EXTENSIONS OF GROUP REPRESENTATIONS OVER NONALGEBRAICALLY CLOSED FIELDS[1]

BY
I. M. ISAACS

Let G be a finite group with $H \triangle G$ a Hall subgroup, let F be any field of characteristic 0 and let Ξ be the character of an irreducible F-representation $\mathfrak{X}$ of H. Suppose Ξ is invariant under the natural action of G on the characters of H. As is well known, if $\mathfrak{X}$ is absolute irreducible, the character Ξ can be extended to G and in fact in this case it is not hard to see that $\mathfrak{X}$ can be extended to an F-representation of G. (For instance, this follows immediately from the results of §2.) This paper is an attempt to prove (at least when H is solvable) that $\mathfrak{X}$ can always be extended to G. We do not succeed in this attempt. We obtain, however, some purely group theoretic conditions which are sufficient to guarantee the extendibility of $\mathfrak{X}$ without any assumptions on F. In particular we prove

THEOREM A. *If H is nilpotent then $\mathfrak{X}$ is extendible to G.*

In proving this we define a property (*) of p-groups such that if H is solvable, $C=G/H$ and every p-subgroup of C satisfies (*) then $\mathfrak{X}$ is extendible. An attempt to find p-groups satisfying (*) yields

THEOREM B. *Let H be solvable and suppose for every prime $p \mid [C : C']$ that a Sylow p-subgroup of C is regular and metabelian where $C=G/H$. Then $\mathfrak{X}$ is extendible to G.*

1. Throughout this section we assume that G is an arbitrary finite group and $H \triangle G$; F is a field of characteristic 0 which is held fixed throughout the whole paper. Let Ξ be the character of an irreducible F-representation of H. Then $\Xi = m \sum_{i=1}^{t} \theta_i$ where $\theta_i \in \mathrm{Irr}(H)$ (the set of absolutely irreducible characters of H) and all the θ_i are distinct and a full set of conjugates under the action of the Galois group $\mathfrak{G} = \mathscr{G}(F(\theta)/F)$ where θ is any one of the θ_i. (See [1, §11].) If Ξ is invariant in G then for any $g \in G$ we have $\theta^g = \theta^\sigma$ for some $\sigma \in \mathfrak{G}$.

(1.1) DEFINITION. Let $\theta \in \mathrm{Irr}(H)$. Then θ is *F-semi-invariant* in G if for every $g \in G$, there exists $\sigma \in \mathscr{G}(F(\theta)/F)$ such that $\theta^g = \theta^\sigma$. If θ is Q-semi-invariant we say it is *semi-invariant*.

It is clear that F-semi-invariance implies semi-invariance.

Received by the editors May 14, 1968 and, in revised form, November 1, 1968.
[1] The research on this paper was partially supported by NSF grant GP-7075.

(1.2) Lemma. *Suppose* $\theta \in \text{Irr}(H)$ *is F-semi-invariant in G. Let* $\mathfrak{G} = \mathscr{G}(F(\theta)/F)$ *and* $I = \mathscr{I}_G(\theta)$, *the inertia group of* θ *in G. Then the mapping* $f\colon G \to \mathfrak{G}$ *defined by* $\theta^g = \theta^{f(g)}$ *is a homomorphism with kernel I. Thus* $I \triangle G$ *and* G/I *is abelian.*

Proof. The $\sigma \in \mathfrak{G}$ such that $\theta^g = \theta^\sigma$ is uniquely defined since $\mathfrak{G}$ acts regularly on the set of conjugates of θ and thus f is well defined. Now

$$\theta^{g_1 g_2} = (\theta^{f(g_1)})^{g_2} = (\theta^{g_2})^{f(g_1)} = \theta^{f(g_2)f(g_1)}$$

since the actions of $\mathfrak{G}$ and G on Irr (H) commute. Now $\mathfrak{G}$ is abelian since $F(\theta) \subseteq F[\varepsilon]$ for some root of unity, ε and thus we have

$$f(g_1 g_2) = f(g_2)f(g_1) = f(g_1)f(g_2)$$

and f is a homomorphism. Clearly, ker $(f) = I$ and we are done.

If θ is a character of any group we may associate with it a linear character by taking the determinant of any representation which affords θ. This well-defined linear character will be denoted by det θ. We denote by $o(\theta)$, the order of det θ when considered as an element of the group of linear characters. We state below an important fact which relates det θ with the extendibility of θ. (This is implicit in [2].)

(1.3) Proposition. *Let* $\theta \in \text{Irr}(H)$ *be invariant in G. Suppose* $o(\theta)$ *and* $\theta(1)$ *are both prime to* $[G : H]$. *Then there exists a unique extension* $\hat{\theta}$ *of* θ *to G such that* $o(\hat{\theta})$ *is prime to* $[G : H]$.

In particular, if H is a Hall subgroup of G and $\theta \in \text{Irr}(H)$ then there is a uniquely defined character $\hat{\theta}$ of $\mathscr{I}_G(\theta)$ which extends θ and has determinantal order as above. We shall generally call this the *canonical* extension of θ.

(1.4) Proposition. *Suppose H is a Hall subgroup and let* $\theta \in \text{Irr}(H)$ *be F-semi-invariant in G. Let* $I = \mathscr{I}_G(\theta)$ *and let* $\hat{\theta}$ *be the canonical extension of* θ *to I. Then,* $\chi = \hat{\theta}^G$ *is irreducible,* $F(\theta) = F(\hat{\theta}) \supseteq F(\chi)$ *and if* $\sigma \in \mathscr{G}(F(\chi)/F)$ *with* $\sigma \neq 1$, *then* $[\chi|H, \chi^\sigma|H] = 0$.

Proof. Since $\hat{\theta}|H = \theta$ we have $I \subseteq \mathscr{I}(\hat{\theta}) \subseteq \mathscr{I}(\theta) = I$ and since $\hat{\theta} \in \text{Irr}(I)$ it follows that $\chi \in \text{Irr}(G)$. Clearly, $F(\hat{\theta}) \supseteq F(\theta)$ so let $\sigma \in \mathscr{G}(F(\hat{\theta})/F(\theta))$. Then $o(\hat{\theta}^\sigma) = o(\hat{\theta})$ and $\hat{\theta}^\sigma|H = \theta^\sigma = \theta$ so $\hat{\theta}^\sigma = \hat{\theta}$ by the uniqueness of $\hat{\theta}$. Thus $\sigma = 1$ and hence we must have $F(\hat{\theta}) = F(\theta)$. Clearly, $F(\chi) = F(\hat{\theta}^G) \subseteq F(\hat{\theta})$.

Finally, let $\sigma \in \mathscr{G}(F(\chi)/F)$ and suppose $[\chi|H, \chi^\sigma|H] \neq 0$. By Clifford's Theorem, both $\chi|H$ and $\chi^\sigma|H$ consist of a single orbit of irreducible characters of H with certain multiplicities. Since they have a constituent in common they must be equal so θ is a constituent of $\chi^\sigma|H$. Choose an irreducible constituent ψ of $\chi^\sigma|I$ with $\psi|H$ containing θ. The degrees and determinantal orders of the constituents of $\chi|I$ are the same as those of $\chi^\sigma|I$ so $o(\psi) = o(\hat{\theta})$ and $\psi|H = \theta$ and so we must have $\psi = \hat{\theta}$. Thus χ^σ is a constituent of $\hat{\theta}^G = \chi$ and this forces $\chi = \chi^\sigma$ and thus $\sigma = 1$. The proof is complete.

2. In this section we assume that $H \triangle G$ is a Hall subgroup and $\mathfrak{X}$ is an irreducible F-representation of H with character Ξ invariant in G. Let θ be an absolutely irreducible constituent of Ξ so θ is F-semi-invariant in G. Let I, $\hat{\theta}$ and χ be as in §1. Here we show

(2.1) PROPOSITION. *The representation $\mathfrak{X}$ is extendible to G if the Schur index, $m_F(\chi)$ is relatively prime to $[G : H]$.*

We begin with two lemmas.

(2.2) LEMMA. *Let $K \subseteq L$ be groups with $\psi \in \mathrm{Irr}(L)$ and $\varphi = \psi | K \in \mathrm{Irr}(K)$. Then $m_F(\psi) | m_F(\varphi)$ with equality if $F(\psi) = F(\varphi)$.*

Proof. Let Λ be the character of an irreducible F-representation of K with absolutely irreducible constituent φ so $[\Lambda, \varphi] = m_F(\varphi)$. Then Λ^L is the character of an F-representation of L and $[\psi, \Lambda^L] = [\varphi, \Lambda] = m_F(\varphi)$. It follows that $m_F(\psi) | m_F(\varphi)$. Now let $\mathfrak{G} = \mathscr{G}(F(\psi)/F)$ so $\Delta = m_F(\psi) \sum_{\sigma \in \mathfrak{G}} \psi^\sigma$ is the character of an irreducible F-representation of L. Now if $F(\psi) = F(\varphi)$ then $\varphi^\sigma \neq \varphi$ whenever $\sigma \neq 1$ so $[\Delta | H, \varphi] = m_F(\psi)$ and hence $m_F(\varphi) | m_F(\psi)$ and we are done.

(2.3) LEMMA. *Let $K \triangle L$, $\varphi \in \mathrm{Irr}(K)$ and $\psi = \varphi^L$. Suppose ψ is irreducible. Then $m_F(\psi) = u m_F(\varphi)$ where $u | [L : K]$.*

Proof. Let Λ be the character of an irreducible F-representation of K with $[\Lambda, \varphi] = m_F(\varphi)$. Then $[\Lambda^L, \psi] = [\Lambda, \psi | H] = v m_F(\varphi)$ where v is the number of elements $xK \in L/K$ such that $\varphi^x = \varphi^\sigma$ for some $\sigma \in \mathscr{G}(F(\varphi)/F)$. If U is the set of such x then U is a subgroup and $K \subseteq U \subseteq L$, $v = [U : K]$ so $v | [L : K]$. Since Λ^L is the character of an F-representation of L, $m_F(\psi) | v m_F(\varphi)$.

On the other hand, let Δ be the character of an irreducible F-representation of L, $\Delta = m_F(\psi) \sum_{\tau \in \mathfrak{G}} \psi^\tau$ where $\mathfrak{G} = \mathscr{G}(F(\psi)/F)$. If $\tau \neq 1$ then φ is not a constituent of $\psi^\tau | K$ since $\varphi^L = \psi \neq \psi^\tau$. Thus $[\Delta | K, \varphi] = m_F(\psi)[\psi | K, \varphi] = m_F(\psi)$ and hence $m_F(\varphi) | m_F(\psi)$. We have then, $m_F(\psi) = u m_F(\varphi)$ and this divides $v m_F(\varphi)$ so $u | v$ and the result follows.

Proof of (2.1). Let Λ be the character of an irreducible F-representation $\mathfrak{Y}$ of G with constituent χ. If $\Lambda | H = \Xi$ then $\mathfrak{Y} | H$ and $\mathfrak{X}$ are F-representations of H which have the same character and thus are similar over F. Effecting this similarlity transformation on $\mathfrak{Y}$ yields an F-representation which is an extension of $\mathfrak{X}$.

Since $\Lambda | H$ has θ as a constituent, Ξ is a constituent of $\Lambda | H$ and since Ξ is invariant in G it follows by Clifford's Theorem that $\Lambda | H = e\Xi$ for some integer e. Thus it suffices to show that $e = 1$ when $m_F(\chi)$ is prime to $[G : H]$. Now $\Lambda = m_F(\chi) \sum_{\sigma \in \mathfrak{G}} \chi^\sigma$ where $\mathfrak{G} = \mathscr{G}(F(\chi)/F)$. If $\sigma \neq 1$ then by (1.4) $[\theta, \chi^\sigma | H] = 0$ so $[\Lambda | H, \theta] = m_F(\chi)[\chi | H, \theta] = m_F(\chi)[\chi | I, \theta^I]$. However, among the irreducible constituents of $\chi | I$, only $\hat{\theta}$ has constituent θ when restricted to H so $[\chi | I, \theta^I] = [\hat{\theta}, \theta^I] = [\theta, \theta] = 1$ and we obtain

$$e m_F(\theta) = [e\Xi, \theta] = [\Lambda | H, \theta] = m_F(\chi).$$

Since $F(\theta)=F(\hat{\theta})$, we have by (2.2) that $m_F(\theta)=m_F(\hat{\theta})$ and by (2.3) that $m_F(\chi)=um_F(\hat{\theta})$ where $u|[G:I]$. Since by assumption $m_F(\chi)$ is prime to $[G:H]$ we must have $u=1$ and it follows that $e=1$ and the proof is complete.

3. In this section and for most of the rest of this paper we shall be concerned with showing that under suitable hypotheses, $m_F(\chi)$ is prime to $[G:H]$ as in (2.1). Here we reduce the problem to the case where G/H is a p-group containing I/H in its Frattini subgroup. We assume that $H \triangle G$ is a Hall subgroup and that $\theta \in \operatorname{Irr}(H)$ is F-semi-invariant in G. Let I, $\hat{\theta}$ and χ be as before.

(3.1) Proposition. *Let $H \subseteq U \subseteq G$ and let $\tilde{\theta}$ be the canonical extension of θ to $I \cap U$ and $\psi=\tilde{\theta}^U$. Then $\psi \in \operatorname{Irr}(U)$. Suppose $m_F(\psi)$ is prime to $[U:H]$. Then if p is any prime divisor of $(m_F(\chi), [G:H])$ we must have $p|[G:IU]$.*

Proof. Clearly $I \cap U=\mathscr{I}_U(\theta)$ so by (1.4) applied to U, $\psi=\tilde{\theta}^U \in \operatorname{Irr}(U)$. By the uniqueness of $\tilde{\theta}$, we have $\hat{\theta}|(I \cap U)=\tilde{\theta}$. Let $\eta=\hat{\theta}^{IU} \in \operatorname{Irr}(IU)$. Then $0 \neq [\eta|(I \cap U), \tilde{\theta}] = [\eta|U, \psi]$ so ψ is an irreducible constituent of $\eta|U$. However, $\psi(1)=[U:I \cap U]\tilde{\theta}(1) =[IU:I]\hat{\theta}(1)=\eta(1)$ and thus $\eta|U=\psi$. It follows from (2.2) that $m_F(\eta)|m_F(\psi)$. Now $m_F(\psi) \mid |U|$ and is prime to $[U:H]$ and thus $m_F(\eta) \mid |H|$. We have $\chi=\eta^G$ and $IU \triangle G$ since G/I is abelian so by (2.3) $m_F(\chi)=um_F(\eta)$ where $u \mid [G:IU]$. If $p \mid (m_F(\chi), [G:H])$ then $p \nmid |H|$ so $p \nmid m_F(\eta)$ and hence $p|u$. The result follows.

Now for any U, $H \subseteq U \subseteq G$, let us denote the ψ of (3.1) by χ_U so $\chi_G=\chi$ and $\chi_H=\theta$. In general, $\chi_U \in \operatorname{Irr}(U)$.

(3.2) Theorem. *Suppose $m_F(\chi_U)$ is prime to $[U:H]$ for every U, $H \subseteq U \subseteq G$, such that U/H is a p-group with $(I \cap U)/H \subseteq \Phi(U/H)$ where Φ denotes the Frattini subgroup. Then $m_F(\chi)$ is prime to $[G:H]$.*

Proof. By induction we may assume for all U with $H \subseteq U < G$ that $m_F(\chi_U)$ is prime to $[U:H]$. If G/H is not a p-group for any prime, let $p|[G:H]$ and let P/H be an S_p subgroup of G/H. Then $P<G$ and thus $m_F(\chi_P)$ is prime to p. By (3.1), any prime q dividing both $[G:H]$ and $m_F(\chi)$ must divide $[G:IP]$ so $p \neq q$. Since p was arbitrary, we conclude that the result follows in this case and we may assume that G/H is a p-group.

If $I/H \subseteq \Phi(G/H)$ there is nothing to prove so we may assume that this is not the case. Thus we can find a subgroup $U/H < G/H$ such that $UI=G$. By the inductive assumption applied to U, $m_F(\chi_U)$ is prime to p and hence by (3.1), any prime dividing $(m_F(\chi), [G:H])$ divides $[G:IU]=1$. The result follows.

Before continuing in §4 to derive conditions which will guarantee that $(m_F(\chi), [G:H])=1$ we give a simple lemma about induced characters.

(3.3) Lemma. *Let L be a group with subgroups A and B such that $L=AB$ and let φ be any class function on A. Then $\varphi^L|B=(\varphi \mid (A \cap B))^B$.*

Proof. Let T be a transversal for the right cosets of $A \cap B$ in B so $AT \supseteq B$ and $AT=AAT \supseteq AB=L$. Also $|T|=[B:A \cap B]=[AB:A]$ so T is a transversal for the right cosets of A in L. Suppose then $x \in B$. We have $\varphi^L(x)=\sum_{t\in T}\varphi^0(txt^{-1})$ and $(\varphi \mid (A \cap B))^B(x)=\sum_{t\in T}\varphi^{00}(txt^{-1})$ where $\varphi^0(y)=\varphi(y)$ if $y \in A$ and 0 otherwise and $\varphi^{00}(y)=\varphi(y)$ if $y \in A \cap B$ and 0 elsewhere in B. Since $T\subseteq B$, $txt^{-1} \in A \cap B$ iff it is in A. Hence for all $t \in T$, $\varphi^0(txt^{-1})=\varphi^{00}(txt^{-1})$ and the result follows.

We note here a result which may be of some interest although we shall not refer to it again.

(3.4) COROLLARY. *If G/H is a split extension of I/H then $m_F(\chi)$ is prime to* $[G : H]$.

Proof. Let U/H be a complement for I/H in G/H. By the Schur-Zassenhaus Theorem, there exists a complement C for the normal Hall subgroup H in U. Clearly, $IC=G$ and $I \cap C=1$ so by the lemma we have

$$[\chi, 1_C^G] = [\chi|C, 1_C] = [(\hat{\theta}|1)^C, 1_C] = \theta(1).$$

Therefore, $m_F(\chi)|\theta(1)$ and the result follows.

4. Let P be a p-group and $B\subseteq P$ with $P'\subseteq B\subseteq \Phi(P)$. We wish to consider the family of groups P such that with any B as above, the following statement is a theorem:

(*) Let $P \in \mathrm{Syl}_p(G)$ where G has a solvable normal p-complement H and let $\theta \in \mathrm{Irr}(H)$ be invariant in HB. Suppose $M\triangle G$, $M<H$ and H/M is a chief factor of G. Suppose further that θ vanishes on $H-M$ and that $\theta|M=e\varphi$ where $\varphi \in \mathrm{Irr}(M)$. Let $\hat{\theta}$ and $\hat{\varphi}$ be the canonical extensions of θ and φ to BH and BM. Then $\hat{\theta}|BM=\psi\hat{\varphi}$ where ψ is a character of BM/M (viewed as one of BM) and $[\psi, 1]$ is prime to p.

Note that if η is any irreducible constituent of $\hat{\theta}|BM$ then φ is a constituent of $\eta|M$ so $\eta=\beta\hat{\varphi}$ where $\beta \in \mathrm{Irr}(BM/M)$ and β is uniquely defined. (See for instance Theorem 2 of [2].) It follows that the ψ of (*) is always a well-defined character and the force of (*) is in the assertion that $p \nmid [\psi, 1]$. In §5 some general methods for calculating ψ will be given and in §6 (*) will be proved for regular, metabelian p-groups. It is conjectured, however, that it is always true. The connection between (*) and the problem of this paper is given in the next theorem. Assume here that H, θ, I, $\hat{\theta}$ and χ have the same meaning as in the previous section.

(4.1) THEOREM. *Suppose $C=G/H$ has the property that for every p-subgroup $P\subseteq C$, that P satisfies (*). Then if H is solvable, $m_F(\chi)$ is prime to* $[G : H]$.

Proof. By (3.2) we may assume that G/H is a p-group and that $I/H\subseteq \Phi(G/H)$. Let $P \in \mathrm{Syl}_p(G)$ and let $B=I \cap P$. Then by assumption (*) holds if it applies and $P'\subseteq B\subseteq \Phi(P)$. We prove the theorem by induction on $|H|$. Let $M<H$, $M\triangle G$ such that H/M is a chief factor of G, so that H/M is an elementary abelian q-group. Let $G_0=PM$ and $I_0=I \cap G_0=BM$. We now consider the various possibilities for $\theta|M$.

Suppose $\theta|M=\varphi\in\mathrm{Irr}(M)$. Then clearly, φ is F-semi-invariant in G_0 and invariant in I_0 so $I_0\subseteq J=\mathscr{I}_{G_0}(\varphi)$. Let $\hat{\varphi}$ be the canonical extension of φ to J and let $\chi_0=\hat{\varphi}^{G_0}$. By the inductive hypothesis, $m_F(\chi_0)=m$ is prime to p. Let $\Xi=m\sum\chi_0^\sigma$ so Ξ is the character of an irreducible F-representation of G_0. Now, using (3.3), we obtain

$$[\Xi^G,\chi] = [\Xi,\chi|G_0] = [\Xi,(\hat{\theta}|I_0)^{G_0}] = [\Xi|I_0,\hat{\theta}|I_0] = [\Xi|I_0,\hat{\varphi}|I_0] = m[\chi_0|I_0,\hat{\varphi}|I_0].$$

The last equality follows because if $\sigma\neq 1$ then $[\chi_0^\sigma|M,\varphi]=0$ by (1.4) and thus $[\chi_0^\sigma|I_0,\hat{\varphi}|I_0]=0$. We have then

$$[\Xi^G,\chi] = m[\chi_0|J,(\hat{\varphi}|I_0)^J].$$

However, $(\hat{\varphi}|I_0)^J=\rho\hat{\varphi}$ where ρ is the regular character of J/I_0. It follows that $(\hat{\varphi}|I_0)^J=\sum_\beta\beta(1)\beta\hat{\varphi}$ where β runs over $\mathrm{Irr}(J/I_0)$. Since each $\beta\hat{\varphi}$ is irreducible, $(\hat{\varphi}|I_0)^J$ has a unique irreducible constituent with degree $\varphi(1)$ and determinantal order prime to p. That constituent is $\hat{\varphi}$ and has multiplicity 1. Since every irreducible constituent of $\chi_0|J$ has this degree and order we have

$$[\Xi^G,\chi] = m[\chi_0|J,\hat{\varphi}] = m.$$

Thus $m_F(\chi)|m$ and we are done in this case.

We suppose then that $\theta|M=e\sum_{i=1}^{u}\varphi_i$ where $eu>1$ and the φ_i are distinct conjugate irreducible characters of M. Now, since H/M is abelian, there exists H_0, $M\subseteq H_0<H$ such that θ is induced from a character of H_0 and thus vanishes on $H-H_0$. Let $L=\bigcap\{H_0 \mid M\subseteq H_0<H \text{ with } \theta \text{ vanishing on } H-H_0\}$. Since θ is semi-invariant in G, the group $L\triangle G$ and thus $L=M$ so θ vanishes on $H-M$ and $ue^2=[\theta|M,\theta|M]=[H:M]$. Now let $T=\mathscr{I}_H(\varphi_1)$. Since $T\triangle H$, T depends only on θ and it follows that $T\triangle G$ so that either $T=H$ or $T=M$. Suppose $T=H$ so $u=1$ and $\theta|M=e\varphi$, $e^2=[H:M]$. Thus $\varphi^H=e\theta$ and it follows that $\mathscr{I}_{G_0}(\varphi)=I_0$ and φ is F-semi-invariant in G_0 so if we set $\chi_0=\hat{\varphi}^{G_0}$ we have by induction that $m=m_F(\chi_0)$ is prime to p. By (*) we have $\hat{\theta}|I_0=\psi\hat{\varphi}$ where $[\psi,1]$ is prime to p. Let $\Xi=m\sum_\sigma\chi_0^\sigma$ where σ runs over $\mathscr{G}(F(\chi_0)/F)$. Then

$$[\Xi^G,\chi] = [\Xi,\chi|G_0] = [\Xi,(\psi\hat{\varphi})^{G_0}] = [\Xi|I_0,\psi\hat{\varphi}] = m\sum[\chi_0^\sigma|I_0,\psi\hat{\varphi}].$$

If β is an irreducible constituent of ψ then $\beta\hat{\varphi}$ is irreducible so the only constituents of $\psi\hat{\varphi}$ with degree equal to $\varphi(1)$ and determinantal order prime to p are the ones of the form $1\cdot\hat{\varphi}$ and the multiplicity of this is $[\psi,1]=r$ which is prime to p. Thus

$$[\Xi^G,\chi] = rm\sum_\sigma[\chi_0^\sigma|I_0,\hat{\varphi}].$$

As before, if $\sigma\neq 1$, $[\chi_0^\sigma|I_0,\hat{\varphi}]=0$ and $[\chi_0|I_0,\hat{\varphi}]=1$ so $[\Xi^G,\chi]=rm$ which is prime to p and divisible by $m_F(\chi)$ and the proof of this case is complete.

We may suppose then that $T=M$ so $u=[H:M]$ and thus $e=1$. Now, I_0/M acts on H/M and on the set $\{\varphi_i\}$ and this action is compatible with the action of H/M on the set so by Glauberman's lemma (Theorem 4 of [3]), some φ_i (say φ_1) is invariant in I_0. If $\varphi_j=\varphi_1^h$ is also invariant in I_0, then for $x\in I_0$ we have $\varphi_1^{x^{-1}hx}=\varphi_1^h$ so $[h, I_0]\subseteq \mathscr{I}_H(\varphi_1)=M$. Thus if φ_1 is not the unique irreducible constituent of $\theta|M$ which is invariant in I_0, then $C_{H/M}(I_0)=C/M>1$. Since $I_0\triangle G_0$, we get $G_0\subseteq N(C)$ and thus $C\triangle G$ so $C=H$. It follows in that case that $I/M=H/M\times I_0/M$ and $I_0\triangle G$. Also every φ_i is invariant in I_0.

Suppose that this situation occurs. Now some irreducible constituent η of $\hat{\theta}|I_0$ satisfies $[\eta|M, \varphi_1]\neq 0$ and thus $\eta=\lambda\hat{\varphi}_1$, where $\lambda\in \text{Irr}(I_0/M)$. Since λ is invariant in I, we have by Clifford's Theorem that $\hat{\theta}|I_0=\lambda\sum_{i=1}^{u}\hat{\varphi}_i$. Thus $\lambda(1)=1$ and by calculating determinants we get $\lambda^{\theta(1)}=1$ so $\lambda=1$ and $\hat{\theta}|I_0=\sum\hat{\varphi}_i$. If any two of the φ_i are Galois conjugate over F, let $S=\{h\in H\mid \exists\sigma\in \mathscr{G}(F(\varphi_1)/F) \text{ with } \varphi_1^h=\varphi_1^\sigma\}$. Then $S>M$ is a subgroup of H which depends only on θ and clearly $S\triangle G$. Thus $S=H$ and hence φ_1 is F-semi-invariant in H. Now if $g\in G$ then $\theta^g=\theta^\tau$ where $\tau\in\mathscr{G}(F(\theta)/F)$ and since $F(\theta)\subseteq F(\varphi_1)$, τ may be extended to τ_1 on $F(\varphi_1)$. Then $\varphi_1^{\tau_1}$ and φ_1^g are both constituents of $\theta^g|M$ so $\varphi_1^{\tau_1^{-1}g}$ is a constituent of $\theta|M$ so that $\varphi_1^{\tau_1^{-1}g}=\varphi_1^\sigma$ where $\sigma\in\mathscr{G}(F(\varphi_1)/F)$. Thus $\varphi_1^g=\varphi_1^{\tau_1\sigma}$ and φ_1 is F-semi-invariant in G. Let $J=\mathscr{I}_G(\varphi_1)$ so $J\triangle G$ and G/J is abelian. Also, $J\cap H=M$ so J/M is a p-group and being normal, it must satisfy $J/M\subseteq G_0/M\in \text{Syl}_p(G/M)$. Thus $G'\subseteq J\subseteq G_0$ so $G_0\triangle G$. Since $\varphi_1^H=\theta$ we have $J\subseteq I$ so we must have $J=I_0$. Now $\hat{\varphi}_1^I=\hat{\theta}$ so $\hat{\varphi}_1^{G_0}=\chi_0$ satisfies $\chi_0^G=\chi$ and by induction $m_F(\chi_0)$ is prime to p. Since $p\nmid[G:G_0]$, it follows that $p\nmid m_F(\chi)$ by (2.3).

Consider now the case where all φ_i are invariant in I_0 but no two of them are Galois conjugate over F. In this case we also have $I_0=\mathscr{I}_{G_0}(\varphi_i)$. We claim that for some i, φ_i is F-semi-invariant in G_0. Let P_0 be a p-subgroup of G of maximal order such that some φ_i is F-semi-invariant in MP_0. Since $P_0^h\subseteq P$ for some $h\in H$ we may replace φ_i by φ_i^h and assume that $P_0\subseteq P$. Assume then that φ_1 is F-semi-invariant in P_0M and suppose that $P_0<P$. Choose $a\in P-P_0$. Then $\theta^a=\theta^\sigma$ for some $\sigma\in\mathscr{G}\cdot(F(\theta)/F)$ and φ_1^a is a constituent of $\theta^\sigma|M$. Extend σ to σ_1 on $F(\varphi_1)$ so that $\varphi_1^{\sigma_1}$ is a constituent of $\theta^\sigma|M$ and thus $\varphi_1^{\sigma_1}=\varphi_1^{ah}$ for some $h\in H$. It follows that φ_1 is F-semi-invariant in $\langle P_0M, ah\rangle=G_1$. Now $ah\notin P_0M$ because for any element of G, the representation as a product of an element of P with an element of H is unique and we conclude that $G_1>P_0M$. However, $G_1\cap H=M$ since no two of the φ_i are Galois conjugate and thus the p-part of $|G_1|$ exceeds $|P_0|$. Taking a Sylow subgroup of G_1 yields a contradiction to the maximality of P_0 and thus we have $\chi_0=\hat{\varphi}_1^{G_0}$ satisfies $m=m_F(\chi_0)$ is prime to p. Now let $\Xi=m\sum_\sigma\chi_0^\sigma$ as before. Then

$$[\Xi^G, \chi] = [\Xi, \chi|G_0] = [\Xi|I_0, \hat{\theta}|I_0] = m\sum[\chi_0^\sigma|I_0, \hat{\theta}|I_0].$$

Suppose $\sigma\neq 1$ and extend σ to σ_1 on $F(\varphi_1)=F(\hat{\varphi}_1)$. If $\hat{\varphi}_i$ is a constituent of $\chi_0^\sigma|I_0$ then $\hat{\varphi}_i$ and $\hat{\varphi}_1^{\sigma_1}$ are conjugate in G_0. Thus $\hat{\varphi}_1^{g\sigma_1}=\hat{\varphi}_i$ for some $g\in G_0$ and hence $\varphi_1^{g\sigma_1}=\varphi_1$. Now $\varphi_1^g=\varphi_1^\tau$ for some $\tau\in\mathscr{G}(F(\varphi_1)/F)$ and thus $\varphi_1^{\tau\sigma_1}=\varphi_i$. This forces

$i=1$ so $\hat{\varphi}_1$ is a constituent of $\chi_0^g|I_0$. However, $\hat{\varphi}_1^{G_0}=\chi_0$ and this is a contradiction. Since $\hat{\theta}|I_0=\sum \hat{\varphi}_i$ we have then

$$[\Xi^G, \chi] = m[\chi_0|I_0, \hat{\theta}|I_0] = m[\chi_0, \chi|G_0] = m[\chi_0^G, \chi].$$

However, $\chi_0^G=\hat{\varphi}_1^G=\hat{\theta}^G=\chi$ so $[\Xi^G, \chi]=m$ and the result follows in this case.

The one remaining case is where exactly one constituent φ_1 of $\theta|M$ is invariant in I_0. Here we have $\hat{\theta}|I_0=\lambda\hat{\varphi}_1+\sum \eta_j$ where $\eta_j \in \mathrm{Irr}\,(I_0)$ with $p|\eta_j(1)$ and $\lambda \in \mathrm{Irr}\,(I_0/M)$. We claim that in fact $\lambda=1$. Since $p\nmid\lambda(1)$ we have $\lambda(1)=1$ and thus $(\lambda\hat{\varphi}_1)^I=\hat{\theta}=\lambda\hat{\varphi}_1^I$ where on the right, λ is viewed as a character of I with kernel containing H. We will have shown that $\lambda=1$ if we show that $\hat{\varphi}_1^I$ has determinantal order prime to p. Let $n=o(\hat{\varphi}_1)$ and consider $(\det \hat{\varphi}_1^I)^n(g)$. This is $\pm$ a product of factors of the form $(\det \hat{\varphi}_1)^n(tgt^{-1})=1$. Hence if the associated sign is $+$ we are done. The sign is determined by the parity of the permutation that $g \in I$ induces on the right cosets of I_0 in I. This parity is the same as that of the action of g on the cosets of G_0 in G. If the action of every element of I were not even, there would exist a subgroup A of index 2 in G with $A\supseteq M$ and $A \cap I<I$ so $AI=G$. If $p\neq 2$ then even if $o(\hat{\varphi}_1^I)=2n$ we may conclude that $\lambda=1$ and the sign above does not matter. We may assume then that $p=2$ so that $A\supseteq H$ and this yields a contradiction to $I/H\subseteq \Phi(G/H)$. Thus in any case our claim is established and $\lambda=1$. Therefore $\hat{\theta}|I_0=\hat{\varphi}_1+\sum \eta_j$.

Now if $g \in G_0$ then φ_1^g is a constituent of $\theta^g|M=\theta^\sigma|M$ for some $\sigma \in \mathscr{G}(F(\theta)/F)$. Thus as before we must have $\varphi_1^g=\varphi_i^{\sigma_1}$ where σ_1 is an extension of σ to $F(\varphi_i)$. However, φ_1^g is invariant in I_0 since $I_0 \triangle G_0$ and so $\varphi_i^{\sigma_1}$ and hence also φ_i is invariant in I_0 and this forces $i=1$ and thus φ_1 is F-semi-invariant in G_0. Since $I_0=\mathscr{I}_{G_0}(\varphi_1)$ we have by induction that $\chi_0=\hat{\varphi}_1^{G_0}$ satisfies $m_F(\chi_0)$ is prime to p. As in the previous cases, we set $\Xi=m\sum_\sigma \chi_0^\sigma$ so $[\Xi^G, \chi]=m\sum [\chi_0^\sigma|I_0, \hat{\theta}|I_0]$ where σ runs over $\mathscr{G}(F(\chi_0)/F)$. Now every constituent of $\chi_0^\sigma|I_0$ has degree $\varphi_1(1)$ and the only constituent of $\hat{\theta}|I_0$ with this degree is $\hat{\varphi}_1$. Also, if $\sigma\neq 1$ then $[\chi_0^\sigma|I_0, \hat{\varphi}_1]=0$ so we have

$$[\Xi^G, \chi] = m[\chi_0|I_0, \hat{\varphi}_1] = m.$$

The result now follows in this case and the entire proof is complete.

5. In this and the next section we attempt to find conditions which will guarantee that (*) will hold so that the previous theorem can be applied. Here we consider a situation somewhat more general than is necessary for this purpose because the results of this section may have some independent interest.

Let G be a finite group and suppose $M<H\triangle G$ with $M\triangle G$ and H/M an elementary abelian q-group where $q\nmid[G : H]$. Let $\theta \in \mathrm{Irr}\,(H)$ satisfy $\theta|M=e\varphi$ where $\varphi \in \mathrm{Irr}\,(M)$ and θ vanishes on $H-M$. Suppose also that θ is invariant in G and that $o(\theta)$ and $\theta(1)$ are prime to $[G : H]$ so that θ has a unique extension $\hat{\theta}$ on G with $(o(\hat{\theta}), [G : H])=1$. Since $(\det \varphi)^e=\det\,(\theta|M)$ and $(e, [G : H])=1$, we may define $\hat{\varphi}$ on U where U/M is a complement for H/M in G/M which exists by the Schur-Zassenhaus Theorem. Then every irreducible constituent of $\hat{\theta}|U$ is of the form

$\beta\hat{\varphi}$ where $\beta \in \operatorname{Irr}(U/M)$ is uniquely determined. Therefore there exists a unique character ψ (possibly reducible) such that $\hat{\theta}|U=\psi\hat{\varphi}$. (As usual we do not distinguish between characters of U/M and characters of U with M contained in their kernels.) It is the object of this section to obtain information about ψ. We first consider the case where G/M is a Frobenius group with cyclic complement U/M. We begin with a lemma.

(5.1) LEMMA. *Let C be a finite group and f a function from C into the rational integers. Suppose that for all $c \in C$ we have*

$$\sum_{x\in C}[f(x)-f(cx)]^2 \leqq 2.$$

Then either f is constant or there exists an integer k and $a \in C$ such that $f(x)=k$ for $x \neq a$ and $f(a)=k\pm 1$.

Proof. For all $x,\ c \in C$, $[f(x)-f(cx)]^2 \leqq 2$ so $|f(x)-f(y)| \leqq 1$ for all $x,\ y \in C$. Suppose f is not constant so that we see f takes on exactly two values and these differ by 1. Let $A,\ B \subseteq C$ be the inverse images of these two values. We have then

$$2(|C|-1) \geqq \sum_{x,c\in C}[f(x)-f(cx)]^2 = \sum_{x,y\in C}[f(x)-f(y)]^2$$
$$= \sum_{x\in A}|B| + \sum_{x\in B}|A| = 2|A|\,|B|.$$

Thus we have $|C|-1 \geqq |A|\,|B|$ and $|A|+|B|=|C|$ and also $1 \leqq |A| \leqq |C|-1$. Since the function $x(|C|-x)$ takes on its minimal value, $|C|-1$, at the endpoints of the interval $1 \leqq x \leqq |C|-1$, the inequality $|C|-1 \geqq |A|(|C|-|A|)$ yields $|A|=1$ or $|A|=|C|-1$ and the result follows.

(5.2) PROPOSITION. *Let G/M be a Frobenius group with G/H cyclic. Then there exists $\varepsilon=\pm 1$ with $[U:M] \mid (e-\varepsilon)$ and one of the following occurs:*

(a) *$\psi=\varepsilon 1+((e-\varepsilon)/[U:M])\rho$ and $[U:M]$ is odd or $(e-\varepsilon)/[U:M]$ is even,*

(b) *$\psi=\varepsilon\mu+((e-\varepsilon)/[U:M])\rho$ and $[U:M]$ is even and $(e-\varepsilon)/[U:M]$ is odd,*

where ρ is the regular character of U/M and in case (b), *μ is the unique linear character of U/M with order* 2.

Proof. Since G/M is a Frobenius group, it is the disjoint union of H/M with the $[H:M]$ conjugates of $(U/M)^{\#}$. Let χ_1 and χ_2 be two class functions of G such that χ_1 vanishes on H. Then

$$|G|[\chi_1, \chi_2] = [H:M]|U|[\chi_1|U, \chi_2|U]$$

and since $[G:U]=[H:M]$ we have $[\chi_1, \chi_2]=[\chi_1|U, \chi_2|U]$.

Now let $\lambda \neq 1$ be a linear character of G/H and $\chi_1=\lambda\hat{\theta}-\hat{\theta}=\chi_2$. Then $\chi_1|H=0$ so $2=[\chi_1, \chi_2]=[(\lambda\hat{\theta}-\hat{\theta})|U, (\lambda\hat{\theta}-\hat{\theta})|U]=[(\lambda\psi-\psi)\hat{\varphi}, (\lambda\psi-\psi)\hat{\varphi}]=[(\lambda\psi-\psi), (\lambda\psi-\psi)]$ where the last equality follows because $[\beta_1\hat{\varphi}, \beta_2\hat{\varphi}]=[\beta_1, \beta_2]$ for $\beta_i \in \operatorname{Irr}(U/M)$. We are freely interpreting λ as a character of G/H or U/M as is convenient. Write

$\psi=\sum_\nu f(\nu)\nu$ where f if a nonnegative integer valued function and ν runs over the group C of linear characters of U/M. Then $\psi-\lambda\psi=\sum_\nu [f(\nu)-f(\lambda^{-1}\nu)]\nu$ so $\sum [f(\nu)-f(\lambda^{-1}\nu)]^2=2$ for all $\lambda \in C$, $\lambda\neq 1$. Clearly f is not constant so by the lemma, there exists $\mu \in C$ and integer k such that $f(\nu)=k$ for $\nu\neq\mu$ and $f(\mu)=k+\varepsilon$ where $\varepsilon=\pm 1$. We can thus write $\psi=\varepsilon\mu+k\rho$ where ρ is the regular character of U/M. Since $\psi(1)=e$, $\rho(1)=[U:M]$ and $\mu(1)=1$ we obtain $k=(e-\varepsilon)/[U:M]$ so all that remains is to determine μ. Since $\det(\psi\hat{\varphi})=(\det\psi)^{\varphi(1)}(\det\hat{\varphi})^{\psi(1)}$ and $\varphi(1)$ is prime to $[G:H]$ as is $o(\hat{\varphi})$, we conclude that $o(\psi)$ is prime to $[G:H]$. To find μ we calculate $\det\rho$. On a generator of U/M, $\det\rho$ has the value $\prod_{i=1}^{[U:M]}\delta^i$ where δ is a primitive $[U:M]$ root of unity. This product is ± 1 where the negative value occurs iff $[U:M]$ is even. Thus if $[U:M]$ is odd or if k is even, $\det(k\rho)=1$ and it follows that $\mu=1$ so we have (a). Otherwise $\det(k\rho)^2=1$ but $\det(k\rho)\neq 1$ and (b) holds. The proof is complete.

(5.3) Proposition. *Suppose $L\triangle G$ with $M\subseteq L\subseteq H$. Then G/H acts on H/L and on the group $(L/M)^*$ of linear characters of L/M. Suppose that these two G/H-modules have no composition factors in common. Then $\theta|L=f\xi$ where $\xi\in \operatorname{Irr}(L)$ is invariant in G and vanishes on $L-M$.*

Proof. Since H/M is abelian, it is clear that both actions are well defined. It is sufficient to show that $\theta|L$ is homogeneous. Let ξ be an irreducible constituent of $\theta|L$. If ξ vanishes on $L-M$, then since $\theta|M$ is homogeneous, we have $\xi|M=a\varphi$ and thus $a^2=[L:M]$ and $\varphi^L=a\xi$. However, every irreducible constituent of $\theta|L$ must be a constituent of φ^L so we are done in this case. Suppose then that ξ does not vanish on $L-M$. Then for some $\lambda\in(L/M)^*$, $\lambda\xi\neq\xi$. However, for any $\lambda\in(L/M)^*$ we have $\lambda\xi|M=\xi|M$ so $\lambda\xi$ is a constituent of φ^L and $(\lambda\xi)^H$ is a constituent of $\varphi^H=e\theta$. Therefore $0\neq[(\lambda\xi)^H,\theta]=[\lambda\xi,\theta|H]$ so $\lambda\xi$ is an irreducible constituent of $\theta|L$. Now let $T=\mathscr{I}_H(\xi)\triangle H$. There is a one-to-one correspondence between the irreducible constituents of $\theta|L$ and the elements of H/T. Thus for each $\lambda\in(L/M)^*$ we can find a unique $Th\in H/T$ such that $\lambda\xi=\xi^h$ and this defines a function $f\colon (L/M)^*\to H/T$. We claim that f is a G/H-homomorphism and since $\ker(f)<(L/M)^*$, we will be done when this claim is established. We have

$$\lambda_1\lambda_2\xi = \lambda_1\xi^{h_2} = (\lambda_1\xi)^{h_2}$$

since $\lambda_1^{h_2}=\lambda_1$. Thus $\lambda_1\lambda_2\xi=\xi^{h_1h_2}$ and $f(\lambda_1\lambda_2)=h_1h_2T$ where $f(\lambda_i)=h_iT$ for $i=1$ or 2. Therefore f is a homomorphism. Now let $g\in G$. We have $\theta^{g^{-1}}=\theta$ so $\xi^{g^{-1}}$ is a constituent of $\theta|L$ and for some $h\in H$, $\xi^{g^{-1}}=\xi^h$. Thus $\xi=\xi^{hg}$ and $\lambda^g\xi=\lambda^g\xi^{hg}$ $=(\lambda\xi^h)^g=(\lambda\xi)^{hg}$ since $\lambda^h=\lambda$. Thus $\lambda^g\xi=\xi^{h_1hg}=\xi^{hh_1g}=\xi^{g^{-1}h_1g}$ and it follows that $f(\lambda^g)=f(\lambda)^g$ and the proof is complete.

(5.4) Theorem. *The character ψ is determined by the action of U/M on H/M. If $x\in U/M$ then $|C_{H/M}(x)|$ is a square and $\psi(x)=\pm(|C_{H/M}(x)|)^{1/2}$. If $[U:M]$ is a power of a prime p then $\psi(x)\equiv e \bmod p$ and this determines the sign if $p\neq 2$.*

Proof. If $H \subseteq G_0 \subseteq G$ and $\tilde{\theta}$ is the canonical extension of θ to G_0, then $\theta | U_0 = (\psi | U_0)\tilde{\varphi}$ where $U_0 = U \cap G_0$ and $\tilde{\varphi}$ is the canonical extension of φ to U_0. By the uniqueness of ψ, it follows that to determine $\psi(x)$, we may set $U_0/M = \langle x \rangle$ so that we may assume that $U/M = \langle x \rangle$ and G/H is cyclic. We proceed by induction on $[G : M]$. Suppose that $C_{U/M}(H/M) = M_0/M > 1$. Then $M_0 \triangle G$. Let $H_0 = HM_0$ and $\tilde{\theta}$ and $\tilde{\varphi}$ be the canonical extensions to H_0 and M_0. Let $\lambda\tilde{\varphi}$ be an irreducible constituent of $\tilde{\theta}|M_0$. Now λ is invariant in H_0 and by the uniqueness of its definition, $\tilde{\varphi}$ is also so we have $\tilde{\theta}|M_0 = e(\lambda\tilde{\varphi})$. By the determinant criterion, $\lambda^{e\varphi(1)} = 1$ and thus we must have $\lambda = 1$ and $\tilde{\theta}|M_0 = e\tilde{\varphi}$. If $M_0 = U$ then $H_0 = G$ and we have $\psi = e \cdot 1$ so $\psi(x) = e = ([H : M])^{1/2}$ and the result follows in this case. We suppose then that $M_0 < U$. Suppose $M_0 > M$. We may apply the theorem to the situation $(G, H_0, M_0, U, \tilde{\theta}, \tilde{\varphi})$ in place of $(G, H, M, U, \theta, \varphi)$ since $[G : M_0] < [G : M]$ so induction applies. The canonical extensions of $\tilde{\theta}$ and $\tilde{\varphi}$ to G and U are $\hat{\theta}$ and $\hat{\varphi}$ and thus the ψ obtained in the inductive situation is the same as the original ψ. The isomorphism between H/M and H_0/M_0 is an isomorphism of U/M-modules so induction applies to tell us that $\psi(x)$ is determined by the action of x on H_0/M_0 which is identical with the action on H/M. In particular, $\psi(x)^2 = |C_{H_0/M_0}(x)| = |C_{H/M}(x)|$ and the result follows in this case.

We now assume that $M_0 = M$ so $C_{U/M}(H/M) = 1$. If U/M acts in a Frobenius manner on H/M then (5.2) applies and we have either (a) $\psi(x) = \varepsilon$ or (b) $\psi(x) = -\varepsilon$ where (a) occurs iff $[U : M]$ is odd or $(e-\varepsilon)/[U : M]$ is even. Thus $\psi(x)^2 = 1 = |C_{H/M}(x)|$. If $[U : M] > 2$, ε is uniquely determined by the condition that $[U : M]$ divides $e - \varepsilon$ and thus $\psi(x)$ is determined by the order of U/M and e. If $[U : M] = 2$ and $\varepsilon = 1$ then $\psi(x) = 1$ iff $(e-1)/2$ is even. If $\varepsilon = -1$ then $\psi(x) = 1$ iff $(e-\varepsilon)/2 = (e+1)/2$ is odd. Thus the value of $\psi(x)$ is independent of the choice of ε. If $[U : M] = p^a$ for an odd prime p then $\psi(x) = \varepsilon$ and since $[U : M]$ divides $e - \varepsilon$ we have $\psi(x) \equiv e \bmod p$. If $p = 2$ this is certainly true and the proof in this situation is complete.

We may suppose then that for some $y \in U/M$ of prime order, that $L/M = C_{H/M}(y) > 1$. However, since $y \notin C_{U/M}(H/M)$, we have $L < H$ and since U/M is abelian, $L \triangle G$. In the action of U/M on $(L/M)^*$, y acts trivially. Now y cannot act trivially on any composition factor of H/L since this would imply that y fixes some element of $H/M - L/M$ because the order of y is a prime not dividing $|H/M|$ and this contradicts the definition of L. It follows that H/L and $(L/M)^*$ have no composition factors in common as U/M-modules. Therefore (5.3) applies and $\theta | L = f\xi$ where $\xi \in \mathrm{Irr}\,(L)$ and ξ vanishes on $L - M$. Let $V = UL$ and let $\hat{\xi}$ be the canonical extension of ξ to V. Then $\hat{\theta}|V = \psi_1\hat{\xi}$ and $\hat{\xi}|U = \psi_2\hat{\varphi}$ and $\psi = \psi_1\psi_2$. By induction, the conclusions of the theorem apply to ψ_1 and ψ_2 since $[G : L] < [G : M]$ and $[V : M] < [G : M]$. We may apply Maschke's Theorem to find a complement, L_0/M for L/M in H/M such that $L_0 \triangle G$. Clearly, $|C_{H/M}(x)| = |C_{L/M}(x)|\ |C_{L_0/M}(x)|$ and the latter factor is equal to $|C_{H/L}(x)|$. We have then, $\psi(x)^2 = \psi_1(x)^2\psi_2(x)^2 = |C_{H/M}(x)|$. Also if $[U : M]$ is a power of a prime p, we have

$\psi_1(x) \equiv f$ and $\psi_2(x) \equiv e/f \bmod p$ and the result follows in this case. The proof is complete.

6. Let G have a solvable normal p-complement H and let H/M be a chief factor of G. Let $\theta \in \mathrm{Irr}(H)$, $\varphi \in \mathrm{Irr}(M)$ and $\theta|M = e\varphi$ where θ vanishes on $H-M$. Let $P \in \mathrm{Syl}_p(G)$ with $P' \subseteq B \subseteq \Phi(P)$ where θ is invariant in HB. Let $\hat{\theta}$ and $\hat{\varphi}$ be the canonical extensions of θ and φ to HB and MB and write $\hat{\theta}|MB = \psi\hat{\varphi}$. We attempt to find conditions on P which will guarantee that it satisfies (*) of §4, that is to show that $[\psi, 1]$ is prime to p. We shall use some of the results of §5 to prove the following theorem. Unfortunately, we do not have a way to utilize the full strength of Theorem (5.4) and so the present result is probably weaker than it must be. In fact it seems reasonable to conjecture that it might be true without any conditions on P at all.

(6.1) THEOREM. *If P is regular and metabelian then $[\psi, 1]$ is prime to p.*

We begin with a lemma.

(6.2) LEMMA. *Let $C = C_P(H/M)$. If the action of B on H/M is homogeneous (all irreducible constituents equivalent) then $B/(B \cap C)$ is cyclic. If P/C is not cyclic and $p \neq 2$ then B acts reducibly and the number of its irreducible constituents is divisible by p.*

Proof. Suppose $B/(B \cap C) \cong BC/C$ is not cyclic. Since $B \subseteq \Phi(P)$, $BC/C \subseteq \Phi(P/C)$ and thus BC/C contains a normal subgroup A/C which is abelian of type (p, p). (See for instance Hilfsatz 7.5, p. 303 of [4].) Since $p^2 \nmid |\mathrm{Aut}\,(A/C)|$, $C_{P/C}(A/C)$ has index $\leq p$ and thus contains $\Phi(P/C) \supseteq BC/C$. Thus $A/C \subseteq \boldsymbol{Z}(BC/C)$ and hence $\boldsymbol{Z}(BC/C)$ is not cyclic. It follows that BC/C cannot have a faithful irreducible representation. Now, BC/C acts faithfully on H/M and thus not all irreducible constituents can have the same kernel and the first statement follows.

Suppose $p \neq 2$. We have P/C acts irreducibly on H/M and is not cyclic so it follows by a theorem of Roquette, (p. 248 of [5]), that there exists a subgroup P_0/C of index p in P/C such that the representation of P_0/C on H/M splits into p irreducible constituents. Since $B \subseteq P_0$, the result follows.

(6.3) PROPOSITION. *Let $C = C_P(H/M)$. If $B/(B \cap C)$ is cyclic then $p \nmid [\psi, 1]$.*

Proof. Let $A = B \cap C$. If $A = B$ then B acts trivially on H/M and it is clear that $\psi = e \cdot 1$ and since $p \nmid e$, there is nothing further to prove in this case. Suppose then $A < B$. We claim that B/A acts in a Frobenius manner on H/M. Suppose then $b \in B$ and $C_{H/M}(b) > 1$. Let $E = \langle b, A\rangle$ so $L/M = C_{H/M}(E) > 1$. Now B/A is cyclic so E/A is characteristic and thus $E \triangle P$. Thus $P \subseteq \boldsymbol{N}(L)$ and by the irreducibility of the action of P on H/M we have $L = H$. Thus $b \in E \subseteq C_B(H/M) = A$ and the action of B/A on H/M is indeed Frobenius.

Now, $AM \triangle BH$ and the action of B on H/M is isomorphic with its action on AH/AM. We may calculate ψ (viewed as a character of B for convenience) by applying (5.2) to BH/AM and we conclude that $\ker \psi \supseteq A$ and $\psi = \varepsilon\mu +$

$(e-\varepsilon)/[B:A]\rho$ where ρ is the regular character of B/A viewed as a character of B and $\mu^2=1$ and $\mu=1$ unless $p=2$ and $(e-\varepsilon)/[B:A]$ is odd.

Suppose $p\neq 2$. We claim that $p[B:A]$ divides $[H:M]-1$. Certainly, $[B:A]|([H:M]-1)$ since the action is Frobenius. If P/C is not cyclic, then by (6.2), B acts reducibly on H/M and the number of irreducible constituents, n, is divisible by p. If each constituent has order k (as an elementary abelian q-group) then since B/A acts fixed point freely on each constituent, $[B:A]|(k-1)$ and $k^n=[H:M]$. Thus

$$[H:M]-1 = k^n-1 = (k-1)(1+k+\cdots+k^{n-1})$$

and to establish the claim it suffices to show that the second factor is divisible by p. However $k\equiv 1 \bmod p$ and so $1+k+\cdots+k^{n-1}\equiv n\equiv 0 \bmod p$. If P/C is cyclic, it acts in a Frobenius manner on H/M and $[P:C]|([H:M]-1)$. Now $[B:A]=[BC:C]<[P:C]$ so the claim is fully established.

Now $[H:M]-1=e^2-1=(e-\varepsilon)(e+\varepsilon)$ and $[B:A]|(e-\varepsilon)$. Since $p\neq 2$ is being assumed, $p\nmid(e+\varepsilon)$ so the full p-part of $[H:M]-1$ divides $e-\varepsilon$ and thus $p[B:A]|(e-\varepsilon)$ so $(e-\varepsilon)/[B:A]$ is divisible by p and we have $[\psi, 1]\equiv\varepsilon\not\equiv 0 \bmod p$ if $p\neq 2$.

Suppose then that $p=2$. If $(e-\varepsilon)/[B:A]$ is even we have $[\psi, 1]\equiv\varepsilon\not\equiv 0 \bmod 2$ and the result follows. If $(e-\varepsilon)/[B:A]$ is odd then $\mu\neq 1$ so $[\psi, 1]=(e-\varepsilon)/[B:A]$ $\not\equiv 0 \bmod 2$ and thus the result follows in this case also and the proof is complete.

We now give the

Proof of (6.1). Let C be as in (6.3). If $p=2$ then since P is assumed to be regular, it is abelian and since P/C is represented irreducibly and faithfully on H/M, P/C is cyclic. Thus $BC/C\cong B/(B\cap C)$ is cyclic and the result follows from (6.3). We assume then that $p\neq 2$. By (6.3) and (6.2) we may assume that B acts reducibly and inhomogeneously on H/M. Since the action is completely reducible, we may let L_i/M, $1\leqq i\leqq p^a$, be the distinct homogeneous constituents of the action of B on H/M. By Clifford's Theorem, the L_i are all conjugate in G. Let $K_i=\prod_{j\neq i} L_j$ so $H/K_i\cong L_i/M$ as B-modules. We claim that H/K_i and $(K_i/M)^*$ have no B-constituent in common, so we may apply (5.3). Since the dual of a homogeneous module is homogeneous, the only way our claim can fail is if $L_i/M\cong(L_j/M)^*$ for some $i\neq j$. If this occurs, then since all L_i are conjugate under the action of P, a pairing will be established among the integers i, $1\leqq i\leqq p^a$ where $i\leftrightarrow j$ iff L_i/M $\cong(L_j/M)^*$. Since p is odd, $i\leftrightarrow i$ for some i and this forces $j\leftrightarrow j$ for all j. Thus $L_i/M\cong(L_j/M)^*\cong L_j/M$ contradicting $i\neq j$ and the claim is established.

From (5.3) it follows that $\theta|K_i=f\xi_i$ and from (5.4) it follows that the class function η_i on B is a rational valued character, where $\eta_i(b)=\pm(|C_{H/K_i}(b)|)^{1/2}$ and the sign is chosen so that $\eta_i(b)\equiv f \bmod p$. Now $|C_{H/M}(b)|=\prod_i |C_{L_i/M}(b)|=\prod_i |C_{H/K_i}(b)|$ and it follows that $\psi(b)=\pm\prod_i \eta_i(b)$. Since $\psi(b)\equiv e=f^{p^a} \bmod p$, the sign is $+$ and $\psi=\prod \eta_i$. Also since the L_i are all conjugate under the action of P it follows that the

η_i are all conjugate. (They are not necessarily all distinct, however.) We put the remainder of the proof into a separate result.

(6.4) PROPOSITION. *Let P be a regular metabelian p-group and suppose that $P' \subseteq B \subseteq \Phi(P)$. Let J be an index set with $|J| = p^a > 1$ and let η_j be a rational valued character of B for each $j \in J$. Suppose that P acts transitively on J in such a manner that $\eta_{j \cdot x} = (\eta_j)^x$ and B is in the kernel of the action. Let $\psi = \prod_{j \in J} \eta_i$ and suppose that $p \nmid \psi(1)$. Then $p \nmid [\psi, 1]$.*

Proof. Choose a particular element, say $1 \in J$ and write $\eta_1 = \sum_{i \in I} a_i \varphi_i^{(1)}$ where the $\varphi_i^{(1)} \in \mathrm{Irr}(B)$ are distinct and I is a suitable index set. For $j \in J$, fix a particular $x_j \in P$ with $1 \cdot x_j = j$ (take $x_1 = 1$) so we have

$$\eta_j = (\eta_1)^{x_j} = \sum_{i \in I} a_i \varphi_i^{(j)} \tag{1}$$

where $\varphi_i^{(j)} = (\varphi_i^{(1)})^{x_j} \in \mathrm{Irr}(B)$.

Let $\mathscr{S}$ be the set of functions $f: J \to I$. Then

$$\psi = \prod_{j \in J} \sum_{i \in I} a_i \varphi_i^{(j)} = \sum_{f \in \mathscr{S}} \prod_{j \in J} a_{f(j)} \varphi_{f(j)}^{(j)}. \tag{2}$$

We write $a_f = \prod_j a_{f(j)}$ so that we obtain

$$\psi = \sum_{f \in \mathscr{S}} a_f \prod_j \varphi_{f(j)}^{(j)}. \tag{3}$$

Now let T be the stabilizer of 1 in the action of P on J so $P' \subseteq B \subseteq T < P$ and $\eta_1^t = \eta_1$ for $t \in T$. Therefore T permutes the irreducible constituents of η_1 and we may define an action of T on I by $\varphi_{i \cdot t}^{(1)} = (\varphi_i^{(1)})^t$. Note that $a_i = a_{i \cdot t}$. Now let T act on $\mathscr{S}$ by $f^t(j) = f(j) \cdot t$. We have then $a_{f^t} = a_f$. Also

$$\varphi_{f^t(j)}^{(j)} = \varphi_{f(j) \cdot t}^{(j)} = (\varphi_{f(j) \cdot t}^{(1)})^{x_j} = (\varphi_{f(j)}^{(1)})^{t x_j} = (\varphi_{f(j)}^{(1)})^{x_j t},$$

where the last equality follows since $P' \subseteq B$. Thus

$$\varphi_{f^t(j)}^{(j)} = (\varphi_{f(j)}^{(j)})^t$$

and

$$a_{f^t} \prod_{j \in J} \varphi_{f^t(j)}^{(j)} = \left(a_f \prod_{j \in J} \varphi_{f(j)}^{(j)} \right)^t. \tag{4}$$

Thus in (3), the contributions of the f and f^t terms are equal when calculating either $\psi(1)$ or $[\psi, 1]$. In evaluating these integers mod p we may therefore neglect all those f which lie in T-orbits of size divisible by p, i.e. all those f which are not fixed by T.

Let $I_0 = \{i \in I \mid i \cdot t = i, \ \forall t \in T\}$, and let $\mathscr{S}_0 = \{f \in \mathscr{S} \mid f(j) \in I_0, \ \forall j \in J\}$. The elements of $\mathscr{S}_0$ are precisely the f which are invariant under T and thus we have $\psi(1) \equiv \psi_0(1)$ and $[\psi, 1] \equiv [\psi_0, 1] \bmod p$ where

$$\psi_0 = \sum_{f \in \mathscr{S}_0} a_f \prod_j \varphi_{f(j)}^{(j)}. \tag{5}$$

Now let P act on $\mathscr{S}_0$ according to the formula $f^y(j)=f(j\cdot y^{-1})$. We have then $a_{f^y}=a_f$. Now suppose $k=j\cdot y$. Then $1\cdot x_k=k=j\cdot y=1\cdot x_j y$ so $x_j y=tx_k$ for some $t\in T$. We obtain for $f\in\mathscr{S}_0$

$$(\varphi_{f(j)}^{(j)})^y = (\varphi_{f(j)}^{(1)})^{x_j y} = (\varphi_{f(j)}^{(1)})^{tx_k} = (\varphi_{f(j)}^{(1)})^{x_k},$$

where the last equality follows because $f(j)\in I_0$. Thus

$$(\varphi_{f(j)}^{(j)})^y = \varphi_{f(j)}^{(k)} = \varphi_{f^y(j\cdot y)}^{(j\cdot y)}$$

and this yields

$$\left(a_f \prod_j \varphi_{f(j)}^{(j)}\right)^y = a_{f^y} \prod_j \varphi_{f^y(j)}^{(j)}. \tag{6}$$

Therefore, reasoning as before we have $\psi(1)\equiv\psi_1(1)$ and $[\psi, 1]\equiv[\psi_1, 1] \bmod p$ where

$$\psi_1 = \sum_{f\in\mathscr{S}_1} a_f \prod_j \varphi_{f(j)}^{(j)} \tag{7}$$

and $\mathscr{S}_1=\{f\in\mathscr{S}_0 \mid f^y=f,\ \forall y\in P\}$. In other words, $\mathscr{S}_1$ consists of the constant functions from J to I_0. Thus we have

$$\psi_1 = \sum_{i\in I_0} (a_i)^{p^a} \prod_j \varphi_i^{(j)}. \tag{8}$$

Let ε be a primitive $|P|$ root of unity and ε_0 a primitive pth root of 1. Let $U=\mathscr{G}(Q(\varepsilon)/Q(\varepsilon_0))$ so U is a p-group and U acts on Irr (B). This induces an action of U on I and since η_1 has rational values, $a_i=a_{i\cdot u}$ for all $u\in U$. Clearly, I_0 is an invariant subset of I so U acts on I_0. We have

$$(\varphi_i^{(j)})^u = (\varphi_i^{(1)})^{x_j u} = (\varphi_i^{(1)})^{ux_j} = \varphi_{i\cdot u}^{(j)}.$$

Let $I_2=\{i\in I_0 \mid i\cdot u=i,\ \forall u\in U\}$ and set

$$\psi_2 = \sum_{i\in I_2} (a_i)^{p^a} \prod_j \varphi_i^{(j)} \tag{9}$$

so that reasoning as before we have $\psi(1)\equiv\psi_2(1)$ and $[\psi, 1]\equiv[\psi_2, 1] \bmod p$. If we can show for $i\in I_2$ that

$$\left[1, \prod_j \varphi_i^{(j)}\right] \equiv \prod_j \varphi_i^{(j)}(1) \mod p, \tag{10}$$

then we will have $\psi_2(1)\equiv[\psi_2, 1] \bmod p$ and the result will follow. We shall show that because of the hypotheses on P, all of these $\varphi_i^{(j)}$ are linear characters and that $\prod_j \varphi_i^{(j)}=1$.

Suppose then that $\varphi\in$ Irr (B), $\varphi(1)>1$ and that φ is invariant under U. It follows that $\boldsymbol{Z}(B/\ker\varphi)$ has order p. Since the hypotheses that P is metabelian and regular are inherited by factor groups, we may assume that $\bigcap_{x\in P}\ker\varphi^x=1$. Now let $z\in\boldsymbol{Z}(B)$ so that $z^p\in\ker\varphi$. It follows that $z^p\in\ker\varphi^x$ for all $x\in P$ so $z^p=1$ and

$Z(B)$ is elementary abelian. Let $C=B\cap Z(\Phi(P))$. Since $\Phi(P)\supseteq B$, $C\subseteq Z(B)<B$, where the latter inequality follows from the assumed existence of a nonlinear φ. Choose $A\triangle P$ with $C<A\subseteq B$ and $[A:C]=p$. Furthermore, since $P'\subseteq B$, if $P'C>C$ we may assume that $A\subseteq P'C$ and since $P'C$ is abelian we have in this case that $C(A)\supseteq P'$. Otherwise, $P'C=C$ so $P'\subseteq Z(B)$ and thus here too we have $P'\subseteq C(A)$. Note that A is abelian. Let $y\in A-C$ and $x\in P$ so $y^x=yc$ for $c\in C$. Thus $1=c^p =(y^{-1}x^{-1}yx)^p=(y^{-1}x^{-p}y)(x^p)u^p$ where $u\in\langle x^{-y},x\rangle'\subseteq\langle x,y\rangle'$ by regularity. However, $\langle x,y\rangle'\subseteq C$ since $[x,y]\in C$ and C is elementary so $u^p=1$. Thus $1=[y,x^p]$ and $x^p\in C(y)$. Also $x^p\in\Phi(P)$ so x^p centralizes C and hence $x^p\in C(A)$ for all $x\in P$. Now $P'\subseteq C(A)$ so $P/C(A)$ is an elementary abelian p-group and $\Phi(P)\subseteq C(A)$ so $A\subseteq B\cap Z(\Phi(P))=C<A$. This contradiction shows that $\varphi(1)>1$ is impossible for $\varphi\in\operatorname{Irr}(B)$, φ invariant under U.

Finally, we suppose that $\varphi_1^{(1)}=\lambda$, a linear character invariant under T and U and we show that $\prod_x\lambda^x=1$ as x runs over a transversal for T in P. Since λ is fixed by U, we have $\lambda^p=1$. Now $[P:T]=p^a>1$ so we may choose $x\in P-T$ and let m be the order of $x \bmod T$. Let W be a transversal for $\langle T,x\rangle$ in P so $\{x^iw \mid 1\leqq i\leqq m,\ w\in W\}$ is a transversal for T in P. Let $\mu=\prod_{i,w}\lambda^{x^iw}$ and pick $b\in B$. We have

$$\mu(b)=\prod_{i,w}\lambda^{x^iw}(b)=\prod_{i,w}\lambda(x^iwbw^{-1}x^{-i}).$$

For given $w\in W$, let $b_0=wbw^{-1}$. We shall show that $\prod_{i=1}^m\lambda(x^ib_0x^{-i})=1$ and it will follow that $\mu(b)=1$. We must show that $\prod_{i=1}^m x^ib_0x^{-i}\in\ker\lambda$. However,

$$(xb_0x^{-1})(x^2b_0x^{-2})\cdots(x^mb_0x^{-m})=(xb_0)^mx^{-m}.$$

Since P is regular, this equals $u^px^mb_0^mx^{-m}$ where $u\in\langle x,b_0\rangle'\subseteq\langle x,B\rangle'\subseteq B$. Since $\lambda^p=1$ we have

$$\prod_{i=1}^m\lambda(x^ib_0x^{-i})=\lambda(u)^p\lambda(x^mb_0x^{-m})^m=1$$

since $p|m$. Thus $\mu(b)=1$ for all $b\in B$ and so $\mu=1$ and the proof is complete.

It is of interest to note that the use of the assumptions that P is regular and metabelian in Theorem (6.1) has been rather minimal. The assumption of regularity was used to handle the case $p=2$ and then was used twice in proving (10) of (6.4). That P is metabelian was used only once, in proving (10).

7. In this section we prove the two theorems which were stated in the introduction. We assume the situation described there.

(7.1) Theorem B. *Let H be solvable and let $C=G/H$. Suppose that for all primes $p|[C:C']$ that a Sylow p-subgroup of C is regular and metabelian. Then $\mathfrak{X}$ is extendible to G.*

Proof. Let θ, I, $\hat\theta$ and χ be as in §1. By (2.1) we are done if we show that $m_F(\chi)$ is prime to $[G:H]$. By (3.2), this will follow if for every U, $H\subseteq U\subseteq G$ such that U/H is a p-group and $I\cap U/H\subseteq\Phi(U/H)$, we have $m_F(\chi_U)$ is prime to p. If

$p\nmid[C:C']$ then since $I/H\supseteq C'$, we must have $U\subseteq I$ whenever U/H is a p-group. Then $U\cap I=U$ and the condition of (3.2) is vacuously satisfied for such primes because $U/H\nsubseteq\Phi(U/H)$ unless $U/H=1$. Suppose then U/H is a p-group for a prime $p|[C:C']$. Then U/H is contained in a regular metabelian p-group by hypothesis, so U/H satisfies these conditions and by (6.1), condition (*) holds for each of its subgroups. The result now follows by (4.1).

(7.2) THEOREM A. *If H is nilpotent then $\mathfrak{X}$ is extendible to G.*

Proof. First, suppose that H is a p-group. Let ε be a primitive $|H|$ root of unity so $F(\theta)\subseteq F(\varepsilon)$ and $\mathscr{G}(F(\theta)/F)$ is cyclic unless $p=2$ in which case it is a 2-group. If I is as in §1, then G/I is isomorphic with a p'-subgroup of $\mathscr{G}(F(\theta)/F)$ so in any case G/I is cyclic. Let $H\subseteq U\subseteq G$ be such that U/H is a q-group and $U\cap I/H\subseteq\Phi(U/H)$. Now $U/U\cap I$ is cyclic and it follows that U/H is cyclic. Then by (6.3), every subgroup satisfies (*). Thus by (4.1) and (3.2), $m_F(\chi)$ is prime to $[G:H]$.

Now we prove by induction on the number of prime divisors of $|H|$ that $m_F(\chi)$ is prime to $[G:H]$ and the result will follow by (2.1). By the above we may assume that H is not a p-group and write $H=H_0\times H_1$ where H_0 is a Sylow subgroup of H and $|H_1|$ is divisible by fewer primes than is $|H|$. Let θ_0 and θ_1 be irreducible characters of H/H_1 and H/H_0 respectively such that $\theta=\theta_0\theta_1$. If $g\in G$ then $\theta^g=\theta_0^g\theta_1^g=\theta^\sigma=\theta_0^\sigma\theta_1^\sigma$ for some $\sigma\in\mathscr{G}(F(\theta)/F)$ since $F(\theta_i)\subseteq F(\theta)$. By the uniqueness of the decomposition, it follows that each θ_i is F-semi-invariant in G. Let $I_i=\mathscr{I}_G(\theta_i)$. Clearly, $I_0\cap I_1=I=\mathscr{I}_G(\theta)$. Let $\hat{\theta}_i$ be the canonical extension of θ_i to I_i and let $\chi_i=\hat{\theta}_i^G$. By the inductive hypothesis we have $m_F(\chi_i)=m_i$ is prime to $[G:H]$ for $i=0, 1$. Let $\mathfrak{G}_i=\mathscr{G}(F(\theta_i)/F)$ and $\Lambda_i=m_i\sum_{\sigma\in\mathfrak{G}_i}\chi_i^\sigma$ so Λ_i is the character of an F-representation of G. It follows that $\Lambda_0\Lambda_1$ is also the character of an F-representation of G so $m_F(\chi)|[\chi, \Lambda_0\Lambda_1]$. Now

$$\Lambda_0\Lambda_1 = m_0m_1\sum_{\sigma\in\mathfrak{G}_0,\tau\in\mathfrak{G}_1}\chi_0^\sigma\chi_1^\tau.$$

We claim that $[\chi, \chi_0^\sigma\chi_1^\tau]=0$ unless $\sigma=1=\tau$. Suppose $0\neq[\chi, \chi_0^\sigma\chi_1^\tau]=[\hat{\theta}, (\chi_0^\sigma|I)(\chi_1^\tau|I)]$. Thus $0\neq[\theta, (\chi_0^\sigma|H)(\chi_1^\tau|H)]$. Now all irreducible constituents of $\chi_0^\sigma|H$ have H_1 in their kernels and similarly for $\chi_1^\tau|H$ and thus the irreducible constituents of $(\chi_0^\sigma|H)(\chi_1^\tau|H)$ are the products of the irreducible constituents of each factor. Since $\theta=\theta_0\theta_1$ is a constituent of this, we must have $[\theta_0, \chi_0^\sigma|H]\neq 0\neq[\theta_1, \chi_1^\tau|H]$ and by (1.4) this yields $\sigma=1=\tau$. Therefore $m_F(\chi)$ divides $m_0m_1[\chi, \chi_0\chi_1]$ and it suffices to show that $[\chi, \chi_0\chi_1]$ is prime to $[G:H]$.

Now $[\chi, \chi_0\chi_1]=[\hat{\theta}, (\chi_0|I)(\chi_1|I)]$ and since $I_i\supseteq I$, we have $\chi_i|I=\sum_x\hat{\theta}_i^x|I$ where x runs over a transversal for I_i in G. Suppose $\hat{\theta}$ is a constituent of $(\hat{\theta}_0^x|I)(\hat{\theta}_1^y|I)$. By considering degrees we have, $\hat{\theta}=(\hat{\theta}_0^x|I)(\hat{\theta}_1^y|I)$ and hence $\theta=\theta_0^x\theta_1^y$. It follows that $\theta_0=\theta_0^x$ and $\theta_1=\theta_1^y$. Thus $x\in I_0$ and $y\in I_1$ and we have

$$[\hat{\theta}, (\chi_0|I)(\chi_1|I)] = [\hat{\theta}, (\hat{\theta}_0|I)(\hat{\theta}_1|I)].$$

However, $(\hat{\theta}_0|I)(\hat{\theta}_1|I)$ is an extension of $\theta = \theta_0\theta_1$ to I with determinantal order prime to $[G : H]$ so by the uniqueness of $\hat{\theta}$ this yields $\hat{\theta} = (\hat{\theta}_0|I)(\hat{\theta}_1|I)$ and it follows that $[\chi, \chi_0\chi_1] = 1$ and the result follows.

References

1. W. Feit, *Characters of finite groups*, Benjamin, New York, 1967.

2. P. X. Gallagher, *Group characters and normal Hall subgroups*, Nagoya Math. J. **21** (1962), 223–230.

3. G. Glauberman, *Fixed points in groups with operator groups*, Math. Z. **84** (1964), 120–125.

4. B. Huppert, *Endliche Gruppen*, Springer-Verlag, Berlin, 1967.

5. P. Roquette, *Realizerung von Darstellungen endlicher nilpotenter Gruppen*, Arch. Math. **9** (1958), 241–250.

University of Chicago,
Chicago, Illinois