[u(1, 1) - u_0] &\leqq -2\beta z \quad \text{for} \quad \pi-\phi_1 \leqq \phi \leqq \pi+\phi_1, \\ -\Im[u(1, 1) - u_0] &\geqq 2\beta z \quad \text{for} \quad \phi_1 \leqq \phi \leqq \pi-\phi_1, \\ -\Im[u(1, 1) - u_0] &\leqq -2\beta z \quad \text{for} \quad \pi+\phi_1 \leqq \phi \leqq 2\pi-\phi_1. \end{aligned} \tag{27}$$

Let us take for P the point (u_0, v_0) and let (a_2, b_2) vary in the square R: $|a_2-u_0| \leqq \epsilon_1$, $|b_2-v_0| \leqq \epsilon_1$. Imagine the complex conjugate quantity $\bar{u}(u_0, v_0,$

$a_2, b_2) - u_0$ as a vector* attached to the variable point (a_2, b_2) of R. Formulas (27) show that $\bar{u}(u_0, v_0, a_2, b_2) - u_0$ forms an angle less than π with the vector

$$a_2 - u_0 + i(b_2 - v_0)$$

which is $\neq 0$, provided $(a_2 - u_0)^2 + (b_2 - v_0)^2 \neq 0$. If we follow $\bar{u}$ along any closed Jordan curve contained in R and such that (u_0, v_0) lies in its interior, the rotation of $\bar{u} - u_0$ after one circuit must equal that of $a_2 - u_0 + i(b_2 - v_0)$, i.e., 2π. On the other hand the vector $\bar{u} - u_0$ depends analytically on a_2, b_2 and has for every subdomain of R only a finite number of singularities. (The same will be true for any vector $\bar{u}(u_0, v_0, a_2, b_2) - \bar{U}$, $\bar{U}$ being a constant.) We conclude that the point $a_2 = u_0$, $b_2 = v_0$ is a singularity of the field $\bar{u}(u_0, v_0, a_2, b_2) - u_0$ with index $+1$.

By (27), the expression $\bar{u} - \bar{U}$ does not vanish along the contour of R if $|U - u_0| < 2\beta\epsilon_1$, since $z = \overline{PQ} \geqq \epsilon_1$ as long as Q varies along the contour of R and P is the center (u_0, v_0) of R. For reasons of continuity, the rotation (along the contour of R) of $\bar{u} - \bar{U}$ equals that of $\bar{u} - u_0$, so that we may say that the vector field $\bar{u} - \bar{U}$ has at least one singularity within R for $|U - u_0| < 2\beta\epsilon_1$, and the sum of the indices of all singularities in R equals $+1$. On the other hand it follows from the lemma below that any such singularity has an essentially positive index, hence there can be at most one singularity, which is the desired result.

LEMMA. *Let, without loss of generality, the origin of an* a, b*-plane be a zero of a complex vector* $\mathfrak{u}(a, b)$, *analytic near the origin and such that the quotient* $\mathfrak{u}_a : \mathfrak{u}_b = \kappa$ *has a negative imaginary part at the origin*: $\mathfrak{I}(\kappa_0) < 0$. *Then the singularity of the origin has a positive index.*

We prove this lemma by developing $\mathfrak{u}$ in a Taylor series in $b + \kappa_0 a$ and $b - \kappa_0 a$. Let $k > 0$ be the smallest degree for which there are non-vanishing terms so that the series begins as follows:

$$\mathfrak{u} = c_1(b - \kappa_0 a)^k + c_2(b - \kappa_0 a)^{k-1}(b + \kappa_0 a) + \cdots + c_{k+1}(b + \kappa_0 a)^k + \cdots .$$

Form the expression $\mathfrak{u}_a - \kappa \mathfrak{u}_b$ and develop it into a power series in a and b; it must vanish identically. On the other hand the terms of lowest degree in $\mathfrak{u}_a - \kappa \mathfrak{u}_b$ have the form

$$-2\kappa_0\left[kc_1(b - \kappa_0 a)^{k-1} + \cdots + c_k(b + \kappa_0 a)^{k-1}\right]$$

so that we conclude $c_1 = c_2 = \cdots = c_k = 0$. Thus the Taylor series of $\mathfrak{u}$ begins with the term $c_{k+1}(b + \kappa_0 a)^k$, $c_{k+1} \neq 0$, and the singularity of $\mathfrak{u}$ at $a = b = 0$ is that of $c_{k+1}(b + \kappa_0 a)^k$ and has the index $k > 0$.

---

* For the notions concerning vector fields used in this paragraph, see for instance W. Fenchel, *Elementare Beweise und Anwendungen einiger Fixpunktsätze*, Matematisk Tidsskrift, (B), 1932, p. 66.

5. The theorem of the preceding section enables us to write the megaline M_1 in the form

$$v = \text{analytic function of } u$$

for u varying within the circle $\Gamma: |u-u_0| < 2\beta\epsilon_1$. Within R none of the partial derivatives of $u(u_0, v_0, a_2, b_2)$ with respect to a_2 or b_2 vanishes. For we conclude from (26) and (24) that there is at least a linear form in u_{a_2} and u_{b_2} which differs from zero. On the other hand the quotient $u_{a_2}:u_{b_2}$ has a positive imaginary part and cannot vanish, which proves that $u_{a_2}\neq 0$, $u_{b_2}\neq 0$.

Hence, the relations (20) and (23) yield the analytic dependence of v, x, p, q on u on the megaline M_1, provided u lies in Γ.

Integration of (13) gives, for $|a_2-u_0| \leqq \epsilon_1$, $|b_2-v_0| \leqq \epsilon_1$,

$$|v - v_0|, \cdots, |q - q_0| \leqq 2^{3/2}\tau\epsilon_1.$$

Let us denote by d/du the differential quotient on M_1 (u in Γ). Cauchy's integral formula yields for $u=u_0$

$$\rho = \frac{dv}{du} = \frac{1}{2\pi i}\oint_\Gamma \frac{(v - v_0)du}{(u - u_0)^2}, \quad \frac{d^n v}{du^n} = \frac{n!}{2\pi i}\oint_\Gamma \frac{(v - v_0)du}{(u - u_0)^{n+1}}, \cdots,$$

$$\frac{d^n q}{du^n} = \frac{n!}{2\pi i}\oint_\Gamma \frac{(q - q_0)du}{(u - u_0)^{n+1}};$$

$$\left|\frac{d^n v}{du^n}\right|, \cdots, \left|\frac{d^n q}{du^n}\right| \leqq \frac{n!}{2\pi}\cdot\frac{2\pi 2^{3/2}\tau\epsilon_1}{(2\beta\epsilon_1)^n} = \frac{n! 2^{3/2}\tau\epsilon_1}{(2\beta\epsilon_1)^n}.$$

The differentiation d/du at the point $u=u_0$ of M_1 is connected with the partial derivatives $\partial/\partial u$ and $\partial/\partial v$ at the point u_0, v_0 by the relation

$$\frac{d}{du} = \frac{\partial}{\partial u} + \rho\frac{\partial}{\partial v}.$$

Thus the above inequalities for the successive derivatives at the point u_0 yield inequalities for the partial derivatives of the given functions $x(u, v)$, $p(u, v)$, $q(u, v)$. And anyone of these inequalities yields two estimates if we take account of the reality of these functions for real u, v.

6. As mentioned in the introduction, the most important of the a priori limitations are those for the third derivatives of x. We easily can establish them at the point u_0, v_0.

We have at u_0, v_0

$$\frac{d^2p}{du^2} = \left(\frac{\partial}{\partial u} + \rho\frac{\partial}{\partial v}\right)^2 p = \left(\frac{\partial}{\partial u} + \rho\frac{\partial}{\partial v}\right)(r + \rho s)$$

$$= r_u + 2\rho s_u + \rho^2 t_u + \frac{d^2 v}{du^2}\, s,$$

$$\frac{d^2q}{du^2} = \left(\frac{\partial}{\partial u} + \rho\frac{\partial}{\partial v}\right)^2 q = \qquad s_u + 2\rho t_u + \rho^2 t_v + \left(\overline{\frac{d^2 v}{du^2}}\right) t,$$

and the conjugate equations

$$\left(\overline{\frac{d^2p}{du^2}}\right) = \left(\frac{\partial}{\partial u} + \bar\rho\frac{\partial}{\partial v}\right)^2 p = r_u + 2\bar\rho s_u + \bar\rho^2 t_u + \left(\overline{\frac{d^2 v}{du^2}}\right) s,$$

$$\left(\overline{\frac{d^2q}{du^2}}\right) = \left(\frac{\partial}{\partial u} + \bar\rho\frac{\partial}{\partial v}\right)^2 q = \qquad s_u + 2\bar\rho t_u + \bar\rho^2 t_v + \left(\overline{\frac{d^2 v}{du^2}}\right) t.$$

The determinant of this system is

$$\begin{vmatrix} 1 & 2\rho & \rho^2 & 0 \\ 1 & 2\bar\rho & \bar\rho^2 & 0 \\ 0 & 1 & 2\rho & \rho^2 \\ 0 & 1 & 2\bar\rho & \bar\rho^2 \end{vmatrix} = (\rho - \bar\rho)^4 = \frac{\Delta^2}{(A + Dt)^4}$$

and is in absolute value $>\alpha^4/[K^4(1+M)^4]$ (see (1)). Moreover $|r|$, $|s|$, $|t|$ $\leqq M$ and

$$|\rho| = \left|\frac{dv}{du}\right| \leqq 2^{1/2}\tau/\beta \quad\text{and}\quad \left|\frac{d^2v}{du^2}\right|, \cdots, \left|\frac{d^2q}{du^2}\right| \leqq 2^{1/2}\tau/(\beta^2\epsilon_1).$$

Hence by solving the above linear system with respect to r_u, s_u, t_u, t_v we easily obtain upper bounds for $|r_u|$, $|s_u|$, $|t_u|$, $|t_v|$, depending on α, ϵ, K, M only. We shall not carry through the computations.

7. The estimates for the higher derivatives are not much more complicated, but for the applications it is important to know that they can be chosen in such a way that they insure a lower a priori bound for the radii of convergence of the Taylor series for x at a point u_0, v_0.

We observe that there are two megalines M_1 and M_2 through the point (u_0, v_0), conjugate to each other. The values of u, v, x, p, q at their points will be referred to by $u(u_0, v_0, a_2, b_2) \equiv u(u_0, v_0; Q), \cdots$ and $u(a_1, b_1, u_0, v_0) \equiv u(P; u_0, v_0)$ respectively. Consider the megaline passing through (u_0, v_0, Q) which consists of points $(a_1, b_1; Q)$ with variable (a_1, b_1) and fixed Q, and the megaline through $(P; u_0, v_0)$ consisting of points $(P; a_2, b_2)$ with variable

(a_2, b_2) and fixed P. The two megalines evidently intersect at (P, Q). Let ξ denote the variable u on M_1 and η the same variable on M_2. If both vary within the circle Γ of "Schlichtheit," i.e., for $|\xi - u_0| < 2\beta\epsilon_1$ and $|\eta - u_0| < 2\beta\epsilon_1$, they are in unique correspondence with Q and P respectively. We therefore may write $u(P, Q), \cdots$ as functions of ξ and η. We say that they are analytic functions of these complex arguments.

We have evidently

$$\xi = \xi(a_2, b_2) = \xi_1 + i\xi_2$$

and the independence of $u_{a_2}:u_{b_2}$ on a_1, b_1 shows that $u_{a_2}:u_{b_2}$ or, what is essentially the same, $u_{\xi_1}:u_{\xi_2}$ are independent on η. Hence we may compute its value by setting $\eta = u_0$. But on M_1 we have Cauchy-Riemann equations $u_{\xi_1}:u_{\xi_2} = -i$. Similarly we find the Cauchy-Riemann equations $u_{\eta_1}:u_{\eta_2} = -i$. Thus $u(\xi, \eta)$ depends analytically on ξ and η for $|\xi - u_0| < 2\beta\epsilon_1$, $|\eta - u_0| < 2\beta\epsilon_1$. The equations (20), (21), (23) show that for any function ω of the set u, v, x, p, q we have

$$\frac{\partial(\omega, u)}{\partial(a_2, b_2)} = 0 \quad \text{and} \quad \frac{\partial(\omega, u)}{\partial(a_1, b_1)} = 0.$$

This yields the analytic dependence of all of them on ξ and η for $|\xi - u_0| < 2\beta\epsilon_1$, $|\eta - u_0| < 2\beta\epsilon_1$. Moreover u, v, x, p, q admit bounds for their absolute values, depending only on ϵ, α, K, M.

On the other hand we find for $\xi = u_0$, $\eta = u_0$ the following *lower* bounds:

$$\frac{\partial u}{\partial \xi} = 1, \quad \frac{\partial u}{\partial \eta} = 1, \quad \left|\frac{\partial v}{\partial \xi}\right| \geqq \frac{\alpha}{2K(1 + M)}, \quad \left|\frac{\partial v}{\partial \eta}\right| \geqq \frac{\alpha}{2K(1 + M)},$$

$$\left|\frac{\partial(u, v)}{\partial(\xi, \eta)}\right| \geqq \frac{\alpha}{K(1 + M)}.$$

We therefore are able to introduce u, v as independent variables* instead of ξ and η within a domain $|u - u_0| < \epsilon_2$, $|v - v_0| < \epsilon_2$, with ϵ_2 depending merely on α, ϵ, K, M. In this domain we may develop x in a power series in $u - u_0$ and $v - v_0$; it will converge there and its coefficients will have absolute bounds in terms of α, ϵ, K, M. Even the majorant series formed with these bounds instead of the derivatives of x at u_0, v_0 will converge for $|u - u_0| < \epsilon_2$, $|v - v_0| < \epsilon_2$. We state the final result in the following form:

THEOREM. *The derivatives of an analytic solution of the analytic Monge-Ampère equation* (3) *existing for* $|u - u_0| < \epsilon$, $|v - v_0| < \epsilon$ *can be developed into a power series in* $u - u_0$ *and* $v - v_0$ *whose associated radii of convergence and*

* This step is justified in §8.

whose coefficients can be bounded, the former from below, the latter from above, by certain numbers which depend only
(i) *on the bound M for the moduli of the second derivatives,*
(ii) *on the value $2\alpha^2$ of $\Delta>0$ at $u=u_0$, $v=v_0$,*
(iii) *on the bound ϵ such that in a neighborhood N_ϵ of the* 10-*dimensional space of complex x, p, q, u, v around u_0, v_0, x_0, p_0, q_0 the coefficients A, B, C, D, E remain regular, and $|\Delta|>\alpha^2$,*
(iv) *on the bound K for these coefficients and their first partial derivatives in N_ϵ, provided we have $p_0^2+q_0^2\leqq 1$. The power series in u and v, formed with the bounds of the coefficients, has the same bounds for its associated radii of convergence.*

8. We have made use of the following lemma concerning the inversion of a system of analytic functions in the neighborhood of the origin.

LEMMA. *Let*

$$u = ax + by + \sum_{i+k\geqq 2} a_{ik}x^iy^k,$$

$$v = cx + dy + \sum_{i+k\geqq 2} c_{ik}x^iy^k.$$

Suppose $|ad-bc|>A>0$, $0<B_1<|a|$, $|b|$, $|c|$, $|d|<B_2$. Suppose furthermore that $\sum a_{ik}x^iy^k$ and $\sum c_{ik}x^iy^k$ converge for $|x|<\rho$, $|y|<\rho$ and that there $|u|$, $|v|<C$. Then there exists a neighborhood of the origin $|u|+|v|<h$, and a fortiori a neighborhood $|u|<h/2$, $|v|<h/2$, for which we can solve with respect to x and y, with h depending only on A, B_1, B_2, C, ρ.

Though the proof of the possibility of inversion in a sufficiently small neighborhood is well known, we were unable to find the above estimate for the neighborhood mentioned in the literature. We therefore sketch a simple method of establishing this estimate.

We have on account of the assumptions

$$|a_{ik}| < \frac{C}{\rho^{i+k}} \quad\text{and}\quad |c_{ik}| < \frac{C}{\rho^{i+k}}\cdot$$

Hence

$$u' = \frac{du - bv}{ad - bc} = x + \sum_{i+k\geqq 2} a_{ik}'\, x^iy^k,$$

$$v' = \frac{-cu + av}{ad - bc} = y + \sum_{i+k\geqq 2} c_{ik}'\, x^iy^k,$$

and $|a_{ik}'|$, $|c_{ik}'|<C'/\rho^{i+k}$ with $C'=C'(A, B_2, C)$. For $|x|+|y|\leqq\rho/2$, we conclude

$$|u' - x| + |v' - y| < \sum_{i+k \geqq 2} \frac{2C'}{\rho^{i+k}} |x|^i |y|^k \leqq (|x| + |y|)^2 M,$$

with $M = M(C', \rho) > 0$. Upon introducing the further restriction

$$|x| + |y| \leqq \min \left\{ \frac{1}{2M}, \frac{\rho}{2} \right\}$$

we get on the boundary of this domain for $|U'| + |V'| < \min \{1/(4M), \rho/4\}$,

$$|u' - x| + |v' - y| < \frac{|x| + |y|}{2},$$

$$|u' - U' - x| + |v' - V' - y| < |x| + |y|.$$

These relations show that the vector of the four-dimensional space with the components $u' - U'$, $v' - V'$ has in $|x| + |y| \leqq \min \{1/(2M), \rho/2\}$ the same sum of indices of its singularities as the vector with components x, y. This sum is $+1$. On the other hand the vector $u' - U'$, $v' - V'$, being analytic, admits only singularities with positive index. Thus there is precisely one solution, x, y, of the equations $u' = U'$, $v' = V'$ for $|U'| + |V'| < \min \{1/(4M), \rho/2\}$. This can be expressed in terms of u, v which proves our lemma.

Brown University,
Providence, R. I.