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IDENTITIES IN IMPLICATIVE SEMILATTICES

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ABSTRACT. An effective procedure is given for deciding whether or not an equation in the theory of implicative semilattices is an identity.

An *implicative semilattice* (Curry [**1**]) is a nonempty set S together with a binary relation $\leqq$ in S, and two binary operations $\wedge$ and $\rightarrow$ in S, satisfying, for all a, b, c in S:

(1) $a \leqq a$.

(2) If $a \leqq b$ and $b \leqq c$, then $a \leqq c$.

(3) $a \wedge b \leqq a$ and $a \wedge b \leqq b$.

(4) If $c \leqq a$ and $c \leqq b$, then $c \leqq a \wedge b$.

(5) If $a \wedge b \leqq c$, then $a \leqq b \rightarrow c$.

(6) $a \wedge (a \rightarrow b) \leqq b$.

(7) If $a \leqq b$ and $b \leqq a$, then $a = b$.

$\leqq$ is a partial order by (1), (2) and (7); $a \wedge b$ is the greatest lower bound of $\{a, b\}$ by (3) and (4).

Let a, b, c, d, e be elements of an implicative semilattice. The following properties are easily derived.

(8) $(a \rightarrow d) \wedge (a \rightarrow e) = a \rightarrow (d \wedge e)$.

(9) $(a \wedge d) \rightarrow e = a \rightarrow (d \rightarrow e)$.

(10) If $a \rightarrow c \leqq a$, then $a \rightarrow c \leqq c$.

A *term* is a meaningful expression built up from variables, $\wedge$ and $\rightarrow$ and parentheses. The *rank* of a term is the number of occurrences of $\wedge$ and $\rightarrow$ in the term. If a and b are terms, then $a = b$ is an *equation*. An equation is an *identity* if and only if it holds in every implicative semilattice, or equivalently, if it follows from (1) through (7). The purpose of this paper is to give an effective procedure for deciding, for any terms a and b, whether or not the equation $a = b$ is an identity.

It is easy to see that the problem of whether or not $a = b$ is an identity reduces to the problem of whether or not $a \leqq b$ and $b \leqq a$ are derivable from (1) through (6).

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The decision procedure makes use of a relation $\subset$, which is not set inclusion, but an extension of $\leqq$. We first set the notation. a, b, c, d, e are terms. v is a variable. R, S, T, U are finite (perhaps empty) sets of terms. (S, T) is the union of S and T. (S, a, b) is the union of S and $\{a, b\}$.

DEFINITION. $S\subset c$ if and only if it is derivable from R1 through R7:

R1. $(S, v)\subset v$.

R2. If $(S, b)\subset c$ and $(S, b)\subset d$, then $(S, b)\subset c\wedge d$.

R3. If $(S, b, a)\subset c$, then $(S, b)\subset a\rightarrow c$.

R4. If $(R, d, e)\subset v$, then $(R, d\wedge e)\subset v$.

R5. If $(R, a\rightarrow d, a\rightarrow e)\subset v$, then $(R, a\rightarrow(d\wedge e))\subset v$.

R6. If $(R, (a\wedge d)\rightarrow e)\subset v$, then $(R, a\rightarrow(d\rightarrow e))\subset v$.

R7. If $(R, a\rightarrow v)\subset a$, then $(R, a\rightarrow v)\subset v$.

LEMMA A. *If $T\subset c$, then $(T, U)\subset c$.*

PROOF. In a given derivation of $T\subset c$, replace each instance $(S, v)\subset v$ of R1 by $(S, U, v)\subset v$. R2 through R7 carry U along, so that the given derivation of $T\subset c$ becomes *mutatis mutandis* a derivation of $(T, U)\subset c$.

LEMMA B. *If $T\subset b$ and $(S, b)\subset c$, then $(T, S)\subset c$.*

PROOF. The proof is by a double strong induction. The primary induction is on the rank of b. The secondary induction is on the number of applications of R1 through R7 in a given derivation of $(S, b)\subset c$. We first consider two special cases whose proofs do not depend on the induction framework.

Case 0. *b is in S.* Then $(S, b)\subset c$ is $S\subset c$, and Lemma A gives $(T, S)\subset c$. From now on it is assumed that Case 0 does not hold.

Case 1. *$(S, b)\subset c$ by* R1. Then c is a variable, and either c is b or S is (S_1, c). If c is b, then Lemma A applied to $T\subset b$ gives $(T, S)\subset b$. If S is (S_1, c), then R1 gives $(T, S_1, c)\subset c$.

Basis. Suppose the rank of b is zero, and the number of applications of R1 through R7 in the given derivation of $(S, b)\subset c$ is one. From the latter it follows that $(S, b)\subset c$ is inferred by R1, and Case 1 applies.

Induction step. Seven cases are considered, according as $(S, b)\subset c$ is inferred by Ri, $1\leqq i\leqq 7$. Case 1 takes care of R1.

Case 2. $(S, b)\subset c$ is inferred by R2. Then c is $c_1\wedge c_2$, and $(S, b)\subset c_1\wedge c_2$ is inferred from $(S, b)\subset c_1$ and $(S, b)\subset c_2$. The SIH (secondary induction hypothesis) applied to $T\subset b$ and $(S, b)\subset c_1$ gives $(T, S)\subset c_1$. Similarly $(T, S)\subset c_2$. Then R2 gives $(T, S)\subset c_1\wedge c_2$.

Case 3. $(S, b)\subset c$ is inferred by R3. Then c is $c_1\rightarrow c_2$, and $(S, b)\subset c_1\rightarrow c_2$ is inferred from $(S, b, c_1)\subset c_2$. The SIH applied to $T\subset b$ and $(S, b, c_1)\subset c_2$ gives $(T, S, c_1)\subset c_2$, and then R3 gives $(T, S)\subset c_1\rightarrow c_2$.

The conclusion of Ri, $4\leqq i\leqq 7$, is $(R, t)\subset v$, where t is a term. Suppose $(S, b)\subset c$ is inferred by Ri, $4\leqq i\leqq 7$. Then c is a variable, and two cases arise, according as b is or is not t. If b is t (and S is R), then Case i below applies. If b is not t, then the proof follows the pattern of Cases 2 and 3: first apply the SIH, and then apply Ri. We illustrate with R4. Suppose $(S, b)\subset c$ is inferred by R4, and b is not the term $d\wedge e$ that explicitly occurs in R4. Then S is $(S_1, d\wedge e)$, and $(S_1, d\wedge e, b)\subset c$ is inferred from $(S_1, d, e, b)\subset c$. The SIH applied to $T\subset b$ and $(S_1, d, e, b)\subset c$ gives $(T, S_1, d, e)\subset c$, and then R4 gives $(T, S_1, d\wedge e)\subset c$.

Case 4. $(S, b)\subset c$ is inferred by R4 and b is $d\wedge e$ and S is R. Then $(S, d\wedge e)\subset c$ is inferred from $(S, d, e)\subset c$. $T\subset b$ is $T\subset d\wedge e$, and is necessarily inferred by R2 from $T\subset d$ and $T\subset e$. The PIH (primary induction hypothesis) applied to $T\subset d$ and $(S, d, e)\subset c$ gives $(T, S, e)\subset c$. Then the PIH applied to $T\subset e$ and $(T, S, e)\subset c$ gives $(T, S)\subset c$.

Case 5. $(S, b)\subset c$ is inferred by R5, and b is $a\to(d\wedge e)$ and S is R. Then $(S, a\to(d\wedge e))\subset c$ is inferred from $(S, a\to d, a\to e)\subset c$. $T\subset b$ is $T\subset a\to(d\wedge e)$, and is necessarily inferred from $(T, a)\subset d\wedge e$ by R3. The latter is necessarily inferred by R2 from $(T, a)\subset d$ and $(T, a)\subset e$. R3 applied to $(T, a)\subset d$ gives $T\subset a\to d$. Similarly $T\subset a\to e$. The PIH applied to $T\subset a\to d$ and $(S, a\to d, a\to e)\subset c$ gives $(T, S, a\to e)\subset c$. Then the PIH applied to $T\subset a\to e$ and $(T, S, a\to e)\subset c$ gives $(T, S)\subset c$.

Case 6. $(S, b)\subset c$ is inferred by R6, and b is $a\to(d\to e)$ and S is R. Then $(S, a\to(d\to e))\subset c$ is inferred from $(S, (a\wedge d)\to e)\subset c$. $T\subset b$ is $T\subset a\to(d\to e)$, and is necessarily inferred by R3 from $(T, a)\subset d\to e$. The latter is necessarily inferred by R3 from $(T, a, d)\subset e$. R4 gives $(T, a\wedge d)\subset e$, and then R3 gives $T\subset(a\wedge d)\to e$. The SIH applied to $T\subset(a\wedge d)\to e$ and $(S, (a\wedge d)\to e)\subset c$ gives $(T, S)\subset c$.

Case 7. $(S, b)\subset c$ is inferred by R7, and b is $a\to c$ and S is R. Then $(S, a\to c)\subset c$ is inferred from $(S, a\to c)\subset a$. $T\subset b$ is $T\subset a\to c$. The SIH applied to $T\subset a\to c$ and $(S, a\to c)\subset a$ gives $(T, S)\subset a$. $T\subset a\to c$ is necessarily inferred by R3 from $(T, a)\subset c$. The PIH applied to $(T, S)\subset a$ and $(T, a)\subset c$ gives $(T, S)\subset c$.

LEMMA C. *If* $(S, d, e)\subset c$, *then* $(S, d\wedge e)\subset c$.

LEMMA D. *If* $(S, a\to d, a\to e)\subset c$, *then* $(S, a\to(d\wedge e))\subset c$.

LEMMA E. *If* $(S, (a\wedge d)\to e)\subset c$, *then* $(S, a\to(d\to e))\subset c$.

PROOFS. Lemma D is proved by strong induction on the rank n of c. If $n=0$, then c is a variable v, and R5 gives the desired result. For the induction step, we consider two cases, according as c is $c_1\wedge c_2$ or c is $c_1\to c_2$. Suppose c is $c_1\wedge c_2$, and $(S, a\to d, a\to e)\subset c_1\wedge c_2$. The latter is necessarily inferred by R2 from $(S, a\to d, a\to e)\subset c_1$ and $(S, a\to d, a\to e)\subset c_2$. The induction hypothesis gives $(S, a\to(d\wedge e))\subset c_1$ and $(S, a\to(d\wedge e))\subset c_2$, and then R2 gives $(S, a\to(d\wedge e))\subset c_1\wedge c_2$. Suppose c is $c_1\to c_2$, and $(S, a\to d, a\to e)\subset c_1\to c_2$. The latter is necessarily inferred by R3 from $(S, a\to d, a\to e, c_1)\subset c_2$. The induction hypothesis gives $(S, a\to(d\wedge e), c_1)\subset c_2$, and then R3 gives $(S, a\to(d\wedge e))\subset c_1\to c_2$. Lemmas C and E are proved in exactly the same way, with the basis steps given by R4 and R6 respectively.

LEMMA F. *$(S, c)\subset c$. If $T\subset a$, then $(T, a\to c)\subset c$.*

PROOF. The two parts are proved simultaneously by strong induction on the rank n of c. If $n=0$, then c is a variable v. R1 gives $(S, v)\subset v$. Suppose $T\subset a$. Then Lemma A gives $(T, a\to v)\subset a$, and R7 gives $(T, a\to v)\subset v$. For the induction step we consider two cases, according as c is $c_1\wedge c_2$ or c is $c_1\to c_2$. Suppose c is $c_1\wedge c_2$. The induction hypothesis gives $(S, c_1, c_2)\subset c_1$ and $(S, c_1, c_2)\subset c_2$, and then R2 gives $(S, c_1, c_2)\subset c_1\wedge c_2$, and then Lemma C gives $(S, c_1\wedge c_2)\subset c_1\wedge c_2$. Suppose $T\subset a$. The induction hypothesis gives $(T, a\to c_1)\subset c_1$. Lemma A gives $(T, a\to c_1, a\to c_2)\subset c_1$. Similarly $(T, a\to c_1, a\to c_2)\subset c_2$. R2 gives $(T, a\to c_1, a\to c_2)\subset c_1\wedge c_2$, and then Lemma D gives $(T, a\to(c_1\wedge c_2))\subset c_1\wedge c_2$. Suppose c is $c_1\to c_2$. The induction hypothesis gives $(S, c_1)\subset c_1$. Then the induction hypothesis gives $(S, c_1, c_1\to c_2)\subset c_2$, and then R3 gives $(S, c_1\to c_2)\subset c_1\to c_2$. Suppose $T\subset a$. Lemma A gives $(T, c_1)\subset a$. The induction hypothesis gives $(T, c_1)\subset c_1$. Then R2 gives $(T, c_1)\subset a\wedge c_1$. Then the induction hypothesis gives $(T, c_1, (a\wedge c_1)\to c_2)\subset c_2$. R3 gives $(T, (a\wedge c_1)\to c_2)\subset c_1\to c_2$, and then Lemma E gives $(T, a\to(c_1\to c_2))\subset c_1\to c_2$.

LEMMA G. *If $a\leqq b$, then $a\subset b$. ($a\subset b$ means $\{a\}\subset b$.)*

PROOF. It suffices to show that (1) through (6) continue to hold if $\leqq$ is replaced throughout by $\subset$. For (1), Lemma F gives $a\subset a$. Lemma B, with $T=\{a\}$ and S empty, gives (2). For (3), Lemma F gives $\{a, b\}\subset a$, and then Lemma C gives $a\wedge b\subset a$. Similarly $a\wedge b\subset b$. R2 gives (4), and R3 gives (5). For (6), Lemma F gives $a\subset a$. Lemma F gives $\{a, a\to b\}\subset b$, and then Lemma C gives $a\wedge(a\to b)\subset b$.

LEMMA H. *If $\{a_1, \cdots, a_k\}\subset b$, then $a_1\wedge\cdots\wedge a_k\leqq b$. In particular, if $a\subset b$, then $a\leqq b$.*

PROOF. It suffices to show that R1 through R7 continue to hold if $\{a_1, \cdots, a_k\}\subset b$ is replaced throughout by $a_1\wedge\cdots\wedge a_k\leqq b$. (3) gives R1 if S is not empty, and (1) gives R1 if S is empty. (4) gives R2. (5) gives R3. R4 is obvious. (8) gives R5; (9) gives R6. R7 follows from (10).

THEOREM. *There is an effective procedure for deciding, for any terms a and b in the theory of implicative semilattices, whether or not the equation $a=b$ is an identity.*

PROOF. By a previous remark, the problem reduces to whether or not $a \leqq b$ and $b \leqq a$. By Lemmas G and H, $a \leqq b$ if and only if $a \subset b$. Let Rk*, $2 \leqq k \leqq 7$, be the inverse of Rk. (Interchange the *if* and *then* parts.) The procedure is to start with $a \subset b$ and work backwards via the inverse rules in an attempt to find a derivation of $a \subset b$. Consider a derivation to be in tree form. The tops are all instances of R1; the bottom is $a \subset b$; R2 gives rise to branching. Call $S \subset c$ an *inequality*. A *branch* is a finite sequence $I_1, \cdots, I_n$ of inequalities such that for each j, $1 \leqq j < n$, I_{j+1} is inferred from I_j by one of R2 through R7. (In the case of R2, I_{j+1} is inferred from I_j and one other inequality.) A derivation is *irredundant* if and only if no inequality occurs more than once in the same branch. (The inequalities $S \subset c$ and $T \subset d$ are the same if and only if c is d and S is the same set as T.) Clearly it suffices to consider only irredundant derivations.

Starting with $a \subset b$, a mixture of R2* and R3* gives rise to one or more branches topped by inequalities of the form $S \subset v$, where v is a variable. Consider one such branch. A mixture of R4* through R6* reduces each term on the left to one or more terms of the form u or $a \rightarrow u$, where u is a variable. If one of these terms is v, then R1 applies and the branch terminates. If none of these terms is v, but one of them is of the form $a \rightarrow v$, then R7* applies, the term a moves to the right, and a new cycle begins. If neither R1 nor R7* applies, then the branch aborts. It may happen that R7* applies in more than one way, giving rise to alternate possible derivations. Each possible derivation must be pursued until one derivation terminates or all possible derivations abort.

It remains to show that the procedure terminates, i.e. each possible derivation is finite, and there are only finitely many possible derivations. Since only irredundant derivations need be considered, it suffices to show that only finitely many inequalities can be generated by starting with $a \subset b$ and applying the inverse rules. To this end, let $v_1, \cdots, v_k$ be all the distinct variables that occur in a or b, and let n be the greater of the ranks of a and b. Let A be the set of all terms of rank not exceeding n which contain no variable not in the list $v_1, \cdots, v_k$. Clearly A is a finite set, say with m elements. Call an inequality $S \subset c$ an *A-inequality* if c and every term in S are in A. Inspection shows that $a \subset b$ is an A-inequality, and each inverse rule applied to an A-inequality yields one or two A-inequalities. Hence only A-inequalities can occur in any possible derivation of $a \subset b$. In an A-inequality $S \subset c$, there are m choices for c since c is in A, and there are 2^m choices for S since S is a finite subset of A. Hence there are only $(m)(2^m)$ A-inequalities, and we are done.

The decision procedure above is based on ideas of Gentzen [2] as set forth in Kleene [3]. Modifications have been made to increase speed and efficiency by minimizing branching and alternate derivations. The chief novelty is that Gentzen's two-premise left $\rightarrow$ rule is replaced by the one-premise rule R7.

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